Dynamical Renormalization Group Approach to the Collective Behavior of Swarms

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We study the critical behavior of a model with nondissipative couplings aimed at describing the collective behavior of natural swarms, using the dynamical renormalization group under a fixed-network approximation. At one loop, we find a crossover between an unstable fixed point, characterized by a dynamical critical exponent $z = d/2$, and a stable fixed point with $z = 2$, a result we confirm through numerical simulations. The crossover is regulated by a length scale given by the ratio between the transport coefficient and the effective friction, so that in finite-size biological systems with low dissipation, dynamics is ruled by the unstable fixed point. In three dimensions this mechanism gives $z = 3/2$, a value significantly closer to the experimental window, $1.0 \leq z \leq 1.3$, than the value $z \approx 2$ numerically found in fully dissipative models, either at or off equilibrium. This result indicates that nondissipative dynamical couplings are necessary to develop a theory of natural swarms fully consistent with experiments.

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Collective behavior in biological systems emerges when local interactions give rise to correlations that significantly exceed the scale of the individuals. This scenario is the ideal hunting ground for statistical physics, and in particular for its most powerful tool, the renormalization group (RG) [1]. A crucial requirement any successful theory of collective phenomena must meet is to reproduce the correlation functions and to predict the correct values of the critical exponents, the calculation of which is the RG task. It is natural, then, that the RG is applied to collective biological systems. The hydrodynamic theory of flocking of Toner and Tu has been a pioneering step in this direction [2], while recent RG studies of neural data are also promising [3,4]. Here, we employ the RG to study the collective dynamics of swarms.

Experiments on large swarms of midges in the field [5] show two things: (i) dynamic correlations of the velocities have an inertial form incompatible with the classic exponential relaxation of overdamped systems, and (ii) critical slowing down, namely, the relation linking relaxation time and correlation length, $\tau \sim \xi^z$, holds with a dynamical critical exponent in the window $1.0 \leq z \leq 1.3$, very unusual for purely dissipative dynamics. Both facts urge for a theoretical explanation. The most far-reaching model of collective biological behavior was introduced by Vicsek and coworkers [6]: individuals are self-propelled particles moving at fixed speed and aligning their velocities to those of their neighbors through a "social force" [7,8]. Vicsek's model is analogous to a ferromagnetic system (velocities playing the role of local magnetizations) with dissipative Langevin dynamics; however, at variance with equilibrium ferromagnets, the interaction network changes in time, due to the self-propulsion of the individuals. Vicsek's model describes both polarized flocks and unpolarized swarms, depending on noise and density; therefore, in [5], dynamical correlations in the swarm phase of the Vicsek model were measured. In contrast with real experiments, though, Vicsek swarms display exponential relaxation and a dynamical critical exponent $z \approx 2$. Moreover, a previous RG study of self-propelled swarms [9] found $z = 1.7$ (in $d = 3$ and under the assumption of incompressibility), which is also quite far from experiments. The fact that in these self-propelled cases the dynamical critical exponent is different from the experimental value suggests that self-propulsion may not be the primary cause of anomalous relaxation in natural swarms. Hence, we focus here on developing a theory with fixed interaction network, but with a novel type of dynamical coupling, leaving the self-propelled generalization to future studies.

An inertial form of the dynamic correlation function suggests that the equations of motion contain conservative terms deriving from a symmetry of the interaction [10]. When this happens, the force between the individuals is mediated by the symmetry generator, which is a conserved momentum conjugate to the primary degree of freedom through some inertia. The inertial spin model (ISM), introduced in [11,12] for the description of information transfer in flocks, contains a nondissipative coupling between the velocities and the generator of rotations, called spin. It was therefore suggested in [5] that the ISM may also describe the inertial form of swarm relaxation. Moreover, nondissipative couplings are known [10] to lower z below its purely dissipative value ≈ 2. Therefore, it seems plausible that the ISM may help with both experimental traits of swarm dynamics. From another point of view, though, this may seem an ill-founded hope. In Hamiltonian systems the spin is strictly conserved [10], while in biological groups it cannot be: the spin is what causes animals to turn; hence it must be dissipated in the absence of interaction or perturbation [12]. Therefore, the ISM contains both nondissipative couplings and friction. Because friction takes over in the hydrodynamic limit, one may expect the ISM to have the same critical dynamics as an overdamped system. Here we resolve this quandary by studying the critical dynamics of the fixed-network ISM through a RG approach. Details of our calculation can be found in [13].

Coarse-graining the microscopic ISM dynamics in the fixed-network case, we obtain

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = -\Gamma_0 \frac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} + g_0 \boldsymbol{\psi} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}, \tag{1}$$

$$\frac{\partial \boldsymbol{s}}{\partial t} = -(\eta_0 - \lambda_0 \nabla^2) \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + g_0 \boldsymbol{\psi} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\zeta}, \tag{2}$$

where the field $\boldsymbol{\psi}(\boldsymbol{x}, t)$ would correspond to the velocity in the self-propelled case, while it is simply a vectorial order parameter in the fixed-network approximation we use here; the conjugate momentum $\boldsymbol{s}(\boldsymbol{x}, t)$ represents the coarse-grained spin, namely, the generator of the rotations acting upon $\boldsymbol{\psi}$ (the Poisson bracket reads $\{\psi_\mu, s_\nu\} = g_0 \epsilon_{\mu\nu\rho} \psi_\rho$, where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita symbol [14]). The uncoupled diagonal terms, $\partial_t \boldsymbol{\psi} \sim \delta\mathcal{H}/\delta\boldsymbol{\psi}$ and $\partial_t \boldsymbol{s} \sim \delta\mathcal{H}/\delta\boldsymbol{s}$, contribute to the dissipative relaxation of the two fields, while the mode-coupling cross terms enforce the conservative nature of the dynamics: $\partial_t \boldsymbol{s} \sim \boldsymbol{\psi} \times \delta\mathcal{H}/\delta\boldsymbol{\psi}$ implies that the force acts on the spin, rather than directly on the order parameter, and in turn the term $\partial_t \boldsymbol{\psi} \sim \boldsymbol{\psi} \times \delta\mathcal{H}/\delta\boldsymbol{s}$ expresses the role of the spin as generating the rotations of the order parameter. The white Gaussian noises $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ have variance $2\Gamma_0$ and $2(\eta_0 + \lambda_0 k^2)$, respectively. The Hamiltonian has the rotationally symmetric form [10]

$$\mathcal{H} = \int d^d x \left(\frac{1}{2} (\nabla \boldsymbol{\psi})^2 + \frac{1}{2} r_0 \psi^2 + u_0 \psi^4 + \frac{s^2}{2\chi_0} \right), \tag{3}$$

where the square gradient enforces the alignment interaction, and instead of the fixed speed constraint, $|\boldsymbol{\psi}|^2 = 1$, of the microscopic model, one has the confining potential, $r_0 \psi^2 + u_0 \psi^4$, where $r_0 < 0$ in the ordered phase. The parameter χ_0 is the effective inertia associated to the spin; at the static level, $\boldsymbol{s}$ is a Gaussian field; hence there are not corrections to the naive scaling dimension of χ_0, so we can fix $\chi_0 = 1$ in the following. The kinetic coefficient Γ_0, the effective friction η_0, the transport coefficient λ_0, and the nondissipative coupling constant g_0 are dynamical parameters. When working in Fourier space, all integrations over k are performed up to a cutoff Λ, which corresponds to the inverse of a microscopic length scale.

Compared to the microscopic ISM [12], there are additional terms in (1) and (2) due to the coarse-graining. The diffusive term $\Gamma_0 \nabla^2 \boldsymbol{\psi}$ (and its coupled noise $\boldsymbol{\theta}$) is subleading in the polarized phase [15], but it is crucial in the near-critical phase. The nondissipative coupling between the two fields is ruled by g_0, to make scaling dimensions explicit. Even though the spin is microscopically dissipated only through the nonconservative friction, η_0, the RG calculation shows that, as an effect of the symmetry, the first correction to the self-energy of $\boldsymbol{s}$ is of order k^2 [13]; this means that the coarse-graining of the microscopic spins produces a conservative transport term of the type $\lambda_0 \nabla^2 \boldsymbol{s}$, which must therefore be included in the field equations. For $\eta_0 = 0$ our model is identical to the Heisenberg antiferromagnet, and, in the planar case, to superfluid helium (models G and E of [10]).

It is interesting to note that the quantity $\sqrt{\lambda_0/\eta_0} \equiv \mathcal{R}_0$ has the physical dimension of a length scale, which is larger the smaller the friction. Intuitively, we can guess that within the scale $\mathcal{R}_0$ nonconservative effects are weaker than the conservative ones generated by the symmetry, while beyond $\mathcal{R}_0$ the opposite occurs and dissipation takes over. As we shall see, the conservation length scale $\mathcal{R}_0$ will indeed rule the critical dynamics of the model, leading to a nontrivial crossover between different critical exponents.

The renormalization group approach unfolds through two stages [1,10]: (i) integration of the short wavelengths details and (ii) rescaling of momentum and frequency. In the first stage, one calculates the effective probability distribution of the long-wavelengths fields $\boldsymbol{\psi}(\boldsymbol{k}, \omega)$, with $k < \Lambda/b$, where b is a scaling factor, by integrating over all short-wavelengths fluctuations with k in the so-called momentum shell, $\Lambda/b < k < \Lambda$. This integration produces corrections to the linear terms in the dynamical equations for the long-wavelengths fields [10]. To calculate these corrections one writes the propagators, which are defined as the inverse of the equations of motions,

$$G_{b,\psi}^{-1}(\mathbf{k}, \omega) = -i\omega + \Gamma_0 (k^2 + r_0) - \Sigma_b(\mathbf{k}, \omega), \tag{4}$$

$$G_{b,s}^{-1}(\mathbf{k},\omega) = -i\omega + \eta_0 + \lambda_0 k^2 - \Pi_b(\mathbf{k},\omega), \tag{5}$$

where the self-energies Σ_b and Π_b arise from the on-shell integration [13]. In the limit $k \to 0$ the $O(k^2)$ terms in the self-energies correct Γ_0 and λ_0, while a term of $O(1)$ in Π_b would correct the effective friction, η_0; however, because of the symmetry there is no such term in Π_b, so that η_0 has no correction from the RG integration [13]. Regarding the dynamical coupling constant g_0, even though the spin is not conserved, the symmetry nevertheless protects the naive scaling dimension of g_0, which therefore does not pick up a perturbative correction [13].

After integrating over the momentum shell, the theory is left with a novel cutoff, Λ/b; hence, the second RG stage consists in rescaling the momentum k in such a way to restore the original cutoff Λ. The key idea of the dynamical renormalization group [10] is that, close to criticality, a rescaling of space entails a rescaling of time, regulated by the dynamical critical exponent z,

$$k \to bk, \qquad \omega \to b^z\omega. \tag{6}$$

By iterating l times the RG step we obtain a set of recursive equations describing the flow of the parameters under successive coarse-grainings. At the critical point the correlation length ξ is infinite and the theory at large distances is scale invariant, so that the fixed points of the RG flow give the effective values of the parameters ruling the system at large distances, i.e., for $k \to 0$ [1]. The RG equations at $\xi = \infty$ are the following [13],

$$\Gamma_{l+1} = \Gamma_l b^{z-2}\left(1 + \frac{2f_l}{1+w_l} X_l \ln b\right), \tag{7}$$

$$\lambda_{l+1} = \lambda_l b^{z-2}\left(1 + \frac{1}{2} f_l \ln b\right), \tag{8}$$

$$\eta_{l+1} = \eta_l b^z, \tag{9}$$

$$g_{l+1} = g_l b^{z-d/2}, \tag{10}$$

where we have introduced the effective running parameters, $f_l = \Lambda^{d-4} K_d g_l^2/(\Gamma_l \lambda_l)$ (K_d is the volume of the d-dimensional unit sphere) and $w_l = \Gamma_l/\lambda_l$, and the crossover factor,

$$X_l = \frac{(1+w_l)(\mathcal{R}_l\Lambda)^2}{1+(1+w_l)(\mathcal{R}_l\Lambda)^2}, \tag{11}$$

where $\mathcal{R}_l = \sqrt{\lambda_l/\eta_l}$ is the (running) conservation length scale. The powers of b in the RG equations are due to the rescaling of k and ω, which renormalizes each parameter by its naive physical dimensions, whereas the $\ln b$ terms derive from the shell integration. For zero friction, $\eta_l = 0$, the conservation length scale diverges, $\mathcal{R}_l = \infty$; this implies $X_l = 1$, and the flow equations become identical to the fully conservative models [16]. However, we see from (9) that for any nonzero initial value of the friction, η_0 grows along the flow: dissipation becomes increasingly relevant at longer scales. From (7)–(10) we can write the closed set of recursive equations for the effective parameters,

$$f_{l+1} = f_l b^{\epsilon}\left[1 - f_l\left(\frac{2X_l}{1+w_l} + \frac{1}{2}\right)\ln b\right], \tag{12}$$

$$w_{l+1} = w_l\left[1 + f_l\left(\frac{2X_l}{1+w_l} - \frac{1}{2}\right)\ln b\right], \tag{13}$$

$$\mathcal{R}_{l+1} = \mathcal{R}_l b^{-1}\left[1 + \frac{1}{4} f_l \ln b\right], \tag{14}$$

where $\epsilon = 4 - d$ is the expansion parameter of the RG series [1]; our calculation is performed at $\mathrm{O}(\epsilon)$, i.e., at one-loop level [13]. Once the fixed points of the RG equations are found, the dynamical critical exponent z, ruling the relaxation of the order parameter, is obtained by imposing that the fixed point of the kinetic coefficient of $\boldsymbol{\psi}$, namely Γ^*, is finite [10]. This condition gives

$$z = 2 - \frac{2f^*}{1+w^*} X^*. \tag{15}$$

The flow equations (12)–(14) have two fixed points; the first one is the same as the fully conservative case [10],

$$f^* = \epsilon, \qquad w^* = 3, \qquad \mathcal{R}^* = \infty, \qquad X^* = 1 \Rightarrow z = d/2. \tag{16}$$

At this fixed point $\mathcal{R}^* = \infty$, implying $\eta^* = 0$: the spin is not damped at all, so that dissipation is irrelevant and the conservation law entailed by the symmetry of the Hamiltonian rules the dynamics at all scales. This fixed point, though, is unstable: any large, but finite, initial value of $\mathcal{R}_0$ decreases under (14), driving the flow to the stable fixed point,

$$f^* = 2\epsilon, \qquad w^* = 0, \qquad \mathcal{R}^* = 0, \qquad X^* = 0 \Rightarrow z = 2. \tag{17}$$

Here dynamics is taken over by dissipation ($\eta^* = \infty$), no trace remains of the nondissipative couplings, and the conservation scale $\mathcal{R}^*$ shrinks to 0 [17].

The RG flow diagram (Fig. 1) shows that, when the starting value of X_0 is close to 1, namely, when the friction η_0 is small, the parameters rapidly converge towards the $z = d/2$ fixed point and linger thereabout for many RG iterations, before eventually crossing over to the $z = 2$ fixed point. This happens because the unstable fixed point is actually stable along the line $X_0 = 1 (\eta_0 = 0)$, thus acting

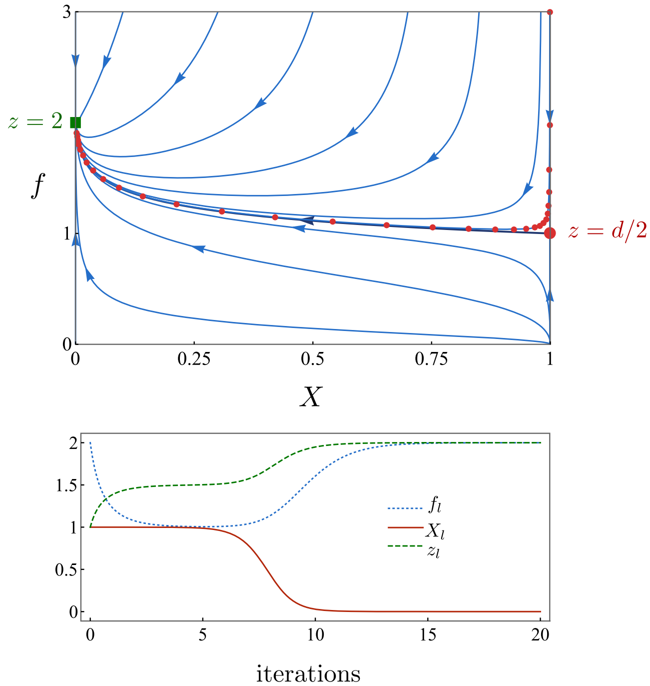

FIG. 1. The renormalization group flow and crossover. Top: Flow diagram on the (X_l, f_l) plane for $d = 3$. When the initial friction η_0 is small, i.e., $X_0 \sim 1$ (red dots), the flow converges towards the unstable fixed point, $z = d/2$, and remains in its neighborhood for many iterations, before crossing over to the $z = 2$ fixed point. Bottom: When plotting the running parameters and the critical exponent as a function of the iteration step along a flow line with small η_0, the RG crossover clearly emerges. The initial values of the parameters are $f_0 = 2$, $X_0 = 0.9999$, $w_0 = 3$.

as a pseudoattractor for low-dissipation dynamics. This RG crossover gives rise to a corresponding crossover in the relaxation of the system, depending on the interplay between the correlation length, ξ, and the conservation length, $\mathcal{R}_0$. Both scales decrease under the RG flow, but while ξ has the regular scaling dimension 1, $\mathcal{R}_0$ develops an anomalous dimension, because close to the unstable fixed point $f_l/4 \sim \epsilon/4$, so that Eq. (14) gives

$$\xi_{l+1} = \xi_l/b, \qquad \mathcal{R}_{l+1} = \mathcal{R}_l/b^{d/4} \tag{18}$$

(the initial value of the correlation length in the flow coincides with its physical value, $\xi_0 = \xi$). The RG iteration must stop when the correlation length has been reduced to the order of the lattice spacing, $\xi_{l+1} \sim 1/\Lambda$, a condition equivalent to $b^l = \xi\Lambda$. In order for the flow to be still in the neighborhood of the $z = d/2$ fixed point when the RG iteration stops, we must have $X_{l+1} \sim 1$, which, given Eq. (11), requires $\mathcal{R}_{l+1} \gg 1/\Lambda$, that is, $\mathcal{R}_0/(b^l)^{d/4} \gg 1/\Lambda$. By plugging into this last relation the RG stop condition, $b^l = \xi\Lambda$, one gets that the $z = d/2$ fixed point rules for $(\xi\Lambda)^{d/4} \ll \Lambda\mathcal{R}_0$. Conversely, for large correlation lengths the system is ruled by the stable fixed point, $z = 2$. In this way we obtain the following dynamical crossover,

$$\xi \ll \mathcal{R}_0^{4/d} \Rightarrow \tau \sim \xi^{d/2}, \tag{19}$$

$$\xi \gg \mathcal{R}_0^{4/d} \Rightarrow \tau \sim \xi^2, \tag{20}$$

where we have set $\Lambda = 1$ for simplicity. Critical slowing down is therefore governed by two different exponents, depending on the scale of the correlation. This result holds for finite ξ at $k = 0$, but an identical crossover occurs at $\xi = \infty$ when varying the scale k [13], namely,

$$k \gg \mathcal{R}_0^{4/d} \Rightarrow \tau \sim k^{-d/2}, \tag{21}$$

$$k \ll \mathcal{R}_0^{4/d} \Rightarrow \tau \sim k^{-2}. \tag{22}$$

We remark that the crossover is regulated by the conservation length scale, $\mathcal{R}_0 = \sqrt{\lambda_0/\eta_0}$: the smaller the effective friction, η_0, the larger the scale up to which critical dynamics is ruled by the conservative exponent, $z = d/2$.

To test our results we run numerical simulations of the microscopic ISM on a fixed lattice with periodic boundary conditions. The dynamical equations are [12]

$$\frac{d\boldsymbol{\psi}_i}{dt} = \frac{1}{\hat{\chi}} \boldsymbol{s}_i \times \boldsymbol{\psi}_i, \tag{23}$$

$$\frac{d\boldsymbol{s}_i}{dt} = -\frac{\hat{\eta}}{\hat{\chi}} \boldsymbol{s}_i + \boldsymbol{\psi}_i \times \hat{J} \sum_{j \in i} n_{ij} \boldsymbol{\psi}_j + \boldsymbol{\psi}_i \times \boldsymbol{\zeta}_i, \tag{24}$$

where $|\boldsymbol{\psi}_i|^2 = 1$, the adjacency matrix n_{ij} corresponds to a $d = 3$ cubic lattice, and $\hat{J}$ is the strength of the alignment interaction (hats distinguish microscopic from coarse-grained parameters). We compute the $k = 0$ relaxation time τ in systems with linear sizes up to $L = 20$ [13]. In order to observe the power-law crossover of (19) and (20) in a τ vs ξ plot, one would need a span of ξ of several orders of magnitude, which is not possible with our maximum size [18]. Therefore, to test the predicted crossover we run simulations at different values of the effective friction: for small enough $\hat{\eta}$ we should have $\mathcal{R}_0 > L$, so that the whole system is within the conservation scale and we expect to observe $\tau \sim \xi^{3/2}$; conversely, for large enough $\hat{\eta}$, we should have $\mathcal{R}_0$ as small as the lattice spacing, giving $\tau \sim \xi^2$. This RG prediction is fully confirmed by numerical simulations [Fig. 2(a)]: the dynamical critical exponent z crosses over from 3/2 to 2 by increasing the effective friction. Moreover, simulations show that the ISM dynamical correlation function in the critical regime has the same inertial form as real swarms, a significant improvement over the Vicsek model [Figs. 2(b) and 2(c)]. In particular, the relaxation form factor, $h(t/\tau) \equiv \dot{C}(t/\tau)/C(t/\tau)$, goes to 1 for overdamped exponential relaxation (Vicsek), while it goes to 0 for the ISM, exactly as in real swarms [5].

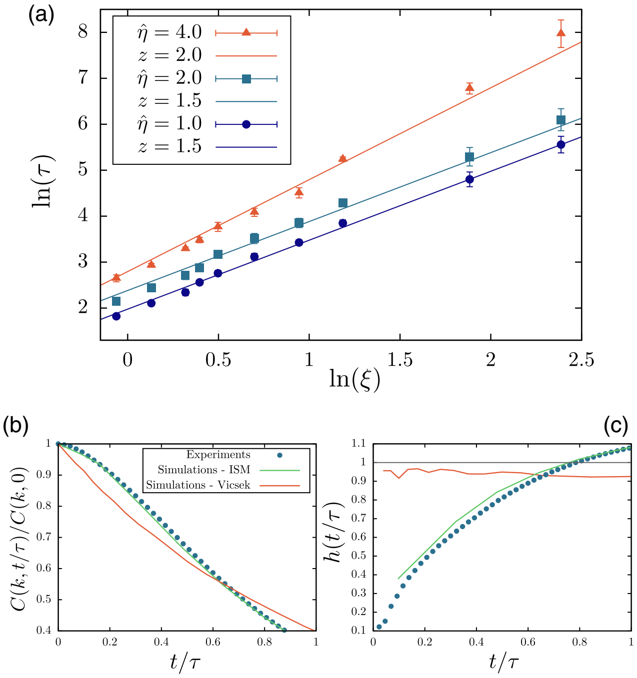

FIG. 2. Numerical simulations. Top: Relaxation time vs correlation length in $d = 3$, at various values of the friction $\hat{\eta}$. Lines are best fit to $z = 3/2$ (blue, low $\hat{\eta}$) and $z = 2$ (orange, large $\hat{\eta}$). Bottom, left: The normalized dynamical correlation function, $C(k,t)/C(k,0)$, at $k = 1/\xi$. Right: The relaxation form factor, $h(t/\tau) \equiv \dot{C}(t/\tau)/C(t/\tau)$, goes to 1 for exponential relaxation, while it goes to 0 for inertial relaxation. Experimental data on swarms and data on the $3d$ Vicsek model close to the ordering transition are from [5].

Experimentally measured correlation functions indicate that dissipation is very small in natural swarms [5]. This fact, plus the obvious observation that real swarms have finite size, makes it plausible that swarms are smaller than the conservation length scale, $\mathcal{R}_0$, so that their dynamics is ruled by the conservative fixed point, namely, by $z = 3/2$. Although this is still about 30% off the experimental barycenter, $z \approx 1.15$, it is a significant improvement over both $z \approx 2$ of the Vicsek model [5] and $z = 1.7$ obtained by one-loop RG on self-propelled incompressible swarms [9], both cases missing mode-coupling terms. We conclude that a theory where the social force acts on the velocities through a nondissipative coupling not only gives much more compelling correlation functions, but also shifts the dynamical critical exponent in the right direction by the most significant amount to date. However, future RG studies will certainly need to include both the nondissipative couplings and the self-propulsion terms neglected here, because both kinds of terms are likely to decrease the critical exponent. Such calculation does not look like a piece of cake, but it seems the only way to fully bridge the gap between theory and experiments in natural swarms.

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[17] Note that $z = 2$ is the Gaussian value of the dynamical critical exponent; at the stable fixed point, the first corrections to z are found at $O(\epsilon^2)$.
[18] Three decades, $L = 10^3$, is the *very* minimum needed to observe a power-law crossover. This gives $N = 10^9$ in $d = 3$, which is hard, considering that $\tau \sim \xi^2 \sim L^2 \sim 10^6$.