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Solving the spherical p-spin model with the cavity method: equivalence with the replica results

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Abstract. The spherical p-spin is a fundamental model for glassy physics, thanks to its analytical solution achievable via the replica method. Unfortunately, the replica method has some drawbacks: it is very hard to apply to diluted models and the assumptions beyond it are not immediately clear. Both drawbacks can be overcome by the use of the cavity method; however, this needs to be applied with care to spherical models. Here, we show how to write the cavity equations for spherical p-spin models, both in the replica symmetric (RS) ansatz (corresponding to belief propagation) and in the one-step replica-symmetry-breaking (1RSB) ansatz (corresponding to survey propagation). The cavity equations can be solved by a Gaussian RS and multivariate Gaussian 1RSB ansatz for the distribution of the cavity fields. We compute the free energy in both ansatzes and check that the results are identical to the replica computation, predicting a phase transition to a 1RSB phase at low temperatures. The advantages of solving the

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model with the cavity method are many. The physical meaning of the ansatz for the cavity marginals is very clear. The cavity method works directly with the distribution of local quantities, which allows us to generalize the method to diluted graphs. What we are presenting here is the first step towards the solution of the diluted version of the spherical p-spin model, which is a fundamental model in the theory of random lasers and interesting per se as an easier-to-simulate version of the classical fully connected p-spin model.

Keywords: cavity and replica method, ergodicity breaking, message-passing algorithms, random graphs, networks

Contents

1. Introduction

Spherical models are made of N real variables $\sigma_i \in \mathbb{R}$ satisfying the global constraint $\sum_i \sigma_i^2 = N$. They play a key role among solvable models in statistical physics, because they usually allow for closed and compact algebraic solutions [1, 2]. Moreover, since the variables are real numbers, the space of configurations is continuous and differentiable, thus allowing one to study several kinds of dynamics in these models (e.g. Langevin dynamics or gradient descent-like relaxations). At variance, models whose variables satisfy local constraints pose more problems. For example, in the Ising and Potts models the variables take discrete values and so the space of configuration is not continuous, while in the $O(n)$ models (e.g. with XY or Heisenberg spins) each variable is continuous, but needs to satisfy a local constraint of unit norm in an n-dimensional space, and this in turn makes the analytic solution much more complicated; see for instance [3–5].

The success of spherical models is well witnessed by the fully connected spherical p-spin model. For $p \geqslant 3$ this model is the most used mean-field model for the glassy dynamics. We learned a lot from it precisely because both the thermodynamics and the dynamics can be easily solved [6–9]. The thermodynamic solution has been obtained via the replica method, and it has a compact analytical form thanks to the spherical constraint: the solution predicts a random first-order transition from a high-temperature paramagnetic phase to a low-temperature spin glass phase. The equilibrium and out-of-equilibrium dynamics have been solved via the generating functional formalism, thanks to the mean-field nature of the model and the spherical constraint [9, 10].

Notwithstanding the success of fully connected spherical models, we are well aware that they have several unrealistic features: fully connectedness is unlikely to happen in any realistic phenomenon and the spherical constraint is just a global surrogate for the actual constraint each variable should satisfy locally. In other words, in realistic models each variable is somehow bounded, and one uses the single global spherical constraint to make computations easier. Although this approximation is extremely useful, it has some drawbacks. For example, when the interactions are diluted, a condensation phenomenon may take place [11–13].

The diluted and sparse versions of a model are particularly interesting, because moving away from the fully connected limit is needed in order to study more realistic phenomena [14–18]. We reserve the word *sparse* for graphs with a mean degree $O(1)$, i.e. not growing with N, while we use the term *diluted* for a graph which is not fully connected, but whose mean degree still grows with N. In sparse models the couplings do not vanish in the large N limit and this implies the solution is deeply non-perturbative. The cavity method has been developed exactly to solve sparse models [19]. In very few cases, such method has been exploited for fully connected lattices, e.g. in the study of models with discrete variables [20] or in the case of linear interactions, as for the planted Sherrington—Kirkpatrick (SK) problem in the context of inference [21]. Diluted models are much less studied in the literature with respect to fully connected and sparse models. Nonetheless, they are very interesting for several aspects. They can be used in numerical simulations as a proxy for fully connected models, which are very demanding in terms of computing resources. They appear in models of random lasers where dilution is induced by the selection rules for the coupling of light modes in random media [22–24].

Depending on the level of dilution, they allow for heterogeneities and local fluctuations in models that can still be solved similarly to the fully connected version; that is, exploiting the fact that couplings are weak and the graph mean degree diverges. We believe it is worth dedicating more effort to studying the realm of diluted models. In the present contribution we aim to set up a framework that would allow us to study diluted models via the cavity method. We are particularly interested in spherical models, because they are models whose solution turns out to be particularly simple and compact. However, spherical models may undergo a condensation transition when the interaction graph is diluted. How the condensation transition can be avoided in a p-spin model by just modifying the spherical constraint is another open problem which we are currently investigating and which will be discussed elsewhere [25].

The study of whether condensation takes place is a delicate matter: this depends on the competition between the functional form of the global constraint, which can even be non-spherical, and the strength of the interactions, the latter depending on both the order of the non-linearity and the amount of dilution in the graph. Working with Hamiltonian models where variables interact via p-body terms and calling $M = O(N^{\delta})$ the number of interaction terms, we would like to single out the threshold exponent δ_{c}, such that for $\delta > \delta_{\mathrm{c}}$ at finite temperature there is no condensation, while for $\delta < \delta_{\mathrm{c}}$ at any temperature the system is in the condensed phase. So far, the situation is clear only for the two boundaries of the interval of possible values for δ. For $\delta = p$, which represents the complete graph, condensation is never found at finite temperature, while the sparse graph, i.e. $\delta = 0$, is always in the condensed phase provided that interactions are non-linear, i.e. $p > 2$. The situation for intermediate values of δ is under current investigation, and we expect the present work to be an important step in this direction. For the moment we focus on the dilution regime where such a condensation phenomenon does not take place.

In the following we present the zero-th order step of the above program by showing how to use the cavity method to solve the fully connected version of spherical spin glass models. Although the cavity method is well known [26], its use in spherical models has not appeared before in the literature (to the best of our knowledge). The application of the cavity method to spherical models is not straightforward, because one has to decide how to convert a *global* constraint in a set of *local* ones. We will discuss this aspect explicitly and propose a standardized solution. Once the cavity equations are written, their solution requires some ansatz for the distribution of local fields. This is one of the advantages of the cavity method with respect to the replica method: all assumptions made in the derivation have a clear and direct physical meaning. By using a Gaussian ansatz for the distribution of local fields (eventually, correlated Gaussian fields in the spin glass phase where the replica symmetry spontaneously breaks down) we are able to obtain the exact solution to the spherical p-spin glass model, that was previously derived via the replica method. We dedicate the main text to the derivation of the saddle point equations, to the illustration of the ansatz for the local field distributions, to the discussion on how to implement the spherical constraint and to reporting the resulting free energies. More technical and lengthy derivations, such as the explicit calculations of the free energy, are postponed to appendices A and B.

In section 2 we explain why a Gaussian ansatz for the cavity marginals is correct in the large degree limit and how to use it to obtain a closure of the belief propagation equations. In particular, in section 2.5 we discuss the two possible choices to implement the spherical constraint in the belief propagation equations, which are equivalent only in the large degree limit. Section 3 is dedicated to the study of survey propagation equations, i.e. the generalization of belief propagation equations in the case of a 1RSB scenario. In section 3.1 we present the multivariate Gaussian ansatz needed for the survey propagation equations, recently introduced in [21], and in section 3.2 we show how the explicit closure of the equations is obtained by means of this ansatz. While the 1RSB expression of the free energy is reported in section 3.3, its explicit derivation in full detail can be found in appendices A and B.

2. Cavity equations with spherical constraint

2.1. Spherical models

We consider models with N real variables $\sigma_i \in \mathbb{R}$ constrained to satisfy the condition

$$\mathcal{A}\left[\boldsymbol{\sigma}\right] \equiv \sum_{i=1}^{N} \sigma_i^2 = N \tag{1}$$

and interacting via p-body interactions

$$\mathcal{H} = -\sum_{a=1}^{M} J_a \prod_{i\in\partial a} \sigma_i, \tag{2}$$

where ∂a is the set of variables entering the ath interaction and we fix $|\partial a| = p$. If the interaction graph is fully connected then $M = \binom{N}{p}$ and the sum runs over all possible p-uplets; otherwise, in diluted models, the M interactions are randomly chosen among the $\binom{N}{p}$ possible p-uplets. The fully connected versions have been solved via the replica method.

For $p = 2$ the model is particularly simple because the energy function has only two minima and the free energy can be computed from the spectrum of the interaction matrix $\boldsymbol{J}$. The model possesses a spin glass phase at low temperatures, but the replica symmetry never breaks down and a replica symmetric (RS) ansatz provides the exact solution [27]. In this case the spherical constraint, although efficient in keeping variables bounded, drastically changes the low-energy physics with respect to models with e.g. Ising variables: indeed, the Sherrington–Kirkpatrick model [28] has a spin glass phase with spontaneous breaking of the replica symmetry [29, 30].

For $p \geqslant 3$ the spherical model is much more interesting since it undergoes a phase transition to a spin glass phase where the replica symmetry is broken just once (1RSB phase) [6] as in the analogous model with Ising variables [31]. More importantly, the thermodynamic phase transition is preceded by a dynamical phase transition [7] which has been connected to the structural glass transition [32, 33] and to the mode coupling

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theory [34]. The spherical p-spin model with $p \geqslant 3$ now represents the most used mean-field model for the random first-order transition [35].

2.2. Self-consistent cavity equations for the local marginals

The replica method allows us to fully characterize the static properties of the spherical p-spin model on complete graphs, as was firstly done in [6]. Our purpose is to study spherical p-spin models, showing that the cavity method is equivalent to replicas on complete graphs. A complete hypergraph can be seen as a bipartite graph made of function nodes, representing the interaction p-uplets, and variable nodes, representing the N spins σ_i's. We will indicate the set of links between function and variable nodes as edges E. A complete graph has $M = \binom{N}{p} = O(N^p)$ function nodes, each of which is linked to p variable nodes. On the other hand, each variable node is linked to $K = \binom{N-1}{p-1} = O(N^{p-1})$ function nodes.

In order to ensure the extensivity of the energy, not only must the N real variables satisfy the spherical constraint in equation (1), the couplings $\{J_a\}$, which are independent and identically distributed quenched random variables, must be properly normalized: in the case of symmetric couplings we have

$$\langle J\rangle = 0, \qquad \langle J^2\rangle = \frac{p!J_2}{2N^{p-1}} \tag{3}$$

with $J_2 = O(1)$ to ensure an extensive energy. Since we have in mind to extend the results of the present study to the case of increasing dilution of the hypergraph, let us start from the statistical ensemble where the partition function of the model, and hence the corresponding thermodynamic potentials, are always well defined, i.e. the *microcanonical ensemble*.

In presence of the spherical constraint written in equation (1) the partition function of the model reads thus

$$\Omega_A(E,N) = \int \prod_{i=1}^{N} \mathrm{d}\sigma_i\ \delta\left(E - \mathcal{H}[\boldsymbol{\sigma}]\right)\ \delta\left(A - \mathcal{A}[\boldsymbol{\sigma}]\right). \tag{4}$$

The first, very important, assumption of the present derivation is the equivalence between the ensemble with hard constraints on both A and E, i.e. the partition function written in equation (4), and the one where the same spherical constraints are realized via a Lagrange multiplier. This means that the study of the partition function in equation (4) is fully equivalent to that of its Laplace transform:

$$\begin{aligned}\mathcal{Z}_\lambda(\beta,N) &= \int_0^\infty \mathrm{d}A\ \mathrm{e}^{-\lambda A} \int_{-\infty}^{+\infty} \mathrm{d}E\ \mathrm{e}^{-\beta E}\ \Omega_A(E,N)\\ &= \int \prod_{i=1}^{N} \mathrm{d}\sigma_i\ \exp\left\{-\lambda \sum_{i=1}^{N} \sigma_i^2 + \beta \sum_{a=1}^{M} J_a \prod_{i\in\partial a} \sigma_i\right\},\end{aligned} \tag{5}$$

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For a given choice of values A and E the ensembles are equivalent if and only if it is possible to find real values of the Lagrange multipliers λ and β such that

$$
\begin{aligned}
A &= -\frac{\partial}{\partial \lambda} \log\left[\mathcal{Z}_\lambda(\beta, N)\right] \\
E &= -\frac{\partial}{\partial \beta} \log\left[\mathcal{Z}_\lambda(\beta, N)\right].
\end{aligned} \tag{6}
$$

In this paper we will consider only choices of $E \propto A \propto N$ such that it is possible to find real positive values of λ and β which solve the equations in equation (6). It is nevertheless important to keep in mind that there are situations where equation (6) does not have a solution in terms of either a real λ or a real β: this is the situation where the equivalence of ensembles breaks down and we expect it to happen in sparse hypergraphs, where condensation takes place. See for instance the recent discussion in [13].

Let us now introduce the cavity approach to solve the analyzed problem. We will introduce two kinds of cavity messages: with $\eta_{i\to a}(\sigma_i)$ we will indicate the variable-to-function cavity message, which indicates the probability that the spin on the i node assumes the value σ_i in the absence of the link between the variable node i and the function node a. Analogously with $\hat{\eta}_{a\to i}(\sigma_i)$ we indicate the function-to-variable cavity message. In the general case $\eta_{i\to a}(\sigma_i)$ will depend on all the messages $\hat{\eta}_{b\to i}(\sigma_i)$, with $b \in \partial i \backslash a$, which are correlated random variables. However, for tree-like graphs they are independent, due to the absence of loops. Loops are negligible at the leading order also on the Bethe lattice, which is locally tree-like (there are loops of size $\log(N)$). A complete graph is not at all locally tree-like, since each spin participates in $\mathcal{O}(N^{p-1})$ interactions, and there are always short loops. Nevertheless, due to the vanishing intensity of coupling constants J_a, i.e. $\langle J^2\rangle \sim 1/N^{p-1}$, $\hat{\eta}_{b\to i}(\sigma_i)$, with $b \in \partial i\backslash a$, behave as independent random variables even on complete graphs.

This allows us to introduce the following *cavity equations*:

$$
\eta_{i\to a}(\sigma_i) = \frac{1}{Z_{i\to a}} \prod_{b\in\partial i\backslash a} \hat{\eta}_{b\to i}(\sigma_i) \tag{7}
$$

$$
\hat{\eta}_{a\to i}(\sigma_i) = \frac{1}{\hat{Z}_{a\to i}}\, \mathrm{e}^{-\frac{\lambda\sigma_i^2}{2\,K}} \int \prod_{j\in\partial a\backslash i} \mathrm{d}\sigma_j\ \eta_{j\to a}(\sigma_j)\ \exp\left\{\beta J_a \sigma_i \prod_{j\in\partial a\backslash i} \sigma_j\right\}, \tag{8}
$$

with $Z_{i\to a}$ and $\hat{Z}_{a\to i}$, which are normalization constants to ensure that the messages are normalized,

$$
\int_{-\infty}^{\infty} \mathrm{d}\sigma_i \eta_{i\to a}(\sigma_i) = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \hat{\eta}_{i\to a}(\sigma_i) = 1.
$$

Let us spend few words to explain the way we have transformed the global spherical constraint into the local terms $\exp(-\frac{\lambda\sigma_i^2}{2K})$ appearing in the equations for the cavity

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marginals $\hat{\eta}_{a\to i}(\sigma_i)$. The factor $1/2$ is convenient for the definition of the Gaussian distributions (the Lagrange multiplier can be changed by a multiplicative factor without changing the physics). Although the most natural place to insert the spherical constraint would be as an external field in the equation for the cavity marginal $\eta_{i\to a}(\sigma_i)$, namely into equation (7), we found it more convenient to put the spherical constraint in the equation for the cavity marginal $\eta_{i\to a}(\sigma_i)$, that is into equation (8). The proof that these two choices are equivalent in the large-N limit is given in section 2.5. We notice that the idea of moving the external field from the variables to the interactions is not new. It is used, for example, in the real space renormalization group.

Once equations (7) and (8) are solved (e.g. in an iterative way as in the belief propagation algorithm), the local marginals for each spin are given by

$$\eta_i(\sigma_i) = \frac{1}{Z_i}\prod_{b\in\partial i}\hat{\eta}_{b\to i}(\sigma_i) \tag{9}$$

with Z_i a new normalization constant.

2.3. The Gaussian ansatz in the large degree limit

In the fully connected model, but also in diluted models, the mean degree grows and diverges in the large N limit. At the same time, the coupling intensities decrease as $N^{-(p-1)/2}$ to ensure well-defined local fields. In this limit we can close the cavity equations with the following Gaussian ansatz for the cavity marginal distribution

$$\eta_{i\to a}(\sigma_i) = \frac{1}{\sqrt{2\pi v_{i\to a}}}\ \exp\left[-\frac{(\sigma_i - m_{i\to a})^2}{2v_{i\to a}}\right] \propto\ \exp\left[\frac{m_{i\to a}}{v_{i\to a}}\sigma_i - \frac{1}{2v_{i\to a}}\sigma_i^2\right]. \tag{10}$$

Since $\langle J^2\rangle \sim 1/N^{p-1}$ the large N limit is equivalent to a small J or high-temperature expansion, known as the Plefka/Georges–Yedidia expansion [36, 37]. Expanding to second order in J, and inserting the ansatz equation (10), we get

$$\begin{aligned}
\hat{\eta}_{a\to i}(\sigma_i) &= \frac{1}{\hat{Z}_{a\to i}}\ \mathrm{e}^{-\frac{\lambda\sigma_i^2}{2\ K}} \int \prod_{j\in\partial a\backslash i} \mathrm{d}\sigma_j\ \eta_{j\to a}(\sigma_j)\ \exp\left\{\beta J_a \sigma_i \prod_{j\in\partial a\backslash i}\sigma_j\right\} \\
&\simeq \frac{1}{\hat{Z}_{a\to i}}\ \mathrm{e}^{-\frac{\lambda\sigma_i^2}{2K}} \left[1 + \beta J_a\sigma_i \prod_{j\in\partial a\backslash i} m_{j\to a} + \frac{\beta^2 J_a^2}{2}\sigma_i^2 \prod_{j\in\partial a\backslash i}(m_{j\to a}^2 + v_{j\to a})\right] \\
&\simeq \frac{1}{\hat{Z}_{a\to i}}\ \mathrm{e}^{-\frac{\lambda\sigma_i^2}{2K}}\ \exp\left\{\beta J_a\sigma_i \prod_{j\in\partial a\backslash i} m_{j\to a} + \frac{\beta^2 J_a^2}{2}\sigma_i^2\right. \\
&\quad \times \left.\left(\prod_{j\in\partial a\backslash i}(m_{j\to a}^2 + v_{j\to a}) - \prod_{j\in\partial a\backslash i} m_{j\to a}^2\right)\right\}
\end{aligned} \tag{11}$$

and

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$$\begin{aligned}\eta_{i\to a}(\sigma_i) &= \frac{1}{Z_{i\to a}} \prod_{b\in\partial i\setminus a} \hat{\eta}_{b\to i}(\sigma_i)\\ &= \frac{1}{Z_{i\to a}}\, \mathrm{e}^{-\frac{\lambda}{2}\sigma_i^2}\ \exp\left\{\beta\sigma_i \sum_{b\in\partial i\setminus a} J_b \prod_{j\in\partial b\setminus i} m_{j\to b} + \frac{\beta^2}{2}\sigma_i^2 \sum_{b\in\partial i\setminus a} J_b^2\right.\\ &\quad \left.\times\left(\prod_{j\in\partial b\setminus i}(m_{j\to b}^2 + v_{j\to b}) - \prod_{j\in\partial b\setminus i} m_{j\to b}^2\right)\right\}. \end{aligned} \tag{12}$$

Comparing equations (10) and (12), one obtains the following self-consistency equations for the means and the variances of the Gaussian marginals:

$$\begin{aligned}\frac{m_{i\to a}}{v_{i\to a}} &= \beta \sum_{b\in\partial i\setminus a} J_b \prod_{j\in\partial b\setminus i} m_{j\to b}\\ \frac{1}{v_{i\to a}} &= \lambda - \beta^2 \sum_{b\in\partial i\setminus a} J_b^2 \left(\prod_{j\in\partial b\setminus i}(m_{j\to b}^2 + v_{j\to b}) - \prod_{j\in\partial b\setminus i} m_{j\to b}^2\right). \end{aligned} \tag{13}$$

The λ parameter has to be fixed in order to satisfy the spherical constraint. $\sum_i \langle\sigma_i^2\rangle = N$, where the average is taken over the marginals defined in equation (9). However, given that we are in a dense system, the cavity marginal and full marginals differ by just terms of order $O(1/N)$, so we can impose the spherical constraint using cavity marginals. These are the replica symmetric cavity equations for dense (fully connected or diluted) spherical p-spin models.

In the limit of large degree (fully connected or diluted models) the two summations appearing in equation (13) are over a large number K of terms. So we can use the law of large numbers and the central limit theorem to simplify the self-consistency equations in (13). Reminding that in the large K limit the couplings scale according to $\langle J\rangle \sim 1/K$ and $\langle J^2\rangle \sim 1/K$, the second equation in (13) concentrates $v_{i\to a}$ close its mean value $v = \mathbb{E}(v_{i\to a})$, while the first equation in (13) implies that the cavity magnetization $m_{i\to a}$ are Gaussian random variables with first moments $m = \mathbb{E}(m_{i\to a})$ and $q = \mathbb{E}(m_{i\to a}^2)$, satisfying the following equations:

$$\frac{m}{v} = \beta\langle J\rangle K\, m^{p-1} \tag{14}$$

$$\frac{q}{v^2} = \beta^2\langle J^2\rangle K\, q^{p-1} + \beta^2\langle J\rangle^2 K^2 m^{2(p-1)} \tag{15}$$

$$\frac{1}{v} = \lambda - \beta^2\langle J^2\rangle K\left((q+v)^{p-1} - q^{p-1}\right). \tag{16}$$

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By imposing the spherical constraint, $\sum_i \langle \sigma_i^2 \rangle = N$, one gets the identity $q + v = 1$ that fixes the Lagrange multiplier and further simplifies the equations:

$$\lambda = \frac{1}{1-q} + \beta^2 \langle J^2 \rangle K (1 - q^{p-1}) \tag{17}$$

$$m = \beta \langle J \rangle K\, m^{p-1} (1-q) \tag{18}$$

$$q = \left[\beta^2 \langle J^2 \rangle K\, q^{p-1} + \beta^2 \langle J \rangle^2 K^2 m^{2(p-1)} \right] (1-q)^2. \tag{19}$$

It can be checked by using this expression for λ that the normalization of messages $\hat{\eta}_{a \to i}(\sigma_i)$ is always well defined in the limit of large N.

2.4. The replica symmetric free energy

We now have all the pieces we need to compute the replica symmetric free energy of the model, which is defined as [26]

$$-\beta F \equiv \beta \left(\sum_{a=1}^{M} \mathbb{F}_a + \sum_{i=1}^{N} \mathbb{F}_i - \sum_{(ai) \in E} \mathbb{F}_{ai} \right) \equiv \sum_{a=1}^{M} \log(Z_a) + \sum_{i=1}^{N} \log(Z_i) - \sum_{(ai) \in E} \log(Z_{(ai)}), \tag{20}$$

where we have respectively

$$Z_a = \int_{-\infty}^{\infty} \prod_{i \in \partial a} \mathrm{d}\sigma_i \; \eta_{i \to a}(\sigma_i) \; \mathrm{e}^{\beta J_a \prod_{i \in \partial a} \sigma_i} \tag{21}$$

$$Z_i = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \prod_{a \in \partial i} \hat{\eta}_{a \to i}(\sigma_i) \tag{22}$$

$$Z_{(ai)} = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \; \hat{\eta}_{a \to i}(\sigma_i) \; \eta_{i \to a}(\sigma_i). \tag{23}$$

The computation of these three terms is reported in appendix A. Here, we just report the final result:

$$-\beta F_{\mathrm{RS}} = \frac{N}{2} \left[\frac{\beta^2}{2} (1 - q^p) J_2 + \log(1-q) + \frac{q}{(1-q)} \right]. \tag{24}$$

The free energy written in equation (24) is identical to that of the spherical p-spin computed with replicas in the replica symmetric case; see equation (4.4) of [6]. From now on we will set $J_2 = 1$.

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2.5. Alternatives for the spherical constraint: equivalence in the large *N* limit

The experienced reader will have probably noticed that the way we have introduced the spherical constraint in the cavity equations is not, perhaps, the most natural one, which would correspond to an *external field* of intensity λ acting on every spin. As such, we should have put

$$\eta_{i\to a}(\sigma_i) \;\propto\; \mathrm{e}^{-\frac{\lambda}{2}\sigma_i^2}, \tag{25}$$

rather than

$$\hat{\eta}_{a\to i}(\sigma_i) \;\propto\; \mathrm{e}^{-\frac{\lambda}{2K}\sigma_i^2}, \tag{26}$$

as we have done in the equations for the cavity marginals in equations (7) and (8). In what follows we show that the choice of where to put the spherical constraint is arbitrary in the large N limit. In practice, we are going to show that either we let the constraint act as an external field in the variable-to-function message $\eta_{i\to a}(\sigma_i)$, as in equation (25), or inside the function-to-variable marginal $\hat{\eta}_{a\to i}(\sigma_i)$, as in equation (26). In both cases we obtain the same expression for the free energy to the leading order in N. The reader must therefore bear in mind that the two ways to put the constraint in the cavity equations *might not be equivalent* in the case of a graph with finite connectivity.

After a trial and error procedure we realized that the choice in equation (26) makes all calculations simpler, so we opted for this one. We have already shown that by doing so we obtain, at high temperature, a free energy which is identical to the one obtained from mean-field replica calculations; see equation (24). We now want to show explicitly that the free energy in the high-temperature ergodic phase is identical for the two choices (equations (25) and (26)) to introduce the constraint.

Let us term $\eta^{(\lambda)}_{i\to a}(\sigma_i)$ and $\hat{\eta}^{(\lambda)}_{a\to i}(\sigma_i)$ the local cavity marginals corresponding to the case where the *field* λ acts directly on the spin:

$$\eta^{(\lambda)}_{i\to a}(\sigma_i) = \frac{1}{Z^{(\lambda)}_{i\to a}}\,\mathrm{e}^{-\frac{\lambda\sigma_i^2}{2}}\prod_{b\in\partial i\setminus a}\hat{\eta}^{(\lambda)}_{b\to i}(\sigma_i) \tag{27}$$

$$\hat{\eta}^{(\lambda)}_{a\to i}(\sigma_i) = \frac{1}{\hat{Z}^{(\lambda)}_{a\to i}}\int_{-\infty}^{\infty}\prod_{j\in\partial a\setminus i}\mathrm{d}\sigma_j\;\eta^{(\lambda)}_{j\to a}(\sigma_j)\;\exp\left\{\beta J_a\sigma_i\prod_{j\in\partial a\setminus i}\sigma_j\right\}. \tag{28}$$

Accordingly, since in the function-to-variable messages there is now no trace of the external field, one has to consider the following modified definition of the entropic term in the local partition functions:

$$Z^{(\lambda)}_a = \int_{-\infty}^{\infty}\prod_{i\in\partial a}\mathrm{d}\sigma_i\;\eta^{(\lambda)}_{i\to a}(\sigma_i)\;\mathrm{e}^{\beta J_a\prod_{i\in\partial a}\sigma_i} \tag{29}$$

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$$Z_i^{(\lambda)} = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \ \mathrm{e}^{-\lambda\sigma_i^2/2} \prod_{a\in\partial i} \hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \tag{30}$$

$$Z_{(ai)}^{(\lambda)} = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \ \hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \ \eta_{i\to a}^{(\lambda)}(\sigma_i). \tag{31}$$

Our task is now to show that

$$\begin{aligned} &\sum_{a=1}^{M} \log(Z_a) + \sum_{i=1}^{N} \log(Z_i) - \sum_{(ai)\in E} \log(Z_{(ai)}) \\ &\quad = \sum_{a=1}^{M} \log(Z_a^{(\lambda)}) + \sum_{i=1}^{N} \log(Z_i^{(\lambda)}) - \sum_{(ai)\in E} \log(Z_{(ai)}^{(\lambda)}). \end{aligned} \tag{32}$$

The key observation is that, in order to have overall consistency, the Gaussian ansatz for the variable-to-function message *must* be the same in both cases; that is

$$\eta_{i\to a}(\sigma_i) = \frac{1}{\sqrt{2\pi v_{i\to a}}} \ \exp\left[-\frac{(\sigma_i - m_{i\to a})^2}{2v_{i\to a}}\right] = \eta_{i\to a}^{(\lambda)}(\sigma_i). \tag{33}$$

The assumption of equation (33) allows us to conclude immediately that $Z_a^{(\lambda)} = Z_a$, so that the identity we need to prove reduces to

$$\sum_{i=1}^{N} \log(Z_i) - \sum_{(ai)\in E} \log(Z_{(ai)}) = \sum_{i=1}^{N} \log(Z_i^{(\lambda)}) - \sum_{(ai)\in E} \log(Z_{(ai)}^{(\lambda)}). \tag{34}$$

By exploiting equation (33), once again we obtain

$$\hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) = \frac{1}{\hat{Z}_{a\to i}^{(\lambda)}} \int_{-\infty}^{\infty} \prod_{j\in\partial a\setminus i} \mathrm{d}\sigma_j \ \eta_{j\to a}(\sigma_j) \ \exp\left\{\beta J_a \sigma_i \prod_{j\in\partial a\setminus i} \sigma_j\right\}, \tag{35}$$

which, by comparison with the definition in equation (8), leads to

$$\hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \ \hat{Z}_{a\to i}^{(\lambda)} = \hat{\eta}_{a\to i}(\sigma_i) \ \hat{Z}_{a\to i} \ \mathrm{e}^{\frac{\lambda\sigma_i^2}{2K}}, \tag{36}$$

so that

$$\hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) = \hat{\eta}_{a\to i}(\sigma_i) \ \frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} \ \mathrm{e}^{\frac{\lambda\sigma_i^2}{2K}}. \tag{37}$$

By inserting equation (37) in the definition of $Z_i^{(\lambda)}$ in equation (30), one finds

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$$
\begin{aligned}
Z_i^{(\lambda)} &= \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \, \mathrm{e}^{-\lambda\sigma_i^2/2} \prod_{a\in\partial i} \hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \\
&= \prod_{a\in\partial i} \left(\frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} \right) \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \prod_{a\in\partial i} \hat{\eta}_{a\to i}(\sigma_i) \\
&= \prod_{a\in\partial i} \left(\frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} \right) Z_i,
\end{aligned}
\tag{38}
$$

so that the identity that we want to prove is further simplified in

$$
\sum_{(ai)\in E} \log(Z_{(ai)}) = \sum_{(ai)\in E} \log(Z_{(ai)}^{(\lambda)}) - \sum_{i=1}^{N} \sum_{a\in\partial i} \log\left(\frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} \right). \tag{39}
$$

Using, once again, equation (33) we can write

$$
\begin{aligned}
Z_{(ai)}^{(\lambda)} &= \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \, \hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \, \eta_{i\to a}^{(\lambda)}(\sigma_i) \\
&= \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \, \hat{\eta}_{a\to i}^{(\lambda)}(\sigma_i) \, \eta_{i\to a}(\sigma_i) \\
&= \frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} \int_{-\infty}^{\infty} \mathrm{d}\sigma_i \, \mathrm{e}^{\frac{\lambda\sigma_i^2}{2K}} \, \hat{\eta}_{a\to i}(\sigma_i) \, \eta_{i\to a}(\sigma_i) \\
&\simeq \frac{\hat{Z}_{a\to i}}{\hat{Z}_{a\to i}^{(\lambda)}} Z_{(ai)},
\end{aligned}
\tag{40}
$$

where the last line equality holds for large N (see equations (A15)–(A17) in appendix A). The $N \to \infty$ limit is equivalent to the $K \to \infty$ limit, since $K \sim N^{p-1}$. Finally, by plugging the result of equation (40) into equation (39) we can conclude that the identity in equation (39) is true in the limit $N \to \infty$. We have thus demonstrated that in the large N limit it is equivalent, and thus just a matter of convenience, to write down explicitly the spherical constraint inside the definition of the function-to-variable message $\hat{\eta}_{a\to i}(\sigma)$, as we have done, or inside the definition of the variable-to-function one, $\hat{\eta}_{i\to a}(\sigma)$.

3. 1RSB solution

In the previous sections we have reviewed the replica symmetric solution that is stable for high temperatures. In this phase we have written closed cavity equations for the marginal distributions of the variables, relying on the assumption that the joint distribution of the cavity variables is factorized as in a single pure state.

However, lowering the temperature, it is known from the replica solution [6] that several metastable glassy states arise on top of the paramagnetic state, their number being exponential in N with a rate Σ called *complexity*. The function $\Sigma(f)$ is in general an increasing function of the state free energy f, with a downwards curvature (for stability reasons as for the entropy).

Comparing the total free energy of the glassy states computed using $\Sigma(f)$ and the paramagnetic free energy [38], one can derive the dynamical critical temperature $T_{\mathrm{d},}$ where the ergodicity breaks down, and the thermodynamic critical temperature, also called the Kauzmann temperature T_{K}, where a phase transition to a replica-symmetry-breaking phase takes place.

Below T_{d} the dynamics of the model are dominated by the states of larger free energy—the so-called threshold states—which are the most abundant and always exponentially many in N (although a more refined picture has been recently presented in [39]).

For $T < T_{\mathrm{d},}$ the Gibbs measure is split over many different states, such that two different equilibrium configurations can be in the same (metastable) state or in different states. Defining the overlap between two different configurations as how close they are to each other, the 1RSB phase is characterized by an overlap q_1 between configurations inside the same pure state (independently of the pure state) and an overlap $q_0 < q_1$ between configurations in two different states.

In formulas, the presence of many metastable pure states yields an additional contribution to the free energy. The complexity $\Sigma(f)$, which counts the number of 'states' (disjoint ergodic components of the phase space) with the same free energy f, can be written as

$$\Sigma(f) = \frac{1}{N}\ \log\left[\sum_{\eta=1}^{\mathcal{N}}\ \delta(f - f_\eta)\right], \tag{41}$$

where $\mathcal{N}$ is the total number of metastable glassy states (formally they can be defined as the non-paramagnetic stationary points of the TAP free energy [40]) and f_η is the free energy of the glassy state η. Please note that the expression in equation (41) is identical to the standard microcanonical definition of entropy, with the only difference being that now we measure the number of phase space regions with the same free energy rather than the volume of phase space with the same energy. The total free energy is thus given by

$$\begin{aligned}\mathcal{F} &= -\frac{1}{\beta N}\ \log Z = -\frac{1}{\beta N}\ \log\left(\sum_\eta \mathrm{e}^{-\beta N f_\eta}\right) = -\frac{1}{\beta N}\ \log\int \mathrm{d}f \sum_\eta\ \delta(f - f_\eta)\ \mathrm{e}^{-\beta N f}\\ &= -\frac{1}{\beta N}\ \log\int \mathrm{d}f\ \mathrm{e}^{-N(\beta f - \Sigma)}.\end{aligned} \tag{42}$$

The problem is that we do not know how to characterize the different states and how to count them to obtain Σ: we are still not able to compute $\mathcal{F}$. In the following we will solve this problem applying the method of real coupled replicas introduced by Monasson in [41] (see also [26] for a rigorous derivation and [38] for a pedagogical review). This method was applied to the spherical p-spin in [42] to compute the 1RSB free energy with a replica computation. The idea of [41] is to introduce x real clones, that we will call replicas, on a single realization of a graph. These replicas will be infinitesimally coupled together in such a way that, even when the coupling between them goes to zero, they will all fall in the same pure state below T_{d}: this cloning method is a way to select a state equivalent to what is usually done in ferromagnetic systems to select a state adding an infinitesimal magnetic field. The free energy $\Phi(x)$ of x replicas in the same state is

$$\begin{aligned}\Phi(x) &= -\frac{1}{\beta N}\ \log\left(\sum_{\eta} \mathrm{e}^{-\beta N x f_{\eta}}\right) = -\frac{1}{\beta N}\ \log\int \mathrm{d}f\ \mathrm{e}^{-N(\beta x f-\Sigma)}\\ &= -\frac{1}{\beta}\max_{f}(\beta x f-\Sigma(f)).\end{aligned} \tag{43}$$

The complexity in this way simply results in the Legendre transform of the free energy of the replicated system. The total free energy in the 1RSB phase is derived passing to the analytical continuation of x to real values and turns out to be $\mathcal{F}=\min_x \frac{\Phi(x)}{x}$. Besides the Monasson–Mezard cloning method, which is mostly useful to study the complexity of systems without quenched disorder, it is worth recalling the physical meaning of the analytic continuation to positive real values of x in a more general setting: it allows us to compute the large deviations of the free energy, e.g. its sample-to-sample fluctuations [43, 44].

In the following we will use this cloning method to write 1RSB closed cavity equations for the spherical p-spin, in a way analogous to what has been done in [21] for the planted SK model. In a situation with many pure states, the factorization of the distribution of the cavity variables is valid only inside a single pure state: we can thus still write some cavity closed equations considering the coupled replicas in the same pure state. Then, we will compute the 1RSB free energy in a cavity approach below T_{d}, obtaining exactly the same expression found with replica computations in [6, 42].

3.1. The ansatz for the distribution of x coupled replicas

For the RS phase, in the dense case, we have written a Gaussian ansatz for the marginal probability of the spin on a given site in equation (10). In the 1RSB phase, we will consider the *joint probability distribution* of x coupled replicas that are all in the same pure state. We will comment in the next sections on the choice and the physical meaning of x. In order to lighten the notation, let us indicate as $\boldsymbol{\sigma}_i=\{\sigma_i^{\alpha}\}$, $\alpha=1,\ldots,x$, the vector of all x replicas on site i. The 1RSB form of the ansatz for the marginal probability $\eta_{i\to a}(\boldsymbol{\sigma}_i)$ amounts to

$$\begin{aligned}\eta_{i\to a}(\boldsymbol{\sigma}_i) =&\int_{-\infty}^{\infty}\mathrm{d}m_{i\to a}\ \frac{1}{\sqrt{2\pi\Delta^{(0)}_{i\to a}}}\ \exp\left(-\frac{(m_{i\to a}-h_{i\to a})^2}{2\Delta^{(0)}_{i\to a}}\right)\\ &\times\frac{1}{\left[\sqrt{2\pi\Delta^{(1)}_{i\to a}}\right]^{x}}\ \exp\left(-\sum_{\alpha=1}^{x}\frac{(\sigma_i^{\alpha}-m_{i\to a})^2}{2\Delta^{(1)}_{i\to a}}\right).\end{aligned} \tag{44}$$

This 1RSB ansatz was firstly introduced in [21]. By shortening the integration measure for the joint probability distribution $\boldsymbol{\sigma}_i$ with the symbol

$$\int \mathcal{D}\boldsymbol{\sigma}_i = \int_{-\infty}^{\infty} \prod_{\alpha=1}^{x} \mathrm{d}\sigma_i^\alpha, \tag{45}$$

and defining the distribution

$$Q_{i\to a}\left(m_{i\to a}\right) \equiv \frac{1}{\sqrt{2\pi\Delta_{i\to a}^{(0)}}}\ \exp\left(-\frac{(m_{i\to a}-h_{i\to a})^2}{2\Delta_{i\to a}^{(0)}}\right), \tag{46}$$

the first diagonal and second moments of the cavity marginal are simply computed as

$$\begin{aligned}
\langle\sigma_i^\alpha\rangle &= \int \mathcal{D}\boldsymbol{\sigma}_i\ \sigma_i^\alpha\ \eta_{i\to a}(\boldsymbol{\sigma}_i) = \int_{-\infty}^{\infty} \mathrm{d}m_{i\to a}\ m_{i\to a}\ Q_{i\to a}\left(m_{i\to a}\right) = h_{i\to a}\\
\langle(\sigma_i^\alpha)^2\rangle &= \int \mathcal{D}\boldsymbol{\sigma}_i\ (\sigma_i^\alpha)^2\ \eta_{i\to a}(\boldsymbol{\sigma}_i) = \int_{-\infty}^{\infty} \mathrm{d}m_{i\to a}\ (\Delta_{i\to a}^{(1)} + m_{i\to a}^2)\ Q_{i\to a}\left(m_{i\to a}\right)\\
&= \Delta_{i\to a}^{(1)} + \Delta_{i\to a}^{(0)} + h_{i\to a}^2\\
\langle\sigma_i^\alpha\sigma_i^\beta\rangle &= \int \mathcal{D}\boldsymbol{\sigma}_i\ \sigma_i^\alpha\ \sigma_i^\beta\ \eta_{i\to a}(\boldsymbol{\sigma}_i) = \int_{-\infty}^{\infty} \mathrm{d}m_{i\to a}\ m_{i\to a}^2\ Q_{i\to a}\left(m_{i\to a}\right)\\
&= \Delta_{i\to a}^{(0)} + h_{i\to a}^2.
\end{aligned} \tag{47}$$

Let us comment briefly on the form of the ansatz. The marginal probability of a single replica in a given state is still a Gaussian, being on a dense graph. If the real replicas are coupled, they will fall in the same state. The only effect of the infinitesimal coupling between the replicas will be that the configurations of the real replicas will be independent variables extracted from the same distribution in each state, once the average m and variance $\Delta^{(1)}$ are given,

$$\eta_{i\to a}^s\left(\sigma_i^\alpha\right) \equiv \frac{1}{\sqrt{2\pi\Delta_{i\to a}^{(1)}}}\ \exp\left(-\frac{(\sigma_i^\alpha - m_{i\to a})^2}{2\Delta_{i\to a}^{(1)}}\right). \tag{48}$$

In the same way, the average magnetizations in different states will be independent variables extracted from the same distribution $Q_{i\to a}(m_{i\to a})$, that will depend on $\Delta^{(0)}$ and h; see, e.g. [45].

With this simple scenario in mind, we can give a simple physical interpretation to the parameters of the distribution in equation (44), rewriting them as

$$\begin{aligned}
h_{i\to a} &= \langle\sigma_i^\alpha\rangle\\
\Delta_{i\to a}^{(1)} &= \langle(\sigma_i^\alpha)^2\rangle - \langle\sigma_i^\alpha\sigma_i^\beta\rangle = 1 - q_1^{i\to a}\\
\Delta_{i\to a}^{(0)} &= \langle\sigma_i^\alpha\sigma_i^\beta\rangle - \langle\sigma_i^\alpha\rangle^2 = q_1^{i\to a} - q_0^{i\to a},
\end{aligned} \tag{49}$$

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where the average is taken with respect to the probability distribution in equation (44). $q_1^{i\to a}$ and $q_0^{i\to a}$ are the local overlap (in the absence of the link from i to a) inside a state and between states that we mentioned at the beginning of this section. Obviously on a complete graph, they will be independent of i and a, as for the only parameter in the RS case (the magnetization) in the homogeneous case. However, we here prefer to write explicitly the dependence on i and a, because in this way the equations we will obtain could be easily applied to non-complete graphs.

3.2. 1RSB cavity equations

We now write the replicated cavity equations for the 1RSB ansatz introduced in the previous section:

$$\eta_{i\to a}(\boldsymbol{\sigma}_i) \propto \prod_{b\in\partial i\setminus a} \hat{\eta}_{b\to i}(\boldsymbol{\sigma}_i) \tag{50}$$

$$\hat{\eta}_{a\to i}(\boldsymbol{\sigma}_i) \propto e^{-\frac{\lambda}{2}\frac{x}{K}\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2} \int_{-\infty}^{\infty} \prod_{k\in\partial a\setminus i} \mathcal{D}\boldsymbol{\sigma}_k \; \eta_{k\to a}(\boldsymbol{\sigma}_k) \\ \times \exp\left\{ \beta J_a \sum_{\alpha=1}^{x} \sigma_i^\alpha \prod_{k\in\partial a\setminus i} \sigma_k^\alpha \right\}. \tag{51}$$

We have omitted the normalization factors that are irrelevant in the subsequent computations. As we did for the RS case, in the dense limit we take the leading term in a small J_a expansion (valid in the large N limit for dense graphs) and in this setting we will close the equations on the parameters of the multivariate Gaussian. That is, we write

$$\int_{-\infty}^{\infty} \prod_{k\in\partial a\setminus i} \mathcal{D}\boldsymbol{\sigma}_k \; \eta_{k\to a}(\boldsymbol{\sigma}_k) \quad \exp\left\{ \beta J_a \sum_{\alpha=1}^{x} \sigma_i^\alpha \prod_{k\in\partial a\setminus i} \sigma_k^\alpha \right\} \\ \simeq \int_{-\infty}^{\infty} \prod_{k\in\partial a\setminus i} \mathcal{D}\boldsymbol{\sigma}_k \; \eta_{k\to a}(\boldsymbol{\sigma}_k) \left[1 + \beta J_a \sum_{\alpha=1}^{x} \sigma_i^\alpha \prod_{k\in\partial a\setminus i} \sigma_k^\alpha + \frac{1}{2}\beta^2 J_a^2 \left(\sum_{\alpha=1}^{x} \sigma_i^\alpha \prod_{k\in\partial a\setminus i} \sigma_k^\alpha \right)^2 \right] \\ = 1 + \beta J_a \left(\prod_{k\in\partial a\setminus i} h_{k\to a} \right) \sum_{\alpha=1}^{x} \sigma_i^\alpha + \frac{1}{2}\beta^2 J_a^2 \left[\prod_{k\in\partial a\setminus i} \left(\Delta_{k\to a}^{(1)} + \Delta_{k\to a}^{(0)} + h_{k\to a}^2 \right) \right] \sum_{\alpha=1}^{x} (\sigma_i^\alpha)^2 \\ + \frac{1}{2}\beta^2 J_a^2 \left[\prod_{k\in\partial a\setminus i} \left(\Delta_{k\to a}^{(0)} + h_{k\to a}^2 \right) \right] \sum_{\alpha\neq\beta}^{x} \sigma_i^\alpha \sigma_i^\beta \\ \simeq \exp\left\{ \hat{A}_{a\to i} \sum_{\alpha=1}^{x} \sigma_i^\alpha - \frac{1}{2}\hat{B}_{a\to i}^{(\mathrm{d})} \sum_{\alpha=1}^{x} (\sigma_i^\alpha)^2 + \frac{1}{2}\hat{B}_{a\to i}^{(\mathrm{nd})} \sum_{\alpha\neq\beta}^{x} \sigma_i^\alpha \sigma_i^\beta \right\}, \tag{52}$$

where the three coefficients are respectively

$$\begin{aligned}
\hat{A}_{a\to i} &= \beta J_a \prod_{k\in\partial a\setminus i} h_{k\to a} \\
\hat{B}^{(\mathrm{d})}_{a\to i} &= \beta^2 J_a^2 \left[\prod_{k\in\partial a\setminus i} h^2_{k\to a} - \prod_{k\in\partial a\setminus i}\left(\Delta^{(1)}_{k\to a} + \Delta^{(0)}_{k\to a} + h^2_{k\to a}\right)\right] \\
\hat{B}^{(\mathrm{nd})}_{a\to i} &= \beta^2 J_a^2 \left[\prod_{k\in\partial a\setminus i}\left(\Delta^{(0)}_{k\to a} + h^2_{k\to a}\right) - \prod_{k\in\partial a\setminus i} h^2_{k\to a}\right].
\end{aligned} \tag{53}$$

The function-to-variable message, expressed by equation (51), therefore reads as

$$\hat{\eta}_{a\to i}(\boldsymbol{\sigma}_i) \propto \exp\left\{\hat{A}_{a\to i}\sum_{\alpha=1}^{x}\sigma_i^\alpha - \frac{1}{2}\left(\hat{B}^{(\mathrm{d})}_{a\to i} + \frac{\lambda}{K}\right)\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2 + \frac{1}{2}\hat{B}^{(\mathrm{nd})}_{a\to i}\sum_{\alpha\neq\beta}^{x}\sigma_i^\alpha\sigma_i^\beta\right\}, \tag{54}$$

while from equation (50) we have that the variable-to-function message reads as

$$\begin{aligned}
\eta_{i\to a}(\boldsymbol{\sigma}_i) \propto \exp&\left\{\left(\sum_{b\in\partial i\setminus a}\hat{A}_{b\to i}\right)\sum_{\alpha=1}^{x}\sigma_i^\alpha - \frac{1}{2}\left(\sum_{b\in\partial i\setminus a}\hat{B}^{(\mathrm{d})}_{b\to i} + \lambda\right)\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2\right. \\
&\left. + \frac{1}{2}\left(\sum_{b\in\partial i\setminus a}\hat{B}^{(\mathrm{nd})}_{b\to i}\right)\sum_{\alpha\neq\beta}^{x}\sigma_i^\alpha\sigma_i^\beta\right\}.
\end{aligned} \tag{55}$$

In order to keep the notation simple, let us define

$$\begin{aligned}
A_{i\to a} &\equiv \sum_{b\in\partial i\setminus a}\hat{A}_{b\to i} = \sum_{b\in\partial i\setminus a}\beta J_b\prod_{k\in\partial b\setminus i} h_{k\to b} \\
B^{(\mathrm{d})}_{i\to a} &\equiv \lambda + \sum_{b\in\partial i\setminus a}\hat{B}^{(\mathrm{d})}_{a\to i} = \lambda - \sum_{b\in\partial i\setminus a}\beta^2 J_b^2 \\
&\quad\times\left[\prod_{k\in\partial b\setminus i}\left(\Delta^{(1)}_{k\to b} + \Delta^{(0)}_{k\to b} + h^2_{k\to b}\right) - \prod_{k\in\partial b\setminus i} h^2_{k\to b}\right] \\
B^{(\mathrm{nd})}_{i\to a} &\equiv \sum_{b\in\partial i\setminus a}\hat{B}^{(\mathrm{nd})}_{a\to i} = \sum_{b\in\partial i\setminus a}\beta^2 J_b^2\left[\prod_{k\in\partial b\setminus i}\left(\Delta^{(0)}_{k\to b} + h^2_{k\to b}\right) - \prod_{k\in\partial b\setminus i} h^2_{k\to b}\right],
\end{aligned} \tag{56}$$

so that equation (50) can be rewritten in the more compact form as

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$$\eta_{i\to a}(\boldsymbol{\sigma}_i) \propto \exp\left\{A_{i\to a}\sum_{\alpha=1}^{x}\sigma_i^\alpha - \frac{1}{2}B_{i\to a}^{(\mathrm{d})}\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2 + \frac{1}{2}B_{i\to a}^{(\mathrm{nd})}\sum_{\alpha\neq\beta}^{x}\sigma_i^\alpha\sigma_i^\beta\right\}. \tag{57}$$

The expression above can be further simplified by introducing the matrix $\mathcal{M}_{\alpha\beta}$ and the vector u_α such that

$$\begin{aligned} u_\alpha &= \frac{A_{i\to a}}{B_{i\to a}^{(\mathrm{d})} - (x-1)B_{i\to a}^{(\mathrm{nd})}} \quad \forall\alpha, \\ \mathcal{M}_{\alpha\beta} &= \delta_{\alpha\beta}\ B_{i\to a}^{(\mathrm{d})} + (1-\delta_{\alpha\beta})\ (-B_{i\to a}^{(\mathrm{nd})}) \end{aligned} \tag{58}$$

and the normalized distribution, written in the standard form for a multivariate Gaussian, reads

$$\eta_{i\to a}(\boldsymbol{\sigma}_i) = \sqrt{\frac{\det\ \mathcal{M}}{(2\pi)^x}}\ \exp\left\{-\frac{1}{2}(\boldsymbol{\sigma}_i - u)^T\mathcal{M}(\boldsymbol{\sigma}_i - u)\right\}. \tag{59}$$

The closed cavity equations, which in the 1RSB case are three rather than two, are simply obtained by taking the averages in equation (49) with respect to the marginal distribution $\eta_{i\to a}(\boldsymbol{\sigma}_i)$:

$$\begin{aligned} h_{i\to a} &= \langle\sigma_i^\alpha\rangle = u_\alpha \\ \Delta_{i\to a}^{(0)} &= \langle\sigma_i^\alpha\sigma_i^\beta\rangle - [\langle\sigma_i^\alpha\rangle]^2 = \mathcal{M}_{\alpha\beta}^{-1} \\ \Delta_{i\to a}^{(1)} &= \langle[\sigma_i^\alpha]^2\rangle - \langle\sigma_i^\alpha\sigma_i^\beta\rangle = \mathcal{M}_{\alpha\alpha}^{-1} - \mathcal{M}_{\alpha\beta}^{-1}, \end{aligned}$$

where the general expression of the inverse matrix element is

$$\mathcal{M}_{\alpha\beta}^{-1} = \frac{1}{\left(B_{i\to a}^{(\mathrm{d})} + B_{i\to a}^{(\mathrm{nd})}\right)}\ \delta_{\alpha\beta} + \frac{B_{i\to a}^{(\mathrm{nd})}}{\left(B_{i\to a}^{(\mathrm{d})} + B_{i\to a}^{(\mathrm{nd})}\right)\left(B_{i\to a}^{(\mathrm{d})} + (1-x)B_{i\to a}^{(\mathrm{nd})}\right)}. \tag{60}$$

For the ease of the reader wishing to implement them in a code, let us write explicitly the closed cavity equations:

$$\begin{aligned} h_{i\to a} &= \frac{\beta\sum_{b\in\partial i\backslash a} J_b \prod_{k\in\partial b\backslash i} h_{k\to b}}{\mathcal{D}_{i\to a}^{(1)} - x\mathcal{D}_{i\to a}^{(0)}} \\ \Delta_{i\to a}^{(0)} &= \frac{\mathcal{D}_{i\to a}^{(0)}}{\mathcal{D}_{i\to a}^{(1)}\left(\mathcal{D}_{i\to a}^{(1)} - x\mathcal{D}_{i\to a}^{(0)}\right)} \\ \Delta_{i\to a}^{(1)} &= \frac{1}{\mathcal{D}_{i\to a}^{(1)}} \end{aligned} \tag{61}$$

with

$$\mathcal{D}_{i\to a}^{(1)} \equiv \lambda - \beta^2\sum_{b\in\partial i\backslash a} J_b^2\left[\prod_{k\in\partial b\backslash i}\left(\Delta_{k\to b}^{(0)} + \Delta_{k\to b}^{(1)} + h_{k\to b}^2\right) - \prod_{k\in\partial b\backslash i}\left(\Delta_{k\to b}^{(0)} + h_{k\to b}^2\right)\right]$$

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$$\mathcal{D}^{(0)}_{i\to a} \equiv \beta^2 \sum_{b\in\partial i\setminus a} J_b^2 \left[\prod_{k\in\partial b\setminus i} \left(\Delta^{(0)}_{k\to b} + h^2_{k\to b} \right) - \prod_{k\in\partial b\setminus i} h^2_{k\to b} \right].$$

While the parameter λ is fixed by the normalization condition, the parameter x is a variational one and has to be chosen in order to extremize the free energy, a quantity that is computed explicitly in the next subsection in the case of a complete graph.

In the limit of large mean degree, the above saddle point equations can be further simplified by noting that both $\mathcal{D}^{(0)}_{i\to a}$ and $\mathcal{D}^{(1)}_{i\to a}$ concentrate to their mean values, which we denote as $\langle \mathcal{D}^{(0)}_{i\to a}\rangle = \mathcal{D}^{(0)}$ and $\langle \mathcal{D}^{(1)}_{i\to a}\rangle = \mathcal{D}^{(1)}$, due to the law of large numbers, while $h_{i\to a}$ becomes a Gaussian variable and it is enough to consider its first two moments.

$$m \equiv \langle h_{i\to a}\rangle = \frac{\beta\langle J\rangle K\ m^{p-1}}{\mathcal{D}^{(1)} - x\mathcal{D}^{(0)}} \tag{62}$$

$$q_0 \equiv \langle h^2_{i\to a}\rangle = \frac{\beta^2\left[\langle J^2\rangle K\ q_0^{p-1} + \langle J\rangle^2 K^2 m^{2(p-1)}\right]}{\left(\mathcal{D}^{(1)} - x\mathcal{D}^{(0)}\right)^2} \tag{63}$$

$$\Delta^{(0)} = \frac{\mathcal{D}^{(0)}}{\mathcal{D}^{(1)}\left(\mathcal{D}^{(1)} - x\mathcal{D}^{(0)}\right)} \tag{64}$$

$$\Delta^{(1)} = \frac{1}{\mathcal{D}^{(1)}}$$

with

$$\mathcal{D}^{(1)} \equiv \lambda - \beta^2\langle J^2\rangle K\left[\left(q_0 + \Delta^{(0)} + \Delta^{(1)}\right)^{p-1} - \left(q_0 + \Delta^{(0)}\right)^{p-1}\right]$$

$$\mathcal{D}^{(0)} \equiv \beta^2\langle J^2\rangle K\left[\left(q_0 + \Delta^{(0)}\right)^{p-1} - q_0^{p-1}\right]$$

and where the symbols $\Delta^{(1)}$ and $\Delta^{(0)}$ represent, respectively, $\Delta^{(1)} = \langle \Delta^{(1)}_{i\to a}\rangle$ and $\Delta^{(0)} = \langle \Delta^{(0)}_{i\to a}\rangle$. By considering the most common model with Gaussian couplings of zero mean ($\langle J\rangle = 0$ and $\langle J^2\rangle = p!/(2N^{p-1})$) and recalling that $\Delta^{(0)} = q_1 - q_0$, $\Delta^{(1)} = 1 - q_1$ and

$$K = \binom{N}{p-1} \sim \frac{N^{p-1}}{(p-1)!}, \quad \Rightarrow \quad \langle J^2\rangle K \sim \frac{p}{2} \tag{65}$$

we are left with the following three closed equations:

$$\beta^2\frac{p}{2}\ q_0^{p-2} = \left[\lambda - p\frac{\beta^2}{2}\left(1 - xq_0^{p-1} + (x-1)q_1^{p-1}\right)\right]^2 \tag{66}$$

$$q_1 - q_0 = \frac{p\frac{\beta^2}{2}\left(q_1^{p-1} - q_0^{p-1}\right)}{\left[\lambda - p\frac{\beta^2}{2}\left(1 - q_1^{p-1}\right)\right]\left[\lambda - p\frac{\beta^2}{2}\left(1 - xq_0^{p-1} + (x-1)q_1^{p-1}\right)\right]} \tag{67}$$

$$1 - q_1 = \frac{1}{\lambda - p\frac{\beta^2}{2}\left(1 - q_1^{p-1}\right)}. \tag{68}$$

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Let us note that equation (68) allows us to easily re-express the spherical constraint parameter λ as a function of q_1 and β, i.e.

$$\lambda = \frac{1}{1-q_1} + p\frac{\beta^2}{2}\left(1 - q_1^{p-1}\right), \tag{69}$$

which will be useful later on. Comparing the expression of the Lagrange multiplier with the one obtained in the RS case in equation (17), we see that they are the same with the substitution $q \to q_1$: in the 1RSB phase the Lagrange multiplier is enforcing the spherical constraint inside each pure state.

3.3. 1RSB free energy

We now want to compute the free energy $\mathcal{F}(x)$ in the presence of a one-step replica symmetric ansatz. The free energy for the replicated system is [26, 38]

$$\Phi(x) = -\left(\sum_{a=1}^{M} \mathbb{F}_a^{\mathrm{RSB}}(x) + \sum_{i=1}^{N} \mathbb{F}_i^{\mathrm{RSB}}(x) - \sum_{(ai)\in E} \mathbb{F}_{ai}^{\mathrm{RSB}}(x)\right). \tag{70}$$

The total free energy of the system is just the free energy of the x coupled replicas, divided by x (and extremized over x). In principle, the three contributions, respectively representing the energetic and the entropic contributions and a normalization, read as

$$\beta\mathbb{F}_a^{\mathrm{RSB}}(x) = \log\left\{\int \prod_{i\in\partial a}\left[\mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a})\right]\ \mathrm{e}^{x\beta\mathbb{F}_a(\{m_{i\to a}\})}\right\} \tag{71}$$

$$\beta\mathbb{F}_i^{\mathrm{RSB}}(x) = \log\left\{\int \prod_{a\in\partial i}\left[\mathrm{d}\hat{m}_{a\to i} \hat{Q}_{a\to i}(\hat{m}_{a\to i})\right]\ \mathrm{e}^{x\beta\mathbb{F}_i(\{\hat{m}_{a\to i}\})}\right\} \tag{72}$$

$$\beta\mathbb{F}_{ai}^{\mathrm{RSB}}(x) = \log\left\{\int \mathrm{d}\hat{m}_{a\to i}\ \mathrm{d}m_{i\to a}\ \hat{Q}_{a\to i}(\hat{m}_{a\to i})\ Q_{i\to a}(m_{i\to a})\ \mathrm{e}^{x\beta\mathbb{F}_{ai}(m_{i\to a},\hat{m}_{a\to i})}\right\} \tag{73}$$

where $\mathbb{F}_a, \mathbb{F}_i$ and $\mathbb{F}_{ai}$ are the RS free energy parts in equation (20), $Q_{i\to a}(m_{i\to a})$ is the distribution of the cavity marginals that in the case of the dense Gaussian 1RSB ansatz has been defined in equation (46), section 3.1, and the distribution $\hat{Q}_{a\to i}(\hat{m}_{a\to i})$ should be the one of the function-to-variable fields.

The detailed computation is reported in appendix B.2; here we just write the result, which is the same as that found with the replica approach [6]:

$$
\begin{aligned}
-x\beta\mathcal{F}(x) &= \sum_{a=1}^{M}\beta\mathbb{F}_a^{\mathrm{RSB}} + \sum_{i=1}^{N}\beta\mathbb{F}_i^{\mathrm{RSB}} - \sum_{(ai)\in E}\beta\mathbb{F}_{ai}^{\mathrm{RSB}} \\
&= \frac{xN}{2}\left\{\frac{\beta^2}{2}\left[1-(1-x)q_1^p - xq_0^p\right] + \frac{q_0}{[1-xq_0-(1-x)q_1]}\right. \\
&\quad \left. + \frac{x-1}{x}\log(1-q_1) + \frac{1}{x}\log[1-xq_0-(1-x)q_1]\right\}.
\end{aligned}
\tag{74}
$$

4. Conclusions

In this paper we have derived the cavity equations for solving diluted spherical p-spin models. Such a cavity-based derivation makes evident the underlying assumption that reflects itself in the distribution of local fields: in the RS ansatz, replicas are uncorrelated and have Gaussian local fields, while in the 1RSB ansatz replicas have correlated Gaussian local fields whose covariance matrix depends on whether the replicas are in the same state or not, as pointed out in [21].

We have derived the cavity equations exploiting the same high-temperature expansion that leads to mean-field approximations [46].

We have solved the cavity equations in the fully connected case. In this case the solution is homogeneous, depends on very few parameters and can be written explicitly, leading to the same expression that was obtained via the replica method.

The approach based on the cavity method has several advantages:

- it makes clear the underlying assumptions;
- it holds also for the diluted version of the model (provided the solution does not condensate);
- it can be converted in message-passing algorithms, the RS belief propagation and the 1RSB survey propagation;
- it allows us to study heterogeneous solutions in diluted models, until the condensation transition.

Our work, besides providing the first complete reference on the equivalence between the replica and the cavity methods for spherical disordered models, paves the way to a more systematic study of *inhomogeneous* glassy phases in diluted mean-field models and represents a reference point for the systematic development of algorithms for combinatorial optimization and inference problems characterized by continuous variables [21, 47–49].

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Appendix A. Replica symmetric free energy

In this section we compute the replica symmetric free energy of the model, which is defined as [26]

$$-\beta F \equiv \beta\left(\sum_{a=1}^{M}\mathbb{F}_a+\sum_{i=1}^{N}\mathbb{F}_i-\sum_{(ai)\in E}\mathbb{F}_{ai}\right)\equiv\sum_{a=1}^{M}\ \log(Z_a)+\sum_{i=1}^{N}\ \log(Z_i)-\sum_{(ai)\in E}\ \log(Z_{(ai)}), \tag{A1}$$

where we have respectively

$$Z_a=\int_{-\infty}^{\infty}\prod_{i\in\partial a}\mathrm{d}\sigma_i\ \eta_{i\to a}(\sigma_i)\ \mathrm{e}^{\beta J_a\prod_{i\in\partial a}\sigma_i} \tag{A2}$$

$$Z_i=\int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \prod_{a\in\partial i}\hat{\eta}_{a\to i}(\sigma_i) \tag{A3}$$

$$Z_{(ai)}=\int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \hat{\eta}_{a\to i}(\sigma_i)\ \eta_{i\to a}(\sigma_i). \tag{A4}$$

Let us now compute the three contributions to the free energy, starting from the simplest, which is the energy per functional node $\log(Z_a)$. By expanding the Boltzmann weight in equation (A2), we get:

$$\begin{aligned}Z_a&\simeq\int\left(\prod_{i\in\partial a}\mathrm{d}\sigma_i\ \eta_{i\to a}(\sigma_i)\right)\ \left[1+\beta J_a\prod_{i\in\partial a}\sigma_i+\frac{\beta^2}{2}J_a^2\prod_{i\in\partial a}\sigma_i^2\right]\\&=\left[1+\beta J_a\prod_{i\in\partial a}m_{i\to a}+\frac{\beta^2}{2}J_a^2\prod_{i\in\partial a}\left(v_{i\to a}+m_{i\to a}^2\right)\right]\\&\simeq\exp\left\{\beta J_a\prod_{i\in\partial a}m_{i\to a}+\frac{\beta^2}{2}J_a^2\left(\prod_{i\in\partial a}\left(v_{i\to a}+m_{i\to a}^2\right)-\prod_{i\in\partial a}m_{i\to a}^2\right)\right\}.\end{aligned} \tag{A5}$$

We have, therefore, that the sum over all the $M=\binom{N}{p}$ interaction nodes reads as

$$
\begin{aligned}
\sum_{a=1}^{M} \log\left(Z_a\right) &= \beta \sum_{a=1}^{M} J_a \prod_{i\in\partial a} m_{i\to a} + \frac{\beta^2}{2}\sum_{a=1}^{M} J_a^2 \left(\prod_{i\in\partial a}\left(v_{i\to a}+m_{i\to a}^2\right) - \prod_{i\in\partial a} m_{i\to a}^2\right)\\
&= \beta m^p M\langle J\rangle + \frac{\beta^2}{2}(1-q^p)M\langle J^2\rangle\\
&= N\frac{\beta^2}{4}(1-q^p)J_2,
\end{aligned}
\tag{A6}
$$

since we have assumed $\langle J\rangle = 0$ and for large N

$$
M\langle J^2\rangle = \frac{MJ_2 p!}{2N^{p-1}} \simeq N\ J_2/2. \tag{A7}
$$

Let us now compute the entropy per spin $\log(Z_i)$. From equation (22) we have that

$$
\begin{aligned}
Z_i = &\int_{-\infty}^{\infty} \mathrm{d}\sigma_i \prod_{a\in\partial i}\left[\frac{1}{\hat{Z}_{a\to i}}\ \exp\left\{-\frac{\lambda}{2K}\sigma_i^2 + \beta J_a\left(\prod_{j\in\partial a\setminus i} m_{i\to a}\right)\ \sigma_i\right.\right.\\
&\left.\left. + \frac{\beta^2}{2}J_a^2\left(\prod_{j\in\partial a\setminus i}\left(v_{j\to a}+m_{j\to a}^2\right) - \prod_{j\in\partial a\setminus i} m_{j\to a}^2\right)\ \sigma_i^2\right\}\right]\\
= &\left[\prod_{a\in\partial i}\frac{1}{\hat{Z}_{a\to i}}\right]\int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \exp\left\{-\frac{\lambda}{2}\sigma_i^2 + \beta\sigma_i\sum_{a=1}^{K} J_a\left(\prod_{j\in\partial a\setminus i} m_{i\to a}\right)\right.\\
&\left. + \frac{\beta^2}{2}\sigma_i^2\sum_{a=1}^{K} J_a^2\left(\prod_{j\in\partial a\setminus i}\left(v_{j\to a}+m_{j\to a}^2\right) - \prod_{j\in\partial a\setminus i} m_{j\to a}^2\right)\right\}.
\end{aligned}
\tag{A8}
$$

Since to the leading order we have

$$
\begin{aligned}
&\sum_{a=1}^{K} J_a^2\left(\prod_{j\in\partial a\setminus i}\left(v_{j\to a}+m_{j\to a}^2\right) - \prod_{j\in\partial a\setminus i} m_{j\to a}^2\right)\\
&\quad\simeq \sum_{a\in\partial i\setminus b} J_a^2\left(\prod_{j\in\partial a\setminus i}\left(v_{j\to a}+m_{j\to a}^2\right) - \prod_{j\in\partial a\setminus i} m_{j\to a}^2\right),
\end{aligned}
\tag{A9}
$$

we can take advantage of the cavity equations (equation (13)) and of the fact that both $m_{j\to a}^2$ and $v_{j\to a}$ are variables which concentrate in the large N limit to write

$$
\beta^2\sum_{a=1}^{K} J_a^2\left(\prod_{j\in\partial a\setminus i}\left(v_{j\to a}+m_{j\to a}^2\right) - \prod_{j\in\partial a\setminus i} m_{j\to a}^2\right) = \lambda - \frac{1}{1-q}, \tag{A10}
$$

which allows us to write

$$Z_i = \left[\prod_{a\in\partial i}\hat{Z}_{a\to i}^{-1}\right]\int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \exp\left\{-\frac{1}{2(1-q)}\ \sigma_i^2+\beta\ \sigma_i\sum_{a=1}^{K}J_a\left(\prod_{j\in\partial a\setminus i}m_{j\to a}\right)\right\}$$
$$= \left[\prod_{a\in\partial i}\hat{Z}_{a\to i}^{-1}\right]\sqrt{2\pi}\ (1-q)^{1/2}\ \exp\left\{\frac{(1-q)}{2}\left[\beta\sum_{a=1}^{K}J_a\left(\prod_{j\in\partial a\setminus i}m_{j\to a}\right)\right]^2\right\}. \tag{A11}$$

By taking the square of the last summation left in the argument of the exponential function in equation (A11), we get

$$\beta^2\left[\sum_{a=1}^{K}J_a\prod_{j\in\partial a\setminus i}m_{j\to a}\right]^2 = \beta^2\sum_{a=1}^{K}J_a^2\prod_{j\in\partial a\setminus i}m_{j\to a}^2+\sum_{a\neq b}J_aJ_b\left(\prod_{j\in\partial a\setminus i}m_{j\to a}\right)$$
$$\times\left(\prod_{k\in\partial a\setminus i}m_{k\to a}\right)$$
$$\simeq \beta^2K\langle J^2\rangle q^{p-1}+K^2\langle J\rangle^2 m^{2p-2} = \frac{q}{(1-q)^2}, \tag{A12}$$

where for the rightmost identity in the last line of equation (A12) we used equation (15) and the definition of the spherical constraint. By then plugging the result of equation (A12) into equation (A11), we get

$$Z_i = \left[\prod_{a\in\partial i}\hat{Z}_{a\to i}^{-1}\right]\sqrt{2\pi}\ (1-q)^{1/2}\ \exp\left\{\frac{1}{2}\frac{q}{(1-q)}\right\}. \tag{A13}$$

From the expression of equation (A13), summing over all spins (and neglecting constant terms), one finds

$$\sum_{i=1}^{N}\ \log(Z_i) = \frac{N}{2}\left[\log(1-q)+\frac{q}{(1-q)}\right]-\sum_{i=1}^{N}\sum_{a\in\partial i}\ \log(\hat{Z}_{a\to i}). \tag{A14}$$

Finally, the contribution to the free energy coming from the edges reads as

$$Z_{(ai)} = \int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \hat{\eta}_{a\to i}(\sigma_i)\ \eta_{i\to a}(\sigma_i)$$
$$= \int_{-\infty}^{\infty}\mathrm{d}\sigma_i\ \frac{\exp\left\{-\frac{(\sigma_i-m_{i\to a})^2}{2v_{i\to a}}-\frac{\lambda}{2K}\ \sigma_i^2+\beta J_a\sigma_i\prod_{j\in\partial a\setminus i}m_{j\to a}+\frac{1}{2}\beta^2J_a^2\left(\prod_{j\in\partial a\setminus i}(v_{j\to a}+m_{j\to a}^2)-\prod_{j\in\partial a\setminus i}m_{j\to a}^2\right)\ \sigma_i^2\right\}}{\hat{Z}_{a\to i}\sqrt{2\pi v_{i\to a}}}. \tag{A15}$$

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Since in the large N limit we have, respectively

$$\begin{aligned} \frac{(\sigma_i - m_{i\to a})^2}{2v_{i\to a}} &= \mathcal{O}(1) \\ |J_a| &= \mathcal{O}\left(\frac{1}{N^{(p-1)/2}}\right) \\ J_a^2 &= \mathcal{O}\left(\frac{1}{N^{p-1}}\right) \\ \frac{1}{K} &= \mathcal{O}\left(\frac{1}{N^{p-1}}\right), \end{aligned} \tag{A16}$$

all terms added inside the argument of the exponential equation (A15) are subleading with respect to the first one, so that one can write

$$Z_{(ai)} \simeq \frac{1}{\hat{Z}_{a\to i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma_i}{\sqrt{2\pi v_{i\to a}}}\, \mathrm{e}^{-\frac{(\sigma_i - m_{i\to a})^2}{2v_{i\to a}}} = \frac{1}{\hat{Z}_{a\to i}}. \tag{A17}$$

Finally, summing over all the edges, we get

$$\sum_{(ai)\in E} \log(Z_{(ai)}) = -\sum_{(ai)\in E} \log(\hat{Z}_{a\to i}) = -\sum_{i=1}^{N} \sum_{a\in\partial i} \log(\hat{Z}_{a\to i}), \tag{A18}$$

which, in turn, allows us to write

$$\begin{aligned} \sum_{i=1}^{N} \log(Z_i) - \sum_{(ai)\in E} \log(Z_{(ai)}) &= \frac{N}{2}\left[\log(1-q) + \frac{q}{(1-q)}\right] - \sum_{i=1}^{N}\sum_{a\in\partial i} \log(\hat{Z}_{a\to i}) \\ &\quad + \sum_{i=1}^{N}\sum_{a\in\partial i} \log(\hat{Z}_{a\to i}) \\ &= \frac{N}{2}\left[\log(1-q) + \frac{q}{(1-q)}\right]. \end{aligned} \tag{A19}$$

By finally taking into account also the contribution of $\sum_{a=1}^{M} \log(Z_a)$, we get

$$-\beta F_{\mathrm{RS}} = \frac{N}{2}\left[\frac{\beta^2}{2}(1-q^p)J_2 + \log(1-q) + \frac{q}{(1-q)}\right], \tag{A20}$$

where we used the fact that $N\sum_{a\in\partial i} = \sum_{(ai)\in E}$.

Appendix B. 1RSB solution

B.1. 1RSB cavity equations

We derive here some useful identities which can be obtained from the 1RSB cavity equations. One has just to equate the coefficients of the diagonal, i.e. $\sum_{\alpha=1}^{x} (\sigma^\alpha)^2$, and

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off-diagonal, i.e. $\sum_{\alpha\neq\beta}^{x}\sigma^\alpha\sigma^\beta$ on the left and on the right of the first of the cavity equations in equation (51). Before doing this let us just write the local 1RSB marginal $\eta(\boldsymbol{\sigma}_i)$ explicitly, i.e. we compute explicitly the integral over $m_{i\to a}$ in equation (44):

$$\begin{aligned}
\eta_{i\to a}(\boldsymbol{\sigma}_i) &= \int_{-\infty}^{\infty}\mathrm{d}m_{i\to a}\,\frac{1}{\sqrt{2\pi\Delta^{(0)}_{i\to a}}}\,\exp\left(-\frac{(m_{i\to a}-h_{i\to a})^2}{2\Delta^{(0)}_{i\to a}}\right)\\
&\quad\times\frac{1}{\left[\sqrt{2\pi\Delta^{(1)}_{i\to a}}\right]^x}\exp\left(-\sum_{\alpha=1}^{x}\frac{(\sigma_i^\alpha-m_{i\to a})^2}{2\Delta^{(1)}_{i\to a}}\right)\\
&\approx\exp\left\{-\frac{1}{2\Delta^{(1)}_{i\to a}}\sum_{\alpha=1}^{x}\sigma_\alpha^2-\frac{h_{i\to a}}{2\Delta^{(0)}_{i\to a}}\right\}\int_{-\infty}^{\infty}\mathrm{d}m_{i\to a}\;\exp\left\{-\frac{1}{2}\left(\frac{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}\Delta^{(0)}_{i\to a}}\right)m_{i\to a}^2\right.\\
&\quad\left.-\left(\frac{h_{i\to a}}{\Delta^{(0)}_{i\to a}}+\frac{1}{\Delta^{(1)}}\sum_{\alpha=1}^{x}\sigma^\alpha\right)m_{i\to a}\right\}\\
&\approx\exp\left\{-\frac{1}{2\Delta^{(1)}_{i\to a}}\sum_{\alpha=1}^{x}\sigma_\alpha^2-\frac{h_{i\to a}}{2\Delta^{(0)}_{i\to a}}+\frac{1}{2}\left(\frac{\Delta^{(1)}_{i\to a}\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right)\left(\frac{h_{i\to a}}{\Delta^{(0)}_{i\to a}}+\frac{1}{\Delta^{(1)}_{i\to a}}\sum_{\alpha=1}^{x}\sigma^\alpha\right)^2\right\}\\
&\approx\exp\left\{-\frac{1}{2\Delta^{(1)}_{i\to a}}\left(1-\frac{\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right)\sum_{\alpha=1}^{x}\sigma_\alpha^2+\frac{1}{2\Delta^{(1)}_{i\to a}}\left(\frac{\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right)\sum_{\alpha\neq\beta}^{x}\sigma_\alpha\sigma_\beta\right.\\
&\quad\left.+\left(\frac{h_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right)\sum_{\alpha=1}^{x}\sigma_\alpha\right\},
\end{aligned}\tag{B1}$$

where, in the last line of equation (B1), we have retained only terms at least linear in σ_i^α. Thus, matching the coefficients of the linear and quadratic in σ_i^α, both diagonal and non-diagonal, appearing in the expression of $\eta(\sigma_i^\alpha)$ in equation (B1) above here and in equation (57), we get a set of three equations:

$$\begin{aligned}
\frac{h_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}} &= \beta\sum_{b\in\partial i\setminus a}J_b\prod_{k\in\partial b\setminus i}h_{k\to b}=A\\
\frac{1}{\Delta^{(1)}_{i\to a}}\left(1-\frac{\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right) &= \lambda+\sum_{b\in\partial i\setminus a}\beta^2J_b^2\left[\prod_{k\in\partial b\setminus i}h^2_{k\to b}-\prod_{k\in\partial b\setminus i}\left(\Delta^{(1)}_{k\to b}+\Delta^{(0)}_{k\to b}+h^2_{k\to b}\right)\right]=B^{(\mathrm{d})}\\
\frac{1}{\Delta^{(1)}_{i\to a}}\left(\frac{\Delta^{(0)}_{i\to a}}{\Delta^{(1)}_{i\to a}+x\Delta^{(0)}_{i\to a}}\right) &= \sum_{b\in\partial i\setminus a}\beta^2J_b^2\left[\prod_{k\in\partial b\setminus i}\left(\Delta^{(0)}_{k\to b}+h^2_{k\to b}\right)-\prod_{k\in\partial b\setminus i}h^2_{k\to b}\right]=B^{(\mathrm{nd})}.
\end{aligned}\tag{B2}$$

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Since we know that the three variables $\Delta^{(1)}_{k\to b}$, $\Delta^{(0)}_{k\to b}$ and $h^2_{k\to b}$ all have a distribution which becomes concentrated in the large N limit, we have that the summations on the right-hand side of the last two lines in equation (B2) read as

$$
\begin{aligned}
\lambda + \sum_{b\in\partial i\setminus a} \beta^2 J_b^2 \left[\prod_{k\in\partial b\setminus i} h^2_{k\to b} - \prod_{k\in\partial b\setminus i}\left(\Delta^{(1)}_{k\to b} + \Delta^{(0)}_{k\to b} + h^2_{k\to b}\right)\right] &= \lambda - \beta^2 K\langle J^2\rangle(1-q_0^{p-1}) = B^{(\mathrm{d})} \\
\sum_{b\in\partial i\setminus a} \beta^2 J_b^2 \left[\prod_{k\in\partial b\setminus i}\left(\Delta^{(0)}_{k\to b} + h^2_{k\to b}\right) - \prod_{k\in\partial b\setminus i} h^2_{k\to b}\right] &= \beta^2 K\langle J^2\rangle(q_1^{p-1} - q_0^{p-1}) = B^{(\mathrm{nd})},
\end{aligned}
\tag{B3}
$$

where we have introduced the symbols $B^{(\mathrm{d})}$ and $B^{(\mathrm{nd})}$ to highlight that dependence on the subscript indices can be dropped. Thus, since expressions on the *right*-hand side of the last two lines of equation (B2) are summations over a very large number of local fields, as a consequence one can drop the site index for the expressions on the *left*-hand side. This means that inside the left-hand members in equation (B2), except the equation in the first line, the fields $\Delta^{(1)}_{i\to a}$, $\Delta^{(0)}_{i\to a}$ and $h^2_{i\to a}$ can be replaced everywhere with their average values:

$$
\begin{aligned}
\langle h^2_{i\to a}\rangle &= q_0 \\
\langle \Delta^{(0)}_{i\to a}\rangle &= q_1 - q_0 \\
\langle \Delta^{(1)}_{i\to a}\rangle &= 1 - q_1,
\end{aligned}
\tag{B4}
$$

leading finally to the following expressions:

$$
\begin{aligned}
B^{(\mathrm{d})} &= \frac{1}{1-q_1}\left[1 - \frac{q_1 - q_0}{1 - xq_0 - (1-x)q_1}\right] \\
B^{(\mathrm{nd})} &= \frac{q_1 - q_0}{1-q_1}\cdot\frac{1}{1 - xq_0 - (1-x)q_1}.
\end{aligned}
\tag{B5}
$$

For what concerns the first line of equation (B2), while it is in general not possible to assume concentration for the distribution of $h_{i\to a}$, this can be done for the distribution of $h^2_{i\to a}$. This fact can be exploited to write useful relations for the squared variable:

$$
\begin{aligned}
A^2_{i\to a} &= \beta^2\left[\sum_{b\in\partial i\setminus a} J_b \prod_{k\in\partial b\setminus i} h_{k\to b}\right]^2 \\
&= \beta^2 \sum_{b\in\partial i\setminus a} J_b^2 \prod_{k\in\partial b\setminus i} h^2_{k\to b} + \beta^2 \sum_{b\neq c\in\partial i\setminus a} J_b J_a \prod_{k\in\partial b\setminus i} h_{k\to b} \prod_{j\in\partial c\setminus i} h_{j\to c} \\
&\simeq \beta^2\langle J^2\rangle\langle h^{2(p-1)}\rangle + \langle J\rangle^2\langle h\rangle^{2(p-1)},
\end{aligned}
\tag{B6}
$$

which tells us that we can also write

$$A_{i\to a}^2 = \left\langle \frac{h_{i\to a}^2}{\left(\Delta_{i\to a}^{(1)} + x\Delta_{i\to a}^{(0)}\right)^2} \right\rangle = \frac{\langle h_{i\to a}^2\rangle}{\left(\langle\Delta_{i\to a}^{(1)}\rangle + x\langle\Delta_{i\to a}^{(0)}\rangle\right)^2} = \frac{q_0}{(1 - xq_0 - (1-x)q_1)^2}. \tag{B7}$$

B.2. 1RSB free energy

Let us reproduce here the exact formulas that we need to compute $\mathcal{F}(x)$ in the presence of a one-step replica symmetry ansatz. The free energy for the replicated system is [26, 38]

$$\Phi(x) = -\left(\sum_{a=1}^{M} \mathbb{F}_a^{\mathrm{RSB}}(x) + \sum_{i=1}^{N} \mathbb{F}_i^{\mathrm{RSB}}(x) - \sum_{(ai)\in E} \mathbb{F}_{ai}^{\mathrm{RSB}}(x)\right). \tag{B8}$$

In principle, the three contributions, respectively representing the energetic and the entropic contributions and a normalization, read as

$$\beta\mathbb{F}_a^{\mathrm{RSB}}(x) = \log\left\{\int \prod_{i\in\partial a} [\mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a})]\ \mathrm{e}^{x\beta\mathbb{F}_a(\{m_{i\to a}\})}\right\} \tag{B9}$$

$$\beta\mathbb{F}_i^{\mathrm{RSB}}(x) = \log\left\{\int \prod_{a\in\partial i} \left[\mathrm{d}\hat{m}_{a\to i}\hat{Q}_{a\to i}(\hat{m}_{a\to i})\right]\ \mathrm{e}^{x\beta\mathbb{F}_i(\{\hat{m}_{a\to i}\})}\right\} \tag{B10}$$

$$\beta\mathbb{F}_{ai}^{\mathrm{RSB}}(x) = \log\left\{\int \mathrm{d}\hat{m}_{a\to i}\ \mathrm{d}m_{i\to a}\ \hat{Q}_{a\to i}(\hat{m}_{a\to i})\ Q_{i\to a}(m_{i\to a})\ \mathrm{e}^{x\beta\mathbb{F}_{ai}(m_{i\to a},\hat{m}_{a\to i})}\right\} \tag{B11}$$

where $\mathbb{F}_a, \mathbb{F}_i$ and $\mathbb{F}_{ai}$ are the RS free energy parts in equation (20), $Q_{i\to a}(m_{i\to a})$ is the distribution of the cavity marginals that in the case of the dense Gaussian 1RSB ansatz has been defined in equation (46), section 3.1, and the distribution $\hat{Q}_{a\to i}(\hat{m}_{a\to i})$ should be the one of the function-to-variable fields. Since we are going to write everything in terms of the variable-to-function fields $m_{i\to a}$, it is convenient to write the probability distribution $\hat{Q}_{a\to i}(\hat{m}_{a\to i})$ of function-to-variable fields $\hat{m}_{a\to i}$ in terms of $Q_{i\to a}(m_{i\to a})$ and $m_{i\to a}$. This can be done taking advantage of the identity [26]:

$$\hat{Q}_{a\to i}(\hat{m}_{a\to i}) = \int \prod_{k\in\partial a\setminus i} [\mathrm{d}m_{k\to a} Q_{k\to a}(m_{k\to a})]\ \hat{Z}_{a\to i}^{x} \\ \times\ (\{m_{k\to a}\})\delta\left[\hat{m}_{a\to i} - f(\{m_{k\to a}\})\right], \tag{B12}$$

where $f(x)$ is shorthand to refer to the cavity equations and $\hat{Z}_{a\to i}$ is the normalization factor introduced in the RS equation (8). By exploiting the last identity, we can rewrite the expression of the 1RSB entropy per site as

$$\beta \mathbb{F}_i^{\mathrm{RSB}}(x) = \log \left\{ \int \prod_{a \in \partial i} \prod_{k \in \partial a \setminus i} \mathrm{d}m_{k\to a} \; Q_{k\to a}(m_{k\to a}) \; \hat{Z}^x_{a\to i} \right.$$
$$\left. \times \; (\{m_{k\to a}\}_{k\in\partial a\setminus i}) \; \mathrm{e}^{x\beta \mathbb{F}_i(\{f(\{m_{k\to a}\})\}_{a\in\partial i})} \right\}. \tag{B13}$$

B.3. 1RSB energy

We compute in what follows the expression of the *energetic* part of the free energy. By plugging into the same expression the definition of the energetic contribution to the local free energy in the RS case, see equation (21), and the 1RSB weighted average over the local messages written in equation (71), we get

$$\begin{aligned} \beta \mathbb{F}_a^{\mathrm{RSB}}(x) &= \log \left\{ \int \prod_{i\in\partial\alpha} [\mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a})] \right. \\ &\quad \left. \times \left(\int_{-\infty}^{\infty} \prod_{i\in\partial a} \mathrm{d}\sigma_i \; \eta^s_{i\to a}(\sigma_i) \; \mathrm{e}^{\beta J_a \prod_{i\in\partial a} \sigma_i} \right)^x \right\} \\ &= \log \left\{ \int \prod_{i\in\partial\alpha} [\mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a})] \right. \\ &\quad \left. \times \int \prod_{i\in\partial a} \prod_{\alpha=1}^{x} \mathrm{d}\sigma_i^\alpha \; \eta^s_{i\to a}(\sigma_i^\alpha) \; \mathrm{e}^{\beta J_a \sum_{\alpha=1}^x \prod_{i\in\partial a} \sigma_i^\alpha} \right\}. \end{aligned} \tag{B14}$$

The expression in equation (B14) is formally correct but completely useless unless an explicit ansatz for the distribution $Q_{i\to a}(m_{i\to a})$ is given. We consider the ansatz in equations (46) and (48), which assumes three parameters, the local magnetization $h_{i\to a}$ and the two parameters $\Delta^{(0)}_{i\to a}$ and $\Delta^{(1)}_{i\to a}$ relative to the coupling between replicas:

$$\begin{aligned} &\int \mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a}) \prod_{\alpha=1}^{x} \eta^s_{i\to a}(\sigma_i^\alpha) \\ &\quad = \int_{-\infty}^{\infty} \mathrm{d}m_{i\to a} \frac{\mathrm{e}^{-(m_{i\to a}-h_{i\to a})^2/(2\Delta^{(0)}_{i\to a})}}{\sqrt{2\pi\Delta^{(0)}_{i\to a}}} \prod_{\alpha=1}^{x} \frac{\mathrm{e}^{-(\sigma_i^\alpha - m_{i\to a})^2/(2\Delta^{(1)}_{i\to a})}}{\sqrt{2\pi\Delta^{(1)}_{i\to a}}}. \end{aligned} \tag{B15}$$

In particular we have that $h_{i\to a}$ and $\Delta^{(0)}_{i\to a}$ are variational parameters for the probability distribution of the fields $m_{i\to a}$ on a given edge, while $\Delta^{(1)}_{i\to a}$ is a variational parameter for the coupling between replicas.

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In order to concretely carry on the calculation, we need first to expand to the leading order the interaction term in the partition function:

$$\exp\left\{\beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \sigma_i^\alpha\right\} \simeq \left(1 + \beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \sigma_i^\alpha + \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\beta=1}^{x} \prod_{i\in\partial a} \sigma_i^\alpha \sigma_i^\beta\right). \tag{B16}$$

By then indicating

$$\int \mathrm{d}m_{i\to a} Q_{i\to a}(m_{i\to a}) \prod_{\alpha=1}^{x} \eta_{i\to a}(\sigma_i^\alpha, m_{i\to a})\ \mathcal{G}(\sigma_i^\alpha) = \left\langle \mathcal{G}(\sigma_i^\alpha)\right\rangle, \tag{B17}$$

we have

$$\begin{aligned}
\beta \mathbb{F}_a^{\mathrm{RSB}}(x) &= \log\left\{\left\langle \mathrm{e}^{\beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \sigma_i^\alpha}\right\rangle\right\} \\
&\simeq \log\left\{1 + \beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha\rangle + \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\beta=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha \sigma_i^\beta\rangle\right\} \\
&\simeq \beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha\rangle + \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\beta=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha \sigma_i^\beta\rangle - \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\beta=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha\rangle\langle \sigma_i^\beta\rangle \\
&= \beta J_a \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha\rangle + \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha=1}^{x} \prod_{i\in\partial a} \langle (\sigma_i^\alpha)^2\rangle + \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\neq\beta}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha \sigma_i^\beta\rangle \\
&\quad - \frac{1}{2}\beta^2 J_a^2 \sum_{\alpha\beta}^{x} \prod_{i\in\partial a} \langle \sigma_i^\alpha\rangle\langle \sigma_i^\beta\rangle.
\end{aligned} \tag{B18}$$

Finally, by making use of the definition of the moments $\langle \sigma_i^\alpha\rangle$, $\langle(\sigma_i^\alpha)^2\rangle$ and $\langle \sigma_i^\alpha \sigma_i^\beta\rangle$ given in equation (47), we can write

$$\begin{aligned}
\beta \mathbb{F}_a^{\mathrm{RSB}}(x) &= x\beta J_a \prod_{i\in\partial a} h_{i\to a} + \frac{x}{2}\beta^2 J_a^2 \prod_{i\in\partial a}\left(\Delta_{i\to a}^{(1)} + \Delta_{i\to a}^{(0)} + h_{i\to a}^2\right) \\
&\quad + \frac{x(x-1)}{2}\beta^2 J_a^2 \prod_{i\in\partial a}\left(\Delta_{i\to a}^{(0)} + h_{i\to a}^2\right) - \frac{x^2}{2} J_a^2 \prod_{i\in\partial a} h_{i\to a}^2.
\end{aligned} \tag{B19}$$

By then summing over all the M interactions we can replace the local messages with their average values:

$$\sum_{a=1}^{M} \beta \mathbb{F}_a^{\mathrm{RSB}}(x) = Mx\left[\beta\langle J\rangle\langle h\rangle^p + \frac{1}{2}\beta^2\langle J^2\rangle + \frac{x-1}{2}\beta^2\langle J^2\rangle q_1^p - \frac{1}{2}\beta^2\langle J^2\rangle q_0^p\right]. \tag{B20}$$

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If we assume the zero field case, i.e. $\langle J\rangle = \langle h\rangle = 0$, recalling then that $\langle J^2\rangle = p!/(2N^{p-1})$ so that $M\langle J^2\rangle \simeq N/2$ we finally obtain

$$\sum_{a=1}^{M} \beta\mathbb{F}_a^{\mathrm{RSB}}(x) = xN\,\frac{\beta^2}{4}\left[1-(1-x)q_1^p - xq_0^p\right]. \tag{B21}$$

B.4. 1RSB entropy

The 1RSB entropy is defined in equation (B13) and we rewrite it here for the ease of the reader:

$$\beta\mathbb{F}_i^{\mathrm{RSB}}(x) = \log\left\{\int \prod_{a\in\partial i}\prod_{k\in\partial a\backslash i} \mathrm{d}m_{k\to a}\; Q_{k\to a}(m_{k\to a})\; \hat{Z}^x_{a\to i}\right.$$
$$\left.\times\; (\{m_{k\to a}\}_{k\in\partial a\backslash i})\; \mathrm{e}^{x\beta\mathbb{F}_i(\{f(\{m_{k\to a}\})\}_{a\in\partial i})}\right\}$$

where the local free energy $\mathbb{F}_i[\{m_{k\to a}\}_{k\in\partial a\backslash i}]$ is the RS one:

$$\exp\left(\beta\mathbb{F}_i\right) = \int_{-\infty}^{\infty} \mathrm{d}\sigma_i\; \mathrm{e}^{-\lambda\sigma_i^2/2}\prod_{a\in\partial i}\left[\hat{Z}^{-1}_{a\to i}(\{m_{k\to a}\}_{k\in\partial a\backslash i})\right.$$
$$\left.\times \int \prod_{k\in\partial a\backslash i} \mathrm{d}\sigma_k\; \eta^s_{k\to a}(\sigma_k)\; \mathrm{e}^{\beta J_a\sigma_i\prod_{k\in\partial a\backslash i}\sigma_k}\right]. \tag{B22}$$

By then assuming that x is integer (the analytic continuation to real values will be taken afterwards) it is straightforward to write

$$\prod_{a\in\partial i} \hat{Z}^x_{a\to i}(\{m_{k\to a}\}_{k\in\partial a\backslash i})\; \mathrm{e}^{x\beta\mathbb{F}_i[\{m_{k\to a}\}_{k\in\partial a\backslash i}]}$$
$$= \int_{-\infty}^{\infty}\prod_{\alpha=1}^{x} d\sigma_i^\alpha\; \mathrm{e}^{-\lambda\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2/2}\prod_{a\in\partial i}$$
$$\times\left[\int\prod_{\alpha=1}^{x}\prod_{k\in\partial a\backslash i}\mathrm{d}\sigma_k^\alpha\;\eta^s_{k\to a}(\sigma_k^\alpha)\;\mathrm{e}^{\beta J_a\sum_{\alpha=1}^{x}\sigma_i^\alpha\prod_{k\in\partial a\backslash i}\sigma_k^\alpha}\right] \tag{B23}$$

The implementation of the 1RSB ansatz comes at this stage very naturally; we just have to insert for each link (ka) the ansatz in equations (46) and (48). Thus, if we take the average over local fields *before* taking the one over local variables—as for any replica calculation, but locally—it is convenient to define

$$\eta_{k\to a}(\boldsymbol{\sigma}_k) = \int \mathrm{d}m_{k\to a}\; Q_{k\to a}(m_{k\to a})\prod_{\alpha=1}^{x}\eta^s_{k\to a}(\sigma_k^\alpha), \tag{B24}$$

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which is precisely the quantity defined in equation (44). The above steps allow us to rewrite the 1RSB entropy simply as

$$\begin{aligned}\beta\mathbb{F}_i^{\mathrm{RSB}} &= \log\left(\int_{-\infty}^{\infty}\prod_{\alpha=1}^{x}\mathrm{d}\sigma_i^\alpha\ \mathrm{e}^{-\lambda\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2/2}\prod_{a\in\partial i}\left[\int_{-\infty}^{\infty}\prod_{k\in\partial a\setminus i}\mathcal{D}\boldsymbol{\sigma}_k\ \eta_{k\to a}(\boldsymbol{\sigma}_k)\ \mathrm{e}^{\beta J_a\sum_{\alpha=1}^{x}\sigma_i^\alpha\prod_{k\in\partial a\setminus i}\sigma_k^\alpha}\right]\right)\\ &= \log\left(\int_{-\infty}^{\infty}\prod_{\alpha=1}^{x}\mathrm{d}\sigma_i^\alpha\ \mathrm{e}^{\left(\sum_{a\in\partial i}\hat{A}_{a\to i}\right)\sum_{\alpha=1}^{x}\sigma_i^\alpha-\frac{1}{2}\left(\lambda+\sum_{a\in\partial i}\hat{B}_{a\to i}^{(\mathrm{d})}\right)\sum_{\alpha=1}^{x}(\sigma_i^\alpha)^2+\frac{1}{2}\left(\sum_{a\in\partial i}\hat{B}_{a\to i}^{(\mathrm{nd})}\right)\sum_{\alpha\neq\beta}^{x}\sigma_i^\alpha\sigma_i^\beta}\right),\end{aligned} \tag{B25}$$

where the quantities $\hat{A}_{a\to i}$, $\hat{B}_{a\to i}^{(\mathrm{d})}$ and $\hat{B}_{a\to i}^{(\mathrm{nd})}$ are those defined in equation (53).

By comparing the definitions in equation (53) of the main text with those in equations (B2) and (B3) of appendix B.1 one gets, at the leading order,

$$\begin{aligned}B^{(\mathrm{d})} &= \lambda+\sum_{a\in\partial i}\hat{B}_{a\to i}^{(\mathrm{d})}\\ B^{(\mathrm{nd})} &= \sum_{a\in\partial i}\hat{B}_{a\to i}^{(\mathrm{nd})},\end{aligned} \tag{B26}$$

where the expressions of $B^{(\mathrm{d})}$ and $B^{(\mathrm{nd})}$ are those given in appendix B.1. According to the definition of $\hat{A}_{a\to i}$ in equation (53) it is then useful to define A as follows:

$$A=\sum_{a\in\partial i}\hat{A}_{a\to i}=\beta\sum_{a\in\partial i}J_a\prod_{j\in\partial a\setminus i}h_{j\to i}, \tag{B27}$$

from which one has

$$A^2=\beta^2\langle J^2\rangle\langle h^{2(p-1)}\rangle+\langle J\rangle^2\langle h\rangle^{2(p-1)}=\frac{q_0}{(1-xq_0-(1-x)q_1)^2}, \tag{B28}$$

where the last equality holds by virtue of equation (B7) when $\langle J\rangle=\langle h\rangle=0$.

We can, therefore, define a vector A and a matrix $\mathcal{M}$ (already introduced in equation (58)) as

$$\begin{aligned}A_\alpha &= A\ ,\quad \forall\alpha\\ \mathcal{M}_{\alpha\beta} &= \delta_{\alpha\beta}B^{(\mathrm{d})}+(1-\delta_{\alpha\beta})(-B^{(\mathrm{nd})}).\\ \alpha,\beta &= 1,\ldots,x\ .\end{aligned} \tag{B29}$$

The 1RSB entropy can be easily written as

$$\begin{aligned}\beta\mathbb{F}_i^{\mathrm{RSB}} &= \log\left(\int\mathcal{D}\boldsymbol{\sigma}\ \exp\left\{\sum_{\alpha=1}^{x}A_\alpha\sigma_\alpha-\frac{1}{2}\sum_{\alpha\beta}\sigma_\alpha\mathcal{M}_{\alpha\beta}\sigma_\beta\right\}\right)\\ &= \sqrt{\frac{(2\pi)^x}{\det\mathcal{M}}}\ \exp\left(\frac{1}{2}A^2\sum_{\alpha\beta}\mathcal{M}_{\alpha\beta}^{-1}\right)\end{aligned} \tag{B30}$$

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where, making use of the definition of the inverse $\mathcal{M}^{-1}$ given in equation (60) and with the help of a little algebra, we get

$$\sum_{\alpha\beta} \mathcal{M}^{-1}_{\alpha\beta} = \frac{x}{B^{(\mathrm{d})} + (1-x)B^{(\mathrm{nd})}} = \frac{x}{\frac{1}{1-q_1} - \frac{x(q_1-q_0)}{(1-q_1)(1-xq_0-(1-x)q_1)}}$$
$$= x\,(1 - xq_0 - (1-x)q_1). \tag{B31}$$

We thus have

$$\frac{1}{2}A^2 \sum_{\alpha\beta} \mathcal{M}^{-1}_{\alpha\beta} = \frac{1}{2}\frac{q_0}{(1-xq_0-(1-x)q_1)^2}\left(\sum_{\alpha\beta} \mathcal{M}^{-1}_{\alpha\beta}\right) = \frac{xq_0}{1-xq_0-(1-x)q_1}. \tag{B32}$$

We also need to compute $\det \mathcal{M}$, that is the determinant of a symmetric $x \times x$ matrix of the form

$$\mathcal{M} = \begin{pmatrix} a_1 & b & b & b & b \\ b & a_2 & b & b & b \\ b & b & a_3 & b & b \\ b & b & b & a_4 & b \\ b & b & b & b & a_5, \end{pmatrix} = \begin{pmatrix} a_1-b & 0 & 0 & 0 & 0 \\ 0 & a_2-b & 0 & 0 & 0 \\ 0 & 0 & a_3-b & 0 & 0 \\ 0 & 0 & 0 & a_4-b & 0 \\ 0 & 0 & 0 & 0 & a_5-b, \end{pmatrix}$$
$$+ \begin{pmatrix} b & b & b & b & b \\ b & b & b & b & b \\ b & b & b & b & b \\ b & b & b & b & b \\ b & b & b & b & b. \end{pmatrix}.$$

The general formula for such a determinant is

$$\det\left[\mathrm{Diag}(a_1-b,\ldots,a_x-b) + b\ \cdot \mathbf{1}_x^T \otimes \mathbf{1}_x\right] = \prod_{i=1}^{x}(a_i-b) + b\sum_{i=1}^{x}\prod^{x}(a_j-b). \tag{B33}$$

In the case of the rank-x symmetric matrix $\mathcal{M}$ defined in equation (B29), where in addition all the elements on the diagonal are identical, from equation (B33) we have that

$$\det(\mathcal{M}) = \left(B^{(\mathrm{d})} + B^{(\mathrm{nd})}\right)^x - x\,B^{(\mathrm{nd})}\,\left(B^{(\mathrm{d})} + B^{(\mathrm{nd})}\right)^{x-1}. \tag{B34}$$

By exploiting then the definition of $B^{(\mathrm{d})}$ and $B^{(\mathrm{nd})}$ in terms of x, q_0 and q_1 written in equation (B5) one gets, after a very simple algebra,

$$\det(\mathcal{M}) = \frac{1}{(1-q_1)^{x-1}} \cdot \frac{1}{(1-xq_0-(1-x)q_1)}, \tag{B35}$$

so that

$$\frac{1}{\sqrt{\det(\mathcal{M})}} = \exp\left\{\frac{1}{2}(x-1)\ \log(1-q_1) + \frac{1}{2}\ \log(1-xq_0-(1-x)q_1)\right\}. \tag{B36}$$

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At this point it is immediate to write (neglecting constant terms) the 1RSB local free entropy term

$$\sum_{i=1}^{N} \beta \mathbb{F}_i^{\mathrm{RSB}} = N \left\{ \frac{x}{2} \frac{q_0}{[1 - xq_0 - (1-x)q_1]} + \frac{x-1}{2} \log(1-q_1) + \frac{1}{2} \log[1 - xq_0 - (1-x)q_1] \right\}. \tag{B37}$$

B.5. Vanishing of the normalization terms $\mathbb{F}_{ai}^{\mathrm{RSB}}$

Let us now show how the terms $\mathbb{F}_{ai}^{\mathrm{RSB}}$ vanish in the limit $N \to \infty$, provided that normalized distributions are used everywhere. The first step is to plug the definition of $\hat{Q}_{a\to i}(\hat{m}_{a\to i})$ of equation (B12) into the definition of $\mathbb{F}_{ai}^{\mathrm{RSB}}$ in equation (73), thus obtaining

$$\exp\left[\beta \mathbb{F}_{ai}^{\mathrm{RSB}}(x)\right] = \int \left[\prod_{k\in\partial a\setminus i} \mathrm{d}m_{k\to a} Q_{k\to a}(m_{k\to a})\right] \mathrm{d}m_{i\to a}\ \hat{Z}_{a\to i}^{x} \times (\{m_{k\to a}\})\ Q_{i\to a}(m_{i\to a})\ \mathrm{e}^{x\beta \mathbb{F}_{ai}(m_{i\to a},\{m_{k\to a}\})}. \tag{B38}$$

From the RS equations (8) and (23) we then have that

$$\begin{aligned} \mathrm{e}^{x\beta \mathbb{F}_{ai}(m_{i\to a},\{m_{k\to a}\})} &= Z_{(ai)}^{x} \\ &= \hat{Z}_{a\to i}^{-x}(\{m_{k\to a}\}) \left[\int_{-\infty}^{\infty} \mathrm{d}\sigma_i\ \eta_{i\to a}(\sigma_i)\ \mathrm{e}^{-\frac{\lambda\sigma_i^2}{2K}} \right. \\ &\quad \left. \times \int_{-\infty}^{\infty} \prod_{k\in\partial a\setminus i} \mathrm{d}\sigma_k\ \eta_{k\to a}(\sigma_k)\ \mathrm{e}^{\beta J_a \sigma_i \prod_{k\in\partial a\setminus i} \sigma_k}\right]^{x} \\ &= \hat{Z}_{a\to i}^{-x}\left[1 + \beta J_a \prod_{k\in\partial a} m_{k\to a} + \mathcal{O}(1/K)\right]^{x}. \end{aligned} \tag{B39}$$

Then, since we have $J_a \sim 1/N^{(p-1)/2}$ and $1/K \sim 1/N^{p-1}$, to the leading order in N we can simply write

$$x\beta \mathbb{F}_{ai}(m_{i\to a}, \{m_{k\to a}\}) = -x\ \log\left[\hat{Z}_{a\to i}(\{m_{k\to a}\})\right] + \mathcal{O}(1/N^{(p-1)/2}), \tag{B40}$$

so that inside the integral of equation (B38) we have

$$\hat{Z}_{a\to i}^{x}(\{m_{k\to a}\})\ \mathrm{e}^{x\beta \mathbb{F}_{ai}(m_{i\to a},\{m_{k\to a}\})} \simeq 1. \tag{B41}$$

In conclusion, to the leading order in N, we can write

$$\exp\left[\beta \mathbb{F}_{ai}^{\mathrm{RSB}}(x)\right] = \int \prod_{k\in\partial a} \mathrm{d}m_{k\to a}\ Q_{k\to a}(m_{k\to a}) = 1, \tag{B42}$$

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by the closure condition over the probability density Q. We have therefore shown that, to the leading order in N, one has $\beta\mathbb{F}_{ai}^{\mathrm{RSB}}(x) = 0$ for each edge (ai).

B.6. 1RSB total free energy

Putting together the results of sections B3 and B4, and remembering that the total free energy of the system is just the free energy of the x-coupled replicas that we computed, divided by x (and extremized over x), we obtain

$$\begin{aligned} -x\beta\mathcal{F}(x) &= \sum_{a=1}^{M} \beta\mathbb{F}_a^{\mathrm{RSB}} + \sum_{i=1}^{N} \beta\mathbb{F}_i^{\mathrm{RSB}} - \sum_{(ai)\in E} \beta\mathbb{F}_{ai}^{\mathrm{RSB}} \\ &= \frac{xN}{2}\left\{\frac{\beta^2}{2}\left[1-(1-x)q_1^p - xq_0^p\right] + \frac{q_0}{[1-xq_0-(1-x)q_1]} \right. \\ &\quad \left. + \frac{x-1}{x}\log\left(1-q_1\right) + \frac{1}{x}\log\left[1-xq_0-(1-x)q_1\right]\right\}. \end{aligned} \tag{B43}$$

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