Singularly perturbed elliptic problems with nonautonomous asymptotically linear nonlinearities

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Abstract

We consider a class of singularly perturbed elliptic problems with nonautonomous asymptotically linear nonlinearities. The dependence on the spatial coordinates comes from the presence of a potential and of a function representing a saturation effect. We investigate the existence of nontrivial nonnegative solutions concentrating around local minima of both the potential and of the saturation function. Necessary conditions to locate the possible concentration points are also given.

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1 Introduction

In this paper we study the existence of positive solutions of the problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u^2 = \dfrac{u^3}{1+s(x)u^2} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

for $N \geq 2$, $\varepsilon > 0$ a small parameter and V, $s : \mathbb{R}^N \to \mathbb{R}$ Hölder continuous functions such that

$$s(x) \geq \alpha > 0 \qquad \forall x \in \mathbb{R}^N, \tag{1.1}$$

$$V(x) \geq \mu > 0, \qquad \forall x \in \mathbb{R}^N. \tag{1.2}$$

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It is well known that every positive solutions u_ε of (P_ε) generates a standing wave, i.e. $\phi_\varepsilon(x,t) = u_\varepsilon(x)e^{-iEt/\hbar}$ solution of

$$i\hbar\partial_t\phi + \frac{\hbar^2}{2m}\Delta\phi - W(x)\phi = \frac{\phi^3}{1+s(x)\phi^2} \tag{1.3}$$

for $W = V + E$, $\varepsilon^2 = \hbar/2m$.
Problem (1.3) represents the propagation of a light pulse along a saturable medium. A typical class of saturable medium is constituted by the photorefractive crystals, one of the most preferable materials to observe the propagation of a light beam, because of their slow response to the propagation, making easier the observation. When a beam passes through these materials its refractive index changes so that the light remains confined and solitons are generated. When observing light propagation through these media one can see a *saturation effect*: it is possible to increase the amplitude of the generated solitons by increasing light intensity up to a critical bound characteristic of the material. This kind of interaction is not well represented by the usual Schrödinger equation, so that this model is replaced by (1.3) where the usual autointeraction represented by the cubic power is prevalent for "small" u, while a linear interaction, $u/s(x)$, is predominant for "large" u. Moreover, aiming to analyze the observation through different materials we admit a possible change of the saturation feature in dependence on the spatial coordinates, which may happen observing the propagation along different material.
An interesting and largely studied class of solutions of (P_ε) is the family of semiclassical states, that are families u_ε with a spike shape concentrating around some points of $\mathbb{R}^N$ for ε sufficiently small. There is a broad variety of contributions concerning the existence of this kind of solutions for the equation

$$-\varepsilon^2\Delta u + V(x)u = f(x,u). \tag{1.4}$$

For $f(x,t) = t^3$, the first contribution on the subject in the one dimensional case is due to Floer and Weinstein [10] who show the existence of a solution u_ε concentrating around any given x_0 nondegenerate critical point of $V(x)$. Their result has been extended in higher dimension in [18, 19] for $f(x,t) = |t|^{p-1}t$ with $1 < p < (N+2)/(N-2)$. The common approach used in this papers is a Lyapunov-Schmidt reduction, consisting in a local bifurcation tuype result, which relies on the uniqueness and nondegeneracy of the ground state solution of the autonomous problem

$$-\Delta v + V(x_0)v = f(x_0,v). \tag{1.5}$$

The Lyapunov-Schmidt procedure or more general finite-dimensional reductions methods have been used to find solutions concentrating around any x_0 isolated minimum (or maximum) point with possibly polynomial degeneration of V in [1], and then around different stable critical points (see [14, 11, 20, 2] and the references therein).
A different approach to this is to find a solution u_ε for ε positive and then study its asymptotic behavior for ε tending to zero. This procedure has been firstly used by Rabinowitz in

[22] assuming that $\inf V(x) < \liminf_{|x|\to+\infty} V(x)$ and proving concentration around a local minimum point of V. This philosophy has been improved in [8, 9], where it is shown, by means of a penaliztion argument, that it is sufficient to assure a local condition on the potential: there exists a bounded open set Λ such that

$$\inf_{\Lambda} V < \inf_{\partial\Lambda} V.$$

As for the reduction method also this procedure has been used to extend the existence and concentration result in many different directions (see [9, 4, 6]).
When passing in (1.5) from $f(t) = k(x)t^3$ to $f(x,t) = t^3/(1+s(x)t^2)$ many differences arises. First of all, thanks to (1.1), we do not have a critical exponent as $|f(t)| < t/\alpha$. Moreover, as f is asymptotically linear, the action functional I_ε, defined in

$$\mathbb{H}^1 = \left\{u \in H^1(\mathbb{R}^N) : V(x)u^2 \in L^1(\mathbb{R}^N)\right\}, \tag{1.6}$$

by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}\left[\varepsilon^2|\nabla u(x)|^2 dx + V(x)u^2(x)\right]dx - \int_{\mathbb{R}^N} F(x,u(x))dx,$$

for $F(x,t)$ given by

$$F(x,t) = \frac{1}{2s(x)}t^2 - \frac{1}{2s^2(x)}\ln(1+s(x)t^2), \tag{1.7}$$

may present different geometric behavior in dependence of V and s, e.g. if $V(x)s(x) > 1$ for every $x \in \mathbb{R}^N$, I_ε is always positive, convex and has only a global minimum at $u \equiv 0$. For V and s constant and such that $Vs < 1$ in [26] it is proved the existence of a positive radially symmetric solution which is showed to be unique according to [24, 25]. Regarding the existence of semiclassical states, in [13] it is studied this kind of problem for general autonomous nonlinearity $f(x,t) = f(t)$, asymptotically linear or not, and it is shown the existence of a positive solution u_ε concentrating around a local minimum of V via variational methods and penalization arguments. Here, being interested in the possible interaction between $V(x)$ and $s(x)$, we will deal with the following autonomous, or frozen, problem

$$\begin{cases} -\Delta u + V(y)u = \dfrac{u^3}{1+s(y)u^2} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{S_y}$$

which has a solution if and only if y belongs to the open set (see [26])

$$\Omega = \{y \in \mathbb{R}^N : V(y)s(y) < 1\}. \tag{1.8}$$

Therefore, the set of possible concentration points is restricted from the beginning. As a further consequence, it is not possible to project every $u \in H^1(\mathbb{R}^N)$ on the Nehari manifold

$$\mathcal{N}_y := \left\{u \in H^1 \setminus \{0\}, : \langle I_y'(u), u\rangle = 0\right\}, \tag{1.9}$$

where I_y is the autonomous, or frozen, functional

$$I_y(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} V(y) \int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} F(y, u(x)) dx. \tag{1.10}$$

Nevertheless, the nonlinearity $f(t) = t^3/(1+st^2)$ is such that $f(t)/t$ is increasing w.r.t. t, ensuring the uniqueness of the projecton whenever it exists.
Another effect of the asymptotically linearity property of f is the loss of the well known Ambrosetti-Rabinowitz condition

$$\exists \theta > 2, \text{ such that } \theta F(x,t) \le f(x,t)t.$$

This condition is useful in proving the boundedness of a Palais-Smale sequence. Here, we overcome this difficulty noticing that f satisfies the so-called nonquadraticity condition

$$f(x,t)t - 2F(x,t) \ge 0, \text{ and } \lim_{|t|\to+\infty} f(x,t)t - 2F(x,t) = +\infty. \tag{1.11}$$

where F is the primitive of f (w.r.t. x) such that $F(x,0) = 0$ (see Lemma 3.1).
This condition enables us to show the boundedness of a Cerami sequence. Our main result concerning sufficient conditions is stated in the following result, where we denote with $B(z,r)$ the open ball centered at z with radius r.

Theorem 1.1 *Assume condition* (1.1), (1.2). *Moreover, suppose that there exists* $z \in \Omega$ *and* $r > 0$ *such that either*

$$\begin{cases} V_0 = V(z) = \min\limits_{B(z,r)} V(x) \le \min\limits_{\partial B(z,r)} V(x), \quad s_0 = s(z) = \min\limits_{B(z,r)} s(x) < \min\limits_{\partial B(z,r)} s(x), \\ \qquad\qquad\qquad\qquad \text{or} \\ V_0 = V(z) = \min\limits_{B(z,r)} V(x) < \min\limits_{\partial B(z,r)} V(x), \quad s_0 = s(z) = \min\limits_{B(z,r)} s(x) \le \min\limits_{\partial B(z,r)} s(x). \end{cases} \tag{1.12}$$

Then there exists $\varepsilon_0 > 0$ *such that, for every* $0 < \varepsilon < \varepsilon_0$, *problem* (P_ε) *admits a nontrivial solution* $u_\varepsilon \in \mathbb{H}^1$, $u_\varepsilon \ge 0$, *such that the following facts hold:*

(i) u_ε *admits exactly one global maximum point* $x_\varepsilon \in B(z,r)$;

(ii) $\lim\limits_{\varepsilon\to 0} V(x_\varepsilon) = V_0$ *and* $\lim\limits_{\varepsilon\to 0} s(x_\varepsilon) = s_0$;

(iii) there exist $\mu_1, \mu_2 > 0$ *such that, for every* $x \in \mathbb{R}^N$,

$$u_\varepsilon(x) \le \mu_1 e^{-\mu_2 \frac{|x-x_\varepsilon|}{\varepsilon}}.$$

Notice that, differently from the most studied case $f(x,t) = k(x)|t|^{p-1}t$ (see [27]), here concentration is produced around minimum points of V and S, moreover notice that the strict inequality is needed only on s or V not on both. Then, for example, one between V or s can be constant. We can also prove an abstract concentration result around minimum points of the function $\Sigma : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\Sigma(y) = \begin{cases} \inf\limits_{\mathcal{N}_y} I_y & y \in \Omega, \\ +\infty & y \notin \Omega. \end{cases} \tag{1.13}$$

Thanks to the uniqueness of the ground state solution of the autonomous problem (1.5) Σ is regular in Ω, nevertheless we cannot obtain an explicit formula for Σ because of the lack of homogeneity of the autonomous problem. More precisely, when $f(x,t) = K(x)|t|^{p-1}t$ one can derive every solution of the equation (1.4) via a change of scale, from the unique positive solution of the equation $-\Delta u + u = |u|^{p-1}u$. Here, this procedure cannot work as there are no nontrivial solutions of the problem $-\Delta u + u = u^3/(1+u^2)$, so that we have no hope to find an explicit function of V and s, the critical points of which consituting the concentration set. In Theorem 2.7 it is given a necessary condition for the concentration to occur. Studying Σ we realize that if z is a concentration point, then the gradient of V and s must be linearly dependent as it results in [23] for a different class of problems. Moreover, in our case the gradients $\nabla V(z)$ and $\nabla s(z)$ point in opposite directions and either z is a common zero of $\nabla V(z)$ and $\nabla s(z)$ or $\nabla V(z), \nabla s(z)$ are both different from zero and still $\nabla \Sigma(z) = 0$.

2 Setting of the Problem and Main Results

In order to study (P_ε) it is natural to introduce the Hilbert space $\mathbb{H}^1$, defined in (1.6), with norm $\|u\|^2_{\mathbb{H}^1} = \|u\|^2_{\varepsilon,V}$, given by

$$\|u\|^2_{\varepsilon,V} = \varepsilon^2 \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x) u^2 dx,$$

where we denote with $\|\cdot\|_p$ the standard norm in $L^p = L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$. Thanks to condition (1.1) we can say that the solutions of problem (P_ε) correspond to the critical points of the C^1 functional $I_\varepsilon : H^1 \to \mathbb{R}$ defined by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2_{\varepsilon,V} - \int_{\mathbb{R}^N} F(x, u(x))dx, \tag{2.1}$$

for $F(x,t)$ given in (1.7).
It is easily checked that I_ε is well defined and of class C^1 on $\mathbb{H}^1$. A nontrivial solution of problem (P_ε) is a $u_\varepsilon \neq 0$ in $\mathbb{H}^1$, critical point of I_ε.
For every $y \in \Omega$ (see (1.8)) we can deduce from [26, 24, 25] that there exists a unique, positive, radially symmetric least energy solution, denoted by Q_y, of the autonomous problem frozen in y (S_y). Q_y is a critical point of the autonomous functional I_y, defined in (1.10) and, denoting with $f(y,t) = \partial_t F(y,t)$, notice that $f(y,t)/t$ is an increasing function with respect to t. This monotonicity property is crucial in proving that the Mountain Pass level equals the minimum on the Nehari manifold (see Proposition 3.11 in [22]). This equivalence will be often used in the sequel.
The first sufficient condition for the concentration Notice that, since for every continuous function $k(x)$

$$\inf_{B(z,r)} k(x) \le \min_{\partial B(z,r)} k(x)$$

in the inequality concerning V in the first alternative in (1.12), it is assumed that the infimum of V in $B(z,r)$ is actually achived in z to occur is contained in Theorem 1.1. With this respect the following comments are in order.

Remark 2.1 In (1.12) it is supposed that the minimum values s_0 and V_0 are achieved at z, the center of the ball. This can always be assumed without loss of generality, indeed condition (1.12) implies that s_0 and V_0 are achieved at a point $z_1 \in B(z,r)\cap\Omega$, as $s(z_1)V(z_1) \le s(z)V(z) < 1$. Therefore, if $z_1 \ne z$ we can replace z with z_1 in the statement of the Theorem and in all the changes of variable in Section 3, obtaining concentration around z_1. On the other hand, it would be interesting to study the case in which the concentration occurs in different critical points of V and s.

Remark 2.2 Notice that, since for every continuous function $k(x)$

$$\inf_{B(z,r)} k(x) \le \min_{\partial B(z,r)} k(x)$$

in the inequality concerning V in the first alternative in (1.12), it is assumed that the infimum of V in $B(z,r)$ is actually achived in z and it may be equal to the minimum of V on the boundary (analogous considerations hold for s in the second alternativ in (1.1)).

We can also prove the following general abstract result.

Theorem 2.3 *Assume condition* (1.1), (1.2). *Moreover, suppose that there exists* $z \in \Omega$ *and* $r > 0$ *such that*

$$\Sigma_0 = \Sigma(z) = \min_{B(z,r)} \Sigma(x) < \min_{\partial B(z,r)} \Sigma(x). \tag{2.2}$$

Then there exists $\varepsilon_0 > 0$ *such that, for every* $0 < \varepsilon < \varepsilon_0$, *problem* (P_ε) *admits a nontrivial solution* $u_\varepsilon \in \mathbb{H}^1$, $u_\varepsilon \ge 0$, *such that the following facts hold:*

(i) u_ε *admits exactly one global maximum point* $x_\varepsilon \in B(z,r)$;

(ii) $\lim_{\varepsilon\to 0} \Sigma(x_\varepsilon) = \Sigma_0$;

(iii) there exist $\mu_1, \mu_2 > 0$ *such that, for every* $x \in \mathbb{R}^N$,

$$u_\varepsilon(x) \le \mu_1 e^{-\mu_2 \frac{|x-x_\varepsilon|}{\varepsilon}}.$$

Remark 2.4 In this abstract result

$$\Sigma_0 = \Sigma(z) = I_z(Q_z),$$

where Q_z is the unique positive least energy critical point (see [26, 24, 25]) of I_z defined in (1.10)

V_0 and s_0 are just the value of the functions $s(x)$ and $V(x)$ on z, and they are not in general related with the minimum values of $s(x)$ and $V(x)$, because it is actually the minimum point of the function Σ that plays the fundamental role. In Theorem 1.1 we have seen that, in the particular case in which V and s attain their minimum in the same point, then this point will be a minimum of Σ. But, in general, this could not be the case, and still we may find a minimum point of Σ.

Remark 2.5 In order to find a sufficient condition in terms of an explicit concentration function, instead of Σ, it is usually crucial to find a change of variable, from the frozen problem to the problem with with all the constants equal to one, that is in this case from (S_y) to

$$\begin{cases} -\Delta u + u^2 = \dfrac{u^3}{1+u^2} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Unfortunately, this procedure is feasible in this context. Since doing that we move from a problem which admits solutions to one which has not any nontrivial solution. This implies that we cannot express a solution as a member of a two-parameters family generating by a fundamental solution as in the most studied case [27, 2, 6]

With respect to the topic of locating the possible concentration points, let us first introduce the concentration set.

Definition 2.6 *The concentration set* $\mathscr{E}$ *for problem* (P_ε)*, is defined by*

$$\mathscr{E} = \Big\{ z \in \mathbb{R}^N \text{ such that there exists a sequence of solutions } \{u_\varepsilon\} \in \mathbb{H}^1 \text{ of } (P_\varepsilon) \text{ with } u_\varepsilon(z+\varepsilon x) \to 0 \text{ as } |x| \to \infty \text{ uniformly w.r.t. } \varepsilon \text{ and } \varepsilon^{-N} J_\varepsilon(u_\varepsilon) \to \Sigma(z) \text{ as } \varepsilon \to 0 \Big\},$$

where Σ *is defined in* (1.13).

We will prove the following result concerning necessary conditions for the concentration to occur.

Theorem 2.7 *Assume* (1.1), (1.2) *and that* $V, s \in C^1(\mathbb{R}^N)$ *such that there exist* $\beta > 0$, $\gamma \geq 0$, *satisfying*

$$|\nabla V| \leq \beta e^{\gamma |x|} \quad \text{and} \quad |\nabla s| \leq \beta e^{\gamma |x|} \quad \forall\, x \in \mathbb{R}^N. \tag{2.3}$$

Then Σ *is of class* $C^1(\Omega)$ *and if* $z \in \mathscr{E}$ *the following facts hold:*
(i) $\nabla V(z)$ *and* $\nabla s(z)$ *are linearly dependent and point in opposite directions.*
(ii) Either $\partial_j V(z) = \partial_j s(z) = 0$ *for every* $j = 1, \dots, N$ *or there exists at least a* $j_0 \in \{1, \dots, N\}$ *such that* $\partial_{j_0} V(z), \partial_{j_0} s(z) \neq 0$ *with still* $\partial_j \Sigma(z) = 0$ *for every* $j = 1, \dots, N$.

Remark 2.8 We can say a little bit more in the last conclusion of the above Theorem. Indeed, as a consequence of conclusion *(i)*, every nontrivial partial derivative of V and s satisfies a precise identity (see for more details Remark 4.4).

3 Proofs of Theorems 1.1 and 2.3

In this section we will prove Theorems 1.1 and 2.3 using the well known penalization procedure introduced in [8, 9]. Let us deal first with the proof of Theorem 1.1.
Recalling that the derivative of $F(x,t)$ with respect to t is given by $f(x,t) = t^3/(1+s(x)t^2)$, fix $0 < \nu < 1/2$ and define the function $\overline{f}(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(x,t) = \begin{cases} \min\{f(x,t), \nu\mu t\} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{3.1}$$

Let $0 < r' < r$ such that

$$\begin{cases} V_0 = V(z) = \min\limits_{B(z,r')} V(x) \leq \min\limits_{\partial B(z,r')} V(x), \quad s_0 = s(z) = \min\limits_{B(z,r')} s(x) < \min\limits_{\partial B(z,r')} s(x), \\ \text{or} \\ V_0 = V(z) = \min\limits_{B(z,r')} V(x) < \min\limits_{\partial B(z,r')} V(x), \quad s_0 = s(z) = \min\limits_{B(z,r')} s(x) \leq \min\limits_{\partial B(z,r')} s(x). \end{cases} \tag{3.2}$$

Note that the existence of r' is a consequence of the continuity of the functions s, V in $\mathbb{R}^N$ and of (1.12). Indeed, let us prove the existence of r' in the case in which the first assumption in (1.12) is satisfied. Arguing by contradiction, it follows that for any $\rho < r$ it holds

$$\inf_{B(z,\rho)} V(x) > \min_{\partial B(z,\rho)} V(x) \text{ or } \inf_{B(z,\rho)} s(x) \geq \min_{\partial B(z,\rho)} s(x).$$

Since, for every continuous function it holds

$$\inf_{B(z,\rho)} k(x) = \min_{\overline{B(z,\rho)}} k(x) \leq \min_{\partial B(z,\rho)} k(x)$$

the first inequality for V cannot be true and we can reduce to the case

$$\inf_{B(z,\rho)} s(x) = \min_{\partial B(z,\rho)} s(x) \qquad \forall \rho < r.$$

Now, let $\{\rho_n\}$ be an increasing sequence such that $\rho_n \to r$. Then we can write

$$\inf_{B(z,\rho_n)} s(x) = \min_{\overline{B(z,\rho_n)}} s(x) = \min_{\partial B(z,\rho_n)} s(x) = s(p_n)$$

with $p_n \in \partial B(z,\rho_n)$.
As $\partial B(z,\rho_n) \subset \overline{B(z,r)}$ and $\rho_n \to r$, it results, up to a subsequence, $p_n \to p \in \partial B(z,r)$ and, passing to the limit,

$$\min_{\overline{B(z,r)}} s(x) = \lim_{n\to+\infty} \min_{\overline{B(z,\rho_n)}} s(x) = \lim_{n\to+\infty} s(p_n) = s(p) \geq \min_{\partial B(z,r)} s(x)$$

and this contradicts (1.12). This proves the existence of $r' < r$ such that (3.2) holds.

Let $\chi \in C^\infty(\mathbb{R}^N)$ be such that

$$\chi(x) = 1 \quad \forall x \in B(z,r'), \quad \chi(x) = 0 \quad \forall x \in \mathbb{R}^N \setminus \overline{B(z,r)}, \tag{3.3}$$

and set

$$g(x,t) = \chi(x) f(x,t) + (1-\chi(x))\overline{f}(x,t),$$

for a.e. $x \in \mathbb{R}^N$ and any $t \in \mathbb{R}$. Having defined $G(x,t) = \int_0^t g(x,\xi)d\xi$, in the light of the above definition, the following result follows.

Lemma 3.1 *Assume conditions* (1.12)*. Then, the following conditions hold for every t in $\mathbb{R}$ and for almost every x in $\mathbb{R}^N$*

$$\lim_{t\to 0} \frac{g(x,t)}{t} = 0, \quad \textit{uniformly in } x \in \mathbb{R}^N \tag{3.4}$$

$$g(x,t)t - 2G(x,t) \geq 0, \ \lim_{|t|\to\infty} g(x,t)t - 2G(x,t) = +\infty, \qquad \forall x \in B(z,r), \tag{3.5}$$

$$0 \leq 2G(x,t) \leq g(x,t)t \leq \nu\mu t^2 \qquad \forall x \notin \overline{B(z,r)} \tag{3.6}$$

Proof. For t going to zero, $\overline{f}(x,t) = f(x,t)$ and from (1.1) it results

$$\frac{g(x,t)}{t} = \frac{f(x,t)}{t} = \frac{t^2}{1+s(x)t^2} \leq t^2,$$

implying (3.4).
In order to show (3.5) and (3.6), let us first show that (1.11) holds because

$$\frac{1}{2} f(x,t)t - F(x,t) = \frac{1}{2s^2(x)}\left[\ln(1+s(x)t^2) - \frac{s(x)t^2}{1+s(x)t^2}\right].$$

Then (3.5) easily follows studying the function $h(t) = \ln(1+t) - t/(1+t)$.
Now, if $x \in B(z,r)$ then

$$g(x,t)t - 2G(x,t) \geq \chi(x)\left[f(x,t)t - 2F(x,t)\right]$$

so that, for every $x \in B(z,r)$, (1.11) implies (3.5) being $\chi(x) > 0$.
For $x \notin \overline{B(z,r)}$, $g(x,t) = \overline{f}(x,t)$ and

$$G(x,t) = \begin{cases} F(x,t) & \text{if } f(x,t) \leq \nu\mu t \\ \dfrac{\nu\mu}{2} t^2 & \text{if } f(x,t) > \nu\mu t \end{cases}$$

and (3.6) easily follows. ∎

The following easy technical lemma will be useful in the sequel.

Lemma 3.2 *The following facts hold:*
i) For any $2 \le q \le 4$, there exists $C = C(q)$ such that $t^2 - \ln(1+t^2) \le C|t|^q$, in $\mathbb{R}$.
ii) $t^2/(1+st) \le C|t|$, for all t in $\mathbb{R}$ and for every $s \in \mathbb{R}^+$.
iii) For every $L \ge 0$, the real function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ defined $h(s) = \dfrac{L}{s} - \dfrac{1}{s^2}\ln(1+Ls)$ is monotone decreasing.

Proof. The proof can be shown by direct calculations. ■

We will study the penalized functional $J_\varepsilon : \mathbb{H}^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_{\varepsilon,V} - \int_{\mathbb{R}^N} G(x, u(x))\,dx \tag{3.7}$$

whose critical points are solutions of the problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(x, u(x)) & \text{in } \mathbb{R}^N, \\ u \in \mathbb{H}^1(\mathbb{R}^N). \end{cases} \tag{3.8}$$

In the study of asymptotically linear problems the usual Palais-Smale condition is substituted by the following Cerami condition introduced in [7].

Definition 3.3 *Let E be a Banach space. A sequence $\{u_n\} \subset E$ is said to be a Cerami sequence for a functional I, $(Ce)_c$ for short, if*

$$I(u_n) \to c, \qquad (1 + \|u_n\|_E)\left\|I'(u_n)\right\|_{E^*} \to 0. \tag{3.9}$$

Moreover, a functional $I \in C^1(E, \mathbb{R})$ is said to satisfy the Cerami condition $(Ce)_c$ if any Cerami sequence possesses a convergent subsequence.

In the next lemma we prove that J_ε satisfies the Cerami condition.

Lemma 3.4 *Assume conditions* (1.1), (1.2) *and* (1.12). *Then, for every $\varepsilon > 0$ fixed, every Cerami sequence for J_ε admits a convergent subsequence.*

Proof. Let us take $\{u_n\}$ a Cerami sequence and let us first prove that $\{u_n\}$ is bounded by contradiction. Assume then, up to a subsequence,

$$\|u_n\|_{\varepsilon,V} \to \infty, \quad J_\varepsilon(u_n) \to c_\varepsilon, \quad \|J'_\varepsilon(u_n)\|_{(\mathbb{H}^1(\mathbb{R}^N))^*}\|u_n\|_{\varepsilon,V} < \frac{1}{n}. \tag{3.10}$$

Arguing as in Lemma 3.30 of [16] it is possible to obtain the following inequalities for every t

$$\int_{\mathbb{R}^N} \frac{1}{2} g(x, u_n(x))u_n(x) - G(x, u_n(x)) \le J_\varepsilon(u_n) + o(1) \le c_\varepsilon + o(1), \tag{3.11}$$

$$\int_{\mathbb{R}^N} G(x, t_n u_n(x)) \ge \frac{t^2}{2}\|u_n\|^2_{\varepsilon,V} - \varepsilon^N c_\varepsilon + o(1), \tag{3.12}$$

$$J_\varepsilon(tu_n) \le J_\varepsilon(u_n) + o(1), \tag{3.13}$$

with $o(1) \to 0$ as $n \to \infty$.
Let us define the function $\phi_n(x) = u_n(z+\varepsilon x)$ and notice that ϕ_n belongs to the Hilbert space

$$\mathbb{H}_{\varepsilon,z} := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(z+\varepsilon x) v^2(x)\,dx < +\infty \right\} \tag{3.14}$$

with norm

$$\|v\|^2_{\varepsilon,z} := \|\nabla v\|_2^2 + \int_{\mathbb{R}^N} V(z+\varepsilon x) v^2(x)\,dx.$$

This class of spaces has been used in [13] and in the rest of this proof (and in the proof of Lemma 3.7) we will adopt some of their arguments. First, notice that proving the result is equivalent to show that there exists a positive constant C possibly depending on ε, satisfying $\|\phi_n\|^2_{\varepsilon,z} \le C$ for every ε fixed, as it results

$$\|\phi_n\|^2_{\varepsilon,z} = \int_{\mathbb{R}^N} \left[\varepsilon^2 |\nabla u_n(z+\varepsilon x)|^2 + V(z+\varepsilon x) u_n^2(z+\varepsilon x)\right] dx = \frac{\|u_n\|^2_{\varepsilon,V}}{\varepsilon^N}. \tag{3.15}$$

Let us study (3.12) in terms of ϕ_n. Since

$$\int_{\mathbb{R}^N} G(x, t u_n(x))\,dx = \varepsilon^N \int_{\mathbb{R}^N} G(z+\varepsilon\xi, t u_n(z+\varepsilon\xi))\,d\xi = \varepsilon^N \int_{\mathbb{R}^N} G(z+\varepsilon\xi, t\phi_n(\xi))\,d\xi,$$

the sequence ϕ_n satisfies the following inequality

$$\int_{\mathbb{R}^N} G(z+\varepsilon\xi, t\phi_n(\xi))\,d\xi \ge \frac{t^2}{2}\|\phi_n\|^2_{\varepsilon,z} - c_\varepsilon + o(1),$$

and choosing $t = t_n = 2\sqrt{c_\varepsilon}/\|\phi_n\|_{\varepsilon,z}$, it follows

$$\int_{\mathbb{R}^N} G(z+\varepsilon\xi, t_n\phi_n(\xi))\,d\xi \ge c_\varepsilon + o(1).$$

Now, we can argue by contradiction, supposing that, up to a subsequence, $\|\phi_n\|_{\varepsilon,z} \to +\infty$ and defining the sequence $\psi_n = t_n\phi_n$, which verifies $\|\psi_n\|_{\varepsilon,z} = 2\sqrt{c_\varepsilon}$, to obtain $\psi \in H^1$ such that ψ_n converges to ψ weakly in H^1, strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ $\forall\, p \in [1, 2^*)$, and almost everywhere. We claim that

$$\limsup_{n\to 0} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\chi(z+\varepsilon x)\psi_n(x)|^2\,dx > 0, \tag{3.16}$$

where χ is introduced in (3.3). By contradiction, if (3.16) were false, it would result

$$\lim_{n\to 0} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\chi(z+\varepsilon x)\psi_n(x)|^2\,dx = 0,$$

then, using the argument of Lemma I.1 in [15], we deduce that, for $2 < p < 2^*$, the sequence $\chi(z+\varepsilon x)\psi_n$ converges to 0 strongly in $L^p(\mathbb{R}^N)$. Let us fix $q \in (2+2/N, \overline{q})$, with

$\overline{q} = \min\{4, N/(N-2)+1\}$ and apply conclusion *i)* in Lemma 3.2, Hölder inequality and (3.3) to obtain for every $L \geq 1$

$$\begin{aligned}(3.17)\qquad \int_{\mathbb{R}^N} |\chi(z+\varepsilon x)F(z+\varepsilon x, L\psi_n(x))|dx &\leq CL^q \int_{\mathbb{R}^N} \chi(z+\varepsilon x)|\psi_n(x)|^q dx \\ &\leq CL^q \left[\int_{\mathbb{R}^N} \chi^p(z+\varepsilon x)|\psi_n|^p\right]^{1/p} \|\psi_n\|_{p'(q-1)}^{(q-1)},\end{aligned}$$

where $p' = p/(p-1)$. Since $q \in (2+2/N, \overline{q})$, $p'(q-1) \in (2, 2^*)$, so that the last integral is uniformly bounded, implying that $\chi(z+\varepsilon x)F(z+\varepsilon x, L\psi_n(x))$ converges to zero in $L^1(\mathbb{R}^N)$. Moreover, for every $L \geq 1$, from (3.1) and (1.2) it follows

$$\int_{\mathbb{R}^N} (1-\chi(z+\varepsilon x))\overline{F}(z+\varepsilon x, L\psi_n) \leq L^2 \frac{\nu\mu}{2} \int_{\mathbb{R}^N} |\psi_n|^2 \leq \frac{\nu L^2}{2} \|\psi_n\|_{\varepsilon,z}^2 \leq c_\varepsilon L^2.$$

Therefore for every $L \geq 1$

$$(3.18)\qquad \tilde{J}_\varepsilon\left(L\psi_n\right) \geq c_\varepsilon L^2 + \mathrm{o}(1),$$

where $\tilde{J}_\varepsilon : \mathbb{H}_{\varepsilon,z} \to \mathbb{R}$ is defined by

$$(3.19)\qquad \tilde{J}_\varepsilon(v) := \frac{1}{2}\|v\|_{\varepsilon,z}^2 - \int_{\mathbb{R}^N} G(z+\varepsilon x, v(x))dx.$$

On the other hand, from (3.13) we deduce that

$$\tilde{J}_\varepsilon(L\psi_n) = \tilde{J}_\varepsilon\left(Lt_n\phi_n\right) = \frac{1}{\varepsilon^N} J_\varepsilon(Lt_n u_n) \leq \frac{1}{\varepsilon^N}\left[J_\varepsilon(u_n) + \mathrm{o}(1)\right] \leq \frac{1}{\varepsilon^N}\left[c_\varepsilon + \mathrm{o}(1)\right].$$

This together with (3.18) produce a contradiction, yielding (3.16).
As in [13], this implies the existence of a number $\gamma > 0$, and of a sequence $\{y_n\}$ with $B(y_n, 1) \cap \mathrm{supp}\,\chi(z+\varepsilon\cdot) \neq \emptyset$ and such that

$$(3.20)\qquad \lim_{n\to+\infty} \int_{B(y_n,1)} |\chi(z+\varepsilon x)\psi_n(x)|^2 dx > 0.$$

Since $B(y_n, 1) \cap \mathrm{supp}\,\chi(z+\varepsilon x) \neq \emptyset$, (3.3) implies that there exists a η satisfying $|y_n - \eta| < 1$ and $|z+\varepsilon\eta - z| < r$, so that $|\varepsilon y_n| \leq \varepsilon|y_n - \eta| + \varepsilon|\eta| < \varepsilon + r$ and we can find x_0 such that

$$(3.21)\qquad \varepsilon y_n \to x_0 \in \overline{B(0, r+\varepsilon)}.$$

Let us now define the functions

$$\overline{\psi}_n(x) = \psi_n\left(y_n + x\right), \qquad \overline{\chi}_n(x) = \chi\left(z + \varepsilon\left(y_n + x\right)\right),$$

and observe that, as $\overline{\psi}_n$ is uniformly bounded in $H^1(\mathbb{R}^N)$, it exists $\overline{\psi} \in H^1(\mathbb{R}^N)$ such that $\overline{\psi}_n$ converges to $\overline{\psi}$ weakly in $H^1(\mathbb{R}^N)$, almost everywhere and strongly in $L^2(B(0,1))$.

Moreover, from (3.21) we deduce that $\overline{\psi}_n(x)\overline{\chi}_n(x) \to \overline{\psi}(x)\chi(z+x_0+\varepsilon x)$ almost everywhere, then (3.20) yields

$$0 < \lim_{n\to\infty}\int_{B(y_n,1)} |\chi(z+\varepsilon\xi)\psi_n(\xi)|^2 dx = \lim_{n\to\infty}\int_{B(0,1)} \left|\psi_n\big(y_n+x\big)\chi\big(z+\varepsilon\big(y_n+x\big)\big)\right|^2 dx$$
$$= \lim_{n\to\infty}\int_{B(0,1)} |\overline{\psi}_n(x)\overline{\chi}_n(x)|^2 dx = \int_{B(0,1)} \chi^2(z+x_0+\varepsilon x)|\overline{\psi}(x)|^2 dx$$

which implies that there exists an open set $A \subset B(0,1)$ such that for every $x \in A$ it holds $|\overline{\psi}(x)| > 0$ and $\chi(z+x_0+\varepsilon x) > 0$. Moreover, it results

$$0 < |\overline{\psi}(x)| = \lim_{n\to\infty}\left|\psi_n\left(y_n+x\right)\right| = 2\sqrt{c_\varepsilon}\lim_{n\to\infty}\frac{\left|\phi_n\left(y_n+x\right)\right|}{\|\phi_n\|_{\varepsilon,z}}$$

then, for every $x \in A$, we have that $|\phi_n(y_n+x)| \to +\infty$ and as $z+\varepsilon(y_n+x) \to z+x_0+\varepsilon x$, with $\chi(z+x_0+\varepsilon x) > 0$, (3.3) yields the existence of n_0 such that, for $n \geq n_0$, $z+\varepsilon(y_n+x) \in B(z,r)$. Then, (3.5) and (3.6) give

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{1}{2}g(z+\varepsilon(y_n+x),\phi_n(y_n+x))\phi_n(y_n+x) - G(z+\varepsilon(y_n+x),\phi_n(y_n+x)) \geq$$
$$\lim_{n\to\infty}\int_A \frac{1}{2}g(z+\varepsilon(y_n+x),\phi_n(y_n+x))\phi_n(y_n+x) - G(z+\varepsilon(y_n+x),\phi_n(y_n+x)) = +\infty$$

But, on the other hand, performing the change of variable $z+\varepsilon(y_n+x) = \xi$, and using (3.11), one derives the desired contradiction. ∎

Remark 3.5 In the proof of the above lemma we use assumption (1.12) only to define the penalization with the function $g(x,t)$ satisfying condition (3.5), (3.6).

Lemma 3.6 *Assume* (1.1), (1.2) *and* (1.12). *Then there exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0,\varepsilon_0)$ J_ε *has a nontrivial critical point* u_ε *satisfying*

$$J_\varepsilon(u_\varepsilon) \leq \varepsilon^N(\Sigma(z) + \mathrm{o}(1)), \tag{3.22}$$

where $\mathrm{o}(1) \to 0$ *as* $\varepsilon \to 0$.

Proof. We will obtain the existence of u_ε by applying the variant of the Mountain Pass Lemma with the Cerami condition (see [7, 3]) to the functional J_ε. Let us first notice that condition (3.4) immediately implies that $v_0 = 0$ is a strict local minimum. In order to show the existence of v^* such that $J_\varepsilon(v^*) < 0$, let us observe that, arguing as in Lemma 2.1 [12], we find w^* such that $I_z(w^*) < 0$, for $I_z(v)$ defined in (1.10). Let us choose $v_\varepsilon^*(x) = \eta(x)w^*\left(\frac{x-z}{\varepsilon}\right)$, with $\eta(x)$ a smooth function compactly supported in $\mathbb{R}^N$ and $\eta(x) \equiv 1$ in $B(z,r)$. Computing $J_\varepsilon(v_\varepsilon^*)$ gives

$$J_\varepsilon(v_\varepsilon^*) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^N}\left\{|\nabla\eta|^2|w^*\big(\frac{x-z}{\varepsilon}\big)|^2 + \frac{2}{\varepsilon}\eta w^*\big(\frac{x-z}{\varepsilon}\big)\nabla\eta\nabla w^*\big(\frac{x-z}{\varepsilon}\big) + \frac{1}{\varepsilon^2}\eta^2|\nabla w^*\big(\frac{x-z}{\varepsilon}\big)|^2\right\}dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^N} V(x)\eta^2|w^*\big(\frac{x-z}{\varepsilon}\big)|^2 dx - \int_{\mathbb{R}^N} G\left(x,\eta w^*\big(\frac{x-z}{\varepsilon}\big)\right)dx,$$

and performing the change of variable $y = (x - z)/\varepsilon$, using the properties of the function η, one gets

$$J_\varepsilon(v_\varepsilon^*) = \varepsilon^N \int_{\mathbb{R}^N} \left[\frac{1}{2}\eta^2(z+\varepsilon y)\big(|\nabla w^*|^2 + V(z+\varepsilon y)|w^*|^2\big) - G\big(z+\varepsilon y, \eta(z+\varepsilon y) w^*\big)\right] + \mathrm{o}\big(\varepsilon^N\big),$$

where $\mathrm{o}(\varepsilon^N)/\varepsilon^N \to 0$ as ε goes to zero. Since the above integral uniformly converges, as ε goes to zero, to $I_z(w^*) < 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $J_\varepsilon(v_\varepsilon^*) < 0$, giving the desired conclusion.

The geometric behavior just observed yields the construction of a Cerami sequence $\{u_n\}$ of J_ε at the Mountain Pass level c_ε, defined by

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{[0,1]} J_\varepsilon(\gamma(t)), \quad \Gamma_\varepsilon = \big\{\gamma \in C([0,1], \mathbb{H}^1(\mathbb{R}^N)),\ \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\big\}. \tag{3.23}$$

Then, Lemma 3.4 allows to pass to the limit and obtain a critical point u_ε with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. In order to show (3.22), we will argue as in Proposition 6.1 in [13]. From Lemma 2.1 in [12] we deduce the existence of a path $\gamma \in C([0,1], H^1(\mathbb{R}^N))$ such that

$$\gamma(0) = 0, \quad I_z(\gamma(1)) < 0, \quad I_z(\gamma(t)) \le \Sigma(z), \quad \max_{t\in[0,1]} I_z(\gamma(t)) = I_z(Q_z) = \Sigma(z), \tag{3.24}$$

where Q_z is the unique positive solution of (S_y) with $y = z$ (see Section 2) and $\Sigma(z)$ is defined in (1.13).

Let us consider a function $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\eta(0) = 1$ and $0 \le \eta(x) \le 1$. First, notice that there exists R_0 such that

$$I_z(\eta(z/R)\gamma(1)) < 0, \quad \forall R \ge R_0 \quad \text{and} \quad \lim_{R\to+\infty} \max_{t\in[0,1]} I_z(\eta(z/R)\gamma(t)) = \Sigma(z) \tag{3.25}$$

Then we define the path

$$\gamma_{R,\varepsilon}(t)(y) = \eta\left(\frac{y}{R}\right)\gamma(t)\left(\frac{y-z}{\varepsilon}\right)$$

so that $\gamma_{R,\varepsilon}(t) : [0,1] \to H^1(\mathbb{R}^N)$. Since $J_\varepsilon\big(\gamma_{R,\varepsilon}(t)\big)/\varepsilon^N$ converges to $I_z\big(\eta(z/R)\gamma(t)\big)$ as ε goes to zero, uniformly with respect to $t \in [0,1]$, (3.25) implies that $\gamma_{R,\varepsilon} \in \Gamma_\varepsilon$. Moreover, from (3.25) it follows

$$c_\varepsilon \le \max_{[0,1]} \frac{J_\varepsilon\big(\gamma_{R,\varepsilon}(t)\big)}{\varepsilon^N} = \max_{[0,1]} I_z\big(\eta(z/R)\gamma(t)\big) + \mathrm{o}(1) \le \Sigma(z) + \mathrm{o}(1),$$

implying (3.22). ■

Lemma 3.7 *There exists a positive constant L and* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$

$$\|u_\varepsilon\|_{\varepsilon,V}^2 \le L\varepsilon^N. \tag{3.26}$$

Proof. We will follow the argument of Lemma 3.4 paying attention to the fact that now ε is not fixed. Arguing as in Lemma 3.30 of [16] it is possible to obtain the following inequalities for every t

$$J_\varepsilon(tu_\varepsilon) \le J_\varepsilon(u_\varepsilon), \quad \int_{\mathbb{R}^N} G(x, tu_\varepsilon(x)) \ge \frac{t^2}{2}\|u_\varepsilon\|_\varepsilon^2 - \varepsilon^N(\Sigma(z) + \mathrm{o}(1)). \tag{3.27}$$

Introducing the function $\phi_\varepsilon(x) = u_\varepsilon(z+\varepsilon x)$ belonging to $\mathbb{H}_{\varepsilon,z}$, defined in (3.14), notice that, (3.15) tells us that proving (3.26) is equivalent to show that there exists $\varepsilon_0 > 0$ such that $\|\phi_\varepsilon\|_{\varepsilon,z}^2 \le L$ for every $\varepsilon \in (0, \varepsilon_0)$.
As in Lemma 3.4 we obtain that the sequence ϕ_ε satisfies the following inequality

$$\int_{\mathbb{R}^N} G(z+\varepsilon\xi, t_\varepsilon\phi_\varepsilon(\xi))\,d\xi \ge \Sigma(z) + \mathrm{o}(1), \tag{3.28}$$

with $t_\varepsilon = 2\sqrt{\Sigma(z)}/\|\phi_\varepsilon\|_{\varepsilon,z}$. Arguing again by contradiction and supposing that, up to a subsequence, $\|\phi_\varepsilon\|_{\varepsilon,z} \to +\infty$, we set $\psi_\varepsilon = t_\varepsilon\phi_\varepsilon$, which verifies $\|\psi_\varepsilon\|_{\varepsilon,z} = 2\sqrt{\Sigma(z)}$. Then ψ_ε converges weakly in H^1 strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $\forall\, p \in [1, 2^*)$, almost everywhere and, as in Lemma 3.4, it satisfies

$$\limsup_{\varepsilon\to 0}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)} |\chi(z+\varepsilon x)\psi_\varepsilon(x)|^2\,dx > 0. \tag{3.29}$$

This implies the existence of a sequence $\{y_\varepsilon\}$ with $B(y_\varepsilon, 1) \cap \operatorname{supp}\chi(z+\varepsilon\cdot) \neq \emptyset$, and such that

$$\lim_{\varepsilon\to 0}\int_{B(y_\varepsilon,1)} |\chi_\varepsilon(z+\varepsilon x)\psi_\varepsilon(x)|^2\,dx > 0. \tag{3.30}$$

Moreover, from (3.3), it follows that $\varepsilon y_\varepsilon \in \{\eta : |\eta| < \varepsilon + r\}$ so that

$$\varepsilon y_\varepsilon \to x_0 \in \overline{B(0,r)}. \tag{3.31}$$

In this case, the functions

$$\overline{\psi}_\varepsilon(x) = \psi_\varepsilon\left(y_\varepsilon + x\right), \qquad \overline{\chi}_\varepsilon(x) = \chi\left(z + \varepsilon\left(y_\varepsilon + x\right)\right)$$

are such that $\overline{\psi}_\varepsilon$ converges to $\overline{\psi}$ weakly in $H^1(\mathbb{R}^N)$, almost everywhere and strongly in $L^2(B(0,1))$. Moreover, from (3.31) we deduce that $\overline{\psi}_\varepsilon(x)\overline{\chi}_\varepsilon(x) \to \overline{\psi}(x)\chi(z+x_0)$ almost everywhere, then (3.30) yields

$$0 < \lim_{\varepsilon\to 0}\int_{B(0,1)} |\overline{\psi}_\varepsilon(x)\overline{\chi}_\varepsilon(x)|^2\,dx = \chi^2(z+x_0)\int_{B(0,1)} |\overline{\psi}(x)|^2\,dx.$$

Then $\chi(z+x_0) > 0$ implying that $x_0 \in B(0,r)$ and there exists an open set $A \subset B(0,1)$ such that for every $x \in A$, $|\overline{\psi}(x)| > 0$. Moreover, it results

$$0 < |\overline{\psi}(x)| = \lim_{\varepsilon\to 0}\left|\psi_\varepsilon\left(y_\varepsilon + x\right)\right| = 4\Sigma(z)\lim_{\varepsilon\to 0}\frac{\left|\phi_\varepsilon\left(y_\varepsilon + x\right)\right|}{\|\phi_\varepsilon\|_{\varepsilon,z}}$$

so that, for every $x \in A \subset B(0,1)$, $|\phi_\varepsilon(y_\varepsilon + x)| \to +\infty$. Moreover, as $z + \varepsilon(y_\varepsilon + x) \to z + x_0 \in B(z,r)$ we can deduce that, for ε sufficiently small, $z + \varepsilon(y_\varepsilon + x) \in B(z,r)$, so that (3.5) and (3.6) give

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \frac{1}{2} g(z+\varepsilon(y_\varepsilon+x), \phi_\varepsilon(y_\varepsilon+x))\phi_\varepsilon(y_\varepsilon+x) - G(z+\varepsilon(y_\varepsilon+x), \phi_\varepsilon(y_\varepsilon+x))\,dx \geq$$
$$\lim_{\varepsilon\to 0}\int_{A} \frac{1}{2} g(z+\varepsilon(y_\varepsilon+x), \phi_\varepsilon(y_\varepsilon+x))\phi_\varepsilon(y_\varepsilon+x) - G(z+\varepsilon(y_\varepsilon+x), \phi_\varepsilon(y_\varepsilon+x))\,dx = +\infty.$$

But, on the other hand, it results

$$\int_{\mathbb{R}^N}\left[\frac{1}{2} g(z+\varepsilon(y_\varepsilon+x), \phi_\varepsilon(y_\varepsilon+x))\phi_\varepsilon(y_\varepsilon+x) - G(z+\varepsilon\xi_\varepsilon, \phi_\varepsilon(y_\varepsilon+x))\right]dx = \tilde{J}_\varepsilon(\phi_\varepsilon)$$

which is uniformly bounded because of (3.22). ■

Proposition 3.8 *Assume* (1.1), (1.2) *and* (1.12). *Then for every* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that*

$$\sup_{0<\varepsilon<\varepsilon_\delta}\ \sup_{x\in\mathbb{R}^N\setminus B(z,r)} u_\varepsilon(x) < \delta. \tag{3.32}$$

Proof. Let us first prove that

$$\lim_{\varepsilon\to 0}\ \sup_{x\in\partial B(z,r)} u_\varepsilon(x) = 0. \tag{3.33}$$

We proceed by contradiction, assuming that there exist a sequence $\{\varepsilon_n\}$ converging to 0 and a sequence $\{x_n\} \subset \partial B(z,r)$ such that, for some positive constant β,

$$u_{\varepsilon_n}(x_n) \geq \beta \qquad \text{for all } n \geq 1. \tag{3.34}$$

Since $\partial B(z,r)$ is a compact set, we can assume that there exists a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, which converges to a point $x_0 \in \partial B(z,r)$. Consider the scaling of u_{ε_n} centered at x_n, that is

$$\phi_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x),$$

which solves the equation

$$-\Delta\phi_n + V(x_n + \varepsilon_n x)\phi_n = g(x_n + \varepsilon_n x, \phi_n), \tag{3.35}$$

so that it is a critical point of the functional $\tilde{J}_n$ defined in $\mathbb{H}^1_{\varepsilon_n, x_n}$ by

$$\tilde{J}_n(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x_n + \varepsilon_n x)u^2\,dx - \int_{\mathbb{R}^N} G(x_n + \varepsilon_n x, u(x))\,dx.$$

Notice that, by a simple change of scale and from (3.26), it is possible to verify that

$$\tilde{J}_n(\phi_n) = \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}), \qquad \|\phi_n\|_{H^1}^2 \leq L. \tag{3.36}$$

As in Lemmas 3.4 3.7 and from elliptic regularity estimates, it results that ϕ_n converges C^2 on compact sets to a function $\phi \in H^1$, which, by (3.34) must be nontrivial. Then, ϕ is a solution of

$$-\Delta\phi + V(x_0)\phi = \overline{f}(x_0, \phi(x)),$$

as $\chi(x_0) = 0$. This is the Euler equation of the functional

$$\overline{I}_{x_0}(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{V(x_0)}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} \overline{F}(x_0, u(x))dx.$$

On the other hand, conditions on G allow us to follow the same arguments of Lemma 2.2 in [8] to deduce that

$$\liminf_{n\to\infty} \tilde{J}_n(\phi_n) \geq \overline{I}_{x_0}(\phi). \tag{3.37}$$

Indeed, consider the function

$$h_n = \frac{1}{2}\left[|\nabla\phi_n|^2 + V(x_n + \varepsilon_n x)\phi_n^2\right] - G(x_n + \varepsilon_n x, \phi_n(x)).$$

Choosing $R > 0$ sufficiently large, from the C^1 convergence of ϕ_n over compacts, and since ϕ belongs to $H^1(\mathbb{R}^N)$ we have, for every $\delta > 0$ fixed,

$$\lim_{n\to\infty} \int_{B(0,R)} h_n \geq \overline{I}_{x_0}(\phi) - \delta.$$

Moreover, taking η_R a smooth cut-off function such that $\eta_R = 0$ on $B(0, R-1)$ and $\eta_R = 1$ on $\mathbb{R}^N \setminus B(0,R)$, and using as test function in (3.35) $w = \eta_R\phi_n$, it is possible to obtain

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0,R)} h_n \geq -\delta,$$

yielding (3.37). Since ϕ is a critical point of $\overline{I}_{x_0}$ and as the nonlinearity $\overline{f}(x_0, t)/t$ is non-decreasing with respect to t, we have

$$\overline{I}_{x_0}(\phi) = \max_{t\geq 0} \overline{I}_{x_0}(t\phi). \tag{3.38}$$

Moreover, it holds $F(x,t) \geq \overline{F}(x,t)$, for every $x \in \mathbb{R}^N$ and for every $t \in \mathbb{R}$, so that, Proposition 3.11 in [22] together with (3.38), implies that

$$\overline{I}_{x_0}(\phi) = \max_{t\geq 0} \overline{I}_{x_0}(t\phi) \geq \inf_{u\in H^1} \sup_{t\geq 0} \overline{I}_{x_0}(tu) \geq \inf_{u\in H^1} \sup_{t\geq 0} I_{x_0}(tu) = \Sigma(x_0). \tag{3.39}$$

This inequality leads to an immediate contradiction in the case in which $x_0 \notin \Omega$, as it would result $\Sigma(x_0) = +\infty$ in this situation. Otherwise we have that $x_0 \in \Omega \cap B(z, r)$ and, assuming the first condition in (1.12) (the other case can be handled analogously) from Lemma 3.2 it follows that $F(x_0, t) < F(z, t)$, moreover, as $V(x_0) \geq V(z)$ we have that $I_{x_0}(v) \geq$

$I_z(v)$ for every $v \in \mathbb{H}^1$, which yields $\Sigma(x_0) > \Sigma(z)$, (for $\Sigma(z)$ defined in (1.13)). This, (3.22), (3.36), (3.37) and (3.39) yield

$$\Sigma(z) < \overline{I}_{x_0}(\phi) \le \liminf_{n\to\infty} \tilde{J}_n(\phi_n) \le \liminf_{n\to+\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \le \Sigma(z), \tag{3.40}$$

which is a contradiction, proving (3.33).

We are now ready to conclude the proof of the result. Let us fix $\delta > 0$; from (3.33) it follows that there exists $\varepsilon_\delta > 0$ such that $0 \le u_\varepsilon(x) < \delta$ for any $x \in \partial B(z,r)$ and $\varepsilon \in (0, \varepsilon_\delta)$. It follows that $(u_\varepsilon - \delta)^+ = 0$ on $\partial B(z,r)$ and hence we can choose

$$\phi_\varepsilon = (u_\varepsilon - \delta)^+ \chi_{\{|x-z|>r\}} \in H^1,$$

as test functions in (3.8). By multiplying and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N \setminus B(z,r)} \left(\varepsilon^2 |\nabla (u_\varepsilon - \delta)^+|^2 + V(x) u_\varepsilon (u_\varepsilon - \delta)^+ - g(x, u_\varepsilon(x))(u_\varepsilon - \delta)^+\right) = 0.$$

Having defined

$$\Upsilon_\varepsilon(x) = \begin{cases} V(x) - \dfrac{g(x, u_\varepsilon(x))}{u_\varepsilon}, & u_\varepsilon(x) \ne 0 \\ 0 & u_\varepsilon(x) = 0 \end{cases}$$

the preceding identity turns into

$$\int_{\mathbb{R}^N \setminus B(z,r)} \left(\varepsilon^2 |\nabla (u_\varepsilon - \delta)^+|^2 + \Upsilon_\varepsilon(x) |(u_\varepsilon - \delta)^+|^2 + \Upsilon_\varepsilon(x) \delta (u_\varepsilon - \delta)^+\right) = 0.$$

By the definition of $g(x,t)$, it is easy to show that $\Upsilon_\varepsilon(x) \ge 3\mu/4$ for all x with $u_\varepsilon(x) > 0$, which implies that $(u_\varepsilon(x) - \delta)^+ = 0$ for every $x \notin B(z,r)$ and every $0 < \varepsilon < \varepsilon_\delta$, namely the assertion. ∎

Remark 3.9 The argument used in Proposition 3.8 has actually a stronger consequence, it implies that for every $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_\delta} \; \sup_{x \in \mathbb{R}^N \setminus B(z,r) \cap \Omega} u_\varepsilon(x) < \delta.$$

Indeed, using (3.33) and following the argument at the beginning of the proof we can prove

$$\lim_{\varepsilon\to 0} \sup_{\partial\Omega \cap B(z,r)} u_\varepsilon = 0$$

arguing again by contradiction and assuming that there exists a sequence $\{\varepsilon_n\}$ converging to 0 and a sequence $\{x_n\} \in \partial\Omega \cap B(z,r)$ such that, for some positive constant β,

$$u_{\varepsilon_n}(x_n) \ge \beta \qquad \text{for all } n \ge 1.$$

As before, $x_n \to x_0 \in \partial\Omega \cap B(z,r)$, and $\phi_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x)$ has a C^2 limit ϕ satisfying

$$-\Delta\phi + V(x_0)\phi = g(x_0, \phi).$$

Since $x_0 \in \partial\Omega$, $s(x_0)V(x_0) = 1$, then, applying Pohozaev identity to the solution ϕ, and recalling (3.1) we obtain

$$\begin{aligned}\frac{2}{2^*}\|\nabla\phi\|_2^2 &= \|\phi\|_2^2 V(x_0)(\chi(x_0)-1) - \frac{\chi(x_0)}{s^2(x_0)}\int_{\mathbb{R}^N} \ln(1+s(x_0)\phi^2) + 2(1-\chi(x_0))\int_{\mathbb{R}^N} \overline{F}(x_0,\phi) \\ &\leq -\frac{\chi(x_0)}{s^2(x_0)}\int_{\mathbb{R}^N} \ln(1+s(x_0)\phi^2) \leq 0\end{aligned}$$

giving the desired contradiction.

Proof of Theorem 1.1.
By virtue of Proposition 3.8, taking into account the definition of G, u_ε turns out to be a solution of (P_ε) for ε sufficiently small. From elliptic regularity theory it follows that u_ε is a positive C^2 function. Let $\xi_\varepsilon \in B(z,r)$ a local maximum point of the function $u_\varepsilon(x)$, then

$$0 \leq -\Delta u_\varepsilon(\xi_\varepsilon) = -V(\xi)u_\varepsilon(\xi_\varepsilon) + f(\xi_\varepsilon, u_\varepsilon(\xi_\varepsilon)) \leq -V(\xi)u_\varepsilon(\xi_\varepsilon) + u_\varepsilon^3(\xi_\varepsilon)$$

which implies that there exists a positive constant σ, independent on ε, such that

$$u_\varepsilon(\xi_\varepsilon) \geq \sigma. \tag{3.41}$$

Let us first prove conclusion *(ii)* of Theorem 1.1 arguing by contradiction. More precisely, consider $\varepsilon_n \to 0$ and $x_n \in B(z,r)$ a local maximum point of u_{ε_n}. Let $x_n \to x^* \in \overline{B}(z,r)$ and consider the sequence $\phi_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x)$, and its limit ϕ, critical point of the autonomous functional I_{x^*}. Thanks to (3.41), $\phi \neq 0$, implying that $x^* \in \Omega$ Moreover, assuming the first alternative in (1.12) (the other situation being similar) $s(x^*) > s_0$ and $V(x^*) \geq V_0$, and from Lemma 3.2 we obtain that $F(x^*, v) < F(z, v)$ so that $I_{x^*}(v) > I_z(v)$ for every $v \in H^1$, yielding the inequality

$$I_{x^*}(\phi) \geq \inf_{u\in H^1}\max_{t>0} I_{x^*}(tu) > \inf_{u\in H^1}\max_{t>0} I_z(tu) = \Sigma(z)$$

which contradicts (3.22), proving *(ii)*. In order to prove conclusion *(i)* of Theorem 1.1, assume by contradiction that there exist a sequence $\{\varepsilon_n\}$ converging to zero and two local maxima x_n^1, $x_n^2 \in \overline{B}(z,r)$, which both satisfy (3.41). We consider the sequence $\phi_n(x) = u_{\varepsilon_n}(x_n^1 + \varepsilon_n x)$ which is a critical point of the functional

$$I_n^1(v) = \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x_n^1 + \varepsilon_n x)v^2\,dx - \int_{\mathbb{R}^N} F(x_n^1 + \varepsilon_n x, v(x))\,dx$$

with critical level

$$I_n^1(\phi_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}). \tag{3.42}$$

Arguing as before, we show that ϕ_n converges in the C^2 sense over compacts to a solution ϕ of (S_y) with $y = x^1$ and from conclusion *(ii)* $s(x^1) = s_0$ and $V(x_1) = V(0)$. From (3.41) we get that $\phi \neq 0$ and from [24, 25] we deduce that ϕ is a nonnegative, radially symmetric function. Then, arguing as in the cubic case i.e. $f(x,t) = t^3$, ϕ has a local non-degenerate maximum point, which, up to translations, is located in the origin. This facts and the C^2 convergence of ϕ_n imply that $x_n = (x_n^1 - x_n^2)/\varepsilon_n \to \infty$. Then we can argue as in the proof of (3.37) to get a contradiction. Indeed, we consider the function

$$h_n = \frac{1}{2}|\nabla\phi_n|^2 + \frac{1}{2}V(x_n^1 + \varepsilon_n x)\phi_n^2 - F(x_n^1 + \varepsilon_n x, \phi_n(x)),$$

and observe that, thanks to the C^2 convergence over compacts of ϕ_n, for every δ we can choose $R > 0$

$$\lim_{n\to\infty} \int_{B(0,R)} h_n(x)\,dx \geq I_{x^1}(\phi) - \delta. \tag{3.43}$$

Moreover, as $x_n \to \infty$ we can fix n_0 sufficiently large such that $B(0,R) \cap B(x_n,R) = \emptyset$. On the other hand, the change of variable $y = x - x_n$ leads to

$$\lim_{n\to\infty} \int_{B(x_n,R)} h_n(x)\,dx = \frac{1}{2}\lim_{n\to\infty} \int_{B(0,R)} |\nabla\psi_n(y)|^2 + V(x_n^2 + \varepsilon_n x)\psi_n^2(y) - 2F(x_n^2 + \varepsilon_n y, \psi_n(y))\,dy$$

where we put $\psi_n(y) = \phi_n(y + x_n)$. Reasoning as in (3.43) and taking into account that $s(x^1) = s(x^2) = s_0$, we get

$$\lim_{n\to\infty} \int_{B(x_n,R)} h_n \geq I_{x^2}(\psi) - \delta = I_{x^1}(\phi) - \delta. \tag{3.44}$$

Then, arguing as in the proof of (3.37) we get

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} I_n^1(\phi_n) \geq 2\Sigma(x^1) = 2\Sigma(z),$$

which is in contradiction with (3.42) and (3.22).

In order to prove the exponential decay, notice that, by Proposition 3.8, u_ε decays to zero at infinity, uniformly with respect to ε. Hence we find $\rho > 0$, $\Theta \in (0, \sqrt{\mu})$ and $\varepsilon_0 > 0$ such that $u_\varepsilon^2 \leq \mu - \Theta^2$, for all $|x - x_\varepsilon| > \varepsilon\rho$ and $0 < \varepsilon < \varepsilon_0$. Let us set

$$\xi_\rho(x) = M_\rho e^{-\Theta(|x - x_\varepsilon|/\varepsilon - \rho)}, \qquad M_\rho = \sup_{(0,\varepsilon_0)} \max_{|x|=\rho}(u_\varepsilon),$$

and introduce the set $\mathscr{A} = \bigcup_{R>\rho} D_R$, where, for any $R > \rho$,

$$D_R = \big\{\rho < |x| < R : \; u_\varepsilon(x) > \xi_\rho(x) \; \text{ for some } \varepsilon \in (0,\varepsilon_0)\big\}.$$

Assume by contradiction that $\mathscr{A} \neq \emptyset$. Then there exist $R_* > \rho$ and $\varepsilon_* \in (0, \varepsilon_0)$ with

$$\varepsilon^2 \Delta(\xi_\rho - u_{\varepsilon_*}) \le \left[\Theta^2 - \frac{2\varepsilon\Theta}{|x - x_\varepsilon|}\right] \xi_\rho - u_{\varepsilon_*}(\mu - u_{\varepsilon_*}^2) \le \Theta^2(\xi_\rho - u_{\varepsilon_*}) < 0,$$

for x in D_R and for all $R \ge R_*$. Hence, by the maximum principle, we get

$$\xi_\rho - u_{\varepsilon_*} \ge \min\left\{\min_{|x|=\rho}(\xi_\rho - u_{\varepsilon_*}), \min_{|x|=R}(\xi_\rho - u_{\varepsilon_*})\right\},$$

in D_R for all $R \ge R_*$. Letting $R \to \infty$ and recalling the definition of ξ_ρ yields

$$\xi_\rho - u_{\varepsilon_*} \ge \min\left\{\min_{|x|=\rho}(\xi_\rho - u_{\varepsilon_*}), 0\right\} \ge 0, \quad \text{in} \bigcup_{R \ge R_*} D_R.$$

In turn, $u_{\varepsilon_*}(x) \le \xi_\rho(x)$ for all x in $\bigcup_{R \ge R_*} D_R$, which yields a contradiction. Whence $\mathscr{A} = \emptyset$, and the desired exponential decay follows. ■

Proof of Theorem 2.3.
Theorem 2.3 can be proved as Theorem 1.1; indeed one can perform the penalization procedure around the minimum point of Σ introduced in (2.2) and make the analogous calculation up to (3.40), as it is readily seen that hypothesis (1.12) is used to obtain condition (2.2). The rest of the proof can be handled in the same way as in the proof of Theorem 1.1. ■

4 Proof of Theorem 2.7

As a first step to prove Theorem 2.7, let us show the following Lemma.

Lemma 4.1 *Let us suppose that V, $s \in C^1(\mathbb{R}^N)$ satisfies (2.3). If $z \in \mathscr{E}$, then*

$$\nabla\left(V(z) - \frac{1}{s(z)}\right)\|Q_z\|_2^2 + \int_{\mathbb{R}^N} \nabla\left(\frac{1}{s^2(z)}\ln(1 + s(z)Q_z^2(x))\right) dx = 0$$

where Q_z is the least energy solution of the autonomous Problem (S_y) for $y = z$.

Proof. We will closely follow the argument in [17] (see also [23]). Let $z \in \mathscr{E}$, a sequence $\{\varepsilon_n\}$ converging to zero and u_{ε_n} a solution of Problem (P_ε), for $\varepsilon = \varepsilon_n$, as in Definition 2.6. Let us define $\varphi_n(x) = u_{\varepsilon_n}(z + \varepsilon_n x)$ and apply the Pucci–Serrin identity [21, Proposition 1] with the lagrangian function $\mathscr{L} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$\mathscr{L}(x, t, \xi) = \frac{1}{2}|\xi|^2 + V(z + \varepsilon_n x)\frac{t^2}{2} - F(z + \varepsilon_n x, t),$$

obtaining

$$\sum_{i,\ell=1}^{N}\int_{\mathbb{R}^N}\partial_{x_i}\boldsymbol{h}^{\ell}\partial_{x_i}\varphi_n\partial_{x_\ell}\varphi_n\,dx=\int_{\mathbb{R}^N}\mathrm{div}\boldsymbol{h}\mathscr{L}(x,\varphi_n,\nabla\varphi_n)+\frac{\varepsilon_n}{2}\int_{\mathbb{R}^N}\boldsymbol{h}\cdot\nabla_x V(z+\varepsilon_n x)\varphi_n^2$$
$$+\varepsilon_n\int_{\mathbb{R}^N}\boldsymbol{h}\cdot\nabla_x F(z+\varepsilon_n x,\varphi_n(x))$$

for all $\boldsymbol{h}\in C^1_{\mathrm{c}}\left(\mathbb{R}^N,\mathbb{R}^N\right)$. Let us choose, for any $\lambda>0$,

$$\boldsymbol{h}_j:\mathbb{R}^N\to\mathbb{R}^N,\qquad \boldsymbol{h}_j^{\ell}(x)=\begin{cases}\Upsilon(\lambda x) & \text{if } \ell=j,\\ 0 & \text{if } \ell\neq j,\end{cases}\qquad \ell=1,\dots,N$$

with $\Upsilon\in C^1_{\mathrm{c}}(\mathbb{R}^N)$, $\Upsilon(x)=1$ if $|x|\le 1$ and $\Upsilon(x)=0$ if $|x|\ge 2$. Then, for $j=1,\dots,N$,

$$\sum_{i=1}^{N}\int_{\mathbb{R}^N}\lambda\partial_{x_i}\Upsilon(\lambda x)\partial_{x_i}\varphi_n\partial_{x_j}\varphi_n=\int_{\mathbb{R}^N}\lambda\partial_{x_j}\Upsilon(\lambda x)\mathscr{L}(x,\varphi_n,\nabla\varphi_n)$$
$$+\frac{\varepsilon_n}{2}\int_{\mathbb{R}^N}\Upsilon(\lambda x)[\partial_{x_j}V(z+\varepsilon_n x)\varphi_n^2-2\partial_{x_j}F(z+\varepsilon_n x,\varphi_n(x))].$$

By the arbitrariness of $\lambda>0$, letting $\lambda\to 0$ and keeping j fixed, we obtain

$$\int_{\mathbb{R}^N}[\partial_{x_j}V(z+\varepsilon_n x)\varphi_n^2-2\partial_{x_j}F(z+\varepsilon_n x,\varphi_n(x))]\,dx=0\qquad j=1,\dots,N.$$

By assumption (2.3), there exists a positive constant β_1 such that, for all $x\in\mathbb{R}^N$ and $j\ge 1$, we get $|\nabla V(z+\varepsilon_n x)|\le\beta_1 e^{\gamma\varepsilon_n|x|}$ and $|\nabla s(z+\varepsilon_n x)|\le\beta_1 e^{\gamma\varepsilon_n|x|}$, so that, invoking the uniform exponential decay of φ_n, letting $n\to\infty$ in the above identity, and recalling that $\varphi\to Q_z$, the least energy solution of (S_y) for $y=z$, we find

$$\int_{\mathbb{R}^N}[\partial_{x_j}V(x)|_z Q_z^2-2\partial_{x_j}F(x,Q_z(x))|_z]\,dx=0\qquad j=1,\dots,N, \tag{4.1}$$

giving the conclusion. ∎

The following result will be useful in studying the function Σ.

Lemma 4.2 *Assume* (1.1) *and that* V, $s\in C^1(\mathbb{R}^N)$. *Then The function* $\mathscr{G}:H^1\times\mathbb{R}^N\to\mathbb{R}$ *defined by*

$$\mathscr{G}(u,y):=\|\nabla u\|_2^2+\left(V(y)-\frac{1}{s(y)}\right)\|u\|_2^2$$

is continuous in y, *for any* $u\in H^1(\mathbb{R}^N)$. *Moreover, if* $\mathscr{G}(u,y)<0$, *then there exists a unique* $\theta(u,y)>0$ *such that* $\theta(u,y)u\in\mathscr{N}_y$, *where* $\mathscr{N}_y$ *is defined in* (1.9). *Finally, the map* θ *is continuous on* $H^1(\mathbb{R}^N)\times\mathbb{R}^N$

Proof. Given u such that $\mathcal{G}(u,y)<0$, we define the function $g:[0,\infty)\times H^1(\mathbb{R}^N)\times\mathbb{R}\to\mathbb{R}$ as follows

$$g(\tau,u,y):=\begin{cases}\|\nabla u\|_2^2+V(y)\|u\|_2^2 & \text{for } \tau=0,\\ \dfrac{1}{\tau^2}\langle (I_y)'(\tau u),\tau u\rangle & \text{for } \tau>0.\end{cases}$$

Condition (1.1) and the regularity properties of the functions V and s imply that g is continuous and Lebesgue Dominate Convergence Theorem yields

$$\lim_{\tau\to+\infty} g(\tau)=\|\nabla u\|_2^2+\left(V(y)-\frac{1}{s(y)}\right)\|u\|_2^2<0.$$

Since g is a continuous function, there exists $\theta(u,y)>0$ such that $g(\theta(u,y))=0$, that is

$$\langle (I_y)'(\theta(u,y)u),\theta(u,y)u\rangle=0,$$

i.e. $\theta(u,y)u\in\mathcal{N}_y$. The uniqueness of $\theta(u,y)$ follows from the fact that $f(y,u)=\partial_u F(y,u)$ satisfies $f(u,y)/u$ is nondecreasing with respect to u.
The continuity of the θ can be deduced from the Implicit function Theorem. ■

In order to prove Theorem 2.7 let us first show the regularity properties of the function Σ.

Proposition 4.3 *Assume* (1.1) *and that* $V, s\in C^1(\mathbb{R}^N)$. *Then, the function* Σ *is of class* $C^1(\Omega)$ *and its gradient is given by*

$$\nabla\Sigma(y)=\nabla_y\left(V(y)-\frac{1}{s(y)}\right)\|Q_y\|_2^2+\int_{\mathbb{R}^N}\nabla_y\left(\frac{1}{s(y)^2}\ln(1+s(y)Q_y^2(x))\right)dx, \tag{4.2}$$

where Q_y *is the least energy solution of* (S_y).

Proof. In order to compute the directional derivative of the function Σ with respect to a unitary vector $\eta\in\mathbb{R}^N$, let $\rho=y+\tau\eta$, and as $\tau\to0$ then $\rho\to y$.
Since $\mathcal{G}(u,y)$ is continuous in y and $\mathcal{G}(Q_y,y)<0$, by Lemma 4.2, there exists a $\delta>0$ such that, for $|\tau|=\|\rho-y\|<\delta$, then $\mathcal{G}(Q_y,\rho)<0$. By Lemma 4.2, there exists $\theta(Q_y,\rho)=\theta(y,\rho)>0$ such that $\theta(y,\rho)Q_y\in\mathcal{N}_\rho$. Using the Mean Value Theorem and the definition (1.13), we have

$$\Sigma(\rho)-\Sigma(y)\le I_\rho(\theta(y,\rho)Q_y)-I_y(\theta(y,y)Q_y)=\tau\eta\cdot\nabla_\xi I_\xi(\theta(y,\xi)Q_y)|_{\xi\in[y,y+\tau\eta]} \tag{4.3}$$

Computing $\nabla_\xi I_\xi(\theta(y,\xi)Q_y)$, we obtain

$$\begin{aligned}\nabla_\xi I_\xi(\theta(y,\xi)Q_y) \;=\;& \frac{\theta(y,\xi)^2}{2}\nabla_\xi\left[V(\xi)-\frac{1}{s(\xi)}\right]\|Q_y\|_2^2\\ &+\frac12\nabla_\xi\left(\frac{1}{s^2(\xi)}\right)\int_{\mathbb{R}^N}\ln(1+s(\xi)\theta^2(y,\xi)Q_y^2)dx\\ &+\frac12\frac{1}{s^2(\xi)}\int_{\mathbb{R}^N}\nabla_\xi\ln(1+s(\xi)\theta^2(y,\xi)Q_y^2)dx\\ &+\nabla_\xi\theta(y,\xi)\theta(y,\xi)\left[\|\nabla Q_y\|_2^2+\left(V(\xi)-\frac{1}{s(\xi)}\right)\|Q_y\|_2^2\right].\end{aligned}$$

Since $\theta(y,\xi)Q_y \in \mathcal{N}_\xi$ it results $\nabla_\xi \theta(y,\xi)\langle (I_\xi)'(\theta(y,\xi)Q_y), \theta(y,\xi)Q_y\rangle/\theta(y,\xi) = 0$ and, on the other hand

$$\langle (I_\xi)'(\theta(y,\xi)Q_y), \theta(y,\xi)Q_y\rangle = \|\theta \nabla Q_y\|_2^2 + \|\theta V(\xi)Q_y\|_2^2 - \int_{\mathbb{R}^N} \frac{\theta^4 Q_y^4}{1+\theta^2 s(\xi) Q_y^2}$$

and substituting above, we get

$$\begin{aligned}
\nabla_\xi I_\xi(\theta(y,\xi)Q_y) \quad &= \quad \frac{\theta(y,\xi)^2}{2}\nabla_\xi \left(V(\xi) - \frac{1}{s(\xi)}\right)\|Q_y\|_2^2 \\
&+ \quad \frac{1}{2}\nabla_\xi \left(\frac{1}{s^2(\xi)}\right)\int_{\mathbb{R}^N} \ln(1+s(\xi)\theta^2(y,\xi)Q_y^2) \\
&+ \quad \frac{1}{2s^2(\xi)}\int_{\mathbb{R}^N} \nabla_\xi \ln(1+s(\xi)\theta^2(y,\xi)Q_y^2) \\
&+ \quad \frac{\nabla_\xi \theta(y,\xi)}{\theta(y,\xi)}\int_{\mathbb{R}^N} \frac{\theta^4(y,\xi)Q_y^4}{1+s(\xi)\theta^2(y,\xi)Q_y^2} - \frac{\theta^2(y,\xi)Q_y^2}{s(\xi)} .
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2s^2(\xi)}\int_{\mathbb{R}^N} \nabla_\xi \ln(1+s(\xi)\theta^2(y,\xi)Q_y^2) \quad &= \quad \frac{\nabla s(\xi)}{2s^2(\xi)}\int_{\mathbb{R}^N} \frac{\theta^2(y,\xi)Q_y^2}{1+s(\xi)\theta^2(y,\xi)Q_y^2} \\
&+ \quad \frac{\nabla_\xi \theta(y,\xi)}{\theta(y,\xi)}\int_{\mathbb{R}^N} \frac{\theta^2(y,\xi)Q_y^2}{s(\xi)(1+s(\xi)\theta^2(y,\xi)Q_y^2)}
\end{aligned}$$

substituting above, we get

$$\begin{aligned}
\nabla_\xi I_\xi(\theta(y,\xi)Q_y) \quad &= \quad \frac{\theta(y,\xi)^2}{2}\nabla_\xi \left(V(\xi) - \frac{1}{s(\xi)}\right)\|Q_y\|_2^2 \\
&+ \frac{1}{2}\nabla_\xi \left(\frac{1}{s^2(\xi)}\right)\int_{\mathbb{R}^N} \ln(1+s(\xi)\theta(y,\xi)^2 Q_y^2) \\
&+ \frac{1}{2}\frac{\nabla_\xi s(\xi)}{s^2(\xi)}\int_{\mathbb{R}^N} \frac{\theta(y,\xi)^2 Q_y^2}{1+s(\xi)\theta(y,\xi)^2 Q_y^2} .
\end{aligned}$$

Using (4.3) we obtain

$$\limsup_{\tau\to 0^+} \frac{\Sigma(y+\tau\eta)-\Sigma(y)}{\tau} \le \limsup_{\tau\to 0^+} \eta \cdot \nabla_\xi I_\xi(\theta(y,\xi)Q_y)|_{\xi\in[y,y+\tau\eta]} .$$

Taking into consideration that $\tau \to 0^+$ implies that $\xi \to y$, applying Lemma 4.2, and observing that $\theta(y,y)Q_y = Q_y \in \mathcal{N}_y$, we obtain

$$\begin{aligned}
\limsup_{\tau\to 0^+} \frac{\Sigma(y+\tau\eta)-\Sigma(y)}{\tau} \quad &\le \quad \eta\cdot\nabla_\xi I_\xi(\theta(y,\xi)Q_y)|_{\xi=y} = \frac{1}{2}\eta\cdot\nabla_y\left(V(y) - \frac{1}{s(y)}\right)\|Q_y\|_2^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^N} \eta\cdot\nabla_y\left(\frac{1}{s(y)^2}\ln(1+s(y)Q_y^2(x))\right) dx .
\end{aligned}$$

On the other hand, using the Mean Value Theorem and the definition (1.13), we have

$$\Sigma(\rho)-\Sigma(y) \geq I_\rho(\theta(\rho,\rho)Q_\rho) - I_y(\theta(y,\rho)Q_\rho) = \tau\eta\cdot\nabla_\xi I_\xi(\theta(\xi,\rho)Q_\rho)|_{\xi\in[y,y+\tau\eta]}\,. \tag{4.4}$$

Performing a similar argument and observing that $\rho = y+\tau\eta \to y$, as $\tau\to 0^+$, yields

$$\begin{aligned}\liminf_{\tau\to 0^+}\frac{\Sigma(y+\tau\eta)-\Sigma(y)}{\tau} &\geq \eta\cdot\nabla_\xi I_\xi(\theta(y,\xi)Q_y)|_{\xi=y} = +\frac{1}{2}\eta\cdot\nabla_y\left(V(y)-\frac{1}{s(y)}\right)\|Q_y\|_2^2 \\ &\quad +\frac{1}{2}\int_{\mathbb{R}^N}\eta\cdot\nabla_y\left(\frac{1}{s(y)^2}\ln(1+s(y)Q_y^2(x))\right)dx\,.\end{aligned}$$

The two inequalities give

$$\left(\frac{\partial\Sigma}{\partial\eta}\right)^+(y) = +\frac{1}{2}\eta\cdot\nabla_y\left(V(y)-\frac{1}{s(y)}\right)\|Q_y\|_2^2+\frac{1}{2}\int_{\mathbb{R}^N}\eta\cdot\nabla_y\left(\frac{1}{s(y)^2}\ln(1+s(y)Q_y^2(x))\right)dx\,.$$

Analogously, we can prove that the same indentity holds for $(\partial\Sigma/\partial\eta)^-(y)$, giving the directional derivative of Σ along a vector η, showing (4.2). The continuity of the gradient follows from the regularity properties of V, s, (2.3) and the exponential decay of Q_y.

■

Proof of Theorem 2.7.
The regularity property of Σ are proved in Proposition 4.3. Let now $z\in\mathscr{E}$, then Lemma 4.1 and Proposition 4.3 imply that z is a critical point of Σ. In order to show that ∇V and ∇s are linearly dependent, let us compute the following partial derivative

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_j\left(\frac{1}{s^2(z)}\ln(1+s(z)Q_z^2(x))\right)dx &= \partial_j\left(\frac{1}{s^2(z)}\right)\int_{\mathbb{R}^N}\ln(1+s(z)Q_z^2(x))dx \\ &\quad+\frac{\partial_j s(z)}{s^2(z)}\int_{\mathbb{R}^N}\frac{Q_z^2(x)}{1+s(z)Q_z^2(x)}dx \\ &= \frac{2}{s(z)}\partial_j\left(\frac{1}{s(z)}\right)\int_{\mathbb{R}^N}\ln(1+s(z)Q_z^2(x))dx \\ &\quad-\partial_j\left(\frac{1}{s(z)}\right)\int_{\mathbb{R}^N}\frac{Q_z^2(x)}{1+s(z)Q_z^2(x)}dx.\end{aligned}$$

Therefore, every $z\in\mathscr{E}$ has to satisfy

$$\partial_j V(z)\|Q_z\|_2^2 = -\frac{\partial_j s(z)}{s^3(z)}\int_{\mathbb{R}^N}\left[s(z)Q_z^2(x)-2\ln(1+s(z)Q_z^2(x))+\frac{s(z)Q_z^2(x)}{1+s(z)Q_z^2(x)}\right]dx,$$

for all $j=1,\ldots,N$, showing the linear dependence of ∇V and ∇s. The proof is complete once one takes into account that the function $h(t)=t-2\ln(1+t)+t/(1+t)$ is positive for $t>0$.

■

Remark 4.4 We can precise Remark 2.8, in the sense that if z is not a critical point of V and s, there exists at least a j_0 such that $\partial_{j_0}V(z)\neq 0$ and $\partial_{j_0}s(z)\neq 0$, nevertheless $\partial_{j_0}V(z)$ and $\partial_{j_0}s(z)$ have to satify (3.42).

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