LARGE DEVIATIONS OF THE EMPIRICAL FLOW FOR CONTINUOUS TIME MARKOV CHAINS

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ABSTRACT. We consider a continuous time Markov chain on a countable state space and prove a joint large deviation principle for the empirical measure and the empirical flow, which accounts for the total number of jumps between pairs of states. We give a direct proof using tilting and an indirect one by contraction from the empirical process.

RÉSUMÉ. On considère une chaîne de Markov en temps continu à espace d'états denómbrable, et on prouve un principe de grandes déviations commune pour la mesure empirique et le courant empirique, qui représente le nombre total de sauts entre les paires d'états. On donne une preuve directe à l'aide d'un *tilting*, et une preuve indirecte par contraction, à partir du processus empirique.

Keywords: Markov chain, Large deviations principle, Entropy, Empirical flow.

AMS 2010 Subject Classification: 60F10, 60J27; Secondary 82C05.

1. INTRODUCTION

One of the most important contribution in the theory of large deviations is the series of papers of Donsker and Varadhan [18] where the authors develop a general approach to the study of large deviations for Markov processes both in continuous and discrete time. They establish large deviations principles (LDP) for the empirical measure and for the empirical process associated to a Markov process. Given a sample path of the process on the finite time window $[0, T]$, the corresponding empirical measure is a probability measure on the state space that associates to any measurable subset the fraction of time spent on it. A LDP for the empirical measure is usually called a *level 2* LDP. Given a sample path, the corresponding empirical process is a probability measure on paths defined on the infinite time window $(-\infty, +\infty)$. More precisely, it is the unique stationary (with respect to time shift) probability measure that gives weight 1 to T–periodic paths such that there exists a period $[t, t+T]$ where they coincide with the original sample path. A LDP for the empirical process is usually called a *level 3* LDP.

The large deviations asymptotic of discrete time Markov chains on a countable state space can be described as follows, see for example [16, 24]. The rate function for the level 3 LDP is the relative entropy per unit of time. The rate function for the level 2 LDP has instead in general only a variational representation, which cannot be solved explicitly even for reversible transition probabilities. A very natural and much studied object is the k–symbols empirical measure. This is a probability measure on strings of symbols with length k obtained from the frequency of appearance in the sample path. With a suitable periodization procedure the k–symbols empirical measures constitute a consistent family of measures that are exactly the

k marginals of the empirical process. For each $k > 1$, and in particular for $k = 2$ the rate function for the LDP associated to the k symbols empirical measure has an explicit expression.

The aim of this paper is to provide an analogous picture for continuous time Markov chains on a countable state space. For the empirical process the rate function is always the relative entropy per unit of time. For the empirical measure the rate function has instead only a variational representation. In the case of reversible Markov chains the corresponding variational problem can be solved and the rate function is related to the Dirichlet form. In the continuous time setting the natural counterpart of the 2–symbols empirical measure is the empirical flow that can be defined as follows. Given a sample path of the Markov chain in the finite time window $[0, T]$, the corresponding empirical flow is the positive measure on the pairs of states assigning to each pair a weight given by the corresponding number of jumps per unit of time.

As in the discrete time setting, the joint rate function for the empirical measure and flow can always be written in a closed form (formula (2.12) below). This joint rate function for the empirical measure and flow first appeared in applied contexts. Originally in information technology [15, 27] and more recently in statistical mechanics [1]. In particular, in [27] it has been used to recover by contraction the Donsker–Varadhan rate function for the empirical measure in the case of a state space with only two elements. Being a LDP intermediate among level 2 and level 3, the authors called it a level 2.5 LDP. Later in [2], motivated by statistical applications, the authors have showed that the contraction on the empirical measure of the rate function proposed by [27] leads to the Donsker-Varadhan rate function in the case of finite state space. In [15] a weak level 2.5 LDP has been proved. Finally in [1] LDPs for flows and currents have been discussed in relation to non equilibrium thermodynamics.

In the present paper we give a rigorous proof of a full LDP for Markov chains on a countable state space. In the case of infinite state space, the empirical flow exhibits novel phenomena with respect to the empirical measure. In particular, the Markov chain could perform very long excursions towards infinity in very short time. Therefore, the exponential tightness of the empirical flow requires additional conditions and poses non trivial topological issues. We have solved these problems by introducing the bounded weak* topology and adding an extra condition with respect to the Donsker-Varadhan ones (see Item (vi) in Condition 2.2). This condition is sharp as it is also necessary for the exponential tightness of the empirical flow in the case of birth and dead processes. Another technical issue is whether the LDP for the empirical measure and flow holds in a stronger topology. For the empirical flow a natural candidate is the strong L^1 topology. However, as shown in the case of birth and dead processes, the rate function has not in general compact level sets in the strong L^1 topology for flows.

We present two different proofs of the LDP of the empirical measure and flow. A direct derivation is obtained using a perturbation of the original Markov measure (under the additional assumption that the graph underlying the Markov chain is locally finite), while an indirect derivation is obtained by contraction from the level 3 LDP. In the last case, the contraction principle (which anyway requires some work as the map is not continuous and the topology is not metrizable) leads to a partial result. In fact, the main point is the identification of the rate function

obtained by contraction with the closed form (2.12). In order to prove this identity - which does require additional conditions with respect to the level 3 LDP - we need a geometrical analysis of divergence–free flows on graphs (see Section 4) which plays a fundamental role also in direct proof of the LDP by exponential tilting. Besides the Donsker-Varadhan conditions, level 2 and 3 LDPs can be proven under hypercontractivity condition, see [17]. Also in this framework, the exponential tightness of the empirical flow requires the additional condition that the inverse of the mean holding time has a finite exponential moment with respect to the invariant measure.

We mention some recent results about fluctuations of currents and flows inspiring and motivating the present work. We already mentioned the paper [1]. In [29, 30] LDPs for the current of the Brownian motion on a compact Riemann manifold are obtained. We mention also the recent preprint [35] on the joint large deviations for the empirical measure and flow for a renewal process on a finite graph. Currents play also a crucial role in biochemical processes, and the study of large fluctuations and related symmetries have recently received much attention (see e.g. [22, 31] and references therein). As development of the result given here, in [6] we recover the LDP for the empirical measure by contraction from the joint LDP proved here. In [7] we shall discuss several applications and consequences of our results like LDPs for currents and Gallavotti–Cohen symmetries. In [7] we will also give sufficient conditions leading to the joint LDP for empirical measure and flow when endowing the flow space of the strong L^1 topology. We also mention that in [3] the scheme proposed here has been extended to the case of continuous time jump processes with an absorbing state, motivated by the study of energy transport in insulators.

We finally outline some possible applications of the LDP for empirical measure and flow in the context of interacting particle systems. (i) The LDP for the total number of jumps per unit of time has been recently analyzed in [11, 12] for some constrained interacting particle systems including the east model. In the limit of infinitely many particles, the associated rate function exhibits a non–trivial zero level set thus leading to second order large deviations. The second order rate function is conjectured and partially proven in [12]. This problem can be attacked, by a purely variational procedure, starting from the joint rate function for empirical measure and flow. (ii) In the context of hydrodynamic scaling limits the LDP of the current has been analyzed in [4, 5, 8, 9]. In this setting a natural problem is the large deviation properties of the time averaged hydrodynamical current in the large time limit. The corresponding rate function exhibits interesting phenomena. On the other hand, one can take the large time limit before the limit of infinitely many particles. As the hydrodynamical current can be written in terms of the empirical flow one can take the scaling limit in the joint LDP for the empirical measure and flows. If all goes well one then recovers the hydrodynamical rate function. In the special case of the one–dimensional boundary driven zero range process, the LDP for the current of particles across an edge of the lattice has been computed by combinatorial techniques in [25] based on a suitable ansatz. In the limit of infinitely many particles it yields the hydrodynamical result. In principle, this problem, including the validity of the ansatz in [25], could be addressed starting from the joint LDP for the empirical measure and flow. (iii) Always in the context of hydrodynamical scaling limit, the LDP for the net flow of particles across a segment of the two-dimensional torus has been analyzed in [10]. In particular, it

is shown that the large deviations asymptotic degenerates due to the occurrence of small vortices near the endpoints of the segment. A non-trivial LDP should hold in a suitable logarithmic rescaling. This phenomenon can be analyzed already for a single random walk for which it becomes a problem on the scaling limit of the rate function here derived.

2. Notation and results

We consider a continuous time Markov chain ξ_t, $t \in \mathbb{R}_+$ on a countable (finite or infinite) state space V. The Markov chain is defined in terms of the *jump rates* $r(x, y)$, $x \neq y$ in V, from which one derives the holding times and the jump chain [40, Section 2.6]. Since the holding time at $x \in V$ is an exponential random variable of parameter $r(x) := \sum_{y \in V} r(x, y)$, we need to assume that $r(x) < +\infty$ for any $x \in V$.

The basic assumptions on the chain are the following:

(A1) for each $x \in V$, $r(x) = \sum_{y \in V} r(x, y)$ is finite and strictly positive;
(A2) for each $x \in V$ the Markov chain ξ_t^x starting from x has no explosion a.s.;
(A3) the Markov chain is irreducible, i.e. for each $x, y \in V$ and $t > 0$ the event $\{\xi_t^x = y\}$ has strictly positive probability;
(A4) there exists a unique invariant probability measure, that is denoted by π.

As in [40], by invariant probability measure π we mean a probability measure on V such that

$$\sum_{y \in V} \pi(x) \, r(x, y) = \sum_{y \in V} \pi(y) \, r(y, x) \qquad \forall \, x \in V \tag{2.1}$$

where we understand $r(x, x) = 0$. We recall some basic facts from [40], see in particular Section 3.5 and Theorem 3.8.1 there. Assuming (A1) and irreducibility (A3), assumptions (A2) and (A4) together are equivalent to the fact that all states are positive recurrent. In (A4) one could remove the assumption of uniqueness of the invariant probability measure, since for an irreducible Markov chain there can be at most only one. Under the above assumptions, $\pi(x) > 0$ for all $x \in V$, the Markov chain starting with distribution π is stationary (i.e. its law is left invariant by time-translations), and the ergodic theorem holds, i.e. for any bounded function $f : V \to \mathbb{R}$ and any initial distribution

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T dt \, f(\xi_t) = \langle \pi, f \rangle \qquad \text{a.s.} \tag{2.2}$$

where $\langle \pi, f \rangle$ denotes the expectation of f with respect to π. Finally, we observe that if V is finite then (A1) and (A2) are automatically satisfied, while (A3) implies (A4).

We consider V endowed with the discrete topology and the associated Borel σ-algebra given by the collection of all the subsets of V. Given $x \in V$, the distribution of the Markov chain ξ_t^x starting from x, is a probability measure on the Skorohod space $D(\mathbb{R}_+; V)$ that we denote by $\mathbb{P}_x$. The expectation with respect to $\mathbb{P}_x$ is denoted by $\mathbb{E}_x$. In the sequel we consider $D(\mathbb{R}_+; V)$ equipped with the Skorohod topology, the associated Borel σ–algebra, and the canonical filtration. The canonical coordinate in $D(\mathbb{R}_+; V)$ is denoted by X_t. The set of probability measures on V is denoted by $\mathcal{P}(V)$ and it is considered endowed with the topology of weak convergence and the associated Borel σ-algebra. Since V has the discrete

topology, the weak convergence of μ_n to μ in $\mathcal{P}(V)$ is equivalent to the pointwise convergence of $\mu_n(x)$ to $\mu(x)$ for any $x \in V$.

2.1. **Empirical measure and empirical flow.** Given $T > 0$ the *empirical measure* $\mu_T \colon D(\mathbb{R}_+; V) \to \mathcal{P}(V)$ is defined by

$$\mu_T(X) = \frac{1}{T} \int_0^T dt\, \delta_{X_t}$$

where δ_y denotes the pointmass at y. Given $x \in V$, the ergodic theorem (2.2) implies that the empirical measure μ_T converges $\mathbb{P}_x$ a.s. to π as $T \to \infty$. In particular, the sequence of probabilities $\{\mathbb{P}_x \circ \mu_T^{-1}\}_{T>0}$ on $\mathcal{P}(V)$ converges to δ_π.

We denote by E the countable set of ordered edges in V with strictly positive jump rate:

$$E := \{(y, z) \in V \times V \,:\, r(y, z) > 0\}.$$

For each $T > 0$ we define the *empirical flow* as the map $Q_T \colon D(\mathbb{R}_+; V) \to [0, +\infty]^E$ given by

$$Q_T(X) := \frac{1}{T} \sum_{t \in [0,T]\,:\, X_{t-} \neq X_t} \delta_{(X_{t-}, X_t)} \tag{2.3}$$

Namely, $T Q_T(X)(y, z)$ is the number of jumps from y to z in the time interval $[0, T]$ of the path X.

Remark 2.1. *By the graphical construction of the Markov chain, the random field $\{T Q_T(y, z)\}_{(y,z) \in E}$ under $\mathbb{P}_x$ is stochastically dominated by the random field $\{\mathcal{Z}_{y,z}\}_{(y,z) \in E}$ given by independent Poisson random variables, $\mathcal{Z}_{y,z}$ having mean $Tr(y, z)$.*

We denote by $L^1(E)$ the collection of absolutely summable functions on E and by $\|\cdot\|$ the associated L^1–norm. The set of nonnegative elements of $L^1(E)$ is denoted by $L_+^1(E)$. Since the chain is not explosive, for each $T > 0$ we have $\mathbb{P}_x$ a.s. that $Q_T \in L_+^1(E)$.

Given a flow $Q \in L_+^1(E)$ we let its *divergence* $\operatorname{div} Q \colon V \to \mathbb{R}$ be the function defined by

$$\operatorname{div} Q(y) = \sum_{z:\,(y,z)\in E} Q(y, z) - \sum_{z:\,(z,y)\in E} Q(z, y), \qquad y \in V. \tag{2.4}$$

Namely, the divergence of the flow Q at y is given by the difference between the flow exiting from y and the flow entering into y. Observe that the divergence maps $L_+^1(E)$ to $L^1(V)$.

Finally, to each probability $\mu \in \mathcal{P}(V)$ we associate the flow $Q^\mu \in \mathbb{R}_+^E$ defined by

$$Q^\mu(y, z) := \mu(y)\, r(y, z) \qquad (y, z) \in E. \tag{2.5}$$

Note that $Q^\mu \in L_+^1(E)$ if and only if $\langle \mu, r \rangle < +\infty$. Moreover, in this case, by (2.1) Q^μ has vanishing divergence if only if μ is invariant for the Markov chain ξ, i.e. $\mu = \pi$.

We now discuss the law of large numbers for the empirical flow. As follows from simple computations (see [39, Lemma II.2.3] and [28, App. 1, Lemma 5.1], which

have to be generalized to the case of unbounded $r(\cdot)$ by means of [40, Sec. 2.8] and Remark 2.1) for each $(y, z) \in E$ the process

$$M_T(y, z) := T\, Q_T(X)(y, z) - \int_0^T dt\, \delta_y(X_t) r(y, z) \qquad (2.6)$$

is a martingale with respect to $\mathbb{P}_x$, $x \in V$. Moreover, the predictable quadratic variation of $M_T(y, z)$, denoted by $\langle M(y, z) \rangle_T$ is given by

$$\langle M(y, z) \rangle_T = \int_0^T dt\, \delta_y(X_t) r(y, z)\,.$$

In view of the ergodic theorem (2.2), we conclude that for each $x \in V$ and $(y, z) \in E$ the family of real random variables $Q_T(y, z)$ converges, in probability with respect to $\mathbb{P}_x$, as $T \to +\infty$ to $Q^\pi(y, z)$. We refer to Remark 3.3 for an alternative proof.

2.2. Compactness conditions.
The classical Donsker-Varadhan theorem [16, 17, 18, 41] describes the LDP associated to the empirical measure. The main purpose of the present paper is to extend this result by considering also the empirical flow.

Below we will state two LDPs (Theorem 2.7 and Theorem 2.10) for the joint process given by the empirical measure and flow. In Theorem 2.7 the flow space is given by $L_1^+(E)$ endowed of the bounded weak* topology and, in order to have some control at infinity in the case of infinite state space V, compactness assumptions are required. In Theorem 2.10 the flow space is given by $[0, +\infty]^E$ endowed of the product topology and weaker assumptions are required (the same of [18]). On the other hand, the rate function has not always a computable form.

Let us now state precisely the compactness conditions under which Theorem 2.7 holds (at least one of the following Conditions 2.2, 2.4 has to be satisfied). To this aim, given $f\colon V \to \mathbb{R}$ such that $\sum_{y \in V} r(x, y)\, |f(y)| < +\infty$ for each $x \in V$, we denote by $Lf\colon V \to \mathbb{R}$ the function defined by

$$Lf(x) := \sum_{y \in V} r(x, y)\big[f(y) - f(x)\big], \qquad x \in V. \qquad (2.7)$$

Condition 2.2. *There exists a sequence of functions $u_n\colon V \to (0, +\infty)$ satisfying the following requirements:*

   (i) *For each $x \in V$ and $n \in \mathbb{N}$ it holds $\sum_{y \in V} r(x, y) u_n(y) < +\infty$. In the sequel $Lu_n\colon V \to \mathbb{R}$ is the function defined by (2.7).*

  (ii) *The sequence u_n is uniformly bounded from below. Namely, there exists $c > 0$ such that $u_n(x) \geq c$ for any $x \in V$ and $n \in \mathbb{N}$.*

 (iii) *The sequence u_n is uniformly bounded from above on compacts. Namely, for each $x \in V$ there exists a constant C_x such that for any $n \in \mathbb{N}$ it holds $u_n(x) \leq C_x$.*

 (iv) *Set $v_n := -Lu_n/u_n$. The sequence $v_n\colon V \to \mathbb{R}$ converges pointwise to some $v\colon V \to \mathbb{R}$.*

  (v) *The function v has compact level sets. Namely, for each $\ell \in \mathbb{R}$ the level set $\big\{x \in V\, :\, v(x) \leq \ell\big\}$ is finite.*

 (vi) *There exist a strictly positive constant σ and a positive constant C such that $v \geq \sigma r - C$.*

Remark 2.3. *Since $u_n > 0$, it holds $v_n(x) = \sum_{y \in V} r(x, y)(1 - u_n(y)/u_n(x)) < r(x)$. Hence the function v in Condition 2.2 must satisfy $v(x) \leq r(x)$ for all $x \in V$. Due to (v), this implies that also r has compact level sets. In particular, when*

considering a Markov chain with infinite state space, the function r must diverge at infinity.

Replacing in Condition 2.2 the strictly positive constant σ with zero one obtains the same assumptions of Donsker and Varadhan for the derivation in [18] of the LDP for the empirical measure of the Markov chain satisfying (A1)–(A4) (shortly, we will say that the *Donsker–Varadhan condition* is satisfied). In particular, the empirical measure satisfies a LDP with rate function $\widehat{I} \colon \mathcal{P}(V) \to [0, +\infty]$ given by

$$\widehat{I}(\mu) = \sup_{u>0} \left\{ - \langle \mu, Lu/u \rangle \right\}. \tag{2.8}$$

Under the same condition, the empirical process satisfies a LDP (see Section 6). Both these results still hold under a suitable compactness condition concerning the hypercontractivity of the underlying Markov semigroup, see [17].

With respect to the hypercontractity condition, in order to establish the exponential tightness of the empirical flow we need extra assumptions. Recall that π is the unique invariant measure of the chain. The maps $P_t f(x) := \mathbb{E}(f(\xi_t^x))$, $t \in \mathbb{R}_+$, define a strongly continuous Markov semigroup on $L^2(V, \pi)$. We write D_π for the Dirichlet form associated to the symmetric part $(\mathcal{L} + \mathcal{L}^*)/2$ of the generator $\mathcal{L}$ in $L^2(V, \pi)$. Since the time–reversed dynamics is described by a Markov chain on V with transition rates $r^*(x, y) := \pi(y) r(y, x)/\pi(x)$, it holds

$$D_\pi(f) = \frac{1}{4} \sum_{x \in V} \sum_{y \in V} \big(\pi(x) r(x, y) + \pi(y) r(y, x) \big) \big(f(y) - f(x) \big)^2, \qquad f \in L^2(V, \pi). \tag{2.9}$$

One can take this expression as definition of D_π, avoiding all technicalities concerning infinitesimal generators. One says that the Markov chain ξ satisfies the logarithmic Sobolev inequality if there exists a constant $c_{\mathrm{LS}} \in (0, +\infty)$ such that for any $\mu \in \mathcal{P}(V)$ it holds (recall that $\pi(x) > 0$ for any $x \in V$)

$$\mathrm{Ent}(\mu | \pi) \leq c_{\mathrm{LS}} \, D_\pi \left(\sqrt{\mu/\pi} \right), \tag{2.10}$$

where $\mathrm{Ent}(\mu | \pi)$ denotes the relative entropy of μ with respect to π.

Condition 2.4.

  (i) *The Markov chain satisfies a logarithmic Sobolev inequality.*
  (ii) *The exit rate r has an exponential moment with respect to the invariant measure. Namely, there exists $\sigma > 0$ such that $\langle \pi, \exp(\sigma \, r) \rangle < +\infty$.*
  (iii) *The graph (V, E) is locally finite, that is for each vertex $y \in V$ the number of incoming and outgoing edges in y is finite.*

Item (iii) is here assumed for technical convenience and it should be possible to drop it. Item (i) is the hypercontractivity condition assumed in [17] to deduce the Donsker-Varadhan theorem for the empirical measure. Item (ii) is here required to prove the exponential tightness of the empirical flow in $L_+^1(E)$.

Remark 2.5. *By taking in (2.10) $\mu = \delta_x$, Condition 2.4–(i) implies that r has compact level sets.*

2.3. **LDP with flow space $L_+^1(E)$ endowed of the bounded weak* topology.** We consider the space $L^1(E)$ equipped with the so-called *bounded weak* topology*. This is defined as follows. Recall that the (countable) set E is the collection of

ordered edges in V with positive jump rate. Let $C_0(E)$ be the collection of the functions $F \colon E \to \mathbb{R}$ vanishing at infinity, that is the closure of the functions with compact support in the uniform topology. The dual of $C_0(E)$ is then identified with $L^1(E)$. The weak* topology on $L^1(E)$ is the smallest topology such that the maps $Q \in L^1(E) \to \langle Q, f \rangle \in \mathbb{R}$ with $f \in C_0(E)$ are continuous. Given $\ell > 0$, let $B_\ell := \left\{ Q \in L^1(E) : \|Q\| \le \ell \right\}$ be the closed ball of radius ℓ in $L^1(E)$ ($\| \cdot \|$ being the standard L^1–norm). In view of the separability of $C_0(E)$ and the Banach-Alaoglu theorem, the set B_ℓ endowed with the weak* topology is a compact Polish space. The bounded weak* topology on $L^1(E)$ is then defined by declaring a set $A \subset L^1(E)$ open if and only if $A \cap B_\ell$ is open in the weak* topology of B_ℓ for any $\ell > 0$. The bounded weak* topology is stronger than the weak* topology (they coincide only when $|E| < +\infty$) and for each $\ell > 0$ the closed ball B_ℓ is compact with respect to the bounded weak* topology. The space $L^1(E)$ endowed with the bounded weak* topology is a locally convex, complete linear topological space and a completely regular space (i.e. for every closed set $C \subset L^1(E)$ and every element $Q \in L^1(E) \setminus C$ there exists a continuous function $f : L^1(E) \to [0,1]$ such that $f(Q) = 1$ and $f(Q') = 0$ for all $Q' \in C$). Moreover, it is metrizable if and only if the set E is finite. We refer to [36, Sec. 2.7] for the proof of the above statements and for further details.

We regard $L^1_+(E)$ as a (closed) subset of $L^1(E)$ and consider it endowed with the relative topology and the associated Borel σ–algebra. Accordingly, the empirical flow Q_T will be considered as a measurable map from $D(\mathbb{R}_+; V)$ to $L^1_+(E)$, defined $\mathbb{P}_x$ a.s., $x \in V$. Recalling that we consider $\mathcal{P}(V)$, the set of probability measures on V, with the topology of weak convergence, we finally consider the product space $\mathcal{P}(V) \times L^1_+(E)$ endowed with the product topology and regard the couple (μ_T, Q_T) where μ_T is the empirical measure and Q_T the empirical flow, as a measurable map from $D(\mathbb{R}_+; V)$ to $\mathcal{P}(V) \times L^1_+(E)$ defined $\mathbb{P}_x$ a.s., $x \in V$.

Below we state the LDP for the family of probability measures on $\mathcal{P}(V) \times L^1_+(E)$ given by $\left\{ \mathbb{P}_x \circ (\mu_T, Q_T)^{-1} \right\}$ as $T \to +\infty$. Before stating precisely the result, we introduce the corresponding rate function. Let $\Phi \colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty]$ be the function defined by

$$\Phi(q,p) := \begin{cases} q \log \dfrac{q}{p} - (q - p) & \text{if } q, p \in (0, +\infty) \\ p & \text{if } q = 0,\, p \in [0, +\infty) \\ +\infty & \text{if } p = 0 \text{ and } q \in (0, +\infty). \end{cases} \qquad (2.11)$$

Since $\Phi(q, p) = \sup_{\lambda \in \mathbb{R}} \left\{ q\lambda - p(e^\lambda - 1) \right\}$, Φ is lower semicontinuos and convex. We point out that, given $p > 0$ and letting N_t, $t \in \mathbb{R}_+$ be a Poisson process with parameter p, the sequence of real random variables $\{N_T / T\}$ satisfies a large deviation principle on $\mathbb{R}$ with rate function $\Phi(\cdot, p)$ as $T \to \infty$. This statement can be easily derived from the Gärtner-Ellis theorem, see e.g [16, Thm. 2.3.6]. Recalling (2.4) and (2.5), we let $I \colon \mathcal{P}(V) \times L^1_+(E) \to [0, +\infty]$ be the functional defined by

$$I(\mu, Q) := \begin{cases} \displaystyle\sum_{(y,z) \in E} \Phi\big(Q(y,z), Q^\mu(y,z)\big) & \text{if } \operatorname{div} Q = 0,\, \langle \mu, r \rangle < +\infty \\ +\infty & \text{otherwise.} \end{cases} \qquad (2.12)$$

Remark 2.6. *In view of the lower semicontinuity and convexity of Φ, I is lower semicontinuous (apply Fatou lemma) and convex. Moreover, as proved in Appendix*

A, if $\langle \mu, r \rangle = +\infty$ the series in (2.12) diverges. Hence the condition $\langle \mu, r \rangle < +\infty$ can be removed from the first line of (2.12).

Theorem 2.7. *Assume the Markov chain satisfies (A1)–(A4) and at least one between Conditions 2.2 and 2.4. Then as $T \to +\infty$ the family of probability measures $\{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\}$ on $\mathcal{P}(V) \times L_+^1(E)$ satisfies a large deviation principle, uniformly for x in compact subsets of V, with good and convex rate function I. Namely, for each not empty compact set $K \subset V$, each closed set $\mathcal{C} \subset \mathcal{P}(V) \times L_+^1(E)$, and each open set $\mathcal{A} \subset \mathcal{P}(V) \times L_+^1(E)$, it holds*

$$\varlimsup_{T \to +\infty} \sup_{x \in K} \frac{1}{T} \log \mathbb{P}_x \Big((\mu_T, Q_T) \in \mathcal{C} \Big) \leq - \inf_{(\mu,Q) \in \mathcal{C}} I(\mu, Q), \qquad (2.13)$$

$$\varliminf_{T \to +\infty} \inf_{x \in K} \frac{1}{T} \log \mathbb{P}_x \Big((\mu_T, Q_T) \in \mathcal{A} \Big) \geq - \inf_{(\mu,Q) \in \mathcal{A}} I(\mu, Q). \qquad (2.14)$$

As discussed in Lemma 3.9, under the assumptions in Theorem 2.7 it holds $\langle \pi, r \rangle < +\infty$. In particular, $I(\mu, Q) = 0$ if and only if $(\mu, Q) = (\pi, Q^\pi)$. Hence, from the LDP one derives the law of large numbers for the empirical flow in $L_+^1(E)$, improving the pointwise version discussed at the end of Section 2.1. In addition, the function I has an affine structure:

Proposition 2.8. *Let $(\mu, Q) \in \mathcal{P}(V) \times L_+^1(E)$ satisfy $I(\mu, Q) < +\infty$. Then*
   (i) *All edges in the support $E(Q)$ of Q connect vertices in the support of μ, i.e. if $Q(y, z) > 0$ then $y, z \in \mathrm{supp}(\mu)$.*
   (ii) *Let $E^u(Q) := \big\{ \{y, z\} : (y, z) \in E(Q) \text{ or } (z, y) \in E(Q) \big\}$. The oriented connected components of the oriented graph $(\mathrm{supp}(\mu), E(Q))$ coincide with the connected components of the unoriented graph $(\mathrm{supp}(\mu), E^u(Q))$.*
   (iii) *$I(\mu, Q)$ has the following affine decomposition. Consider the oriented graph $(\mathrm{supp}(\mu), E(Q))$ and let K_j, $j \in J$, be the family of its oriented connected components. Consider the probability measure $\mu_j(\cdot) := \mu(\cdot | K_j)$ and the flow $Q_j \in L_+^1(E)$ defined as*

$$Q_j(y, z) = \begin{cases} \frac{Q(y,z)}{\mu(K_j)} & \text{if } (y, z) \in E, \ y, z \in K_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $(\mu, Q) = \sum_{j \in J} \mu(K_j)(\mu_j, Q_j)$ and

$$I(\mu, Q) = \sum_{j \in J} \mu(K_j) I(\mu_j, Q_j). \qquad (2.15)$$

For the unfamiliar reader, the definition of (oriented) connected components is recalled after Remark 4.2. Note that the oriented components of $(\mathrm{supp}(\mu), E(Q))$ coincide with the irreducible classes of the Markov chain on $\mathrm{supp}(\mu)$ with transition rates $r(y, z) := Q(y, z)/\mu(y)$. Moreover, note that due to Item (i) the graph $(\mathrm{supp}(\mu), E(Q))$ is well defined. The proof of Proposition 2.8 is given in Section 4.

2.4. **LDP with flow space $[0, +\infty]^E$ endowed of the product topology.** When considering the product topology on $[0, +\infty]^E$ we take $[0, +\infty]$ endowed of the metric making the map $x \to \frac{x}{1+x} \in [0, 1]$ an isometry. Namely, on $[0, +\infty]$ we take the metric $d(\cdot, \cdot)$ defined as $d(x, y) = \big| x/(1+x) - y/(1+y) \big|$. It is standard to define on the space $[0, +\infty]^E$ a metric $D(\cdot, \cdot)$ inducing the product topology: enumerating the edges in E as $e_1, e_2, \ldots$ we set $D(Q, Q') := \sum_{n=1}^{|E|} 2^{-n} d\big(Q(e_n), Q'(e_n) \big)$.

We write $\mathcal{M}_S$ for the space of stationary probabilities on $D(\mathbb{R}; V)$ endowed of the weak topology. Given $R \in \mathcal{M}_S$ we denote by $\widehat{\mu}(R) \in \mathcal{P}(V)$ the marginal of R at a given time and by $\widehat{Q}(R)$ the flow in $[0, +\infty]^E$ defined as $\widehat{Q}(R)(y, z) := \mathbb{E}_R[Q_T(y, z)]$ for all $(y, z) \in E$, where $\mathbb{E}_R$ denotes the expectation with respect to R. It is simple to check that this expectation does not depend on the time $T > 0$ (see Lemma 2.9). We point out that jumps between a pair of states non belonging to E could take place with positive R–probability. In particular, the flow $\widehat{Q}(R)$ does not correspond to the complete flow associated to R.

Lemma 2.9. *Given an edge $(y, z) \in E$ and a stationary process $R \in \mathcal{M}_S$, the expectation $\mathbb{E}_R[Q_T(y, z)] \in [0, +\infty]$ does not depend on $T > 0$.*

Proof. Since R is stationary, fixed $t \in \mathbb{R}$ it holds $R(X_t \neq X_{t-}) = 0$. In particular, given $T > 0$ and an integer n, R–a.e. it holds

$$Q_T(X)(y, z) = \frac{1}{n} \sum_{j=0}^{n-1} Q_{T/n}(\theta_{jT/n} X)(y, z).$$

Above we have used the notation $(\theta_s X)_t := X_{s+t}$. From this identity and the stationarity of R, taking the expectation w.r.t. R one gets $f(T) = f(T/n)$, where $f(T) := \mathbb{E}_R[Q_T(y, z)]$. Then by standard arguments one gets that $f(T) = f(1)$ as T varies among the positive rational numbers. Since for $0 < t_1 \leq T \leq t_2$ it holds $t_1 f(t_1) \leq T f(T) \leq t_2 f(t_2)$ it is trivial to conclude that $f(T)$ is constant as T varies among the positive real numbers. □

We can now state our second main result:

Theorem 2.10. *Assume the Markov chain satisfies (A1)–(A4) together with the Donsker–Varadhan condition. Consider the space $\mathcal{P}(V) \times [0, +\infty]^E$, with $\mathcal{P}(V)$ endowed of the weak topology and $[0, +\infty]^E$ endowed of the product topology. Then the following holds:*

(i) *As $T \to +\infty$ the family of probability measures $\{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\}$ on $\mathcal{P}(V) \times [0, +\infty]^E$ satisfies a large deviation principle with good rate function*

$$\widetilde{I}(\mu, Q) := \inf \left\{ H(R) \, : \, R \in \mathcal{M}_S \, , \, \widehat{\mu}(R) = \mu \, , \, \widehat{Q}(R) = Q \right\}. \qquad (2.16)$$

Above $H(R)$ denotes the entropy of R with respect to the Markov chain ξ as defined in $[18]$–(IV) (see Section 6). Moreover we have

$$\begin{cases} \widetilde{I}(\mu, Q) = I(\mu, Q) & \text{if } Q \in L^1_+(E), \\ \widetilde{I}(\mu, Q) = +\infty & \text{if } Q \notin [0, +\infty)^E. \end{cases} \qquad (2.17)$$

(ii) *If in addition Condition 2.2 is satisfied, then the rate function $\widetilde{I}$ is given by*

$$\widetilde{I}(\mu, Q) := \begin{cases} I(\mu, Q) & \text{if } Q \in L^1_+(E), \\ +\infty & \text{otherwise}. \end{cases} \qquad (2.18)$$

Since Condition 2.2 implies the Donsker–Varadhan condition, Theorem 2.10 under Condition 2.2 implies the variational characterization

$$I(\mu, Q) = \inf \left\{ H(R) : R \in \mathcal{M}_S \, , \, \widehat{\mu}(R) = \mu \, , \, \widehat{Q}(R) = Q \right\}, \; (\mu, Q) \in \mathcal{P}(V) \times L^1_+(E).$$

In addition, note that (2.17) does not cover the case $Q \in [0, +\infty)^E \setminus L^1_+(E)$.

2.5. **Outline.** The rest of the paper is devoted to the proofs of Theorems 2.7 and 2.10, and of Proposition 2.8. Sections 3 and 4 contain preliminary results and the proof of Proposition 2.8. Then in Section 5 we give a direct proof of Theorem 2.7. For this proof it is necessary to add the condition that the graph (V, E) is locally finite.

In Sections 6, 7 and 8 we remove the above condition and prove both Theorems 2.7 and 2.10 by projection from the large deviations principle for the empirical process proven by Donsker and Varadhan in [18]–(IV). We discuss the details only for the Donsker-Varadhan type compactness conditions. For this reason, we added item (iii) as a separate requirement in the hypercontractivity type Condition 2.4. By using similar arguments to the ones here presented, it should be possible to remove it from Theorem 2.7 and prove the first statement in Theorem 2.10 by assuming only items (i) and (ii) in Condition 2.4.

Finally, in Section 9 we discuss some examples from birth and death processes and compare the different compactness conditions.

3. Exponential estimates

In this section we collect some preliminary results that will enter in the proof of Theorems 2.7 and 2.10. Between other, we prove the exponential tightness in $L^1_+(E)$ of the empirical flow when at least one between Conditions 2.2 and 2.4 holds.

3.1. **Exponential local martingales.** We start by comparing our Markov chain with a perturbed one. Let $\widehat{\xi}$ be a continuous time Markov chain on V with jump rates $\widehat{r}(y, z)$, $y \neq z$ in V. We assume that $\widehat{r}(y) := \sum_{z \in V} \widehat{r}(y, z) < +\infty$ for all $y \in V$, thus implying that the Markov chain $\widehat{\xi}$ is well defined at cost to add a coffin state ∂ to the state space in case of explosion [40, Ch. 2]. We write $\widehat{\mathbb{P}}_x$ for the law on $D(\mathbb{R}_+, V \cup \{\partial\})$ of the above Markov chain $\widehat{\xi}$ starting at $x \in V$. We denote by ρ_T the map $\rho_T : D(\mathbb{R}_+, V \cup \{\partial\}) \to D([0, T], V \cup \{\partial\})$ given by restriction of the path to the time interval $[0, T]$. We now assume that $\widehat{r}(y, z) = 0$ if $(y, z) \notin E$. Then, restricting the probability measures $\mathbb{P}_x \circ \rho_T^{-1}$ and $\widehat{\mathbb{P}}_x \circ \rho_T^{-1}$ to the set $D([0, T], V)$ (no explosion takes place in the interval $[0, T]$), we obtain two reciprocally absolutely continuous measures with Radon–Nykodim derivative

$$\frac{d\widehat{\mathbb{P}}_x \circ \rho_T^{-1}}{d\mathbb{P}_x \circ \rho_T^{-1}}\bigg|_{D([0,T],V)} = \exp\left\{-T\langle \mu_T, \widehat{r} - r\rangle\right\} \prod_{(y,z) \in E} \left[\frac{\widehat{r}(y,z)}{r(y,z)}\right]^{TQ_T(y,z)}. \qquad (3.1)$$

This formula can be checked very easily. Indeed, calling $\tau_1(X) < \tau_2(X) < \tau_{N(X)}(X)$ the jump times of the path X in $[0, T]$ (below $N(X) < +\infty$ almost surely) we have

$$\mathbb{P}_x \circ \rho_T^{-1}\left(N(X) = n, \ X(\tau_i) = x_i, \ \tau_i \in (t_i, t_i + dt_i) \ \forall i : 1 \leq i \leq n\right)$$

$$= \left[\prod_{i=0}^{n-1} e^{-r(x_i)(t_{i+1}-t_i)} r(x_i, x_{i+1})\right] e^{-r(x_n)(T-t_n)} dt_1 \cdots dt_n,$$

where $t_0 := 0$ and $x_0 := x$, $0 \leq t_1 < t_2 < \cdots < t_n \leq T$, $n = 0, 1, 2, \ldots$. Since a similar formula holds also for the law $\widehat{\mathbb{P}}_x \circ \rho_T^{-1}$, one gets (3.1).

As immediate consequence of the Radon–Nykodim derivative (3.1) we get the following result:

Lemma 3.1. *Let $F: E \to \mathbb{R}$ be such that $r^F(y) := \sum_z r(y,z)e^{F(y,z)} < +\infty$ for any $y \in V$. For $t \geq 0$ define $\mathbb{M}_t^F : D(\mathbb{R}_+, V) \to (0, +\infty)$ as*

$$\mathbb{M}_t^F := \exp\left\{ t\left[\langle Q_t, F \rangle - \langle \mu_t, r^F - r \rangle \right] \right\} \tag{3.2}$$

where $\langle Q_t, F \rangle = \sum_{(y,z) \in E} Q_t(y,z)F(y,z)$. Then for each $x \in V$ and $t \in \mathbb{R}_+$ it holds $\mathbb{E}_x(\mathbb{M}_t^F) \leq 1$.

Proof. By (3.1) $(\widehat{r}(y,z) := r(y,z)e^{F(y,z)})$, $\mathbb{E}_x(\mathbb{M}_t^F) = \widehat{\mathbb{P}}_x(D([0,t]; V)) \leq 1$. $\qquad \square$

Remark 3.2. *It is simple to check that the process $\mathbb{M}^F$ is a positive local martingale and a supermartingale with respect to $\mathbb{P}_x$, $x \in V$.*

Remark 3.3. *Fixed $(y,z) \in E$, taking in Lemma 3.1 $F := \pm\lambda\delta_{y,z}$ with $\lambda > 0$ and applying Chebyshev inequality, one gets for $\delta > 0$ that the events $\{Q_t(y,z) > \mu_t(y)r(y,z)(e^\lambda - 1)/\lambda + \delta\}$ and $\{Q_t(y,z) < \mu_t(y)r(y,z)(1 - e^{-\lambda})/\lambda - \delta\}$ have $\mathbb{P}_x$-probability bounded by $e^{-t\delta\lambda}$. Using that $(e^{\pm\lambda} - 1)/\lambda = \pm 1 + o(1)$ and since $\mu_t(y) \to \pi(y)$ as $t \to +\infty$ $\mathbb{P}_x$-a.s. by the ergodic theorem (2.2), taking the limit $t \to +\infty$ and afterwards taking δ, λ arbitrarily small, one recovers the LLN of $Q_t(y,z)$ towards $\pi(y)r(y,z)$ discussed in Section 2.1.*

The next statement is deduced from Lemma 3.1 by choosing there $F(y,z) = \log[u(z)/u(y)]$, $(y,z) \in E$ for some $u: V \to (0, +\infty)$.

Lemma 3.4. *Let $u: V \to (0, +\infty)$ be such that $\sum_z r(y,z)u(z) < +\infty$ for any $y \in V$. For $t \geq 0$ define $\mathcal{M}_t^u : D(\mathbb{R}_+, V) \to (0, +\infty)$ as*

$$\mathcal{M}_t^u := \frac{u(X_t)}{u(X_0)} \exp\left\{ t\left\langle \mu_t, -\frac{Lu}{u} \right\rangle \right\}. \tag{3.3}$$

Then for each $x \in V$ and $t \in \mathbb{R}_+$ it holds $\mathbb{E}_x(\mathcal{M}_t^u) \leq 1$.

3.2. **Exponential tightness.** We shall prove separately the exponential tightness of the empirical measure and of the empirical flow. We first discuss the case in which Condition 2.2 holds. Then the proof of the exponential tightness of the empirical measure is essentially a rewriting of the argument in [18] in the present setting. On the other hand, the proof of the exponential tightness of the empirical flow depends on the extra assumption $\sigma > 0$ in item (vi) of Condition 2.2.

Lemma 3.5. *Assume Condition 2.2 to hold and let the function v and the constants c, C_x, C, σ be as in Condition 2.2. Then for each $x \in V$ it holds*

$$\mathbb{E}_x\left(e^{T\langle \mu_T, v \rangle} \right) \leq \frac{C_x}{c}, \qquad \mathbb{E}_x\left(e^{T\sigma\langle \mu_T, r \rangle} \right) \leq e^{TC}\frac{C_x}{c}. \tag{3.4}$$

Proof. The second bound in (3.4) follows trivially from the first one and item (vi) in Condition 2.2. To prove the first bound, let u_n be the sequence of functions on V provided by Condition 2.2 and recall that $v_n = -Lu_n/u_n$. In view of the pointwise convergence of v_n to v and Fatou lemma

$$\mathbb{E}_x\left(e^{T\langle \mu_T, v \rangle} \right) \leq \varliminf_n \mathbb{E}_x\left(e^{T\langle \mu_T, v_n \rangle} \right) = \varliminf_n \mathbb{E}_x\left(\exp\left\{ T\left\langle \mu_T, -\frac{Lu_n}{u_n} \right\rangle \right\} \right) \leq \frac{C_x}{c}$$

where the last step follows from Lemma 3.4 and items (ii)–(iii) in Condition 2.2. $\quad \square$

The following provides the exponential tightness of the empirical measure and the empirical flow.

Proposition 3.6. *Assume Condition 2.2. For each $x \in V$ there exists a sequence $\{\mathcal{K}_\ell\}$ of compacts in $\mathcal{P}(V)$ and a real sequence $A_\ell \uparrow +\infty$ such that for any $\ell \in \mathbb{N}$*

$$\varlimsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x\big(\mu_T \notin \mathcal{K}_\ell\big) \leq -\ell\,, \tag{3.5}$$

$$\varlimsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x\big(\|Q_T\| > A_\ell\big) \leq -\ell\,. \tag{3.6}$$

In particular, the empirical measure and flow are exponentially tight.

Proof. We first prove (3.5). For a sequence $a_\ell \uparrow +\infty$ to be chosen later, set $W_\ell := \{x \in V : v(x) \leq a_\ell\}$. In view of item (v) in Condition 2.2, W_ℓ is a compact subset of V. Set now

$$\mathcal{K}_\ell := \bigcap_{m \geq \ell} \Big\{ \mu \in \mathcal{P}(V) \,:\, \mu\big(W_m^{\mathrm{c}}\big) \leq \frac{1}{m} \Big\}$$

and observe that, by Prohorov theorem, $\mathcal{K}_\ell$ is a compact subset of $\mathcal{P}(V)$.

From item (vi) in Condition 2.2 (for this step we only need it with $\sigma = 0$) and the definition of W_ℓ we deduce $v \geq a_\ell \mathbb{1}_{W_\ell^{\mathrm{c}}} - C$. By the exponential Chebyshev inequality and Lemma 3.5 we then get

$$\mathbb{P}_x\Big(\mu_T\big(W_\ell^{\mathrm{c}}\big) > \frac{1}{\ell}\Big) \leq \mathbb{P}_x\Big(\langle \mu_T, v \rangle > \frac{a_\ell}{\ell} - C\Big)$$

$$\leq \exp\Big\{ -T\Big[\frac{a_\ell}{\ell} - C\Big]\Big\} \mathbb{E}_x\Big(e^{T\langle \mu_T, v \rangle}\Big) \leq \frac{C_x}{c} \exp\Big\{ -T\Big[\frac{a_\ell}{\ell} - C\Big]\Big\}.$$

By choosing $a_\ell = \ell^2 + C\ell$ the proof is now easily concluded.

Let us now prove (3.6). By the second bound in Lemma 3.5 and Chebyshev inequality, $\mathbb{P}_x\big(\langle \mu_T, r \rangle > \lambda\big) \leq \frac{C_x}{c}\, e^{-T(\sigma\lambda - C)}$ for any $\lambda > 0$. In particular we obtain that

$$\mathbb{P}_x\big(\langle \mu_T, r \rangle > A_\ell'\big) \leq \frac{C_x}{c} e^{-T\ell}\,, \qquad A_\ell' := \sigma^{-1}(\ell + C)\,.$$

Hence, it is enough to show that for each $x \in V$ there exists a sequence $A_\ell \uparrow +\infty$ such that for any $T > 0$ and any $\ell \in \mathbb{N}$

$$\mathbb{P}_x\Big(\|Q_T\| > A_\ell\,,\ \langle \mu_T, r \rangle \leq A_\ell'\Big) \leq e^{-T\ell}\,. \tag{3.7}$$

We consider the exponential local martingale of Lemma 3.1 choosing there $F \colon E \to \mathbb{R}$ constant, $F(x,y) = \lambda \in (0, +\infty)$ for any $(x,y) \in E$. We deduce

$$\mathbb{P}_x\Big(\|Q_T\| > A_\ell\,,\ \langle \mu_T, r \rangle \leq A_\ell'\Big)$$

$$= \mathbb{E}_x\Big(e^{-T\big[\lambda\|Q_T\| - (e^\lambda - 1)\langle \mu_T, r \rangle\big]}\, \mathbb{M}_T^F\, \mathbb{1}_{\{\|Q_T\| > A_\ell\}}\, \mathbb{1}_{\{\langle \mu_T, r \rangle \leq A_\ell'\}}\Big)$$

$$\leq \exp\big\{ -T\big[\lambda A_\ell - (e^\lambda - 1)A_\ell'\big]\big\}$$

where we used Lemma 3.1 in the last step. The proof of (3.7) is now completed by choosing $A_\ell = \lambda^{-1}\ell + \lambda^{-1}(e^\lambda - 1)A_\ell'$.

Recalling that the closed ball in $L_+^1(E)$ is compact with respect to the bounded weak* topology, the exponential tightness of the empirical flow is due to (3.6). $\square$

We next discuss the exponential tightness when Condition 2.4 is assumed.

Proposition 3.7. *Fix $x \in V$. If item (i) in Condition 2.4 holds then the sequence of probabilities $\{\mathbb{P}_x \circ \mu_T^{-1}\}$ on $\mathcal{P}(V)$ is exponentially tight. If furthermore it holds also item (ii) in Condition 2.4, then the sequence of probabilities $\{\mathbb{P}_x \circ Q_T^{-1}\}$ on $L_+^1(E)$ is exponentially tight.*

While the first statement is a consequence of the general results in [17], we next give a direct and alternative proof also of this result. We premise an elementary lemma whose proof is omitted.

Lemma 3.8. *Let $\pi \in \mathcal{P}(V)$ be such that $\pi(x) > 0$ for any $x \in V$. There exists a decreasing function $\psi_\pi \colon (0,1) \to (0,+\infty)$ such that $\lim_{s\downarrow 0} \psi_\pi(s) = +\infty$ and*

$$\sum_{x \in V} \pi(x)\, \psi_\pi(\pi(x)) < +\infty.$$

Proof of Proposition 3.7. We prove first the exponential tightness of the empirical measure. Let π be the invariant measure of the chain, ψ_π be the function provided by Lemma 3.8 and $\alpha := \sum_x \pi(x)\psi_\pi(\pi(x)) < +\infty$. We define $v \colon V \to (0,+\infty)$ as

$$v(x) := \log \frac{\psi_\pi(\pi(x))}{\alpha}\,, \qquad x \in V.$$

Then, in view of Lemma 3.8, v has compact level sets and $\langle \pi, e^v \rangle = 1$.

By the proof of Proposition 3.6, it is enough to show the following bound. For each $x \in V$ there exist constants $\lambda, C_x > 0$ such that for any $T > 0$

$$\mathbb{E}_x\left(e^{\lambda T \langle \mu_T, v \rangle}\right) \le C_x. \tag{3.8}$$

Since the function v diverges at infinity, it is bounded from below and has finite level sets $V_n := \{x \in V : v(x) \le n\}$. We define $v_n(x) := v(x)\mathbb{1}_{x \in V_n}$ and set for $x, y \in V$

$$r_n(x,y) := \begin{cases} r(x,y) & \text{if } x \in V_n \\ r(x,y)/r(x) & \text{if } x \notin V_n \end{cases} \qquad \pi_n(x) := \begin{cases} \frac{\pi(x)}{Z_n} & \text{if } x \in V_n \\ \frac{\pi(x)r(x)}{Z_n} & \text{if } x \notin V_n \end{cases}$$

where Z_n is the normalizing constant making π_n a probability measure on V. Due to Condition 2.4 it holds $\langle \pi, r \rangle < +\infty$, thus implying that Z_n is well defined and that $\lim_{n\to\infty} Z_n = 1$.

We notice that the function $r_n \colon V \to (0,+\infty)$, $r_n(x) := \sum_{y \in V} r_n(x,y)$, is bounded from above. We then consider the continuous–time Markov chain $\xi^{(n)}$ in V with transition rates $r_n(\cdot, \cdot)$. Since $\pi_n(x)r_n(x,y) = \pi(x)r(x,y)/Z_n$, we derive that π_n is the unique invariant distribution of $\xi^{(n)}$. We denote by $\mathbb{E}_x^{(n)}$ the expectation w.r.t. the law of the Markov chain $\xi^{(n)}$ starting at x and by $\mathcal{A}_n$ the subset of $D([0,T];V)$ defined as $\mathcal{A}_n = \{X : X_t \in V_n \ \forall t \in [0,T]\}$. Then we have

$$\mathbb{E}_x\left(e^{\lambda T \langle \mu_T, v \rangle}\right) = \lim_{n\to\infty} \mathbb{E}_x\left(e^{\lambda T \langle \mu_T, v_n \rangle}\mathbb{1}_{\mathcal{A}_n}\right) = \lim_{n\to\infty} \mathbb{E}_x^{(n)}\left(e^{\lambda T \langle \mu_T, v_n \rangle}\mathbb{1}_{\mathcal{A}_n}\right) \tag{3.9}$$

(the first identity follows from the monotone convergence theorem). Since v_n and r_n are bounded function, we can apply [28, App. 1, Lemma 7.2] and get

$$\begin{aligned} \mathbb{E}_x^{(n)}\left(e^{\lambda T \langle \mu_T, v_n \rangle}\mathbb{1}_{\mathcal{A}_n}\right) &\le \frac{1}{\pi_n(x)}\mathbb{E}_{\pi_n}^{(n)}\left(e^{\lambda T \langle \mu_T, v_n \rangle}\right) \\ &\le \frac{1}{\pi_n(x)} \exp\left\{T \sup_{f \,:\, \pi_n(f^2)=1}\left[-D_{\pi_n}^{(n)}(f) + \lambda \langle \pi_n, f^2\, v_n \rangle\right]\right\}. \end{aligned} \tag{3.10}$$

Since v_n vanishes on V_n^{c} and $v_n \le v$ we have $\langle \pi_n, f^2 v_n \rangle \le Z_n^{-1} \langle \pi, f^2 v \rangle$, while from the identity $\pi_n(x) r_n(x, y) = \pi(x) r(x, y)/Z_n$ we get $D_{\pi_n}^{(n)}(f) = Z_n^{-1} D_\pi(f)$. Hence

$$- D_{\pi_n}^{(n)}(f) + \lambda \langle \pi_n, f^2 v_n \rangle \le \frac{1}{Z_n} \big[- D_\pi(f) + \lambda \langle \pi, f^2 v \rangle \big]. \tag{3.11}$$

We next show that if $\lambda \in (0, 1/c_{\mathrm{LS}})$ then the right hand side in (3.11) is bounded above by zero whenever $\pi(f^2) < +\infty$ (note that, in view of Remark 2.5 and since $\pi_n(f^2) = 1$, it holds $\pi(f^2) < +\infty$). To this aim let $f_* := f/\sqrt{\pi(f^2)}$, hence $\pi(f_*^2) = 1$. The basic entropy inequality yields

$$\langle \pi, f_*^2 v \rangle \le \log \langle \pi, e^v \rangle + \mathrm{Ent}(\mu|\pi), \qquad \mu = f_*^2 \pi.$$

Recalling that $\langle \pi, e^v \rangle = 1$, the logarithmic Sobolev inequality (2.10) implies that $\langle \pi, f_*^2 v \rangle \le c_{\mathrm{LS}} D_\pi(f_*)$, hence our claim. The bound (3.8) follows, thus concluding the proof of the exponential tightness of the empirical measure.

To prove the exponential tightness of the empirical flow, we first observe that the only properties of v used to derive (3.8) are that v has compact level sets and satisfies $\langle \pi, e^v \rangle = 1$. By Remark 2.5 and item (ii) in Condition 2.4, the function $\widetilde{v} = \sigma r - \log \langle \pi, e^{\sigma r} \rangle$ meets these two requirements. Hence the bound (3.8) holds for $\widetilde{v}$ which implies

$$\mathbb{E}_x \Big(e^{T\sigma' \langle \mu_T, r \rangle} \Big) \le e^{TC} C_x. \tag{3.12}$$

for some $\sigma', C, C_x > 0$. In view of (3.8) and (3.12), the proof of the exponential tightness of the empirical flow is achieved by the argument leading to (3.6). $\square$

We conclude with a simple observation on the stationary flow:

Lemma 3.9. *Assume at least one between Conditions 2.2 and 2.4 to hold. Then $\langle \pi, r \rangle < +\infty$, equivalently $Q^\pi \in L_+^1(E)$.*

Proof. The thesis is trivially true under Condition 2.4. Let us assume Condition 2.2. By Lemma 3.5 we have $\mathbb{E}_x \big(e^{T\sigma \langle \mu_T, r \rangle} \big) \le e^{TC} C_x/c$. We restrict to V infinite, the finite case being obvious. Enumerating the points in V as $\{x_n\}_{n \ge 0}$, by the ergodic theorem (2.2) fixed N there exists a time $T_0 = T_0(N) > 0$ and a Borel set $\mathcal{A} \subset D(\mathbb{R}_+; V)$ such that (i) $\mathbb{P}_x(\mathcal{A}) \ge 1/2$ and (ii) $\mu_T(x_n) \ge \pi(x_n)/2$ for all $T \ge T_0$ and $n \le N$ $\mathbb{P}_x$–a.s. on $\mathcal{A}$. Hence, for all $T \ge T_0$ it holds

$$e^{T\sigma \sum_{n=0}^N \pi(x_n) r(x_n)/2}/2 \le \mathbb{E}_x \big(e^{T\sigma \sum_{n=0}^N \mu_T(x_n) r(x_n)}; \mathcal{A} \big) \le \mathbb{E}_x \big(e^{T\sigma \langle \mu_T, r \rangle} \big) \le e^{TC} C_x/c.$$

This implies that $\sum_{n=0}^N \pi(x_n) r(x_n) \le 2C/\sigma$. To conclude it is enough to take the limit $N \to +\infty$. $\square$

4. Structure of divergence–free flows in $L_+^1(E)$

In this section we show that any divergence–free flow in $L_+^1(E)$, and more in general any divergence–free flow in $\mathbb{R}_+^E$ with zero flux towards infinity, can be written as superposition of flows along self avoiding finite cycles. See [23] for other problems related to cyclic decompositions of divergence–free flows on graphs and [38] for similar decompositions for divergence–free vector valued measures on $\mathbb{R}^d$.

We first introduce some key graphical structures. A *finite cycle C* in the oriented graph (V, E) is a sequence $(x_1, \dots, x_k)$ of elements of V such that $(x_i, x_{i+1}) \in E$ when $i = 1, \dots, k$ and the sum in the indices is modulo k. A finite cycle is *self avoiding* if for $i \ne j$ it holds $x_i \ne x_j$. Given $(x, y) \in E$, if there exists an index $i = 1, \dots, k$ such that $(x, y) = (x_i, x_{i+1})$ we write $(x, y) \in C$. Similarly, given

$x \in V$, if there exists an index $i = 1, \ldots, k$ such that $x = x_i$ we say that $x \in C$. The collection of all the self avoiding finite cycles in (V, E) is a countable set which we denote by $\mathcal{C}$. In the sequel we shall mostly regard elements $C \in \mathcal{C}$ as finite subsets of E and denote by $|C|$ the corresponding cardinality. Consider an invading sequence $V_n \nearrow V$ of finite subsets V_n. This means a sequence such that $|V_n| < +\infty$, $V_n \subset V_{n+1}$ and moreover $\cup_n V_n = V$. For any fixed n we define

$$E_n := \{(y, z) \in E \,:\, y, z \in V_n\} \,, \tag{4.1}$$

and observe that it is an invading sequence of edges. Given a flow $Q \in \mathbb{R}_+^E$, we define

$$E(Q) := \{(y, z) \in E \,:\, Q(y, z) > 0\} \,, \tag{4.2}$$

$$M_n(Q) := \max_{(y, z) \in E_n} Q(y, z) \,, \tag{4.3}$$

$$\phi_n^+(Q) := \sum_{y \in V_n, z \notin V_n} Q(y, z) \,, \tag{4.4}$$

$$\phi_n^-(Q) := \sum_{y \notin V_n, z \in V_n} Q(y, z) \,. \tag{4.5}$$

The definition (2.4) of the divergence of a flow Q is well posed also if $Q \notin L_+^1(E)$ provided the incoming and outgoing fluxes are finite at every vertex. In this case the series in (4.4) and (4.5) are convergent. By a divergence–free flow $Q \in \mathbb{R}_+^E$ we mean that Q has well defined vanishing divergence. Moreover, we say that Q has zero flux towards infinity if there exists an invading sequence $V_n \nearrow V$ of finite subsets V_n such that

$$\lim_{n \to +\infty} \phi_n^+(Q) = 0 \,. \tag{4.6}$$

Finally, we say that Q admits a cyclic decomposition if there are constants $\widehat{Q}(C) \geq 0$, $C \in \mathcal{C}$ such that

$$Q = \sum_{C \in \mathcal{C}} \widehat{Q}(C) \mathbb{1}_C \,. \tag{4.7}$$

Namely, for each $(y, z) \in E$ it holds $Q(y, z) = \sum_{C \in \mathcal{C}, \, C \ni (y, z)} \widehat{Q}(C)$. We emphasize that the constants $\widehat{Q}(C)$, $C \in \mathcal{C}$, are not uniquely determined by the flow Q.

Lemma 4.1. *Let $Q \in \mathbb{R}_+^E$ be a divergence–free flow having zero flux towards infinity. Then Q admits a cyclic decomposition* (4.7). *In particular, any divergence–free flow $Q \in L_+^1(E)$ has a cyclic decomposition.*

Proof. Since (4.6) holds for any invading sequence of vertices if $Q \in L_+^1(E)$, the second statement follows directly from the former on which we concentrate.

On a finite graph any divergence–free flow admits a cyclic decomposition. The proof follows classical arguments (see e.g. [23, 33]). If Q has finite support, i.e. if $|E(Q)| < +\infty$, the thesis follows directly by the analogous result on finite graphs. We will then consider only the case of infinite support, using below the result in the finite case. Let V_n be an invading sequence satisfying (4.6).

We assume $|E(Q)| = +\infty$ and $\operatorname{div} Q = 0$. Due to the zero divergence condition, a discrete version of the Gauss theorem guarantees that $\phi_n^+(Q) = \phi_n^-(Q)$. We define by an iterative procedure a sequence of flows Q^i, $i \geq 0$, with infinite support and having zero flux towards infinity as follows. We set $Q^0 := Q$ and explain how to define Q^{i+1} knowing Q^i. First, we define $n_i := \inf \{n \in \mathbb{N} \,:\, M_n(Q^i) > \phi_n^+(Q^i)\}$.

Since $Q^i \neq 0$, it must be $n_i < +\infty$. Indeed, $\phi_n^+(Q^i)$ is a sequence in n converging to zero, while $M_n(Q^i)$ is a non decreasing sequence not identically zero. Let g be a ghost site and define the flow Q_g^i on a finite graph having vertices $V_{n_i} \cup \{g\}$ as

$$
\begin{cases}
Q_g^i(y,z) := Q^i(y,z), & (y,z) \in E_{n_i}, \\
Q_g^i(y,g) := \sum_{z \notin V_{n_i}} Q^i(y,z), & y \in V_{n_i}, \\
Q_g^i(g,y) := \sum_{z \notin V_{n_i}} Q^i(z,y), & y \in V_{n_i}.
\end{cases}
$$

Roughly speaking, the flow Q_g^i is obtained from Q^i by collapsing all vertices outside V_{n_i} into a single vertex, called g. By construction we have $\operatorname{div} Q_g^i = 0$. Calling $\mathcal{C}_{n_i}^g$ the collection of self avoiding cycles of the finite graph and using the validity of the cyclic decomposition in the finite case, we have

$$
Q_g^i = \sum_{C \in \mathcal{C}_{n_i}^g} \widehat{Q}_g^i(C) \mathbb{I}_C. \tag{4.8}
$$

We claim that in the decomposition (4.8) there exists a self avoiding cycle C_i not visiting the ghost site g and such that $\widehat{Q}_g^i(C_i) > 0$. Let us suppose by contradiction that our claim is false and let $(x^*, y^*) \in E_{n_i}$ be such that $Q^i(x^*, y^*) = M_{n_i}(Q^i)$. Then we have

$$
M_{n_i}(Q^i) = Q^i(x^*, y^*) = \sum_{C \in \mathcal{C}_{n_i}^g} \widehat{Q}_g^i(C) \mathbb{I}_{(x^*,y^*) \in C} \leq \sum_{C \in \mathcal{C}_{n_i}^g} \widehat{Q}_g^i(C) = \phi_{n_i}^+(Q^i).
$$

The last equality follows by the fact that any cycle with positive weight in $\mathcal{C}_{n_i}^g$ has to contain necessarily the ghost site g. This contradicts the definition of n_i, thus proving our claim.

At this point, we know that there exists a self avoiding cycle $C_i := (x_1, \ldots, x_k)$ such that $x_j \in V_{n_i}$ and $Q^i(x_j, x_{j+1}) > 0$ for any j (the sum in the indices is modulo k). We fix $m_i := \min_{j=1,\ldots,k} Q^i(x_j, x_{j+1})$ and define

$$
Q^{i+1} := Q^i - m_i \mathbb{I}_{C_i} = Q - \sum_{j=0}^i m_j \mathbb{I}_{C_j}.
$$

With this definition we have that Q^{i+1} is an element of $\mathbb{R}_+^E$, it satisfies $\operatorname{div} Q^{i+1} = 0$, it has zero flux towards infinity, and infinite support. Moreover

$$
|E_{n_i} \cap E(Q^{i+1})| \leq |E_{n_i} \cap E(Q^i)| - 1. \tag{4.9}
$$

Condition (4.9) implies that $\lim_{i \to +\infty} n_i = +\infty$. Hence, fixed any $(y,z) \in E$, for i large it holds

$$
Q^i(y,z) \leq M_{n_i-1}(Q^i) \leq \phi_{n_i-1}^+(Q^i) \leq \phi_{n_i-1}^+(Q) \tag{4.10}
$$

(for the first inequality note that $(y,z) \in E_{n_i-1}$ for i large, for the second one use the definition of n_i, for the third one observe that by construction $Q^i \leq Q$).

Since the right hand side of (4.10) converges to zero when i diverges we obtain $\lim_{i \to +\infty} Q^i(y,z) = 0$ for any $(y,z) \in E$. Finally we get

$$
\lim_{i \to +\infty} \left(Q(y,z) - \sum_{j=0}^i m_j \mathbb{I}_{C_j}(y,z) \right) = \lim_{i \to +\infty} Q^{i+1}(y,z) = 0.
$$

This trivially implies that $Q = \sum_{j=0}^\infty m_j \mathbb{I}_{C_j}$. $\qquad\square$

Remark 4.2. *It is easy to see that Lemma 4.1 remains valid if the condition of zero flux towards infinity is satisfied just by the reduced flow $q \in \mathbb{R}_+^E$ defined as*

$$q(y,z) := \begin{cases} Q(y,z) & (z,y) \notin E, \\ Q(y,z) - \min\{Q(y,z), Q(z,y)\} & (z,y) \in E. \end{cases}$$

Given an oriented graph $(\mathcal{V}, \mathcal{E})$ with countable $\mathcal{V}, \mathcal{E}$ we say that it is connected if for any $y, z \in \mathcal{V}$ there exist $x_1, \ldots, x_n$ such that $x_1 = y$, $x_n = z$ and $(x_i, x_{i+1}) \in \mathcal{E}$, $i = 1, \ldots, n-1$. To every oriented graph we can associate an unoriented graph $(\mathcal{V}, \mathcal{E}^u)$ for which $\{y, z\} \in \mathcal{E}^u$ if at least one among (y, z) and (z, y) belongs to $\mathcal{E}$. We say that the unoriented graph $(\mathcal{V}, \mathcal{E}^u)$ is connected if for any $y, z \in \mathcal{V}$ there exist $x_1, \ldots, x_n$ such that $x_1 = y$, $x_n = z$ and $\{x_i, x_{i+1}\} \in \mathcal{E}^u$, $i = 1, \ldots, n-1$.

The following lemma will be useful.

Lemma 4.3. *Let $(\mathcal{V}, \mathcal{E})$ be a countable oriented graph. Then $(\mathcal{V}, \mathcal{E})$ is connected if and only if* (i) *$(\mathcal{V}, \mathcal{E}^u)$ is connected and* (ii) *there exists a flow $Q \in L_+^1(\mathcal{E})$ with $Q(y,z) > 0$ for any $(y,z) \in \mathcal{E}$ and $\operatorname{div} Q = 0$.*

Proof. Suppose first that $(\mathcal{V}, \mathcal{E})$ is connected. Then $(\mathcal{V}, \mathcal{E}^u)$ is also trivially connected. To show property (ii), since $\mathcal{C}$ is countable. we can find a sequence $\{\alpha_C, C \in \mathcal{C}\}$ with $\alpha_C > 0$ and $\sum_C \alpha_C < +\infty$. We then define $Q = \sum_C \alpha_C \mathbb{1}_C$ which is summable and divergence–free. It remains to check that $Q(y,z) > 0$ for any $(y,z) \in \mathcal{E}$. Since $(\mathcal{V}, \mathcal{E})$ is connected, we can add to (y,z) an oriented path from z to y obtaining a finite cycle $C \ni (y,z)$.

We now prove the converse implication. In view of Lemma 4.1, the flow Q in (ii) admits a cyclic decomposition (4.7). If $(y,z) \in \mathcal{E}$ then $0 < Q(y,z) = \sum_{C \ni (y,z)} \widehat{Q}(C)$. Thus there exists a finite cycle C containing (y,z), hence there exists an oriented path from z to y. This shows that neighbors in $(\mathcal{V}, \mathcal{E}^u)$ are connected in $(\mathcal{V}, \mathcal{E})$. $\qquad\square$

We can now give the proof of Proposition 2.8.

Proof of Proposition 2.8.
Proof of (i). Fix $(y,z) \in E$ with $Q(y,z) > 0$. From the definition of $I(\mu, Q)$ and Φ we deduce that $\mu(y)r(y,z) > 0$ and therefore $\mu(y) > 0$. Since $\operatorname{div} Q = 0$ and the ingoing flow in z is strictly positive then there exists $(z, y') \in E$ with $Q(z, y') > 0$ hence, by what just proven, $\mu(z) > 0$.
Proof of (ii). It is an immediate consequence of Lemma 4.3.
Proof of (iii). To this aim we first observe that $\operatorname{div} Q_j = 0$. Indeed, the following property (P) holds: given $y \in V$, if $Q(y,z) > 0$ or $Q(z,y) > 0$ then z belongs to the same oriented connected component of y (apply Item (ii)). This property and the zero divergence of Q imply that $\operatorname{div} Q_j = 0$. By definition (2.12) and Remark 2.6,

$$I(\mu_j, Q_j) = \sum_{(y,z) \in E \cap (K_j \times K_j)} \Phi\big(Q_j(y,z), Q^{\mu_j}(y,z)\big) + \sum_{(y,z) \in E \cap (K_j \times K_j^c)} Q^{\mu_j}(y,z). \tag{4.11}$$

Always property (P) implies that

$$I(\mu, Q) = \sum_j \left\{ \sum_{(y,z) \in E \cap (K_j \times K_j)} \Phi\big(Q(y,z), Q^\mu(y,z)\big) + \sum_{(y,z) \in E \cap (K_j \times K_j^c)} Q^\mu(y,z) \right\}. \tag{4.12}$$

To conclude compare (4.11) with (4.12) using that $Q(y,z) = \mu(K_j)Q_j(y,z)$ and $Q^\mu(y,z) = \mu(K_j)Q^{\mu_j}(y,z)$ if $(y,z) \in E$ with $y \in K_j$. $\qquad\square$

4.1. An approximation result for the function $I(\mu,Q)$.

Let $\mathcal{S}$ be the subset of $\mathcal{P}(V) \times L^1_+(E)$ given by the elements (μ,Q) with $I(\mu,Q) < +\infty$ and such that the graph $(\mathrm{supp}(\mu), E(Q))$ is finite and connected.

Proposition 4.4. *Fix $(\mu,Q) \in \mathcal{P}(V) \times L^1_+(E)$. There exists a sequence $\{(\mu_n,Q_n)\}$ in $\mathcal{S}$ such that $(\mu_n,Q_n) \to (\mu,Q)$ in $\mathcal{P}(V) \times L^1_+(E)$ and*

$$\varlimsup_{n \to +\infty} I(\mu_n,Q_n) \leq I(\mu,Q). \tag{4.13}$$

As proven below, the convergence $(\mu_n,Q_n) \to (\mu,Q)$ in $\mathcal{P}(V) \times L^1_+(E)$ holds also with $L^1_+(E)$ endowed with the L^1–norm (strong topology).

Proof. We consider only elements (μ,Q) such that $I(\mu,Q) < +\infty$, otherwise the thesis is trivially true. In particular, $\mathrm{div}\, Q = 0$. Denote by $\mathcal{S}^*$ the set of elements $(\mu,Q) \in \mathcal{P}(V) \times L^1_+(E)$ with finite support (i.e. with finite $\mathrm{supp}(\mu)$ and $E(Q)$) and $\mathrm{div}\, Q = 0$. We first show that (4.13) holds for $(\mu,Q) \in \mathcal{S}^*$

Let $(\mu,Q) \in \mathcal{S}^*$. Then there exists a finite connected oriented subgraph (V^*, E^*) of (V,E) which contains $(\mathrm{supp}(\mu), E(Q))$ (add to $(\mathrm{supp}(\mu), E(Q))$ suitable paths joining the connected components of $(\mathrm{supp}(\mu), E(Q))$). Denote by r^* the restriction of r to E^* and let π^* be the (unique) invariant probability of the chain with rates r^* on the graph (V^*, E^*). Set also $Q^*(y,z) := \pi^*(y)r^*(y,z)$ and extend π^*, Q^* to functions on V, E by setting them equal to zero outside V^*, E^*. Due to the invariance of π^*, $\mathrm{div}\, Q^* = 0$. Moreover, it holds

$$I(\pi^*, Q^*) = \sum_{(y,z) \in E^*} \Phi(Q^*(y,z), \pi^*(y)r(y,z)) + \sum_{(y,z) \notin E^*} \pi^*(y)r(y,z).$$

As the first sum is a finite sum of finite terms and the second one is bounded by $\langle \pi^*, r \rangle$, we deduce $I(\pi^*, Q^*) < +\infty$. We define the sequence $\left\{(1 - \frac{1}{n})(\mu,Q) + \frac{1}{n}(\pi^*, Q^*)\right\}$ which belongs to $\mathcal{S}$ and converges to (μ,Q) (even with $L^1_+(E)$ endowed of the strong topology). By the convexity of I stated in Remark 2.6

$$\varlimsup_{n \to \infty} I\left((1 - \frac{1}{n})(\mu,Q) + \frac{1}{n}(\pi^*, Q^*)\right) \leq I(\mu,Q).$$

Given $(\mu,Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $\mathrm{div}\, Q = 0$, we now show that there exists a sequence $\{(\mu_n,Q_n)\} \subset \mathcal{S}^*$ such that (4.13) holds. The thesis then follows by a diagonal argument. We fix (μ,Q) with $\mathrm{div}\, Q = 0$ and $I(\mu,Q) < +\infty$. By Lemma 4.1 the cyclic decomposition (4.7) of Q holds. We fix an invading sequence $V_n \nearrow V$ of finite subsets and call E_n the edges in E connecting vertices in V_n (recall (4.1)). Finally, we construct the sequence $(\mu_n,Q_n) \in \mathcal{S}^*$ by

$$\mu_n := \frac{\mu\big|_{V_n}}{\mu(V_n)}, \qquad Q_n := \sum_{\{C \in \mathcal{C}\,:\,C \subset E_n\}} \widehat{Q}(C) \mathbb{1}_C.$$

For n large $\mu(V_n) > 0$ and the definition is well posed. Clearly (μ_n,Q_n) converges to (μ,Q) (also considering the strong topology of $L^1_+(E)$). It remains to show (4.13). By construction $\mathrm{div}\, Q_n = 0$ and $\langle \mu_n, r \rangle < +\infty$, hence, recalling (2.12),

$$I(\mu_n,Q_n) = \sum_{y \in V_n} \sum_{z \in V:(y,z) \in E} \Phi\big(Q_n(y,z), Q^{\mu_n}(y,z)\big). \tag{4.14}$$

We claim that $\Phi\big(Q_n(y,z), Q^{\mu_n}(y,z)\big) = 0$ if (y,z) is as in the above sum and $Q^{\mu_n}(y,z) = 0$. Since $y \in V_n$ then $Q^\mu(y,z) = \mu(V_n)Q^{\mu_n}(y,z) = 0$. As $I(\mu, Q) < +\infty$ it follows $Q(y,z) = 0$ and therefore $Q_n(y,z) = 0$, which concludes the proof of the claim. As a consequence, we can restrict the sum in (4.14) to $Q^{\mu_n}(y,z) > 0$

Recall the definition of Φ given in (2.11). Given $0 \le q' \le q$ and $p' \ge p > 0$, let $\alpha, \beta \ge 0$ be respectively defined by $q' = q(1-\alpha)$ and $p' = p(1+\beta)$. Then we have

$$\Phi(q',p') - \Phi(q,p) = q'\Big(\log\frac{q'}{p'} - \log\frac{q}{p}\Big) + (q'-q)\log\frac{q}{p} + (q-q') + (p'-p)$$

$$\le (q'-q)\log\frac{q}{p} + (q-q') + (p'-p) = -\alpha\,\Phi(q,p) + (\alpha+\beta)\,p \le (\alpha+\beta)\,p.$$
$$(4.15)$$

By construction, it holds $\mu_n(y) \ge \mu(y)$ for $y \in V_n$ and $Q_n(y,z) \le Q(y,z)$ for $(y,z) \in E_n$. We set $\beta_n := [\mu(V_n)]^{-1} - 1$ and $\alpha_n \colon E_n \to [0,1]$ be defined by $Q_n(y,z) = Q(y,z)\big[1 - \alpha_n(y,z)\big]$ when $(y,z) \in E(Q)$. From (4.15) we then obtain

$$I(\mu_n, Q_n) \le I(\mu, Q) + \sum_{y \in V_n\; z \in V:(y,z) \in E} \big[\beta_n + \alpha_n(y,z)\big]\,\mu(y)r(y,z).$$

Since $I(\mu, Q) < +\infty$ then it holds $\langle \mu, r \rangle < +\infty$. Since $\beta_n, \alpha_n(y,z) \downarrow 0$ and the maps $\alpha_n(\cdot)$ are uniformly bounded, by dominated convergence we conclude the proof of (4.13). $\qquad\square$

5. Direct proof of Theorem 2.7

In this section we give a direct proof of Theorem 2.7, independent from the LDP for the empirical process. As already mentioned, the proof works only under the additional condition that the graph (V, E) is *locally finite* (cf. Condition 2.4–(iii)). This assumption implies that, given $\phi \in C_0(V)$, the function $\nabla\phi \colon E \to \mathbb{R}$ defined as $\nabla\phi(y,z) = \phi(y) - \phi(z)$ belongs to $C_0(E)$. As a consequence, the map

$$L^1_+(E) \ni Q \to \langle \phi, \operatorname{div} Q \rangle = -\langle \nabla\phi, Q \rangle \in \mathbb{R} \qquad (5.1)$$

is continuous. Since a linear functional on $L^1_+(E)$ is continuous w.r.t. the bounded weak* topology if and only if it is continuous w.r.t. the weak* topology [36], by definition of weak* topology the map defined in (5.1) is continuous (w.r.t. the bounded weak* topology) if and only if $\nabla\phi \in C_0(E)$. Hence, our additional condition is equivalent to the fact that (5.1) is continuous for any $\phi \in C_0(V)$. An explicit example of a not locally finite graph where (5.1) becomes not continuous for $\phi = \mathbb{1}_x$, $x \in V$, is given in Appendix B.

5.1. **Upper bound.** Given $\phi \in C_0(V)$ and $F \in C_c(E)$ (i.e. ϕ vanishes at infinity and F is nonzero only on a finite set) let $I_{\phi,F} \colon \mathcal{P}(V) \times L^1_+(E) \to \mathbb{R}$ be the map defined by

$$I_{\phi,F}(\mu, Q) := \langle \phi, \operatorname{div} Q \rangle + \langle Q, F \rangle - \langle \mu, r^F - r \rangle \qquad (5.2)$$

where $r^F \colon V \to (0, +\infty)$ is defined by $r^F(y) = \sum_{z \in V} r(y,z)e^{F(y,z)}$ and $\langle \phi, \operatorname{div} Q \rangle = \sum_{y \in V} \phi(y)\operatorname{div} Q(y)$.

Lemma 5.1. *Fix $x \in V$. For each $\phi \in C_0(V)$, $F \in C_c(E)$, and each measurable $\mathcal{B} \subset \mathcal{P}(V) \times L^1_+(E)$, it holds*

$$\varlimsup_{T \to +\infty} \frac{1}{T}\log\mathbb{P}_x\Big((\mu_T, Q_T) \in \mathcal{B}\Big) \le -\inf_{(\mu,Q) \in \mathcal{B}} I_{\phi,F}(\mu, Q).$$

Proof. Fix $x \in V$ and observe that the following pathways continuity equation holds $\mathbb{P}_x$ a.s.

$$\delta_y(X_T) - \delta_y(X_0) + T \operatorname{div} Q_T(X)(y) = 0 \qquad \forall \, y \in V. \tag{5.3}$$

Fix $F \in C_c(E)$ and $\phi \in C_0(V)$ and recall the semimartingale $\mathbb{M}^F$ introduced in Lemma 3.1. In view of (5.2) and (5.3), for each $T > 0$ and each measurable set $\mathcal{B} \subset \mathcal{P}(V) \times L^1_+(E)$

$$\mathbb{P}_x\big((\mu_T, Q_T) \in \mathcal{B}\big)$$

$$= \mathbb{E}_x\Big(\exp\big\{ -T \, I_{\phi,F}(\mu_T, Q_T) - [\phi(X_T) - \phi(x)] \big\} \, \mathbb{M}_T^F \, \mathbb{1}_{\mathcal{B}}(\mu_T, Q_T) \Big)$$

$$\leq \sup_{(\mu,Q) \in \mathcal{B}} e^{-T \, I_{\phi,F}(\mu,Q)} \, \mathbb{E}_x\Big(\exp\big\{ -[\phi(X_T) - \phi(x)] \big\} \, \mathbb{M}_T^F \, \mathbb{1}_{\mathcal{B}}(\mu_T, Q_T) \Big).$$

Since ϕ is bounded, the proof is now achieved by using Lemma 3.1. □

We can conclude the proof of the upper bound in Theorem 2.7. In view of the exponential tightness proven in Subsection 3.2, it is enough to prove (2.13) for compacts. Since the graph (V, E) is locally finite the map $I_{\phi,F}$ is continuous. Fix $x \in V$. By Lemma 5.1 and the min-max lemma in [28, App. 2, Lemma 3.3] for each compact $\mathcal{K} \subset \mathcal{P}(V) \times L^1_+(E)$ it holds

$$\varlimsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x\Big((\mu_T, Q_T) \in \mathcal{K}\Big) \leq - \inf_{(\mu,Q) \in \mathcal{K}} \sup_{\phi, F} I_{\phi,F}(\mu, Q)$$

where the supremum is carried out over all $\phi \in C_0(V)$ and $F \in C_c(E)$. Recalling (2.12), it is now simple to check (see Appendix A) that for each $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ it holds

$$I(\mu, Q) = \sup_{\phi, F} I_{\phi,F}(\mu, Q), \tag{5.4}$$

which concludes the proof of the upper bound.

5.2. **Lower bound.** Recall the following general result concerning the large deviation lower bound.

Lemma 5.2. *Let $\{P_n\}$ be a sequence of probability measures on a completely regular topological space $\mathcal{X}$. Fix $J \colon \mathcal{X} \to [0, +\infty]$ and assume that for each $x \in \mathcal{X}$ there exists a sequence of probability measures $\{\widetilde{P}_n^x\}$ weakly convergent to δ_x and such that*

$$\varlimsup_{n \to \infty} \frac{1}{n} \operatorname{Ent}\big(\widetilde{P}_n^x \big| P_n\big) \leq J(x). \tag{5.5}$$

Then the sequence $\{P_n\}$ satisfies the large deviation lower bound with rate function given by $\mathrm{sc}^- J$, the lower semicontinuous envelope of J, i.e.

$$\big(\mathrm{sc}^- J\big)(x) := \sup_{U \in \mathcal{N}_x} \inf_{y \in U} J(y)$$

where $\mathcal{N}_x$ denotes the collection of the open neighborhoods of x.

This lemma has been originally proven in [26, Prop. 4.1] in a Polish space setting. The proof given in [34, Prop. 1.2.4] applies also to the present setting of a completely regular topological space.

Recall the definition of the set $\mathcal{S}$ given before Proposition 4.4: $\mathcal{S}$ is given by the elements $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $I(\mu, Q) < +\infty$ and such that the graph $(\operatorname{supp}(\mu), E(Q))$ is finite and connected.

First we prove the entropy bound (5.5) with J given by the restriction of I, as defined in (2.12), to $\mathcal{S}$, that is

$$J(\mu, Q) := \begin{cases} I(\mu, Q) & \text{if } (\mu, Q) \in \mathcal{S} \\ +\infty & \text{otherwise.} \end{cases} \tag{5.6}$$

Then we complete the proof of the lower bound (2.14) by showing that the lower semicontinuous envelope of J coincides with I.

Lemma 5.3. *Fix $x \in V$ and set $P_T := \mathbb{P}_x \circ (\mu_T, Q_T)^{-1}$. For each $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ there exists a sequence $\{\widetilde{P}_T^{(\mu,Q)}\}$ of probability measures on $\mathcal{P}(V) \times L^1_+(E)$ weakly convergent to $\delta_{(\mu,Q)}$ and such that*

$$\varlimsup_{T \to +\infty} \frac{1}{T} \operatorname{Ent} \left(\widetilde{P}_T^{(\mu,Q)} \big| P_T \right) \leq J(\mu, Q).$$

Proof. By definition (5.6) of J, we can restrict to $(\mu, Q) \in \mathcal{S}$. First we discuss the case when $x \in K := \operatorname{supp}(\mu)$. We denote by $\widetilde{\mathbb{P}}_x^{(\mu,Q)}$ the distribution of the Markov chain $\widetilde{\xi}^x$ on V starting from x and having jump rates

$$\widetilde{r}(y, z) := \begin{cases} \frac{Q(y,z)}{\mu(y)} & \text{if } (y, z) \in E(Q) \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Observe that this perturbed chain can be thought of as an irreducible chain on the finite state space K. Moreover, the condition $\operatorname{div} Q = 0$ implies that μ is the invariant probability measure.

Set $\widetilde{P}_T^{(\mu,Q)} := \widetilde{\mathbb{P}}_x^{(\mu,Q)} \circ (\mu_T, Q_T)^{-1}$. The ergodic theorem for finite state Markov chains and the law of large numbers for the empirical flow discussed in Section 2.1 imply that $\{\widetilde{P}_T^{(\mu,Q)}\}$ converges weakly to $\delta_{(\mu,Q)}$. We observe that

$$\frac{1}{T} \operatorname{Ent} \left(\widetilde{P}_T^{(\mu,Q)} \big| P_T \right) \leq \frac{1}{T} \operatorname{Ent} \left(\widetilde{\mathbb{P}}_x^{(\mu,Q)} \big|_{[0,T]} \, \Big| \, \mathbb{P}_x \big|_{[0,T]} \right)$$

$$= \sum_{y \in K, \, z:(y,z) \in E} \widetilde{\mathbb{E}}_x^{(\mu,Q)} \left(Q_T(y,z) \log \frac{Q(y,z)}{\mu(y)r(y,z)} - \mu_T(y) \left[\frac{Q(y,z)}{\mu(y)} - r(y,z) \right] \right) \tag{5.8}$$

where the subscript $[0,T]$ denotes the restriction to the interval $[0,T]$ (above we used the convention $0 \log 0 := 0$). Indeed, the first inequality follows from the variational characterization of the relative entropy (see [18, Sec. 2]–(IV)) and the second from a straightforward computation of the Radon-Nikodym density (recall (3.1)). Since $T \widetilde{\mathbb{E}}_x^{(\mu,Q)} \left(Q_T(y,z) \right) = \widetilde{\mathbb{E}}_x^{(\mu,Q)} \left(\langle \mu_T, \widetilde{r} \rangle \right)$ (adapt (2.6) to the present setting) and since $\mu_T(y) \to \mu(y)$ $\widetilde{\mathbb{P}}_x^{(\mu,Q)}$–a.s. by ergodicity, the r.h.s. of (5.8) converges in the limit $T \to +\infty$ to

$$\sum_{y,z \in K:(y,z) \in E} \left(Q(y,z) \log \frac{Q(y,z)}{\mu(y)r(y,z)} + \mu(y)r(y,z) - Q(y,z) \right) + \sum_{y \in K} \mu(y) \sum_{z \notin K} r(y,z),$$

that is $I(\mu, Q)$.

When $x \notin K$ then there exists an oriented path on (V, E) from x to K since (V, E) is connected. In this case the perturbed Markov chain $\widetilde{\xi}^x$ is defined with rates (5.7) with exception that $\widetilde{r}(y, z) := r(y, z)$ for any (y, z) belonging to the oriented path from x to K (fixed once for all). Since after a finite number of jumps

that Markov chain reach the component K, it is easy conclude the proof by the same computations as before.   □

Recall (2.12) and (5.6). Since I is lower semicontinuous and convex on $\mathcal{P}(V) \times L_+^1(E)$ (see Remark 2.6), the inequality $\mathrm{sc}^- J \geq I$ holds. The proof of the equality $I = \mathrm{sc}^- J$ is therefore completed by Proposition 4.4.

6. Projection from the empirical process: proof of Theorems 2.7, 2.10

We recall the definition of the *empirical process* referring to [18]–(IV), [41] for more details. We consider the space $D(\mathbb{R}; V)$ endowed of the Skorohod topology and write X for a generic element of $D(\mathbb{R}; V)$. Given $X \in D(\mathbb{R}_+; V)$ and $t > 0$, $X^t \in D(\mathbb{R}; V)$ is the t–periodic path which coincides with X on $[0, t)$, that is

$$\begin{cases} X_s^t := X_s \text{ for } 0 \leq s < t \,, \\ X_{s+t}^t := X_s^t \text{ for } s \in \mathbb{R} \,. \end{cases}$$

Writing $\mathcal{M}_S$ for the space of stationary probabilities on $D(\mathbb{R}; V)$ endowed of the weak topology, given $X \in D(\mathbb{R}_+; V)$ and $t > 0$ we denote by $\mathcal{R}_{t,X}$ the element in $\mathcal{M}_S$ such that

$$\mathcal{R}_{t,X}(A) = \frac{1}{t} \int_0^t \chi_A(\theta_s X^t) ds \,, \qquad \forall A \subset D(\mathbb{R}; V) \text{ Borel} \,,$$

where $(\theta_s X^t)_u := X_{s+u}^t$. Since $X \to \mathcal{R}_{t,X}$ is a Borel map from $D(\mathbb{R}_+; V)$ to $\mathcal{M}_S$, for each $x \in V$ it induces a probability measure $\Gamma_{t,x}$ on $\mathcal{M}_S$ defined as $\Gamma_{t,x} := \mathbb{P}_x \circ \mathcal{R}_{t,X}^{-1}$. The above distribution $\Gamma_{t,x}$ corresponds to the t–periodized empirical process.

Let us denote by $\bar{R}$ the stationary process in $\mathcal{M}_S$ associated to the Markov chain ξ and having π as marginal distribution. By the ergodic theorem (2.2), $\Gamma_{t,x}$ weakly converges to $\delta_{\bar{R}}$ as $t \to +\infty$, for each $x \in V$. As proven in [18]–(IV), under the Donsker–Varadhan condition, for each $x \in V$ as $t \to +\infty$ the family of probability measures $\Gamma_{t,x}$ satisfies a LDP with rate t and rate function given by the relative entropy per unit of time H w.r.t. the Markov chain ξ.

We briefly recall the definition of H and some of its properties, referring to [18]–(IV) for more details. Given $-\infty \leq s \leq t \leq \infty$, let $\mathcal{F}_t^s$ be the σ–algebra in $D(\mathbb{R}; V)$ generated by the functions $(X_r)_{s \leq r \leq t}$. Let $R \in \mathcal{M}_S$ and $R_{0,X}$ be the regular conditional probability distribution of R given $\mathcal{F}_0^{-\infty}$, evaluated on the path X. Then $H(R) \in [0, \infty]$ is the only constant such that $H(t, R) = tH(R)$ for all $t > 0$, where

$$H(t, R) := \mathbb{E}_R \left[H_{\mathcal{F}_t^0}(R_{0,X} \mid \mathbb{P}_{X_0}) \right] , \tag{6.1}$$

$H_{\mathcal{F}_t^0}(R_{0,X} \mid \mathbb{P}_{X_0})$ being the relative entropy of $R_{0,X}$ w.r.t. $\mathbb{P}_{X_0}$ thought of as probability measures on the measure space $D(\mathbb{R}; V)$ with measurable sets varying in the σ–subalgebra $\mathcal{F}_t^0$. The entropy $H(R)$ can be also characterized as the limit $H(R) = \lim_{t \to \infty} \bar{H}(t, R)/t$, where

$$\bar{H}(t, R) := \sup_{\varphi \in \mathcal{B}(\mathcal{F}_t^0)} \left[\mathbb{E}_R(\varphi) - \mathbb{E}_R(\log \mathbb{E}_{X_0}(e^\varphi)) \right] \tag{6.2}$$

and $\mathcal{B}(\mathcal{F}_t^0)$ denotes the family of bounded $\mathcal{F}_t^0$–measurable functions on $D(\mathbb{R};V)$. Below we will frequently use that

$$tH(R) = H(t,R) \geq \bar{H}(t,R) = \sup_{\varphi \in Y_1(t)} \mathbb{E}_R(\varphi), \qquad (6.3)$$

where $Y_1(t)$ is the family of functions $\varphi \in \mathcal{B}(\mathcal{F}_t^0)$ such that $\mathbb{E}_x(e^\varphi) \leq 1$ for all $x \in V$ (the last identity is an immediate restatement of (6.2)).

In the following proposition we investigate some key identities concerning the map $R \to (\widehat{\mu}(R), \widehat{Q}(R))$. Recall the definitions of $\widehat{\mu}(R)$ and $\widehat{Q}(R)$ given before Lemma 2.9.

Proposition 6.1. *Assume the Markov chain satisfies (A1)–(A4). Then $\widehat{\mu}(\mathcal{R}_{T,X}) = \mu_T(X)$ and $\widehat{Q}(\mathcal{R}_{T,X}) = Q_T(X^T) \in L_+^1(E)$ for $\mathbb{P}_x$–a.e. $X \in D(\mathbb{R}_+;V)$.*

Proof. The fact that $\widehat{\mu}(\mathcal{R}_{T,X}) = \mu_T(X)$ $\mathbb{P}_x$–a.s. has already been observed in [18]–(IV). Let us prove that $\widehat{Q}(\mathcal{R}_{T,X}) = Q_T(X^T)$ $\mathbb{P}_x$–a.s. It is convenient to introduce the following notation: given $(y,z) \in E$, $X \in D(\mathbb{R}_+;V)$ and $I \subset \mathbb{R}_+$ we write $N_I(X)(y,z)$ for the number of jumps along (y,z) performed by the path X at some time in I. In addition we write $N_T(X)(y,z)$ for $N_{[0,T]}(X)(y,z)$. Equivalently, $N_T(X)(y,z) = TQ_T(X)(y,z)$. Given $T > 0$, fix $a \in (0,T)$. We then have

$$\widehat{Q}(\mathcal{R}_{T,X})(y,z) = \frac{1}{a}\mathbb{E}_{\mathcal{R}_{T,X}}(N_a(y,z)) = \frac{1}{aT}\int_0^T N_a(\theta_s X^T)(y,z)\,ds$$

$$= \frac{1}{aT}\int_0^T N_{[s,s+a]}(X^T)(y,z)\,ds\,.$$

Let us write $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ for the times in $[0,T]$ at which the path X^T jumps from y to z. Note that $n = N_T(X^T)(y,z)$. We denote by $\pi_T : \mathbb{R} \to \mathbb{R}/T\mathbb{Z}$ the canonical projection of $\mathbb{R}$ on the circle of length T. It maps bijectively $[0,T)$ on $\mathbb{R}/T\mathbb{Z}$. Moreover, we define the set $\Theta_T(X^T)(y,z) := \{\pi_T(t_1), \pi_T(t_2), \ldots, \pi_T(t_n)\}$. Since $T > a$ the number $N_{[s,s+a]}(X^T)(y,z)$ of jumps from y to z made by X^T in the time interval $[s,s+a]$ coincides with the cardinality of $\Theta_T(X^T)(y,z) \cap \pi_T([s,s+a])$. Hence

$$\widehat{Q}(\mathcal{R}_{T,X})(y,z) = \frac{1}{aT}\int_0^T \left|\Theta_T(X^T)(y,z) \cap \pi_T([s,s+a])\right|\,ds =$$

$$\sum_{k=1}^n \frac{1}{aT}\int_0^T \mathbb{1}\left(\pi_T(y_k) \in \pi_T([s,s+a])\right)ds = \sum_{k=1}^n \frac{1}{T} = Q_T(X^T)(y,z). \quad (6.4)$$

$\square$

Note that, since $\mathbb{P}_x$–a.s. time T is not a jump time, it holds

$$Q_T(X^T)(y,z) = \begin{cases} Q_T(X)(y,z) + \frac{1}{T} & \text{if } (X_{T-},X_0) = (y,z) \in E\,, \\ Q_T(X)(y,z) & \text{otherwise}\,, \end{cases} \quad \mathbb{P}_x\text{–a.s.} \quad (6.5)$$

In what follows, in order to allow a better overview of the proof of Theorems 2.7 and 2.10, we focus on the main steps, postponing some technical details in subsequent sections. We start with Theorem 2.10, since the product topology on the flow space is simpler.

6.1. **Proof of Theorem 2.10.** The proof is based on the generalized contraction principle related to the concept of exponential approximation discussed in [16, Sec. 4.2.2]. To this aim, given $\varepsilon \in (0, 1/2)$, we fix a continuous function $\varphi_\varepsilon : \mathbb{R} \to [0, 1]$ such that $\varphi_\varepsilon(x) = 0$ if $x \notin (0, 1)$ and $\varphi_\varepsilon(x) = 1$ if $x \in [\varepsilon, 1 - \varepsilon]$. For each $(y, z) \in E$ we consider the continuous and bounded function $F_{y,z}^\varepsilon : D(\mathbb{R}; V) \to \mathbb{R}$ defined as

$$F_{y,z}^\varepsilon(X) := \Big\{ \sum_{s \in [0,1]} \varphi_\varepsilon(s) \mathbb{1}\big(X_{s-} = y, \, X_s = z\big) \Big\} \wedge \varepsilon^{-1}.$$

Then, we define $\widehat{Q}_\varepsilon : \mathcal{M}_S \to [0, +\infty]^E$ as $\widehat{Q}_\varepsilon(R)(y, z) := \mathbb{E}_R\big(F_{y,z}^\varepsilon\big)$. Note that $\widehat{Q}_\varepsilon$ maps $\mathcal{M}_S$ into $[0, \varepsilon^{-1}]^E$.

Proposition 6.2. *Assume the Markov chain satisfies (A1)–(A4). Consider the space $[0, +\infty]^E$ endowed of the product topology and the Borel σ–algebra. Then the following holds:*

(i) *The map $(\widehat{\mu}, \widehat{Q}) : \mathcal{M}_S \to \mathcal{P}(V) \times [0, +\infty]^E$ is measurable and the map $\widehat{\mu} : \mathcal{M}_S \to \mathcal{P}(V)$ is continuous.*

(ii) *The maps $\widehat{Q}_\varepsilon : \mathcal{M}_S \to [0, +\infty]^E$, parameterized by $\varepsilon \in (0, 1/2)$, are continuous and satisfy*

$$\lim_{\varepsilon \downarrow 0} \sup_{R \in \mathcal{M}_S \, : \, H(R) \leq \alpha} \big|\widehat{Q}(R)\,(y, z) - \widehat{Q}_\varepsilon(R)\,(y, z)\big| = 0, \tag{6.6}$$

$$\lim_{\varepsilon \downarrow 0} \varlimsup_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T,x} \Big(\big|\widehat{Q}(y, z) - \widehat{Q}_\varepsilon(y, z)\big| > \delta \Big) = -\infty, \tag{6.7}$$

for any $x \in V$, $\alpha > 0$, $\delta > 0$ and any edge $(y, z) \in E$.

As shown below, if $H(R) < +\infty$ then $\widehat{Q}(R) \in \mathbb{R}_+^E$. In addition $\widehat{Q}_\varepsilon$ always assumes finite values. In particular, the quantities appearing in (6.6) and (6.7) are finite and the subtraction is meaningful. We postpone the proof of Proposition 6.2 to Section 7 and conclude the proof of Theorem 2.10.

To prove item (i) up to (2.16) we apply Theorem 4.2.23 in [16]. Identity (6.6) corresponds to formula (4.2.24) there, while identity (6.7) states, following the terminology in [16], that the family of probability measures $\big\{ \Gamma_{T,x} \circ (\widehat{\mu}, \widehat{Q}_\varepsilon)^{-1} \big\}$ is an exponentially good approximation of the family $\big\{ \Gamma_{T,x} \circ (\widehat{\mu}, \widehat{Q})^{-1} \big\}$. Combining the last observations with the LDP of the empirical process proved in [18]–(IV), one gets the thesis for the family of probability measures $\{ \mathbb{P}_x \circ (\mu_T, \widetilde{Q}_T)^{-1} \}$ on $\mathcal{P}(V) \times [0, +\infty]^E$ where $\widetilde{Q}_T(X) := Q_T(X^T)$ (use Proposition 6.1). At this point, due to Theorem 4.2.13 in [16], we only need to prove that the families of probability measures $\{ \mathbb{P}_x \circ (\mu_T, Q_T)^{-1} \}$ and $\{ \mathbb{P}_x \circ (\mu_T, \widetilde{Q}_T)^{-1} \}$ are exponentially equivalent. It is enough to show that for each $\delta > 0$ it holds

$$\varlimsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x \big(D(\widetilde{Q}_T, Q_T) > \delta \big) = -\infty, \tag{6.8}$$

where $D(\cdot, \cdot)$ denotes the metric of $[0, +\infty]^E$ introduced at the beginning of Subsection 2.4. By (6.5) $\widetilde{Q}_T(y, z) = Q_T(y, z)$ with exception of at most one edge (y, z) where it holds $\widetilde{Q}_T(y, z) = Q_T(y, z) + 1/T$. Since $|a/(1+a) - (a+\Delta)/(1+a+\Delta)| \leq \Delta$ for $a, \Delta \geq 0$, we conclude that $D(\widetilde{Q}_T, Q_T) \leq 1/T$, thus allowing to end the proof.

6.2. **Proof of** (2.17).

6.2.1. *Proof of* (2.17) *for* $Q \notin [0, +\infty)^E$. Let $Q \in [0, +\infty]^E$ be such that $Q(y, z) = +\infty$ for some $(y, z) \in E$. We need to show $\widetilde{I}(\mu, Q) = +\infty$. By Remark 2.1 (stochastic domination), it holds $C := \sup_{x \in V} \mathbb{E}_x \left(e^{Q_T(y,z)} \right) < +\infty$. Hence for $\lambda > 0$ the function $\varphi(X) := Q_T(X)(y, z)\mathbb{1}(Q_T(X)(y, z) \leq \lambda) - \log C$ belongs to $Y_1(T)$. By (6.3) we get

$$TH(R) \geq \bar{H}(T, R) \geq \mathbb{E}_R(\varphi)$$

and we conclude by taking the limit $\lambda \to \infty$.

6.2.2. *Proof that* $I(\mu, Q) \leq \widetilde{I}(\mu, Q)$ *for* $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$.

Given $y \neq z$ in V define $Q_T(X)(y, z)$ as the T times the number of jumps up to time T along (y, z) in the trajectory X.

Lemma 6.3. *If* $R \in \mathcal{M}_S$ *and* $H(R) < +\infty$ *then* $R\big(Q_T(y, z) > 0\big) = 0$ *for all* $T \geq 0$ *and* $(y, z) \in (V \times V) \setminus E$ *with* $y \neq z$.

Proof. Take the function $\varphi(X) := \lambda\mathbb{1}(Q_T(y, z) > 0)$ for fixed $\lambda > 0$. Note that $\varphi \in Y_1(T)$ since $\varphi \equiv 0$ $\mathbb{P}_x$–a.s. Hence by (6.3) we get

$$TH(R) \geq \bar{H}(T, R) \geq \mathbb{E}_R(\varphi) = \lambda R\left(Q_T(y, z) > 0\right).$$

Since $H(R) < +\infty$ the thesis follows by taking λ arbitrarily large. $\qquad\square$

Lemma 6.4. *Given* $R \in \mathcal{M}_S$ *with* $H(R) < +\infty$, *it holds*

$$\sum_{z \,:\, (y,z) \in E} \widehat{Q}(y, z) = \sum_{z \,:\, (z,y) \in E} \widehat{Q}(z, y), \qquad \widehat{Q} = \widehat{Q}(R).$$

Proof. The thesis follows by using Lemma 6.3 and considering the R–expectation of the following identity on $D([0, T]; V)$:

$$\mathbb{1}(X_T = y) + \sum_{z \,:\, z \neq y} TQ_T(X)(y, z) = \mathbb{1}(X_0 = y) + \sum_{z \,:\, z \neq y} TQ_T(X)(z, y). \quad\square$$

Fix $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$. By Lemma 6.4, if $\operatorname{div} Q \neq 0$ then $\widetilde{I}(\mu, Q) = +\infty = I(\mu, Q)$. Hence, from now on we can restrict to $\operatorname{div} Q = 0$. Fix $R \in \mathcal{M}_S$ such that $Q = \widehat{Q}(R)$ and $\mu = \widehat{\mu}(R)$ (the absence of such an R would imply $\widetilde{I}(\mu, Q) = +\infty$ and there would be nothing to prove).

We first consider the case that there is some edge $(y, z) \in E$ with $Q(y, z) > 0$ and $\mu(y) = 0$. Trivially in this case $I(\mu, Q) = +\infty$. Let us prove that $\widetilde{I}(\mu, Q) = +\infty$. To this aim, given $\varepsilon > 0$, we define the function $F_\varepsilon : E \to \mathbb{R}$ as $F_\varepsilon(u, v) = \log \frac{Q(y,z)}{\varepsilon r(y,z)}\mathbb{1}\big((u, v) = (y, z)\big)$. Let $e^{\varphi_\varepsilon} := \mathbb{M}_T^{F_\varepsilon}$ be the supermartingale introduced in Lemma 3.1:

$$\varphi_\varepsilon = TQ_T(y, z)\log\frac{Q(y, z)}{\varepsilon r(y, z)} - T\mu_T(y)\left[\frac{Q(y, z)}{\varepsilon} - r(y, z)\right]. \tag{6.9}$$

We take ε small enough so that $\log\frac{Q(y,z)}{\varepsilon r(y,z)} > 0$ and define for $\ell > 0$ the new function $\varphi_{\varepsilon,\ell}$ as in the r.h.s. of (6.9) with $Q_T(y, z)$ replaced by $Q_T(y, z) \wedge \ell$. Then $\varphi_{\varepsilon,\ell} \leq \varphi_\varepsilon$ and by Lemma 3.1 we conclude that $\varphi_{\varepsilon,\ell} \in Y_1(T)$. Applying (6.3) we conclude that

$$H(R) \geq \mathbb{E}_R\big(\varphi_{\varepsilon,\ell}\big)/T = \mathbb{E}_R(Q_T(y, z) \wedge \ell)\log\frac{Q(y, z)}{\varepsilon r(y, z)}.$$

Taking first the limit $\ell \to +\infty$ and afterwards $\varepsilon \to 0$, we get that $H(R) = +\infty$, thus implying $\widetilde{I}(\mu, Q) = +\infty$.

Due to the previous result, we restrict to the case that $\mu(y) > 0$ if $Q(y, z) > 0$, with $(y, z) \in E$. Then we fix an invading sequence $E_n \nearrow E$ of finite subsets of E and consider the function $F_n : E \to \mathbb{R}$ defined as

$$r^{F_n}(y, z) = r(y, z)e^{F_n(y,z)} := \begin{cases} \frac{Q(y,z)}{\mu(y)}, & \text{if } (y, z) \in E_n, \\ r(y, z) & \text{otherwise}. \end{cases}$$

with the convention that $0/0 = 0$. Note that the above ratio is well defined since $\mu(y) > 0$ if $Q(y, z) > 0$. Let $e^{\varphi_n} := \mathbb{M}_T^{F_n}$ be the supermartingale introduced in Lemma 3.1:

$$\varphi_n = T \sum_{(y,z)\in E_n} \left\{ Q_T(y, z) \log \frac{Q(y, z)}{\mu(y)r(y, z)} - \mu_T(y)r(y, z)\left[\frac{Q(y, z)}{\mu(y)r(y, z)} - 1\right] \right\}.$$

Since φ_n is unbounded, for $\ell > 0$ we consider the cut–off

$$\varphi_{n,\ell} := \begin{cases} \varphi_n & \text{if } |\varphi_n| \leq \ell, \\ \frac{\varphi_n}{|\varphi_n|}\ell & \text{if } |\varphi_n| > \ell. \end{cases}$$

We stress that the sum in the definition of φ_n is finite. Since $|\varphi_{n,\ell}| \leq |\varphi_n| \in L^1(R)$ (recall that $Q = \widehat{Q}(R) \in L^1_+(E)$), by the Dominated Convergence Theorem it holds $\lim_{\ell \to +\infty} \mathbb{E}_R(\varphi_{n,\ell}) = \mathbb{E}_R(\varphi_n)$. Moreover, there exist positive constants A_n, B_n depending only on n such that

$$|\varphi_{n,\ell}| \leq |\varphi_n| \leq A_n \sum_{(y,z)\in E_n} TQ_T(y, z) + B_n.$$

By Remark 2.1, this implies that $\log \mathbb{E}_x(e^{\varphi_{n,\ell}})$ is bounded uniformly in $x \in V$. Therefore, by dominated convergence and Lemma 3.1, we conclude that

$$\lim_{\ell \to +\infty} \mathbb{E}_R \log \mathbb{E}_{X_0}(e^{\varphi_{n,\ell}}) = \lim_{\ell \to +\infty} \sum_{x\in V} \mu(x) \log \mathbb{E}_x(e^{\varphi_{n,\ell}})$$

$$= \sum_{x\in V} \mu(x) \log \mathbb{E}_x(e^{\varphi_n}) \leq 0.$$

As a consequence

$$\lim_{\ell \to \infty} \left\{ \mathbb{E}_R(\varphi_{n,\ell}) - \mathbb{E}_R \log \mathbb{E}_{X_0}(e^{\varphi_{n,\ell}}) \right\} \geq \mathbb{E}_R(\varphi_n).$$

Combining the above estimate, (6.2) and (6.3), we conclude that

$$H(R) \geq \bar{H}(T, R)/T \geq \mathbb{E}_R(\varphi_n)/T = \sum_{(y,z)\in E_n} \Phi(Q(y, z), Q^\mu(y, z)). \qquad (6.10)$$

To conclude we take the limit $n \to +\infty$, obtaining $H(R) \geq I(\mu, Q)$ for each $R \in \mathcal{M}_S$ such that $\widehat{\mu}(R) = \mu$, $\widehat{Q}(R) = Q$. This implies that $\widetilde{I}(\mu, Q) \geq I(\mu, Q)$.

6.2.3. *Proof that $I(\mu, Q) \geq \widetilde{I}(\mu, Q)$ for $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$.* As a consequence of the first part of Theorem 2.10 (already proved), the function $\widetilde{I}$ is lower semicontinuous. Consider the sequence $\{(\mu_n, Q_n)\}_{n \geq 0}$ in $\mathcal{S}$ converging to (μ, Q) as stated in Proposition 4.4. The set $\mathcal{S}$ has been defined in Section 4.1 as the subset of $\mathcal{P}(V) \times L^1_+(E)$ given by the elements (μ, Q) with $I(\mu, Q) < +\infty$ and such that the graph $(\mathrm{supp}(\mu), E(Q))$ is finite and connected. For each n we consider the continuous time Markov chain $\xi^{(n)}$ on V with jump rates $r_n(y, z) = Q_n(y, z)/\mu_n(y)$ with the convention $0/0 = 0$. Since $I(\mu_n, Q_n) < +\infty$ it cannot be $Q_n(y, z) > 0$ and $\mu_n(y) = 0$, hence the above ratio is well defined. Since μ_n and Q_n have finite support, the Markov chain $\xi^{(n)}$ has finite effective state space. In particular, explosion does not take place. The bound $I(\mu_n, Q_n) < +\infty$ implies also that $\mathrm{div}\, Q_n = 0$, hence we get that μ_n is an invariant measure for $\xi^{(n)}$. We define R_n as the stationary Markov chain $\xi^{(n)}$ with marginal μ_n, then $\widehat{Q}(R_n) = Q_n$. By the Radon–Nykodim derivative (3.1) and the definition of the entropy $H(\cdot)$, we get that $\widetilde{I}(\mu_n, Q_n) \leq H(R_n) = I(\mu_n, Q_n)$. Invoking the lower semicontinuity of $\widetilde{I}$ and Proposition 4.4, we get the thesis.

6.3. **Proof of** (2.18). Let us take (μ, Q) with $\mu \in \mathcal{P}(V)$ and $Q \in \mathbb{R}^E_+ \setminus L^1_+(E)$. We need to prove that $\widetilde{I}(\mu, Q) = +\infty$. Let $R \in \mathcal{M}_S$ be such that $\widehat{\mu}(R) = \mu$ and $\widehat{Q}(R) = Q$ (we assume R exists, otherwise the thesis is trivially true). We fix an invading sequence $V_n \nearrow V$ of finite sets, define $E_n := \{(y, z) \in E : y, z \in V_n\}$ and $F_n(y, z) := \mathbb{1}((y, z) \in E_n)$ for $(y, z) \in E$. Then we know that $\mathbb{E}_x\left(\exp\{\mathbb{M}^{F_n}_T\}\right) \leq 1$ for all $x \in V$, using the same notation of Lemma 3.1. Again we need to work with functions in $\mathcal{B}(\mathcal{F}^0_T)$. To this aim, given $\ell > 0$ we define $\mathbb{M}^{F_n}_{T,\ell}$ as the supermartingale $\mathbb{M}^{F_n}_T$ except that the empirical flow $Q_T(y, z)$ is replaced by $Q_T(y, z) \wedge \ell$ for all edges (y, z). Then (note that $r^{F_n} \geq r$) $\mathbb{M}^{F_n}_{T,\ell} \in \mathcal{B}(\mathcal{F}^0_T)$ and $\mathbb{M}^{F_n}_{T,\ell} \leq \mathbb{M}^{F_n}_T$, thus implying that $\mathbb{M}^{F_n}_{T,\ell} \in Y_1(T)$. By (6.3) this implies that

$$H(R) \geq \bar{H}(T, R)/T \geq \varlimsup_{\ell \to \infty} \mathbb{E}_R\left(\mathbb{M}^{F_n}_{T,\ell}\right)/T = \sum_{(y,z) \in E_n} Q(y, z) - \mathbb{E}_R(\mu_T(r^{F_n} - r)).$$

(6.11)

The conclusion then follows from the next result:

Lemma 6.5. *Assume Condition 2.2 (where the constants σ, C are defined). Then for each $R \in \mathcal{M}_S$ it holds*

$$\|\widehat{Q}(R)\| \leq H(R)(1 + e/\sigma) + C\, e/\sigma.$$

(6.12)

Proof. Let us first prove (6.12) knowing that $H(R) \geq \mathbb{E}_R(v(X_0))$ (this will be proved later). We come back to (6.11) and take first the limit $T \to +\infty$ and afterwards the limit $n \to +\infty$. Since $F_n(y, z) = \mathbb{1}((y, z) \in E_n)$, then $0 \leq r^{F_n} - r \leq er$. By Fubini–Tonelli and stationarity, $\mathbb{E}_R(\mu_T(r)) = \mathbb{E}_R(r(X_0))$. We then conclude that

$$\|\widehat{Q}\| = \|Q\| = \lim_{n \to +\infty} \sum_{(y,z) \in E_n} Q(y, z) \leq H(R) + e\mathbb{E}_R(r(X_0)).$$

By Condition 2.2, $\mathbb{E}_R(r(X_0)) \leq \mathbb{E}_R(v(X_0))/\sigma + C/\sigma$. Combining with $H(R) \geq \mathbb{E}_R(v(X_0))$ we get the thesis.

Let us now prove that $H(R) \geq \mathbb{E}_R(v(X_0))$. Since both $H(R)$ and $E_R(v(X_0))$ are affine in R (see [18]–(IV)) and since all stationary processes are convex combinations

of ergodic stationary processes, it is enough to prove the claim for an ergodic $R \in \mathcal{M}_S$. Given $k, T > 0$ and $W \subset V$ we define $v^{(k)} := v \wedge k$ and $\varphi(X) := \mathbb{1}(X_0 \in W) \int_0^T v^{(k)}(X_s)ds$. Trivially, $\varphi \in \mathcal{B}(\mathcal{F}_T^0)$. Then, by the definition of $\bar{H}(T, R)$, it holds

$$
TH(R) \geq \bar{H}(T, R) \geq \mathbb{E}_R(\varphi) - \mathbb{E}_R\big(\log \mathbb{E}_{X_0}(e^\varphi)\big)
$$
$$
\geq \mathbb{E}_R\Big(\int_0^T v^{(k)}(X_s)ds; X_0 \in W\Big) - \max_{x \in W} \log(C_x/c). \tag{6.13}
$$

In the last inequality we have used Lemma 3.5 and the inequality $v^{(k)} \leq v$. At this point, we divide (6.13) by T. Since R is ergodic, by Birkhoff ergodic theorem (note that $v^{(k)}(X_0) \in L^1(R)$ since $v^{(k)}$ is bounded) we know that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T v^{(k)}(X_s)ds = \mathbb{E}_R\big(v^{(k)}(X_0)\big), \qquad R\text{–a.s.}
$$

Taking the limit $T \to \infty$ and applying the Dominated Convergence Theorem we conclude that

$$
H(R) \geq \mathbb{E}_R\big(v^{(k)}(X_0)\big)\mathbb{R}(X_0 \in W).
$$

At this point it is enough to take the limit $k \to \infty$ and afterwards to take W arbitrarily large and invading all V. $\qquad\square$

6.4. **Proof of Theorem 2.7.** The proof uses the results of [20], where the notion of exponentially good approximation and the contraction principle are extended to the case of completely regular space as image space of the projection. To this aim we recall some further properties of the bounded weak* topology on $L_+^1(E)$.

We define $\mathcal{A}$ as the set of sequences $\mathfrak{a} = (a_n)_{n \geq 1}$ of functions in $C_0(E)$ such that $\|a_n\|_\infty \to 0$. Given $\mathfrak{a} \in \mathcal{A}$ we introduce the pseudometric $d_\mathfrak{a}$ on $L_+^1(E)$ as

$$
d_\mathfrak{a}(Q, Q') := \sup_{n \geq 1} \langle Q - Q', a_n \rangle.
$$

Writing $B_\mathfrak{a}(Q, r) := \{Q' \in L_+^1(E) : d_\mathfrak{a}(Q, Q') < r\}$, the family of sets $\{B_\mathfrak{a}(Q, r)\}$, with $\mathfrak{a} \in \mathcal{A}$, $Q \in L_+^1(E)$ and $r > 0$, forms a basis for $L_+^1(E)$. This follows from Def. 2.7.1 and Cor. 2.7.4 in [36]. In addition, the family $\mathcal{D}$ of pseudometrics $\{d_\mathfrak{a} : \mathfrak{a} \in C_0(E)\}$ is separating, i.e. given $Q \neq Q'$ in $L_+^1(E)$ there exists $\mathfrak{a} \in \mathcal{A}$ such that $d_\mathfrak{a}(Q, Q') > 0$. The above two properties (basis and separating family of pseudometrics) make $L_+^1(E)$ a so called *gauge space*. Indeed, one can prove that the concepts of completely regular space and gauge space are equivalent [19, Ch. IX].

Due to the above observations on the gauge structure of $L_+^1(E)$ we are in the same settings of [20]. In what follows we restrict to the case $|V| = +\infty$, thus implying $|E| = +\infty$ due to the irreducibility of the Markov chain ξ (the finite case is much simpler). Fix an enumeration $(e_n)_{n \geq 1}$ of E. Consider the maps $\widehat{Q}, \widehat{Q}_\varepsilon$ entering in Proposition 6.2 and define the maps $\bar{Q}, \bar{Q}_\varepsilon : \mathcal{M}_S \to L_+^1(E)$ by

$$
\bar{Q}(R) = \begin{cases} \widehat{Q}(R) & \text{if } \widehat{Q}(R) \in L_+^1(E), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\bar{Q}_\varepsilon(R)(e_n) = \begin{cases} \widehat{Q}_\varepsilon(R)(e_n) & \text{if } n \leq \varepsilon^{-1}, \\ 0 & \text{otherwise}. \end{cases}
$$

Proposition 6.6. *Assume the Markov chain satisfies (A1)–(A4) and Condition 2.2. Consider the space $L^1_+(E)$ endowed of the bounded weak* topology and the Borel σ–algebra. Then the following holds:*

(i) *The map $\bar{Q} : \mathcal{M}_S \to L^1_+(E)$ is measurable while the maps $\bar{Q}_\varepsilon : \mathcal{M}_S \to L^1_+(E)$ are continuous.*

(ii) *For each $\mathfrak{a} \in \mathcal{A}$*

$$\lim_{\varepsilon \downarrow 0} \sup_{R \in \mathcal{M}_S \,:\, H(R) \leq \alpha} d_\mathfrak{a}\big(\bar{Q}(R), \bar{Q}_\varepsilon(R)\big) = 0 , \tag{6.14}$$

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{T \uparrow \infty}} \frac{1}{T} \log \Gamma_{T,x}\big(d_\mathfrak{a}(\bar{Q}, \bar{Q}_\varepsilon) > \delta\big) = -\infty , \tag{6.15}$$

for any $x \in V$, $\alpha > 0$, $\delta > 0$.

The proof is given in Section 8.

As byproduct of Proposition 6.6, the extended contraction principle in [20], the LDP of the empirical process and Theorem 2.10–(ii) we can conclude the proof of Theorem 2.7. Let us be more precise. We apply Theorem 1.13 in [20]. Formula (6.14) corresponds to formula (1.14) in [20], while formula (6.15) means that the family of probability measures $\big\{\Gamma_{T,x} \circ \big(\widehat{\mu}, \bar{Q}_\varepsilon\big)^{-1}\big\}$ is a $(d_\mathfrak{a})_{\mathfrak{a} \in \mathcal{A}}$–exponentially good approximation of the family $\big\{\Gamma_{T,x} \circ \big(\widehat{\mu}, \bar{Q}\big)^{-1}\big\}$. On the other hand, we have that $\bar{Q} = \widehat{Q} \in L^1_+(E)$ $\Gamma_{T,x}$–a.s., while by Proposition 6.1 the random variable $\widehat{Q}$ sampled according to $\Gamma_{T,x}$ has the same law of $\widetilde{Q}_T(X) := Q_T(X^T)$ with $X \in D(\mathbb{R}_+; V)$ sampled according to $\mathbb{P}_x$. Hence, by Corollary 1.10 in [20] we only need to prove that the families of probability measures $\{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\}$ and $\{\mathbb{P}_x \circ (\mu_T, \widetilde{Q}_T)^{-1}\}$ are $(d_\mathfrak{a})_{\mathfrak{a} \in \mathcal{A}}$–exponentially equivalent on $\mathcal{P}(V) \times L^1_+(E)$. It is enough to show for each $\delta > 0$ and $\mathfrak{a} \in \mathcal{A}$ that

$$\overline{\lim_{T \to +\infty}} \frac{1}{T} \log \mathbb{P}_x\big(d_\mathfrak{a}(\widetilde{Q}_T, Q_T) > \delta\big) = -\infty , \tag{6.16}$$

Since by (6.5) $d_\mathfrak{a}(\widetilde{Q}_T, Q_T) \leq \|\mathfrak{a}\|_\infty / T$, we get the thesis.

7. Exponential approximations: Proof of Proposition 6.2

Item (i) is straightforward. We concentrate on item (ii). Since $\mathcal{M}_S$ is endowed of the weak topology and since $F^\varepsilon_{y,z}$ is a continuous bounded function on $D(\mathbb{R}; V)$ we conclude that $\widehat{Q}_\varepsilon$ is continuous.

7.1. **Proof of** (6.6). As already proved in the previous section (independently from the content of Proposition 6.2), $\widetilde{I}(\mu, Q) = +\infty$ if $Q \notin [0, +\infty)^E$. Hence, given $R \in \mathcal{M}_S$ with $H(R) < +\infty$, it must be $\widehat{Q}(R)(y, z) < \infty$ for all $(y, z) \in E$. Below $R \in \mathcal{M}_S$ is such that $H(R) \leq \alpha$.

Recall the definition of $N_I(y, z)$ and $N_T(y, z)$ given in the proof of Proposition 6.1. We can estimate

$$\big|\widehat{Q}(R)(y, z) - \widehat{Q}_\varepsilon(R)(y, z)\big|$$
$$\leq \mathbb{E}_R\big(N_1(y, z); N_1(y, z) \geq \varepsilon^{-1}\big) + \mathbb{E}_R\big(N_{[0,\varepsilon] \cup [1-\varepsilon, 1]}(y, z)\big). \tag{7.1}$$

By stationarity (see the proof of Lemma 2.9)

$$\mathbb{E}_R\big(N_{[0,\varepsilon]}(y, z)\big) = \mathbb{E}_R\big(N_{[1-\varepsilon, 1]}(y, z)\big) = \varepsilon \mathbb{E}_R\big(N_1(y, z)\big) = \varepsilon \widehat{Q}(R)(y, z) .$$

Consider $\ell \in \mathbb{R}_+$ and apply (6.3) with $t = 1$ and $\varphi = N_1(y, z) \wedge \ell - r(y, z)(e - 1)$ (note that $\varphi \in Y_1(t)$ by Remark 2.1). We get for $R \in \mathcal{M}_S$ such that $H(R) \leq \alpha$

$$\alpha + r(y, z)(e - 1) \geq H(R) + r(y, z)(e - 1) \geq \mathbb{E}_R(N_1(y, z) \wedge \ell). \qquad (7.2)$$

Since by the monotone convergence $\lim_{\ell \to +\infty} \mathbb{E}_R(N_1(y, z) \wedge \ell) = \widehat{Q}(R)(y, z)$, taking the limit $\ell \to +\infty$ on both extreme sides of (7.2) we deduce

$$\alpha + r(y, z)(e - 1) \geq \widehat{Q}(R)(y, z).$$

From this inequality we get that the last term in (7.1) converges uniformly to zero on $\{R \in \mathcal{M}_S : H(R) \leq \alpha\}$ as $\varepsilon \downarrow 0$. To conclude, it remains to prove that $\lim_{\varepsilon \downarrow 0} \mathbb{E}_R(N_1(y, z); N_1(y, z) \geq \varepsilon^{-1}) = 0$. To this aim, given $\gamma, \ell > 0$ we define on $D(\mathbb{R}; V)$ the function

$$\varphi_{\gamma, \ell, \varepsilon} := \gamma N_1(y, z) \mathbb{I}(\ell \geq N_1(y, z) \geq \varepsilon^{-1}) - C(\gamma, \varepsilon)$$

where $C(\gamma, \varepsilon) := \sup_{x \in V} \log \mathbb{E}_x\left(e^{\gamma N_1(y, z)\, \mathbb{I}\left(N_1(y, z) \geq \varepsilon^{-1}\right)}\right)$. Due to Remark 2.1 we get $C(\gamma, \varepsilon) < +\infty$ and $\lim_{\varepsilon \downarrow 0} C(\gamma, \varepsilon) = 0$. By construction $\varphi_{\gamma, \ell, \varepsilon} \in Y_1(t)$ for $t \geq 1$. Applying (6.3) we get for $t \geq 1$ that

$$\mathbb{E}_R(\varphi_{\gamma, \ell, \varepsilon}) \leq \bar{H}(t, R) \leq tH(R) \leq t\alpha.$$

Taking $\ell \to \infty$, we conclude that $\mathbb{E}_R(N_1(y, z); N_1(y, z) \geq \varepsilon^{-1}) \leq t\alpha/\gamma + C(\gamma, \varepsilon)/\gamma$. Taking first the limit $\varepsilon \downarrow 0$ and afterwards the limit $\gamma \uparrow \infty$, we conclude that the expectation $\mathbb{E}_R(N_1(y, z); N_1(y, z) \geq \varepsilon^{-1})$ is negligible as $\varepsilon \downarrow 0$. $\qquad \square$

7.2. **Proof of** (6.7)**.** We restrict to $T > 1$ (the generic case could be treated by the same arguments of the proof of Proposition 6.1). Recall the definition of the projection π_T and set $\Theta_T(X^T)(y, z)$ given there. $\mathbb{P}_x$–a.s. it holds

$$\widehat{Q}_\varepsilon(\mathcal{R}_{T, X})(y, z) = \frac{1}{T} \int_0^T \left\{ \sum_{\substack{u \in [s, s+1]: \\ \pi_T(u) \in \Theta_T(X^T)(y, z)}} \varphi_\varepsilon(u - s) \right\} \wedge \varepsilon^{-1} ds. \qquad (7.3)$$

For each $(y, z) \in E$ and $\varepsilon > 0$ we define the functions $G_\varepsilon(y, z)$ and $H_\varepsilon(y, z)$ on $D(\mathbb{R}; V)$ as

$$G_\varepsilon(X)(y, z) := \frac{1}{T} \int_0^T \left|\Theta_T(X^T)(y, z) \cap \pi_T\left([s + \varepsilon, s + 1 - \varepsilon]\right)\right| \wedge \varepsilon^{-1} ds$$

$$H_\varepsilon(X)(y, z) := \frac{1}{T} \int_0^T \left|\Theta_T(X^T)(y, z) \cap \pi_T\left([s + \varepsilon, s + 1 - \varepsilon]\right)\right| ds.$$

By the same argument used in identity (6.4), it holds

$$H_\varepsilon(X)(y, z) = (1 - 2\varepsilon)Q_T(X^T)(y, z) = (1 - 2\varepsilon)\widehat{Q}(\mathcal{R}_{T, X})(y, z). \qquad (7.4)$$

Trivially, it holds $\widehat{Q}(\mathcal{R}_{T,X})(y,z) \geq \widehat{Q}_\varepsilon(\mathcal{R}_{T,X})(y,z) \geq G_\varepsilon(X)(y,z)$. Using (7.4) and the last bounds, we can estimate

$$\mathbb{P}_x\left(\widehat{Q}(\mathcal{R}_{T,\cdot})(y,z) - \widehat{Q}_\varepsilon(\mathcal{R}_{T,\cdot})(y,z) \geq \delta\right)$$

$$\leq \mathbb{P}_x\left(\widehat{Q}(\mathcal{R}_{T,\cdot})(y,z) - G_\varepsilon(y,z) \geq \delta\right)$$

$$\leq \mathbb{P}_x\left(\widehat{Q}(\mathcal{R}_{T,\cdot})(y,z) - H_\varepsilon(y,z) \geq \delta/2\right) + \mathbb{P}_x\left(H_\varepsilon(y,z) - G_\varepsilon(y,z) \geq \delta/2\right)$$

$$= \mathbb{P}_x\left(2\varepsilon Q_T(X^T)(y,z) \geq \delta/2\right) + \mathbb{P}_x\left(H_\varepsilon(y,z) - G_\varepsilon(y,z) \geq \delta/2\right).$$

$$(7.5)$$

In order to prove the super-exponential estimate (6.7) it is enough to prove a super-exponential estimate for both terms in the last line of (7.5).

Since, by the graphical construction, under $\mathbb{P}_x$ the process $\{|TQ_T(X)(y,z)|\}_{T\in\mathbb{R}_+}$ is dominated by a Poisson process $\{Z_T\}_{T\in\mathbb{R}_+}$ with parameter $r(y,z)$ we have

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{T\to+\infty}} \frac{1}{T} \log\left[\mathbb{P}_x\left(2\varepsilon Q_T(X^T)(y,z) \geq \delta/2\right)\right]$$

$$\leq \lim_{\varepsilon\downarrow 0} \overline{\lim_{T\to+\infty}} \frac{1}{T} \log\left[\mathbb{P}\left(2\varepsilon(Z_T+1)/T \geq \delta/2\right)\right]$$

$$\leq \lim_{\varepsilon\downarrow 0} -\Phi\left(\frac{\delta}{4\varepsilon}, r(y,z)\right) = -\infty.$$

We used a LDP for the Poisson process (the extra $1/T$ term is irrelevant) and the explicit form of the rate functional.

It remains to bound the last term in (7.5). For simplicity of notation we restrict to T integer (the general case can be treated similarly). We define $\psi_\varepsilon(r) = r\mathbb{1}(r > \varepsilon^{-1})$. Given $j = 0, 1, \ldots, T-1$ and $s \in [j, j+1)$ we have

$$\left|\Theta_T(X^T)(y,z) \cap \pi_T\left([s+\varepsilon, s+1-\varepsilon]\right)\right|$$

$$- \left|\Theta_T(X^T)(y,z) \cap \pi_T\left([s+\varepsilon, s+1-\varepsilon]\right)\right| \wedge \varepsilon^{-1}$$

$$\leq \psi_\varepsilon\left(\left|\Theta_T(X^T)(y,z) \cap \pi_T\left([j, j+2)\right)\right|\right).$$

Hence, we can estimate

$$H_\varepsilon(X)(y,z) - G_\varepsilon(X)(y,z) \leq \frac{1}{T}\sum_{j=0}^{T-1} \psi_\varepsilon\left(\left|\Theta_T(X^T)(y,z) \cap \pi_T\left([j, j+2)\right)\right|\right). \quad (7.6)$$

By the graphical construction of Markov chains, under $\mathbb{P}_x$ the set of jump times for a jump from y to z can be identified with a suitable subset of an homogeneous Poisson point process on $\mathbb{R}_+$ with intensity $r(y,z)$. In particular, it is possible to define a probability measure $\mathcal{P}$ on the product space $D(\mathbb{R}_+; V) \times D(\mathbb{R}_+; \mathbb{N})$ such that

(i) the marginal of $\mathcal{P}$ on $D(\mathbb{R}_+; V)$ equals $\mathbb{P}_x$;
(ii) the marginal of $\mathcal{P}$ on $D(\mathbb{R}_+; \mathbb{N})$ is the law of a Poisson process with parameter $r(x,y)$,
(iii) calling $(X_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ the generic elements of respectively $D(\mathbb{R}_+; V)$ and $D(\mathbb{R}_+; \mathbb{N})$, it holds $\mathcal{P}$–a.s.

$$N_{[a,b]}(X)(y,z) \leq Z_b - Z_a, \qquad \forall a < b \text{ in } \mathbb{R}_+.$$

Due to the above coupling and since on the interval $[0, T]$ the paths X and X^T can differ at most in T, we can estimate $\mathcal{P}$–a.s.

$$\psi_\varepsilon\left(\left|\Theta_T(X^T)(y, z) \cap \pi_T\big([j, j+2)\big)\right|\right)$$
$$\leq \begin{cases} \psi_\varepsilon(Z_{j+2} - Z_j) & \text{if } 0 \leq j \leq T - 2, \\ \psi_\varepsilon([Z_T - Z_{T-1}] + Z_1 + 1) & \text{if } j = T - 1. \end{cases} \quad (7.7)$$

Now we introduce the nondecreasing function $\widehat{\psi}_\varepsilon(r) := 2r\mathbb{1}(r > \varepsilon^{-1}/2)$ satisfying the inequality $\psi_\varepsilon(a + b) \leq \widehat{\psi}_\varepsilon(a) + \widehat{\psi}_\varepsilon(b)$. Then (7.6) and (7.7) imply $\mathcal{P}$–a.s. that

$$H_\varepsilon(X)(y, z) - G_\varepsilon(X)(y, z) \leq \frac{2}{T} \sum_{j=0}^{T-1} \widehat{\psi}_\varepsilon(Z_{j+1} - Z_j + 1).$$

At this point we recall that under $\mathcal{P}$ the random variables $\left(Z_{j+1} - Z_j\right)_{0 \leq j \leq T-1}$ are independent Poisson random variables with parameter $r(y, z)$. Hence we can estimate

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{T \to +\infty}} \frac{1}{T} \log\left[\mathbb{P}_x\Big(H_\varepsilon(y, z) - G_\varepsilon(y, z) \geq \delta/2\Big)\right]$$
$$= \lim_{\varepsilon \downarrow 0} \overline{\lim_{T \to +\infty}} \frac{1}{T} \log\left[\mathcal{P}\Big(H_\varepsilon(y, z) - G_\varepsilon(y, z) \geq \delta/2\Big)\right]$$
$$\leq \lim_{\varepsilon \downarrow 0} \overline{\lim_{T \to +\infty}} \frac{1}{T} \log\left[\mathcal{P}\Big(\frac{2}{T} \sum_{j=0}^{T-1} \widehat{\psi}_\varepsilon(Z_{j+1} - Z_j + 1) \geq \delta/2\Big)\right]$$
$$\leq \lim_{\varepsilon \downarrow 0} -I_\varepsilon(\delta/2) = -\infty. \quad (7.8)$$

In the above chain of inequalities we used Cramer Theorem for the sum of the independent random variables $2\widehat{\psi}_\varepsilon(Z_{j+1} - Z_j + 1)$ calling I_ε the associated rate function. The divergence in the last line follows by the following argument. Let $\Lambda_\varepsilon(\lambda) := \log \mathbb{E}\left(e^{\lambda 2\widehat{\psi}_\varepsilon(Z_1 - Z_0)}\right)$. By the Monotone Convergence Theorem $\Lambda_\varepsilon(\lambda)$ converges to zero for each $\lambda \in \mathbb{R}$ as ε goes to zero. Since the rate function I_ε is the Legendre transform of Λ_ε, we get for each fixed $\lambda \in \mathbb{R}$ that

$$I_\varepsilon(\delta/2) \geq \frac{\delta\lambda}{2} - \Lambda_\varepsilon(\lambda).$$

Hence, $\liminf_{\varepsilon \downarrow 0} I_\varepsilon(\delta/2) \geq \delta\lambda/2$. By the arbitrariness of λ we get the thesis.

8. Exponential approximations: Proof of Proposition 6.6

The measurability of $\bar{Q}$ can be checked by straightforward arguments. Let us prove that $\bar{Q}_\varepsilon$ is continuous w.r.t. the bounded weak* topology of $L^1_+(E)$. As stated in Prop. 6.2 each map $\widehat{Q}_\varepsilon(y, z) : \mathcal{M}_S \to [0, \varepsilon^{-1}]$ is continuous and bounded. In addition it holds $\|\bar{Q}_\varepsilon(R)\| \leq \varepsilon^{-2}$ for all $R \in \mathcal{M}_S$. The thesis then follows from Corollary 2.7.3 in [36].

8.1. **Proof of** (6.15). Due to Proposition 6.1 the law of $\widehat{Q}$ under $\Gamma_{T,x}$ is the same of the law of $Q_T(X^T)$ under $\mathbb{P}_x$. Moreover, it holds $Q_T(X^T) \in L^1_+(E)$ $\mathbb{P}_x$–a.s. In particular, we get that $\widehat{Q} = \bar{Q}$ $\Gamma_{T,x}$–a.s. In addition, by Proposition 3.6, we have

$$\lim_{\ell\uparrow+\infty} \overline{\lim_{T\uparrow+\infty}} \frac{1}{T} \log \Gamma_{T,x}\left(\|\widehat{Q}\| \geq \ell\right) = -\infty. \tag{8.1}$$

Due to (8.1) in order to prove (6.15) we only need to show for any $\ell > 0$ that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{T\uparrow\infty}} \frac{1}{T} \log \Gamma_{T,x}\left(d_{\mathfrak{a}}(\bar{Q}, \bar{Q}_\varepsilon) > \delta,\, \|\widehat{Q}\| \leq \ell\right) = -\infty. \tag{8.2}$$

Since $\mathfrak{a} \in \mathcal{A}$, there exits $\bar{n} \geq 1$ such that $\|a_n\|_\infty \leq \delta/(2\ell)$ for all $n \geq \bar{n}$. Note that, since $\widehat{Q}(y,z)(R) \geq \bar{Q}_\varepsilon(y,z)(R)$, it holds $\|\widehat{Q}(R)\| \geq \|\bar{Q}_\varepsilon(R)\|$ and $\|\widehat{Q}(R)\| \geq \|\widehat{Q}(R) - \bar{Q}_\varepsilon(R)\|$ for any $R \in \mathcal{M}_S$. Then for any $n \geq \bar{n}$ we have $| < \widehat{Q}(R) - \bar{Q}_\varepsilon(R), a_n > | \leq \delta/2$ if $\|\widehat{Q}(R)\| \leq \ell$. Therefore, in order to prove (8.2) we only need to show for any $\ell > 0$ that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{T\uparrow\infty}} \frac{1}{T} \log \Gamma_{T,x}\left(\exists n : 1 \leq n \leq \bar{n} \text{ s.t. } | < \widehat{Q} - \bar{Q}_\varepsilon, a_n > | > \delta/2,\, \|\widehat{Q}\| \leq \ell\right) = -\infty \tag{8.3}$$

Since $a_n \in C_0(E)$ we can find a finite subset $E' \subset E$ such that $|a_n(e)| \leq \delta/4\ell$ for all $n : 1 \leq n \leq \bar{n}$ and $e \in E \setminus E'$. Estimating

$$| < \widehat{Q} - \bar{Q}_\varepsilon, a_n > | \leq \sum_{(y,z)\in E'} \left|(\widehat{Q}(y,z) - \bar{Q}_\varepsilon(y,z))a_n(y,z)\right| + \|\widehat{Q} - \bar{Q}_\varepsilon\| \sup_{e\in E\setminus E'} |a_n(e)|,$$

we reduce the proof of (8.3) to the proof of

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{T\uparrow\infty}} \frac{1}{T} \log \Gamma_{T,x}\left(|\widehat{Q}(y,z) - \bar{Q}_\varepsilon(y,z)| > \beta\right) = -\infty, \qquad \forall(y,z) \in E,\, \forall\beta > 0. \tag{8.4}$$

This follows from (6.7).

8.2. **Proof of** (6.14). By arguments similar to the ones used in the previous proof the thesis follows thanks to the bound (6.12) in Lemma 6.5 and (6.6).

9. BIRTH AND DEATH PROCESSES

Birth and death processes are nearest–neighbor continuous time Markov chains on $\mathbb{Z}_+$ with jump rates $r(k, k+1) = b_k$ and $r(k+1, k) = d_{k+1}$, $k \geq 0$. We assume the birth rate b_k and the death rate d_k to be strictly positive. We also assume

$$Z := \sum_{k=0}^{+\infty} \frac{b_0 b_1 \cdots b_{k-1}}{d_1 d_2 \cdots d_k} < +\infty \tag{9.1}$$

and

$$\sum_{k=0}^{+\infty} \frac{d_1 d_2 \cdots d_k}{b_1 b_2 \cdots b_k} = +\infty. \tag{9.2}$$

Then assumptions (A1)–(A4) holds. Indeed, (A1) and (A3) are trivially satisfied. Due to the presence of a leftmost point (the origin), equation (2.1) reduces to the detailed balance equation and admits normalizable solutions if and only if (9.1) is fulfilled. In particular, one obtains a unique invariant probability given by

$$\pi(0) = \frac{1}{Z}, \qquad \pi(k) = \frac{1}{Z} \frac{b_0 b_1 \cdots b_{k-1}}{d_1 d_2 \cdots d_k} \qquad k \geq 1. \tag{9.3}$$

Having (9.1), condition (9.2) is equivalent to non–explosion (A2) (combine Corollary 3.18 in [14] with (9.2)) and can be rewritten as $\sum_{k=1}^{\infty} 1/(\pi(k)b_k) = +\infty$. Note that condition (9.2) is equivalent to recurrence (combine [40, Ex. 1.3.4] with [40, Th. 3.4.1]. Under the above assumptions, the logarithmic Sobolev inequality holds if and only if (see Table 1.4 in [13, Ch. 1])

$$\sup_{k \geq 1} \pi([k, +\infty)) \log \left(\frac{1}{\pi([k, +\infty))} \right) \sum_{j=0}^{k-1} \frac{1}{\pi(j)b_j} < +\infty. \tag{9.4}$$

Possible absence of exponential tightness of the empirical measure. We first discuss a case in which the empirical measure fails to be exponentially tight. Consider constant birth and death rates, i.e. $b_k = \beta$ and $d_k = \delta$. Then (9.1) and (9.2) together are equivalent to the condition $\gamma := \beta/\delta \in (0,1)$. In particular, π is geometric with parameter γ, i.e. $\pi(k) = (1 - \gamma)\gamma^k$. Consider an event in which in the time interval $[0, T]$ there are $O(T)$ jumps (typical behavior) but all the jumps are to the right (atypical behavior). The probability of such an event is "only" exponentially small in T and therefore the empirical measure cannot be exponentially tight. To be more precise, we write N_T for the number of jumps performed in the time interval $[0, T]$. Since the holding time at site k is exponential of parameter β if $k = 0$ and $\beta + \delta$ if $k \geq 1$, N_T stochastically dominates [is stochastically dominated by] a Poisson random variable with mean βT $[(\beta + \delta)T]$. Hence, with probability $1 - o(1)$, N_T has value in $I := [\beta T/2, 2(\beta + \delta)T]$. By conditioning on N_T, it is then simple to check that with probability at least $(1 - o(1))[\beta/(\beta + \delta)]^{2(\beta+\delta)T-1}$ the following event $\mathcal{A}_T$ takes place: the random variable N_T has value in I and all the jumps are to the right. The event $\mathcal{A}_T$ implies $\mu_T = \sum_{i=0}^{N_T} \delta_i/T$. Take now a compact set $\mathcal{K} \subset \mathcal{P}(V)$. By Prohorov's theorem, $\mathcal{K}$ is a tight family of probability measures and therefore, given $\varepsilon > 0$, there exists a compact (finite) set $K \subset V$ such that $\mu(K^c) \leq \varepsilon$ for all $\mu \in \mathcal{K}$. Taking T large enough, under the event $\mathcal{A}_T$ the empirical measure μ_T cannot fulfills the above requirement. Hence

$$\mathbb{P}_0(\mu_T \notin \mathcal{K}) \geq \mathbb{P}_0(\mu_T(K^c) > \varepsilon) \geq \mathbb{P}_0(\mathcal{A}_T) \geq (1 - o(1))[\beta/(\beta + \delta)]^{2(\beta+\delta)T-1}.$$

This estimate proves that the empirical measure cannot be exponentially tight. In particular neither Condition 2.2 nor 2.4 holds (even with $\sigma = 0$).

Condition 2.2. Assume now

$$\lim_{k \to \infty} d_k = +\infty, \qquad \overline{\lim_{k \to \infty}} \frac{b_k}{d_k} < 1. \tag{9.5}$$

Trivially, (9.1) and (9.2) are satisfied. We show that Condition 2.2 holds. As u_n we pick the constant sequence $u(k) = A^k$, $k \in \mathbb{Z}_+$ for some $A > 1$ to be chosen later. Since u_n does not depend on n, it is enough to check Condition 2.2. Items (i)–(iv) then hold trivially; moreover setting $d_0 := 0$ we get

$$v(k) = -\frac{Lu}{u}(k) = d_k \left(1 - \frac{1}{A} \right) + b_k(1 - A), \qquad k \in \mathbb{Z}_+.$$

Since $r(k) = b_k + d_k$, for each $\sigma \in (0, 1)$ we can write $v(k) = \sigma r(k) + d_k(1 - \sigma - 1/A) - b_k(A - 1 + \sigma)$. By (9.5), choosing A large items (v) and (vi) hold. Observe that (9.5) is satisfied when $d_k = k$ and $b_k = \lambda \in (0, +\infty)$. In this case π is Poisson with parameter λ. This implies that $e^{-\lambda}\lambda^k/k! \leq \pi([k, +\infty)) \leq \lambda^k/k!$ (for the last bound estimate $\pi(i) \leq e^{-\lambda}\lambda^i/(k - i)!$ for $i \geq k$). Using these bounds, by simple computations one can check from (9.4) that the logarithmic Sobolev inequality

(2.10) does not hold. This shows there are cases in which Condition 2.2 holds but Condition 2.4 does not.

Condition 2.4. Let now focus our attention on Condition 2.4. As already mentioned, the validity of the logarithmic Sobolev inequality is equivalent to (9.4) (assuming (9.1) and (9.2)).

We next exhibit a choice in which Condition 2.4 holds. We take $b_k = (k+1)$ and $d_{k+1} = 2b_k$ for $k \geq 0$. Observe that such rates satisfy (9.5), and therefore (9.1) and (9.2). The invariant probability π is $\pi(k) = 2^{-k-1}$. In remains to estimate $\sum_{j=0}^{k-1} (\pi(j)b_j)^{-1} = \sum_{j=1}^{k} 2^j/j$. Supposing for simplicity k even, we observe that $\sum_{j=1}^{k/2} 2^j/j \leq (k/2)2^{k/2}$ while $\sum_{j=k/2}^{k} 2^j/j \leq (2/k) \sum_{j=k/2}^{k} 2^j = (2/k)2^{k/2} \sum_{j=0}^{k/2-1} 2^j$ $= (2/k)2^{k/2}(2^{k/2} - 1)$. Hence $\sum_{j=0}^{k-1} (\pi(j)b_j)^{-1} \leq Ck2^{k/2} + C2^k/k$. From these bounds it is immediate to get (9.4). In addition, since $r(k) \sim k$ we deduce immediately that also item (ii) in Condition 2.4 holds, thus completing the check of Condition 2.4.

Violation of the LDP in the strong topology of $L_+^1(E)$. By exhibiting a concrete example, we show that - under Condition 2.2 - Theorem 2.7 does not hold in the strong topology of $L_+^1(E)$. We choose the birth and death rates as $b_k = (k+1)/2$ and $d_k = k$; in particular π is geometric with parameter $1/2$. Since (9.5) holds, Condition 2.2 is satisfied. We shall show that the level sets of I in (2.12) are not compact in the strong topology of $L_+^1(E)$. Set

$$\mu^n := \left(1 - \tfrac{1}{n}\right) \pi + \tfrac{1}{2n}\left[\delta_n + \delta_{n+1}\right]$$
$$Q^n := \left(1 - \tfrac{1}{n}\right) Q^\pi + \tfrac{1}{2}\left[\delta_{(n,n+1)} + \delta_{(n+1,n)}\right].$$

While $\{\mu^n\}$ converges to π in $\mathcal{P}(\mathbb{Z}_+)$, observe that $\{Q^n\}$ converges to Q^π in the bounded weak* topology of $L_+^1(E)$ but it is not compact in the strong topology of $L_+^1(E)$. Since $\operatorname{div} Q^n = 0$, it is simple to check that $\overline{\lim}_n I(\mu^n, Q^n) < +\infty$. This implies that the level sets of I are not compact in the strong topology of $L_+^1(E)$.

APPENDIX A. PROOF OF (5.4)

We call $\bar{I}(\mu, Q)$ the r.h.s. of (5.4). Trivially it holds $\bar{I}(\mu, Q) = +\infty = I(\mu, Q)$ if $\operatorname{div} Q \neq 0$. In the sequel we assume $\operatorname{div} Q = 0$. Then, equation (5.4) reads $I(\mu, Q) = \sup_{F \in C_c(E)} I_F(\mu, Q)$ where $I_F(\mu, Q) := \langle Q, F \rangle - \langle \mu, r^F - r \rangle$. If for some $y \in V$ and $(y, z) \in E$ it holds $\mu(y) = 0$ and $Q(y, z) > 0$, then taking $F = \lambda \delta_{(y,z)}$ with $\lambda \to +\infty$ we obtain that $\bar{I}(\mu, Q) = \infty$. On the other hand

$$I(\mu, Q) \geq \Phi(Q(y, z), Q^\mu(y, z)) = \Phi(Q(y, z), 0) = +\infty.$$

As a consequence, from now on we can restrict to (μ, Q) such that $\operatorname{div} Q = 0$ and $Q(y, z) = 0$ for all $(y, z) \in E$ with $\mu(y) = 0$. Calling $E_+ := \{(y, z) \in E : \mu(y) > 0\}$ we get that

$$I_F(\mu, Q) = \sum_{(y,z) \in E_+} \left\{ Q(y, z)F(y, z) - \mu(y)r(y, z)(e^{F(y,z)} - 1) \right\}.$$

At this point, it is simple to check that, varying $F(y, z)$, the supremum of the above addendum is given by $\Phi(Q(y, z), Q^\mu(y, z))$ and the value of the above addendum for $F(y, z) = 0$ is zero. Hence,

$$\bar{I}(\mu, Q) = \sum_{(y,z) \in E_+} \Phi(Q(y, z), Q^\mu(y, z)) = \sum_{(y,z) \in E} \Phi(Q(y, z), Q^\mu(y, z)).$$

We now claim that the above expression is $+\infty$ if $\langle \mu, r \rangle = +\infty$, thus concluding the proof. To this aim we observe that for $0 \le q < p/2$ it holds $\Phi(q, p) \ge p(1 - \log 2)/2$. Indeed, the thesis is trivially true if $q = 0$, while for $q > 0$ we can write $\Phi(q, p) = pf(q/p)$ where $f(x) = x \log x + 1 - x$. Since $f(x)$ is decreasing for $0 < x < 1$, one has $\Phi(q, p) \ge pf(1/2)$ for $0 \le q < p/2$. Hence, setting $c := 2/(1 - \log 2)$, our claim follows from the bound

$$
\begin{aligned}
\langle \mu, r \rangle &= \sum_{(y,z) \in E} Q^{\mu}(y, z) \\
&\le \sum_{\substack{(y,z) \in E : \\ Q(y,z) < Q^{\mu}(y,z)/2}} c\ \Phi(Q(y, z), Q^{\mu}(y, z)) + \sum_{\substack{(y,z) \in E : \\ Q(y,z) \ge Q^{\mu}(y,z)/2}} 2Q(y, z) \\
&\le \sum_{(y,z) \in E} c\ \Phi(Q(y, z), Q^{\mu}(y, z)) + 2\|Q\|_1 .
\end{aligned}
$$

Appendix B. An example with discontinuous divergence

Consider the oriented graph (V, E) where $V = \mathbb{N} \cup \{v, w\}$ and E is given by the oriented bonds of the form (v, n), (n, w),(w, v) for some $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define $Q^{(n)}$ as the flow of unitary flux associated to the cycle (v, n, w, v), i.e. $Q^{(n)} = \mathbb{I}_{(v,n)} + \mathbb{I}_{(n,w)} + \mathbb{I}_{(w,v)}$. We claim that $Q^{(n)}$ converges to $Q := \mathbb{I}_{(w,v)}$ in $L^1_+(E)$ (endowed of the bounded weak* topology). Since $\|Q^{(n)}\| = 3$, the sequence $\left(Q^{(n)}\right)_{n \in \mathbb{N}}$ is bounded in the strong topology of $L^1_+(E)$. In particular, $Q^{(n)} \to Q$ in the bounded weak* topology if and only if $Q^{(n)} \to Q$ in the weak* topology, and therefore if and only if $\langle \phi, Q^{(n)} \rangle \to \langle \phi, Q \rangle$ for each $\phi \in C_0(E)$. By construction we have

$$
\langle \phi, Q^{(n)} \rangle = \phi(v, n) + \phi(n, w) + \phi(w, v) \to \phi(w, v) = \langle \phi, Q \rangle ,
$$

thus concluding the proof of our claim.

We observe that, despite $\operatorname{div} Q^{(n)} = 0$ for all $n \in \mathbb{N}$, it holds $\operatorname{div} Q \ne 0$. This example shows that the map $L^1_+(E) \ni Q \to \operatorname{div} Q(x) \in \mathbb{R}$, with $x \in V$, is not in general a continuous map.

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