Numerical methods for Hamilton–Jacobi type equations

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Abstract

We give an overview of numerical methods for first order Hamilton–Jacobi equations. After a short presentation of the theory of viscosity solutions, we show their link with entropy solutions of conservation laws. Then, we review theory and construction of monotone numerical methods in finite difference and semi-Lagrangian form, also providing a numerical test which shows the main features of this class of schemes. Finally, we sketch the main ideas behind high-order methods and more recent developments.

Keywords. Hamilton–Jacobi equations, viscosity solutions, numerical methods, convergence

1 Introduction and motivations

The analysis and approximation of first order partial differential equations of Hamilton–Jacobi (HJ) type have an important role in a number of fields such as fluid dynamics, optimal control and differential games, image processing and material science just to mention a few. In the 80s, the notion of weak solutions in the viscosity sense, introduced by Crandall and Lions [CL83], has had a crucial impact in giving a sound theoretical framework for both the analytical and the numerical study. The goal of this chapter is to sketch this theory and give some introductory material on the construction of approximation schemes for viscosity solutions. Due to space restrictions, we will only provide the main concepts on both theory and numerical methods – for a more complete exposition of the analytical theory, we refer the interested readers to the books by P.L. Lions [Li82], Barles [Ba98], Bardi–Capuzzo Dolcetta [BCD97] and Evans [E10]. A detailed survey on the related applications and numerical methods can be found in the books by Sethian [Se96], Osher–Fedkiw [OF03] and Falcone–Ferretti [FF14]. In many applications is necessary to have efficient algorithms which can give quick responses, this has motivated a research activity on Fast Marching and Fast Sweeping methods based on the schemes presented in this chapter (the interested reader can find more informations in [Se96, QZZ07]).

We start by presenting two typical examples of Hamilton–Jacobi equations, arising in front propagation and optimal control problems.

Front propagation via level set method The evolutive problem related to the *level set formulation of a front propagating in the normal direction with a (known) velocity* $c : \mathbb{R}^d \to \mathbb{R}$ is

$$\begin{cases} v_t + c(x)|Dv| = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\ v(x,0) = v_0(x) & x \in \mathbb{R}^d \end{cases} \tag{1}$$

where c is typically required to be strictly positive, and $v_0 : \mathbb{R}^d \to \mathbb{R}$ is a proper representation of the initial front Γ_0. For simplicity, assume that Γ_0 is a piecewise smooth surface, boundary of a compact domain Ω_0. Then, the initial condition v_0 must change sign on Γ_0, so that

$$\begin{cases} v_0(x) < 0 & x \in \Omega^0 \\ v_0(x) = 0 & x \in \Gamma_0 \\ v_0(x) > 0 & x \in \mathbb{R}^d \setminus \Omega_0 \end{cases}$$

and, for any $t \geq 0$, the front is identified with the 0-level set of $v(x,t)$. This approach allows for topology changes of the front, and can be extended to more general situations, such as the case of a curvature-related propagation speed (see [Se96, OF03] for an extensive presentation).

The infinite horizon problem Consider the controlled system of ordinary differential equations

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)) \\ y(0) = x \in \mathbb{R}^d, \end{cases} \tag{2}$$

where $\alpha \in \mathcal{A} := \{\alpha : [0, +\infty[\to A, \alpha \text{ measurable}\}$ and $A \subset \mathbb{R}^M$ is compact. Define a cost functional as

$$J_x(\alpha) := \int_0^\infty g(y(s), \alpha(s))\mathrm{e}^{-\lambda s} ds,$$

where $\lambda > 0$ is a discount factor for the costs. The *value function* is defined as $v(x) := \inf_{\alpha \in \mathcal{A}} J_x(\alpha)$, and via the Dynamic Programming principle one can derive the stationary Hamilton–Jacobi equation

$$\lambda v(x) + \max_{a \in A}\{-f(x,a) \cdot \nabla u(x) - g(x,a)\} = 0. \tag{3}$$

A classical result shows that, under general assumptions, the value function is the unique viscosity solution of (3). A similar characterization can be obtained for the value function of the *finite horizon problem* of optimal control theory, leading to an evolutive HJ equation (see, e.g., [BCD97] for more details).

In what follows, we will mainly treat the case of first-order HJ equations with a convex Hamiltonian H. Nonetheless, non-convex Hamiltonians may occur in various relevant applications, such as differential games [I65, FSo89]. On the other hand, stochastic optimal control problems lead to consider second order HJ equations, for which the books by Kushner–Dupuis [KuD01] and Fleming–Soner [FS93] provide classical references.

2 Basics on viscosity solutions

Let us start with the stationary model problem of Dirichlet type in an open subset $\Omega \subset \mathbb{R}^d$,

$$\begin{cases} H(x, v, Dv) = 0 & x \in \Omega \\ v(x) = b(x) & x \in \partial\Omega, \end{cases} \tag{4}$$

where b is a given boundary condition, and $H : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the em Hamiltonian function which will be required to satisfy the basic assumptions

(A1) $H(\cdot, \cdot, \cdot)$ is uniformly continuous

(A2) $H(x, v, \cdot)$ is convex

(A3) $H(x, \cdot, p)$ is monotone

In order to show that no smooth solution is expected to exist in general, we consider the following Dirichlet problem for the *eikonal equation* in one space dimension:

$$\begin{cases} |v_x| = 1 & x \in \Omega = (-1, 1) \\ v(x) = 0 & x = \pm 1 \end{cases} \tag{5}$$

Clearly, $v_1(x) = x$ and $v_2(x) = -x$ satisfy the equation, but not the boundary conditions, and a C^1 solution cannot exist due to Rolle's Theorem. However, both functions $v_3(x) = |x| - 1$ and $v_4(x) = 1 - |x|$ satisfy (almost everywhere) the equation along with the boundary conditions. In fact, there exist infinitely many a.e. solutions of the equation, which may be constructed as piecewise affine functions with slope ± 1, satisfying the boundary conditions. Then, it is clear that the notion of "a.e. solution" is unsuitable for a uniqueness result. One possibility to single out a solution is to perform an *elliptic regularization* of the equation, in the form

$$-\varepsilon v_{xx} + |v_x| = 1$$

(with the same boundary conditions as before) and pass to the limit for $\varepsilon \to 0$. This problem has a regular solution $v^\varepsilon \in C^2(-1, 1)$ for every positive ε, and passing to the limit for vanishing ε, we get

$$\lim_{\varepsilon \to 0} v^\varepsilon(x) = \overline{v}(x) = 1 - |x|, \tag{6}$$

which will be defined to be the weak solution of our problem. This is the so-called "vanishing viscosity method", and is the origin for the name "viscosity solution".

What is now considered as the usual definition of viscosity solution makes no longer any reference to a regularization and/or a limit. In what follows, $BUC(\Omega)$ will denote the space of bounded and uniformly continuous functions over the open set Ω.

Definition 2.1 *A function $v \in BUC(\Omega)$ is a viscosity solution of* (4) *if and only if, for any $\varphi \in C^1(\Omega)$, the following conditions hold:*

(i) at every local maximum *point $x_0 \in \Omega$ for $v - \varphi$,*

$$H(x_0, v(x_0), D\varphi(x_0)) \le 0$$

(ii) at every local minimum *point $x_0 \in \Omega$ for $v - \varphi$,*

$$H(x_0, v(x_0), D\varphi(x_0)) \ge 0$$

We say that v *is a viscosity sub(super)-solution* if *(i)* (resp. *(ii)*) is satisfied. Resuming the previous example, it can be proved that $\overline{v}$ is the unique solution according to this definition.

For solutions defined in this form, some good properties may be proved. First, if v is a classical C^1 solution (i.e., it satisfies the equation pointwise), then it is also a viscosity solution. Vice versa, if v is a regular viscosity solution, then it is also a classical solution. The viscosity solution v is the maximal sub-solution, and is the vanishing viscosity limit of the elliptic regularization.

The crucial point when dealing with viscosity solutions is to prove uniqueness. This difficulty is typically overcome by using a *comparison principle* (also called *maximum principle*), stating that if $u, v \in BUC(\Omega)$ are respectively a sub- and a super-solution for (4) and $u(x) \le v(x)$ for any $x \in \partial\Omega$, then $u(x) \le v(x)$ for any $x \in \Omega$. Indeed, let u and v be two viscosity solutions of (4). Clearly, they are both sub- and super-solutions, so that we have

$$u(x) \le v(x) \text{ for any } x \in \Omega$$

and, reverting the role of u and v, we also have

$$u(x) \ge v(x) \text{ for any } x \in \Omega$$

which implies $u(x) = v(x)$ in Ω.

A sufficient conditions for uniqueness may be given as follows. Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be continuous and ω be a modulus of continuity. Assume that, for any $x, y \in \Omega$, $u \in [-R, R]$ and $p \in \mathbb{R}^n$,

(A4) $|H(x,u,p) - H(y,u,p)| \leq \omega(|x-y|(1+|p|))Q_R(x,y,u,p)$,

with $Q_R(x,y,u,p) = \max\left(\varphi(H(x,u,p)), \varphi(H(y,u,p)\right)$. Then, we have the following

Theorem 2.2 *Let the assumptions (A1)–(A4) be satisfied. Then, the comparison principle holds for* (4)*, i.e., the viscosity solution is unique.*

We mention that in general *boundary conditions* should be stated in a suitable weak sense. We refer to [Ba98] for more details and for other types of boundary conditions (e.g., Neumann and state constraints).

For reader's convenience, we add the definition of viscosity solution, adapted for the evolutive case.

Definition 2.3 *(Evolutive case) A function $v \in BUC(\Omega \times (0,T))$ is a viscosity solution of*

$$v_t + H(x, v, Dv) = 0$$

in $\Omega \times (0,T)$ if and only if, for any $\varphi \in C^1(\Omega \times (0,T))$ the following conditions hold:

(i) at every local maximum *point $(x_0, t_0) \in \Omega \times (0,T)$ for $u - \varphi$*

$$\varphi_t(x_0,t_0) + (H(x_0, v(x_0,t_0), D\varphi(x_0,t_0)) \leq 0;$$

(ii) at every local minimum *point $(x_0, t_0) \in \Omega \times (0,T)$ for $u - \varphi$*

$$\varphi_t(x_0,t_0) + (H(x_0, u(x_0,t_0), D\varphi(x_0,t_0)) \geq 0.$$

In some special cases, a representation formula for the viscosity solution can be constructed. This is the case for the problem

$$\begin{cases} v_t + H(Dv) = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\ v(x,0) = v_0(x) & x \in \mathbb{R}^d \end{cases} \tag{7}$$

where the Hamiltonian $H : \mathbb{R}^d \to \mathbb{R}$ is continuous and convex. Assuming that H is also coercive, i.e.,

$$\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty,$$

the *Legendre–Fenchel conjugate* of H may be defined, for $p \in \mathbb{R}^d$, as

$$H^*(p) := \sup_{q \in \mathbb{R}^n} \{p \cdot q - H(q)\}.$$

In this particular case, the solution of (7) is given by the Hopf–Lax representation formula as

$$v(x,t) = \inf_{a \in \mathbb{R}^d} \left\{ v_0(x - ta) + tH^*(a) \right\}. \tag{8}$$

We will see later that this formula can also be used for numerical purposes.

Last, we examine the link between entropy solutions and viscosity solutions. This link is exploited to set up numerical methods originating from conservation laws, and, by a splitting argument, to derive multidimensional schemes [To06]). Consider the following two problems: the evolutive Hamilton–Jacobi equation

$$\begin{cases} v_t + H(v_x) = 0 & (x,t) \in \mathbb{R} \times (0,T), \\ v(x,0) = v_0(x) & x \in \mathbb{R}, \end{cases} \tag{9}$$

and the associated conservation law

$$\begin{cases} u_t + H(u)_x = 0 & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{10}$$

and define

$$v_0(x) := \int_{-\infty}^{x} u_0(\xi) d\xi.$$

It can be proved (see [CFN95]) that, if u is the entropy solution of (10), then

$$v(x,t) = \int_{-\infty}^{x} u(\xi,t) d\xi$$

is the unique viscosity solution of (9), and vice versa. The previous relationship also admits a multidimensional analogue. In fact, the viscosity solution of the Cauchy problem

$$\begin{cases} v_t + H(\nabla v) = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\ v(x,0) = v_0(x) & x \in \mathbb{R}^d, \end{cases} \tag{11}$$

is equivalent to the entropic solution of the system of conservation laws

$$\begin{cases} \boldsymbol{p}_t + \nabla H(\boldsymbol{p}) = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\ \boldsymbol{p}(x,0) = \boldsymbol{p}_0(x) = \nabla v_0(x) & x \in \mathbb{R}^d \end{cases} \tag{12}$$

where $\boldsymbol{p} := \nabla u$ (see [JX98] for a sketch of the proof).

2.1 Convergence results

To state the main convergence results for the approximation of viscosity solutions, we carry out the discretization in the usual difference scheme framework. Time is discretized with a (fixed) time step Δt, so that $t_k = k\Delta t$; space is discretized with a fixed space step Δx. A generic node will be denoted by $x_j = j\Delta x$, $j \in \mathbb{Z}^d$. We also define $\Delta = (\Delta x, \Delta t)$. In some cases, more general options can be considered, e.g., variable time steps and/or unstructured grids, but we will not treat such situations in detail.

We write the scheme in compact form as

$$V^{n+1} = S(\Delta; V^n) \tag{13}$$

where S may be defined in terms of its components S_j, $j \in \mathbb{Z}^d$. We denote by v_j^n the desired approximation of $v(x_j, t_n)$, by V^n the set of nodal values for the numerical solution at time t_n, by U (respectively, $U(t)$) that for the exact solution $v(x)$ (resp. $v(x,t)$). We also denote by W and Φ (resp. $W(t)$ and $\Phi(t)$) the sets of nodal values of generic functions $w(x)$ and $\phi(x)$ (resp. $w(x,t)$ and $\phi(x,t)$). In general, we will refer to the set of nodal values as to a (possibly infinite) vector.

In this section we collect two key results, which make use of monotonicity as a stability assumption. The *Crandall–Lions theorem* is inspired by the convergence result of monotone conservative schemes for conservation laws, and assumes that the scheme structure parallels that of conservative schemes. The

Barles–Souganidis theorem is suitable for more general situations, including second-order HJ equations, provided a comparison principle holds, and does not assume any particular structure for the scheme.

We present the result of Crandall–Lions in two space dimensions, the extension to an arbitrary number of dimensions being straightforward. Let us explicitly write the evolutive HJ equation as:

$$v_t + H(v_{x_1}, v_{x_2}) = 0. \tag{14}$$

We define an approximation of the partial derivative v_{x_i} at the point x_j by the right (partial) incremental ratio, that is

$$D_{i,j}[V] = \frac{v_{j+e_i} - v_j}{\Delta x}, \qquad i = 1, 2.$$

In agreement with the definition of schemes in *conservative form* for conservation laws, we define here the class of schemes in *differenced form.*

Definition 2.4 *A scheme S is said to be* in differenced form *if it has the form*

$$v_j^{n+1} = S_j(V^n) := v_j^n - \Delta t \mathcal{H}\left(D_{1,j-p}[V^n], \ldots, D_{1,j+q}[V^n]; D_{2,j-p}[V^n], \ldots, D_{2,j+q}[V^n]\right), \tag{15}$$

for two multiindices p and q with positive components, and for a Lipschitz continuous function $\mathcal{H}$ (called the numerical Hamiltonian*).*

In practice, (15) defines schemes where the dependence on V^n appears only through its finite differences, computed on a rectangular stencil of points around the node x_j. The differenced form of a scheme lends itself to an easier formulation of the consistency condition, which is given in the following definition.

Definition 2.5 *A scheme in differenced form is* consistent *if, for any $a, b \in \mathbb{R}$,*

$$\mathcal{H}(a, \ldots, a; b, \ldots, b) = H(a, b). \tag{16}$$

On the other hand, a *monotone scheme* is defined as follows.

Definition 2.6 *The scheme S is said to be* monotone *if*

$$S(\Delta; V) - S(\Delta; W) \geq 0 \tag{17}$$

for any pair of vectors V and W such that $V - W \geq 0$, this inequality to be intended component by component.

In the nonlinear case, we expect that monotonicity may or may not hold depending on the solution propagation speed, which is related to the Lipschitz constant of v_0. We will say that the scheme is *monotone on* $[-R, R]$ if Definition 2.6 is satisfied for any V and W such that $|D_{i,j}[V]|, |D_{i,j}[W]| \leq R$.

We can now state the Crandall–Lions convergence theorem (see [CL84] for the proof).

Theorem 2.7 *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be continuous, the initial condition v_0 be bounded and Lipschitz continuous (with Lipschitz constant L) on $\mathbb{R}^2$, and $v_j^0 = v_0(x_j)$. Let the scheme* (15) *be monotone on $[-(L+1), L+1]$ and consistent, for a locally Lipschitz continuous numerical Hamiltonian $\mathcal{H}$. Then, there exists a constant C such that, for any $n \leq T/\Delta t$,*

$$\left|v_j^n - v(x_j, t_n)\right| \leq C \Delta t^{1/2} \tag{18}$$

for $\Delta t \to 0$, $\Delta x = \lambda \Delta t$.

While it still requires monotonicity, the *Barles–Souganidis convergence theorem* [BaS91] gives a more abstract and general framework for convergence of schemes, including the possibility of treating second-order, degenerate and singular equations. Roughly speaking, this theory states that any monotone, stable and consistent scheme converges to the exact solution *provided there exists a comparison principle for the limiting equation.* The Cauchy problem under consideration is:

$$\begin{cases} v_t + H(x, v, Dv) = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\ v(x,0) = v_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{19}$$

The function $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a continuous Hamiltonian for all $w \in \mathbb{R}$, $x, p \in \mathbb{R}^d$. We assume that a *comparison principle* holds true for (19).

Consider a scheme in the general form (13). We assume the following *generalized consistency condition*:

Definition 2.8 *Let $\Delta_m = (\Delta x_m, \Delta t_m)$ be a generic sequence of discretization parameters, (x_{j_m}, t_{n_m}) be a generic sequence of nodes in the space–time grid such that, for $m \to \infty$,*

$$(\Delta x_m, \Delta t_m) \to 0 \quad \text{and} \quad (x_{j_m}, t_{n_m}) \to (x,t). \tag{20}$$

Let $\phi \in C^\infty(\mathbb{R}^d \times (0,T])$. Then, the scheme S is said to be consistent *if*

$$\liminf_{m\to\infty} \frac{\phi(x_{j_m}, t_{n_m}) - S_{j_m}(\Delta_m; \Phi(t_{n_m-1}))}{\Delta t_m} \geq \phi_t(x,t) + \underline{H}(x, \phi(x,t), D\phi(x,t)), \tag{21}$$

$$\limsup_{m\to\infty} \frac{\phi(x_{j_m}, t_{n_m}) - S_{j_m}(\Delta_m; \Phi(t_{n_m-1}))}{\Delta t_m} \leq \phi_t(x,t) + \overline{H}(x, \phi(x,t), D\phi(x,t)), \tag{22}$$

where $\underline{H}$ and $\overline{H}$ denote respectively the lower and upper semicontinuous envelopes of H. In (21)–(22), the index of the sequence is m, while j_m and n_m denote the corresponding node indices with respect to the mth space-time grid; we also recall that Φ or $\Phi(t)$ denote the vector of node values for respectively $\phi(x)$ and $\phi(x,t)$. Note that, if a scheme is consistent in the usual sense, it also satisfies (21)–(22).

The standard definition of monotonicity is replaced by the following *generalized monotonicity* assumption.

Definition 2.9 *Let $(\Delta x_m, \Delta t_m)$ and (x_{j_m}, t_{n_m}) be generic sequences satisfying* (20)*. Then, the scheme S is said to be* monotone *(in the generalized sense) if it satisfies the following conditions:*

$$\text{if} \quad v_{j_m} \leq \phi_{j_m} \quad \text{then} \quad S_{j_m}(\Delta_m; V) \leq S_{j_m}(\Delta_m; \Phi) + o(\Delta t_m) \tag{23}$$

$$\text{if} \quad \phi_{j_m} \leq v_{j_m} \quad \text{then} \quad S_{j_m}(\Delta_m; \Phi) \leq S_{j_m}(\Delta_m; V) + o(\Delta t_m). \tag{24}$$

for any smooth function $\phi(x)$.

Also in this case, if a scheme is monotone in the sense of Definition 2.6, then it also satisfies (23)–(24).

Given a numerical solutions V^n, we define its piecewise constant (in time) interpolation $v^{\Delta t}$ as:

$$v^{\Delta t}(x,t) = \begin{cases} I[V^n](x) & \text{if } t \in [t_n, t_{n+1}), \\ v_0(x) & \text{if } t \in [0, \Delta t), \end{cases}$$

where $I[V^n](x)$ denotes an interpolation of the node values in V, computed at x. We remark that whenever an interpolation operator is used, as is the case in the definition of $v^{\Delta t}$ or in the semi-Lagrangian approach, the interpolation operator has to satisfy some monotonicity (or relaxed monotonicity) properties to obtain a monotone scheme.

We can now state the extended version of the convergence result given in [BaS91]:

Theorem 2.10 *Let $v(x,t)$ be the unique viscosity solution of* (19)*. Assume that* (21)*–*(22) *and* (23)*–*(24) *hold. Assume in addition that the family $v^{\Delta t}$ is uniformly bounded in L^∞. Then, $v^{\Delta t}(x,t) \to v(x,t)$ locally uniformly on $\mathbb{R}^d \times [0,T]$ as $\Delta \to 0$.*

3 Evolutive problems

To introduce the schemes for time-dependent Hamilton–Jacobi equations, we refer to the basic problem

$$\begin{cases} v_t(x,t) + H(v_x(x,t)) = 0, & (x,t) \in \mathbb{R} \times [0,T] \\ v(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{25}$$

We will make the standing assumption that H is convex, and that there exists $\alpha_0 \in \mathbb{R}$ such that

$$\begin{cases} H'(\alpha) \le 0 & \text{if } \alpha \le \alpha_0, \\ H'(\alpha) \ge 0 & \text{if } \alpha \ge \alpha_0. \end{cases} \tag{26}$$

We also define:

$$M_{H'}(L) = \max_{[-L,L]} |H'| \tag{27}$$

i.e., the maximum propagation speed of a solution with Lipschitz constant L.

3.1 Monotone schemes in differenced form

The construction outlined will follow the guidelines of [CL84]. Note that, by construction, schemes in differenced form are necessarily invariant for the addition of constants. Therefore, l^∞ stability follows from monotonicity.

3.1.1 Upwind discretization

In adapting the upwind scheme to the nonlinear case, it should be taken into consideration that $H'(v_x)$ is the propagation speed of the solution. While it is perfectly clear how to construct an upwind scheme for a speed of constant sign, care should be taken at points where the speed changes sign, in order to obtain a monotone scheme.

Construction of the scheme The differenced form of the upwind scheme is

$$v_j^{n+1} = v_j^n - \Delta t \mathcal{H}^{Up} \left(D_{j-1}[V^n], D_j[V^n] \right), \tag{28}$$

where the numerical Hamiltonian $\mathcal{H}^{Up}$ is typically defined by

$$\mathcal{H}^{Up}(\alpha,\beta) = \begin{cases} H(\alpha) & \text{if } \alpha, \beta \ge \alpha_0, \\ H(\beta) + H(\alpha) - H(\alpha_0) & \text{if } \alpha \ge \alpha_0, \beta \le \alpha_0, \\ H(\alpha_0) & \text{if } \alpha \le \alpha_0, \beta \ge \alpha_0, \\ H(\beta) & \text{if } \alpha, \beta \le \alpha_0. \end{cases} \tag{29}$$

Note that the situation in which speed changes sign is subject to a different handling, depending on whether characteristics converge or diverge.

Consistency Since the scheme is in differenced form, it actually suffices to apply the consistency condition in Definition 2.5. If $\alpha = \beta = a$, then the numerical Hamiltonian (29) satisfies

$$\mathcal{H}^{Up}(a,a) = H(a). \tag{30}$$

Note that, in (29), the second and third cases only occur if $a = \alpha_0$.

Monotonicity The partial derivative of the jth component of the scheme is written as

$$\frac{\partial}{\partial v_i} S_j^{Up}(\Delta; V) = \delta_{ij} - \Delta t \left[\frac{\partial \mathcal{H}^{Up}}{\partial \alpha} \frac{\partial D_{j-1}[V]}{\partial v_i} + \frac{\partial \mathcal{H}^{Up}}{\partial \beta} \frac{\partial D_j[V]}{\partial v_i} \right] \tag{31}$$

where α and β are the dummy variables used in the definition (29), and δ_{ij} is the Kronecker symbol. A simple computation shows that, if $i \neq j$, then the monotonicity condition

$$\frac{\partial}{\partial v_i} S_j^{Up}(\Delta; V) \geq 0 \tag{32}$$

is always satisfied, whereas, for $i = j$, it is satisfied provided

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{2M_{H'}(L_V)}, \tag{33}$$

where L_V denotes the Lipschitz constant of the sequence V. In contrast to the linear case, this condition is more restrictive than the CFL condition.

3.1.2 Central discretization

Rather than using more general forms of the scheme, we will restrict here to the particular form that directly generalizes the linear Lax–Friedrichs (LF) scheme.

Construction of the scheme The simplest way to recast Lax–Friedrichs scheme for the HJ equation is to define it in the form

$$v_j^{n+1} = \frac{v_{j-1}^n + v_{j+1}^n}{2} - \Delta t H\left(D_j^c[V^n]\right), \tag{34}$$

where $D_j^c[V^n]$ is the centered difference at x_j defined by

$$D_j^c[V^n] = \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = \frac{D_{j-1}[V^n] + D_j[V^n]}{2}. \tag{35}$$

This definition of the LF scheme completely parallels the linear case, and is also suitable to be treated in the framework of the Crandall–Lions theorem. Indeed, keeping in mind that

$$\frac{v_{j-1}^n + v_{j+1}^n}{2} = v_j^n + \frac{\Delta x}{2}\left(D_j[V^n] - D_{j-1}[V^n]\right),$$

(34) can be written in the differenced form

$$v_j^{n+1} = v_j^n - \Delta t \mathcal{H}^{LF}\left(D_{j-1}[V^n], D_j[V^n]\right) \tag{36}$$

by setting

$$\mathcal{H}^{LF}(\alpha, \beta) = H\left(\frac{\alpha + \beta}{2}\right) - \frac{\Delta x}{2\Delta t}(\beta - \alpha).$$

Consistency The Lax–Friedrichs scheme (34) satisfies condition (16), and in fact

$$\mathcal{H}^{LF}(a, a) = H\left(\frac{a + a}{2}\right) = H(a). \tag{37}$$

Consistency is therefore satisfied.

Monotonicity In examining monotonicity, it is convenient to refer to the LF scheme in the form (34). Clearly, the jth component $S_j^{LF}(\Delta; V)$ depends only on the values $v_{j\pm1}$, so that

$$\frac{\partial}{\partial v_i} S_j^{LF}(\Delta; V) = 0 \qquad (i \neq j \pm 1)$$

On the other hand, if $i = j \pm 1$, we have

$$\frac{\partial}{\partial v_{j\pm1}} S_j^{LF}(\Delta; V) = \frac{1}{2} \mp \frac{\Delta t}{2\Delta x} H'\left(D_j^c[V]\right).$$

Therefore, if L_V is the Lipschitz constant of the sequence V, the scheme is monotone provided

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{M_{H'}(L_V)}. \tag{38}$$

3.2 Semi–Lagrangian discretization

We analyze here the monotone version of the Semi-Lagrangian (SL) scheme, that is, the version obtained with $\mathbb{P}_1$ interpolation (see [FF14]).

Construction of the scheme In the SL discretization of the HJ equation, what is really discretized is the representation formula for the solution. In the case of convex HJ equations, the formula under consideration is the Hopf–Lax formula (8). Once rewritten in a single space dimension, and at a point (x_j, t_{n+1}) of the space–time grid, it reads:

$$\begin{aligned} u(x_j, t + \Delta t) &= \min_{a\in\mathbb{R}} \{\Delta t H^*(a) + u(x_j - a\Delta t, t)\} \\ &= \Delta t H^*(\bar{a}_j) + u(x_j - \bar{a}_j \Delta t, t), \end{aligned} \tag{39}$$

where $\bar{a}_j$ denotes the minimizer at the node x_j in (39). In the special case of (25), characteristics are straight lines, so that no special care should be taken about the accuracy of time discretization (we will comment on this later). The value $u(x_j - a\Delta t, t)$ should be reconstructed by a monotone space interpolation, for example in piecewise linear ($\mathbb{P}_1$) form. The resulting scheme is therefore:

$$\begin{cases} v_j^{n+1} = \min\limits_{\alpha\in\mathbb{R}} \{\Delta t H^*(\alpha) + I_1[V^n](x_j - \alpha\Delta t)\} \\ v_j^0 = u_0(x_j), \end{cases} \tag{40}$$

in which $I_1[V](x)$ denotes the $\mathbb{P}_1$-interpolate of the sequence V, computed at the point x.

Since the SL scheme is not in differenced form, the convergence analysis is carried out in the framework of the Barles–Souganidis theorem. Given the invariance of the $\mathbb{P}_1$ interpolation for the sum of constants, it suffices to check consistency and monotonicity.

Consistency The scheme is compared with the representation formula (8). Denoting by $\bar{\alpha}_j$ the minimizer at the node x_j in (40), we define

$$S_j^{SL}(\Delta; V) = \Delta t H^*(\bar{\alpha}_j) + I_1[V](x_j - \bar{\alpha}_j \Delta t).$$

Let u be a smooth solution of (25), and U the sequence of its node values. First, recall that

$$|u(x) - I_1[U](x)| \leq C\Delta x^2 \min_{m\in\mathbb{Z}} \frac{x - x_m}{\Delta x}. \tag{41}$$

Writing now $u(x_j, t+\Delta t)$ by means of (39), and using (41), via two unilateral estimates the consistency error is bounded as

$$\begin{aligned} \left|L_j^{SL}(\Delta; t, U(t))\right| &= \left|\frac{1}{\Delta t}\left[u(x_j, t+\Delta t) - S_j^{SL}(\Delta; t, U(t))\right]\right| \\ &\le C \min\left(\Delta x, \frac{\Delta x^2}{\Delta t}\right), \end{aligned}$$

which implies consistency for any $\Delta x/\Delta t$ relationship. Note that this estimate would suggest that the scheme achieves its best result when going to the final time in a single time step. In practice, in situation in which characteristics are not straight lines, errors in characteristics tracking should also be taken into consideration.

Monotonicity First, note that the SL scheme is invariant for the addition of constants since, for a Lagrange interpolation of any order, $I[V+c](x) \equiv I[V](x) + c$.

To check that the SL scheme is monotone, consider two sequences V and W such that $V - W \ge 0$ componentwise. A simple computation gives:

$$S_j^{SL}(\Delta; V) - S_j^{SL}(\Delta; W) \ge I_1[V](x_j - \bar{\alpha}_j \Delta t) - I_1[W](x_j - \bar{\alpha}_j \Delta t),$$

where $\bar{\alpha}_j$ is the minimizer obtained for the sequenca V. Since $\mathbb{P}_1$ interpolation is monotone itself (that is, $I_1[V] - I_1[W] \ge 0$), we get

$$S_j^{SL}(\Delta; V) - S_j^{SL}(\Delta; W) \ge 0. \tag{42}$$

Last, since the scheme is invariant for the addition of constants, L^∞ stability is implied by monotonicity.

3.3 Convergence

For the monotone approximations outlined above, it is possible to prove explicit a priori error estimates. More precisely:

- For monotone schemes in differenced form, the Crandall–Lions theorem applies, providing a theoretical convergence rate of order 1/2 under a linear CFL condition. We have therefore [CL84] the following

  Theorem 3.1 *Let H satisfy the basic assumptions, $u_0 \in W^{1,\infty}$, u be the solution of (25) with L as its Lipschitz constant and v_j^n be defined by (28) (respectively, (36)) with $v_j^0 = v_0(x_j)$. Then, for any $j \in \mathbb{Z}$ and $n \in [1, T/\Delta t]$, and for some positive constant C,*

  $$\left|v_j^n - v(x_j, t_n)\right| \le C \Delta t^{1/2} \tag{43}$$

  as $\Delta t \to 0$, with $2M_{H'}(L+1)\Delta t \le \Delta x$ (resp., $M_{H'}(L+1)\Delta t \le \Delta x$).

- For monotone Semi-Lagrangian schemes, convergence by Barles–Souganidis theorem would not provide an explicit convergence rate. An ad hoc convergence proof leads [FF14] to the following

  Theorem 3.2 *Let H satisfy the basic assumptions, $u_0 \in W^{1,\infty}$, u be the solution of (25) with L as its Lipschitz constant and v_j^n be defined by (40) with $v_j^0 = v_0(x_j)$. Then, for any $j \in \mathbb{Z}$ and $n \in [1, T/\Delta t]$, and for some positive constant C,*

  $$\left|v_j^n - v(x_j, t_n)\right| \le C \frac{\Delta x}{\Delta t} \tag{44}$$

  as $\Delta x, \Delta t \to 0$, with $\Delta x = o(\Delta t)$.

We point out that, as soon as characteristics are no longer straight lines, this convergence estimate becomes

$$\left|v_j^n - v(x_j, t_n)\right| \le C\left(\Delta t^\gamma + \frac{\Delta x}{\Delta t}\right)$$

with γ denoting the order of approximation of characteristics (see [FF94, FF14]).

Last, it is observed in the numerical practice that, if the solution is uniformly semiconcave in $[0,T]$, then the actual convergence rate improves. In fact, in this case, singularities of the gradient are generated by characteristics coming from regular regions of the solution, and this causes the propagation of smaller numerical errors, as shown by the following numerical exampe.

A numerical example We present a simple numerical example, using the one-dimensional model problem

$$\begin{cases} u_t(x,t) + \frac{1}{2}|u_x(x,t)|^2 = 0 & (x,t) \in (0,1)\times(0,T) \\ u(x,0) = u_0(x) \end{cases} \tag{45}$$

with $T = 0.05$ and two different Lipschitz continuous initial conditions u_0 with bounded support. The first is:

$$u_0(x) = \max(1 - 16(x-0.25)^2, 0), \tag{46}$$

whereas the second, obtained by a simple change of sign, is also semiconcave:

$$u_0(x) = -\max(1 - 16(x-0.25)^2, 0). \tag{47}$$

Using the initial condition (46), the solution eventually develops a singularity with nonempty superdifferential. After the onset of the singularity, the exact solution reads

$$u(x,t) = \begin{cases} \frac{\left(\left|x-\frac{1}{2}\right|-\frac{1}{4}\right)^2}{2t} & \text{if } \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{else} \end{cases}$$

On the other hand, using the initial condition (47), the solution has the expression

$$u(x,t) = \min\left(\frac{\left|x-\frac{1}{2}\right|^2}{2t+\frac{1}{16}} - 1, 0\right),$$

and is uniformly semiconcave for $t > 0$. The test is performed with Upwind, Lax–Friedrichs and Semi-Lagrangian schemes. In this case, the refinement has been carried out with $\Delta t = \Delta x/40$ for Upwind scheme, $\Delta t = \Delta x/20$ for Lax–Friedrichs scheme and $\Delta t = 0.01$ (fixed) for Semi-Lagrangian scheme. Figure 1 compares exact with numerical solutions for the first initial condition, whereas Table 1 shows numerical errors in the ∞-norm, showing that the theoretical convergence rates are optimal in lack of uniform semiconcavity, but improve in the semiconcave case. Among the schemes in differenced form, the LF scheme has an apparently higher numerical viscosity, but similar convergence rate with respect to the upwind scheme.

4 Stationary problems

In adapting the various schemes to stationary HJ equations, we refer to the stationary model which in some sense parallels (25), that is

$$\lambda v(x) + H(v_x(x)) = g(x) \qquad x \in \mathbb{R}, \tag{48}$$

for $\lambda > 0$. A typical setting to discretize (48) is to consider time-marching schemes, either in differenced form or of Semi-Lagrangian type. This amounts to look for fixed points of the numerical scheme.

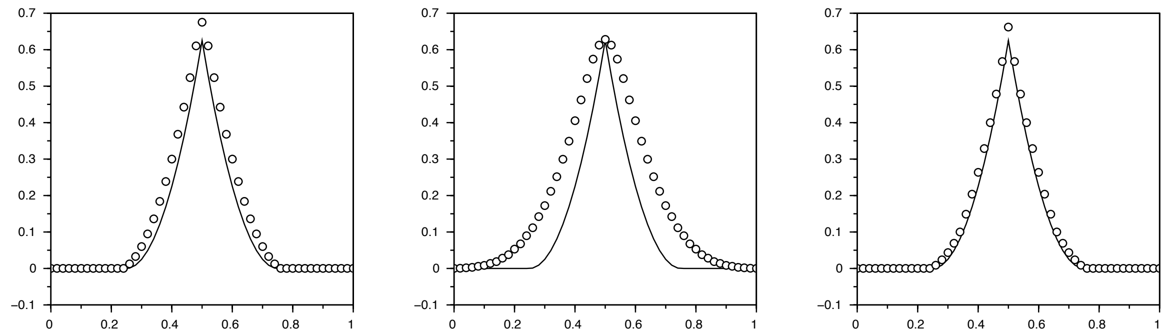

Figure 1: Numerical results for problem (45)–(46), obtained via Upwind (left), Lax–Friedrichs (center) and Semi-Lagrangian (right) schemes, 50 nodes.

	$W^{1,\infty}$ initial condition			Semiconcave initial condition		
n_n	Upwind	LF	SL	Upwind	LF	SL
25	$1.13 \cdot 10^{-1}$	$2.84 \cdot 10^{-1}$	$8.82 \cdot 10^{-2}$	$6.42 \cdot 10^{-2}$	$3.64 \cdot 10^{-1}$	$2.11 \cdot 10^{-2}$
50	$1.01 \cdot 10^{-1}$	$2.51 \cdot 10^{-1}$	$3.53 \cdot 10^{-2}$	$3.58 \cdot 10^{-2}$	$1.97 \cdot 10^{-1}$	$5.02 \cdot 10^{-3}$
100	$6.62 \cdot 10^{-2}$	$1.89 \cdot 10^{-1}$	$1.81 \cdot 10^{-2}$	$1.92 \cdot 10^{-2}$	$9.76 \cdot 10^{-2}$	$1.24 \cdot 10^{-3}$
200	$4.08 \cdot 10^{-2}$	$1.27 \cdot 10^{-1}$	$8.26 \cdot 10^{-3}$	$9.97 \cdot 10^{-3}$	$4.83 \cdot 10^{-2}$	$3.07 \cdot 10^{-4}$
400	$2.42 \cdot 10^{-2}$	$8.0 \cdot 10^{-2}$	$3.94 \cdot 10^{-3}$	$5.08 \cdot 10^{-3}$	$2.4 \cdot 10^{-2}$	$7.63 \cdot 10^{-5}$
rate	0.56	0.46	1.12	0.91	0.98	2.03

Table 1: Errors in the ∞-norm for problem (45), Upwind, Lax–Friedrichs and Semi-Lagrangian schemes.

Discretization in differenced form In differenced form time-marching schemes, the schemes are applied to the evolutive equation

$$v_t + \lambda v + H(v_x) = g,$$

whose solution converges to a regime state satisfying (48). Keeping the scheme in its most general form, and adding the zeroth order term, we have

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \lambda v_j^n + \mathcal{H}\left(D_{j-p}[V^n], \ldots, D_{j+q}[V^n]\right) = g(x_j),$$

which is clearly a consistent scheme. Hence, replacing the time index with an iteration index k, we obtain

$$v_j^{(k+1)} = (1 - \lambda\Delta t)v_j^{(k)} - \Delta t\mathcal{H}\left(D_{j-p}\left[V^{(k)}\right], \ldots, D_{j+q}\left[V^{(k)}\right]\right) + \Delta t g(x_j). \tag{49}$$

Semi-Lagrangian discretization In the case of Semi-Lagrangian discretization, we apply a generalized form of the Hopf–Lax formula, which applies to the solution of (48), in the form

$$v(x) = \inf_{\alpha\in\mathcal{A}}\left\{\int_0^\tau \left[g(y_x(s;\alpha)) + H^*(\alpha(s))\right] e^{-\lambda s} ds + e^{-\lambda\tau} v(y_x(\tau;\alpha))\right\},$$

where $\mathcal{A}$ is the set of measurable functions mapping $[0, +\infty)$ into $\mathbb{R}^M$, and $y_x(s;\alpha)$ satisfies

$$\begin{cases} \dot{y}_x(s;\alpha) = \alpha(s) \\ y_x(0;\alpha) = x. \end{cases}$$

Then, a SL type discretization can be written in iterative form as:

$$v_j^{(k+1)} = \min_{a\in\mathbb{R}}\left\{\left(1 - e^{-\lambda\Delta t}\right) H^*(a) + e^{-\lambda\Delta t} I_1\left[V^{(k)}\right](x_j - a\Delta t)\right\} + \Delta t g(x_j). \tag{50}$$

Let us conclude this section mentioning that a different approximation scheme for stationary Hamilton-Jacobi-Bellman equations also based on a control interpretation has been proposed in [BR06].

Convergence and a priori error estimates It can be easily proved that the right-hand side of both (49) and (50) is a contraction in l^∞ (this follows from monotonicity, as shown in [BFFKZ14]). Therefore, the iteration converges towards a unique fixed point. Although relatively inefficient (the contraction coefficient is $L_S = 1 - O(\Delta t)$), this procedure is simple and robust.

With some further work, it could be proved that the adaptation of monotone schemes to the stationary forms (49)–(50) satisfies both Barles–Souganidis and (in the differenced form) Crandall–Lions theory, with the due changes necessary to treat the stationary case. In both cases, the convergence estimates parallel the estimates of the time-dependent case, i.e.,

$$|v_j - v(x_j)| \le C\Delta t^{1/2} \tag{51}$$

for the schemes in differenced form, and

$$|v_j - v(x_j)| \le C\left(\Delta t + \frac{\Delta x}{\Delta t}\right) \tag{52}$$

for the SL scheme.

5 High-order approximation methods

In this section, we sketch some basic ideas about high-order approximations for HJ equations. The topic is undergoing a fast development, and these notes are intended only as a general introduction.

5.1 Theoretical tools

Out of the framework of monotone schemes, the convergence theory for approximations of HJ equations becomes less classical, and no general recipe has been singled out yet. However, in the last years a couple of techniques have been proved to be viable devices in the convergence analysis of high-order numerical scheme. One is the relaxation of the monotonicity assumption to quasi-monotonicity, the other is semi-concave stability.

ε-monotonicity Despite being usually applied to strictly monotone schemes, the Barles–Souganidis theorem allows for some small (more precisely, $o(\Delta t)$) monotonicity defect. Within this margin, it is sometimes possible to prove convergence for quasi-monotone schemes. Notably, this technique has been applied to high-order SL schemes and to filtered schemes (for which the monotonicity defect can be set a priori). Applications of this framework are given in [AA00, FF14, BFFKZ14].

Lin–Tadmor theory In Lin–Tadmor convergence theory (which derives from the *Lip*′-stability theory for conservation laws, see [LT01]), a different concept of stability is singled out, i.e., the concept of *semi-concave stability*:

Definition 5.1 *A family of approximate solutions v^ϵ of* (7) *is said to be* semi-concave stable *if there exists a function $k(t) \in L^1([0,T])$ such that*

$$D^2 u^\epsilon(x,t) \le k(t) I \tag{53}$$

for $t \in [0,T]$.

More explicitly, condition (53) means that the matrix

$$D^2 u^\epsilon(x,t) - k(t) I$$

is negative semidefinite, that is (since we are dealing with symmetric matrices), that all eigenvalues of $D^2 u$ are bounded from above by $k(t)$. In practice, the semi-concave stability is replaced by a bound on the second directional incremental ratios in the form:

$$\frac{v^\Delta(x+\delta,t) - 2v^\Delta(x,t) + v^\Delta(x-\delta,t)}{|\delta|^2} \le k(t). \tag{54}$$

Here, the function $k(t) \in L^1([0,T])$ plays the same role as in the original definition, and δ is a vector whose norm should remain bounded away from zero, and more precisely

$$|\delta| \ge C\Delta x. \tag{55}$$

The core of the theory is an abstract result of convergence for perturbed semi-concave stable solutions.

Theorem 5.2 *Consider problem* (7) *for a semi-concave initial condition v_0 with compact support, and assume the family v^ϵ is semi-concave stable. Define the truncation error associated to v^ϵ as*

$$F(x,t) = v^\epsilon_t(x,t) + H(\nabla v^\epsilon(x,t)). \tag{56}$$

Then, for any $t \in [0,T]$,

$$\|v(t) - v^\epsilon(t)\|_{L^1(\mathbb{R}^d)} \le C_1 \|v_0 - v^\epsilon(0)\|_{L^1(\mathbb{R}^d)} + C_2 \|F\|_{L^1(\mathbb{R}^d \times [0,T])}. \tag{57}$$

The second ingredient of the Lin–Tadmor theory, i.e., the L^1 estimation of the truncation error, may be difficult in general, since it requires a reversed approach in measuring the truncation error, that is, by plugging the numerical solution into the exact equation. An easier expression can be derived for Godunov-type schemes, taking into account that in this case numerical errors are generated only by the projection step, and not by the evolution operator, which is in principle exact. A practical application of this theory to a Godunov-type scheme is presented in [LT01].

5.2 High order FD schemes

The basic strategy for constructing high-order finite difference methods has been first proposed in [OS91], and passes through the intermediate step of a semi-discrete scheme. In a second step, a TVD Runge–Kutta method is applied to the semi-discrete scheme, to obtain a fully discrete approximation.

The semi-discrete scheme is constructed using a monotone numerical Hamiltonian, in the form

$$\dot{v}_j = -\mathcal{H}\left(D_j^-[V], D_j^+[V]\right), \tag{58}$$

in which $\mathcal{H}(\cdot,\cdot)$ is increasing with respect to its first argument, and decreasing with respect to the second. In (58), $D_j^{\pm}[V]$ denote high order approximations of the right/left derivative at the node x_j, which replace in the numerical Hamiltonian the mere right/left incremental ratios. In the most extensively studied versions of the scheme, these estimates are usually obtained via non-oscillatory (ENO/WENO) techniques (see [OS91, BL03, JP00, KNP01, KP06]).

5.3 High order SL schemes

The Semi-Lagrangian scheme (40) is easily extended to a higher consistency rate by replacing the $\mathbb{P}_1$ space interpolation I_1 with an interpolation of higher accuracy [FF02, CFR05]. In more general cases, in which characteristics are not straight lines, a more accurate method of characteristics tracking is also desirable [FF94].

In some model cases (a single space dimension, no x-dependence of the Hamiltonian) convergence of high-order SL schemes, for both the evolutive and the stationary case, can be proved by showing their quasi-monotonicity. Here, a crucial role is played by the Lipschitz stability of the scheme, along with the inverse CFL condition $\Delta x = O(\Delta t^2)$ (see [Fe02, FF14, BFFKZ14])

5.4 Discontinuous Galerkin

The application to HJ equations of Discontinuous Galerkin (DG) methods uses the relationship with conservation laws. In fact, what is discretized in this case is the CL (or system of CLs, see (12) and [JX98]) associated to the HJ equation. Following [HS99], the approximate solution is constructed in the space

$$V_{\Delta x}^k = \left\{ w : w|_{I_j} \in \mathbb{P}_k(I_j) \right\}, \tag{59}$$

in which I_j denotes the j-th element of the computational domain, for example the interval

$$[x_{j-1/2}, x_{j+1/2}] = [x_j - \Delta x/2, x_j + \Delta x/2]$$

in a one-dimensional uniform grid. A DG scheme of order k for the equation (25) is defined by looking for the function $w \in V_{\Delta x}^k$ such that

$$\int_{I_j} w_{xt}\phi dx - \int_{I_j} H(w_x)\phi_x dx + \mathcal{H}_{j+1/2}\phi_{j+1/2}^- - \mathcal{H}_{j-1/2}\phi_{j-1/2}^+ = 0$$

for all j and all $\phi \in V_{\Delta x}^{k-1}$. In (59), the values $\mathcal{H}_{j-1/2}$ are defined by

$$\mathcal{H}_{j\pm 1/2} := \mathcal{H}\left(w_x(x_{j\pm 1/2}^-), w_x(x_{j\pm 1/2}^+)\right),$$

and $\mathcal{H}$ is as usual a monotone numerical Hamiltonian. Note that the outcome of the scheme is the derivative w_x (or, in the multi-dimensional case, all the partial derivatives). Suitable techniques allow to recover the approximate solution w from this information.

5.5 Filtered schemes

The general idea behind the construction of filtered scheme is to provide a clever coupling between a monotone and a high-order scheme. The lack of regularity may cause high-order schemes to introduce spurious oscillations, so the idea is to apply the high-order scheme only where the solution is regular enough, this being accomplished by a suitable "filter" function.

The construction of a filtered scheme needs three ingredients: a monotone scheme (denoted by S^M), a high-order scheme (denoted by S^{HO}) and a bounded (not necessarily smooth) *filter function*, $F : \mathbb{R} \to \mathbb{R}$. The filtered scheme S^F is then defined as

$$v_j^{n+1} = S_j^F(V^n) := S_j^M(V^n) + \epsilon \Delta t F\left(\frac{S_j^{HO}(V^n) - S_j^M(V^n)}{\epsilon \Delta t}\right), \tag{60}$$

where $\epsilon = \epsilon(\Delta) > 0$ is a parameter vanishing for $\Delta t, \Delta x \to 0$, whose choice controls the monotonicity defect of the filtered scheme (more hints on the choice of ϵ can be found in [FO13, BFS16]). A typical filter function is given by

$$F(x) = \mathrm{sign}(x)\max(1 - ||x| - 1|, 0) = \begin{cases} x & |x| \le 1. \\ 0 & |x| \ge 2. \\ -x+2 & 1 \le x \le 2. \\ -x-2 & -2 \le x \le -1. \end{cases}$$

This definition of the filter function blends the two schemes according to the ratio $\rho = (S^A - S^M)/(\Delta t \epsilon)$. If $|\rho| \le 1$, then $S^F = S^M + \Delta t \epsilon F(\rho) \equiv S^{HO}$, whereas if $|\rho| \ge 2$, then $F(\rho) = 0$ and $S^F \equiv S^M$, i.e., the scheme coincides with the monotone scheme. It can be shown that, for a suitable choice of ϵ, the filtered scheme converges to the viscosity solution. Moreover, a high-order consistency rate can be proved in the regular case, although globally the filtered scheme is not expected to have more than an $O(\sqrt{\Delta x})$ rate of convergence on Lipschitz continuous solutions.

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