

Construction of soliton solutions of the matrix modified Korteweg-de Vries equation

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Abstract. An explicit solution formula for the matrix modified KdV equation is presented, which comprises the solutions given in [7]. In fact, the solutions in [7] are part of a subclass studied in detail by the authors in a forthcoming publication. Here several solutions beyond this subclass are constructed and discussed with respect to qualitative properties.

Keywords: Matrix modified KdV equation, explicit solution formulas, soliton solutions.

1 Introduction

The present article is a sequel to [7], where a general approach to the solution theory of the matrix MKdV is outlined and certain solutions are explicitly constructed. Actually these solutions are part of a family for which a complete classification will be given in the forthcoming article [8].

Here the focus is on solutions beyond the setting of [7,8]. In a somewhat experimental spirit, we will examine ways to weaken the assumptions in [8], and initialize the study of some novel solution classes. The emphasis does not lie on completeness, but on a qualitative study of phenomena, discussed mainly for the first interesting cases.

The result to start from is a general solution formula presented in Theorem 1, building on work in [4,5]. Solution formulas of this kind have been studied for a quite a while, see [3,15,18,20] and the references therein. Here the use of Bäcklund techniques should also be mentioned [2,13,16,17]. Closely related formulas for scalar equations are known to generate very large solution classes, roughly speaking the solutions accessible by the standard inverse scattering method [1]. As the situation for matrix equations is much less transparent, the case studies made here are also meant as a step towards better understanding the range of our methods. Finally we mention some alternative approaches to matrix solutions in [10,12,14,19].

2 An explicit solution class of the $\mathbf{d} \times \mathbf{d}$ -matrix Korteweg-de Vries equation depending on matrix parameters and N -solitons

We start with stating an explicit solution class for the modified Korteweg-de Vries equation with values in the $\mathbf{d} \times \mathbf{d}$ -matrices,

$$V_t = V_{xxx} + 3\{V^2, V_x\}, \quad (1)$$

depending on matrix parameters.

Theorem 1. *For $N \in \mathbb{N}$, let k_1, \dots, k_N be complex numbers such that $k_i + k_j \neq 0$ for all i, j , and let B_1, \dots, B_N be arbitrary $\mathbf{d} \times \mathbf{d}$ -matrices.*

Define the $N\mathbf{d} \times N\mathbf{d}$ -matrix function $L = L(x, t)$ as block matrix $L = (L_{ij})_{i,j=1}^N$ with the $\mathbf{d} \times \mathbf{d}$ -blocks

$$L_{ij} = \frac{\ell_i}{k_i + k_j} B_j,$$

where $\ell_i = \ell_i(x, t) = \exp(k_i x + k_i^3 t)$.

Then

$$V = (B_1 \ B_2 \ \dots \ B_N) \left(I_{N\mathbf{d}} + L^2 \right)^{-1} \begin{pmatrix} \ell_1 I_{\mathbf{d}} \\ \vdots \\ \ell_N I_{\mathbf{d}} \end{pmatrix}$$

is a solution of the matrix modified KdV equation (1) with values in the $\mathbf{d} \times \mathbf{d}$ -matrices on every domain Ω on which $\det(I_{N\mathbf{d}} + L^2) \neq 0$.

The proof of Theorem 1, which is based on results in [4,5], is provided in [8]. Here we focus on applications and discuss some interesting examples.

Remark 1.

- a) In [5] it is shown that the solution class for the matrix KdV equation which corresponds to the class in Theorem 1 comprises the N -soliton solutions as derived by the inverse scattering method in [12].
- b) In contrast to (1), the non-commutative mKdV in the form

$$V_t = V_{xxx} + 3 (VV^T V_x + V_x V^T V) = 0,$$

(as for example derived from reduction of the non-commutative AKNS system) admits also non-square matrix interpretation. We refer to [20] for a fairly complete asymptotics of 2-solitons in the vector case.

3 Explicit solutions

Motivated by [7], the subclass of solutions arising from choosing $k_1, \dots, k_N \in \mathbb{R}$ and $B_1 = \dots = B_N =: B$ (up to a common real multiple) where both B and its Jordan canonical form are real, is discussed thoroughly in [8], the main result being a complete classification of this subclass up to a possible (common) change of coordinates. It should be stressed that all solutions in [7] belong to this subclass.

In the present section a variety of solutions beyond this case are presented.

3.1 The matrix parameter B does not have a real Jordan form

A prototypical example for a real matrix without real Jordan form in the case $d = 2$ are rotations. Consider

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

the rotation with the angle $\frac{\pi}{4}$, and let $N = 1$.

The corresponding solution according to Theorem 1 is

$$\begin{aligned} V &= B \left(I_2 + \left(\frac{1}{2k} \ell B \right)^2 \right)^{-1} \ell I_2 \\ &= 2k g B \left(I_2 + (gB)^2 \right)^{-1}, \end{aligned}$$

where $g = \ell/(2k)$. Since B^2 is the clockwise rotation by $\pi/2$, i.e. $B^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this is very easily computed explicitly, giving

$$\begin{aligned} V &= \sqrt{2}k g \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & g^2 \\ -g^2 & 1 \end{pmatrix}^{-1} = \sqrt{2}k \frac{g}{1+g^4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -g^2 \\ g^2 & 1 \end{pmatrix} \\ &= \sqrt{2}k \frac{g}{1+g^4} \begin{pmatrix} 1+g^2 & 1-g^2 \\ -(1-g^2) & 1+g^2 \end{pmatrix} \end{aligned}$$

Note that this solution is regular and moves, without changing shape, with velocity constant $-k^2$.

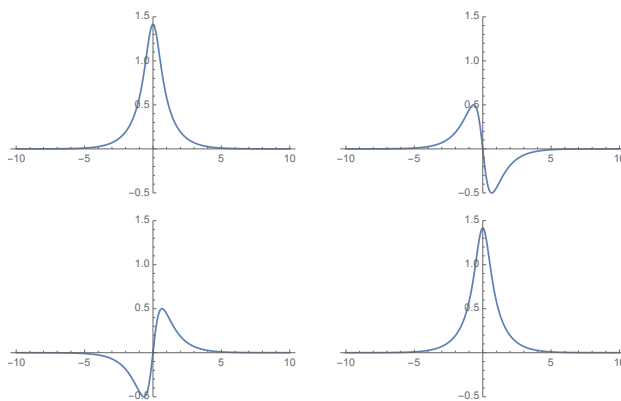


Fig. 1. Snapshot of the solution in Subsection 3.1 for $k = 1$ at $t = 0$.

3.2 Unequal matrix parameters

Next we consider the case $N = 2$ with matrix parameters B_1, B_2 such that $B_1 \neq cB_2$ (for all $c \in \mathbb{R}$). In this case we get from Theorem 1

$$V = (B_1 \ B_2) \left(I_{2d} + M^2 \right)^{-1} \begin{pmatrix} \ell_1 I_d \\ \ell_2 I_d \end{pmatrix} \text{ with } M = \begin{pmatrix} \frac{1}{2k_1} \ell_1 B_1 & \frac{1}{k_1+k_2} \ell_1 B_2 \\ \frac{1}{k_1+k_2} \ell_2 B_1 & \frac{1}{2k_2} \ell_2 B_2 \end{pmatrix}.$$

Let us first focus on the case that

$$B_1 B_2 = 0 = B_2 B_1.$$

Using this assumption, it is straightforward to verify that

$$(B_1 \ B_2) M = (B_1 \ B_2) R, \quad (2)$$

$$M^2 = MR, \quad (3)$$

where

$$R = \begin{pmatrix} \frac{1}{2k_1} \ell_1 B_1 & 0 \\ 0 & \frac{1}{2k_2} \ell_2 B_2 \end{pmatrix}.$$

From (3) we get $M^3 R = M M^2 R = M(MR)R = M^2 R^2$. Thus, $M^2(I_{2d} + R^2) = (I_{2d} + M^2)MR$, showing $(I_{2d} + M^2)^{-1}M^2 = MR(I_{2d} + R^2)^{-1}$, and hence

$$(I_{2d} + M^2)^{-1} = I_{2d} - (I_{2d} + M^2)^{-1}M^2 = I_{2d} - MR(I_{2d} + R^2)^{-1}.$$

Together with (2), this implies

$$\begin{aligned} (B_1 \ B_2) (I_{2d} + M^2)^{-1} &= (B_1 \ B_2) \left(I_{2d} - MR(I_{2d} + R^2)^{-1} \right) \\ &= (B_1 \ B_2) \left(I_{2d} - R^2(I_{2d} + R^2)^{-1} \right) = (B_1 \ B_2) (I_{2d} + R^2)^{-1} \end{aligned}$$

As a result,

$$\begin{aligned} V &= (B_1 \ B_2) \begin{pmatrix} \left(I_d + \frac{1}{(2k_1)^2} \ell_1^2 B_1^2 \right)^{-1} & 0 \\ 0 & \left(I_d + \frac{1}{(2k_2)^2} \ell_2^2 B_2^2 \right)^{-1} \end{pmatrix} \begin{pmatrix} \ell_1 I_d \\ \ell_2 I_d \end{pmatrix} \\ &= \sum_{j=1,2} \ell_j B_j \left(I_d + \frac{1}{(2k_j)^2} \ell_j^2 B_j^2 \right)^{-1} \\ &=: \sum_{j=1,2} V_j. \end{aligned}$$

Observe that V_j is precisely the solution one obtains from the input data $N = 1$ with parameters k_j, B_j in Theorem 1. In this sense, V_j can be interpreted as a matrix 1-soliton. In the case $B_1 B_2 = 0 = B_2 B_1$, the solution V therefore is a *linear* superposition of the two matrix 1-solitons.

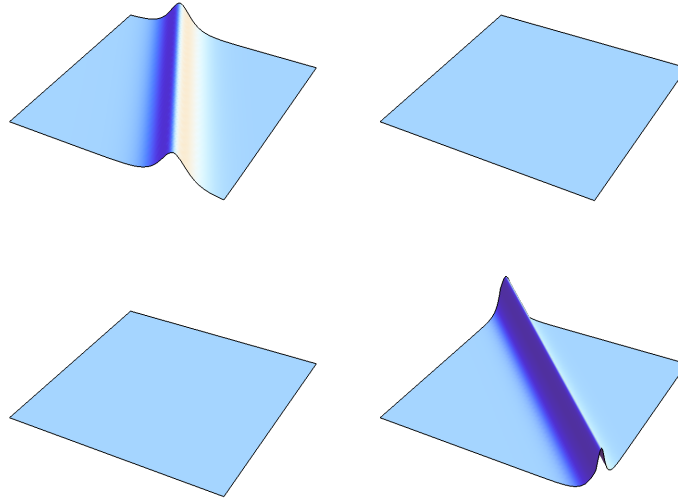


Fig. 2. The solution in Example 1a) depicted for $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$ with plot range between $-\sqrt{2}$ and $\sqrt{2}$.

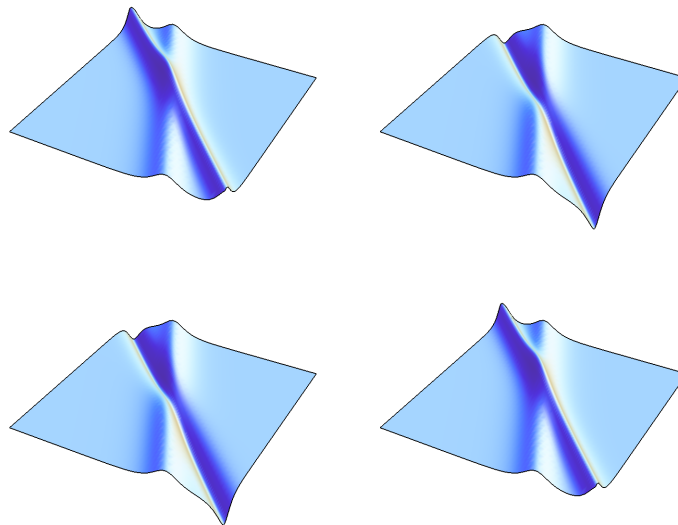


Fig. 3. The solution in Example 1b) depicted for $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$ with plot range between $-\sqrt{2}$ and $\sqrt{2}$.

Example 1. In Figures 2 and 3, the solution is depicted in the case $d = 2$, for $k_1 = 1$, $k_2 = \sqrt{2}$, and the matrix parameters are

- a) FIGURE 2 $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,
 b) FIGURE 3 $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Of course there is a huge variety of solutions not covered by the cases above. We conclude this subsection with one additional example.

Example 2. In Figure 4 the solution is depicted in the case $d = 2$, for $k_1 = 1$, $k_2 = \sqrt{2}$, and with the matrix parameters

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

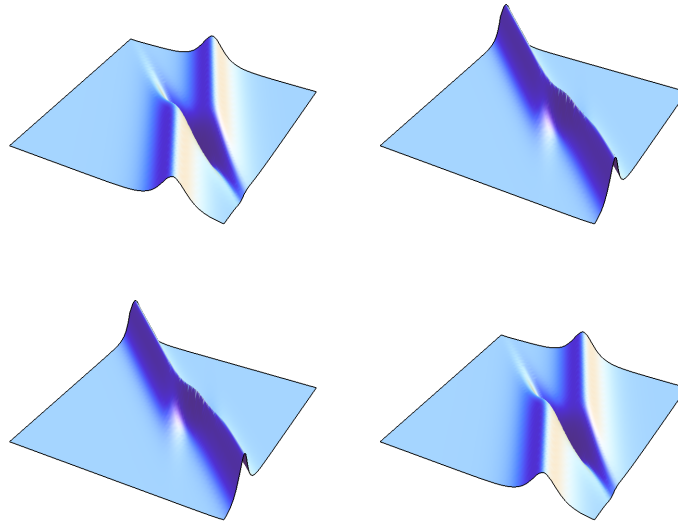
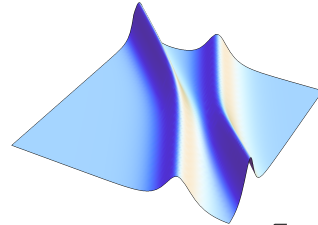


Fig. 4. The solution in Example 2 depicted for $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$ with plot range between $-\sqrt{2}$ and $\sqrt{2}$.

For comparison, we also depict the *scalar* 2-soliton⁴. The frame is the same as in Figure 4.



⁴ As generated in the case $d = 1$ with the input data $N = 2$, $k_1 = 1$, $k_2 = \sqrt{2}$, and $b_1 = b_2 = 1$ in Theorem 1.

Remark 2. In [11], a Bäcklund chart of KdV-type equations is introduced, linking in particular the KdV with its singularity equation (For a generalization of this link to the operator level we refer to [9], see also [6]). It is then indicating how this can be used to generate explicit solutions. It would be interesting to compare the solution in Example 2, see Figure 4, with the (scalar) interacting soliton in [11], see Figure 3.

3.3 Complex parameters: A breather solution

Finally we would like to mention that also complex parameters can lead to real solutions. This is well-known in the case of the scalar modified KdV equation where the input data k, \bar{k}, b, \bar{b} results in a breather⁵, a solution consisting of a bound state of a soliton and an antisoliton [21]. The same holds true in the case of the matrix modified KdV equation as the following argument shows.

Starting from Theorem 1 with $N = 2$ and the parameters chosen to as $k_1 = k, k_2 = \bar{k}$ (such that $\ell_1 = \ell, \ell_2 = \bar{\ell}$) and $B_1 = B, B_2 = \bar{B}$, our solution reads

$$V = (B \bar{B}) \left(I_{2d} + M^2 \right)^{-1} \begin{pmatrix} \ell I_d \\ \bar{\ell} I_d \end{pmatrix} \quad \text{with } M = \begin{pmatrix} \frac{1}{2k} \ell B & \frac{1}{k+\bar{k}} \ell \bar{B} \\ \frac{1}{k+\bar{k}} \bar{\ell} B & \frac{1}{2\bar{k}} \bar{\ell} \bar{B} \end{pmatrix}.$$

Introducing $D = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$, such that

$$(B \bar{B}) = (\bar{B} B) D \quad \text{and} \quad \begin{pmatrix} \ell I_d \\ \bar{\ell} I_d \end{pmatrix} = \begin{pmatrix} \bar{\ell} I_d \\ \ell I_d \end{pmatrix}$$

and

$$\begin{aligned} DMD &= \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2k} \ell B & \frac{1}{k+\bar{k}} \ell \bar{B} \\ \frac{1}{k+\bar{k}} \bar{\ell} B & \frac{1}{2\bar{k}} \bar{\ell} \bar{B} \end{pmatrix} \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2k} \ell B & \frac{1}{k+\bar{k}} \ell \bar{B} \\ \frac{1}{k+\bar{k}} \bar{\ell} B & \frac{1}{2\bar{k}} \bar{\ell} \bar{B} \end{pmatrix}. \end{aligned}$$

Observe that $D^{-1} = D$. Hence, since $D(I_{2d} + M^2)^{-1}D = (I_{2d} + DM^2D)^{-1} = (I_{2d} + (DMD)^2)^{-1}$, we find

$$V = (\bar{B} B) \left(I_{2d} + \begin{pmatrix} \frac{1}{2k} \ell B & \frac{1}{k+\bar{k}} \ell \bar{B} \\ \frac{1}{k+\bar{k}} \bar{\ell} B & \frac{1}{2\bar{k}} \bar{\ell} \bar{B} \end{pmatrix}^2 \right)^{-1} \begin{pmatrix} \ell I_d \\ \bar{\ell} I_d \end{pmatrix} = \bar{V},$$

showing that the solution V is real.

Example 3. For illustration, we add two random examples. In both examples $k = 1 + i$. For the corresponding scalar breather this implies velocity = 2, and hence the plots are drawn for $(x, x + 2t)$ giving a stationary picture. The matrix parameter is

⁵ Here \bar{k} denotes the complex conjugate of k .

FIGURE 6 $B = \begin{pmatrix} i & -2 \\ 1+i & 2-i \end{pmatrix},$

FIGURE 7 $B = \begin{pmatrix} i & -2i \\ 3i-1 & -1 \end{pmatrix}.$

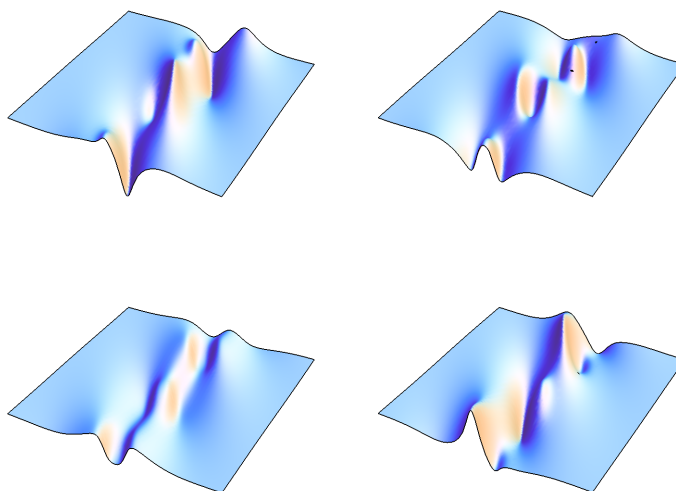


Fig. 5. The solution in Example 3a) is depicted for $-5 \leq x \leq 5$ and $0 \leq t \leq 2$ with plot range between -3.5 and 3.5 .

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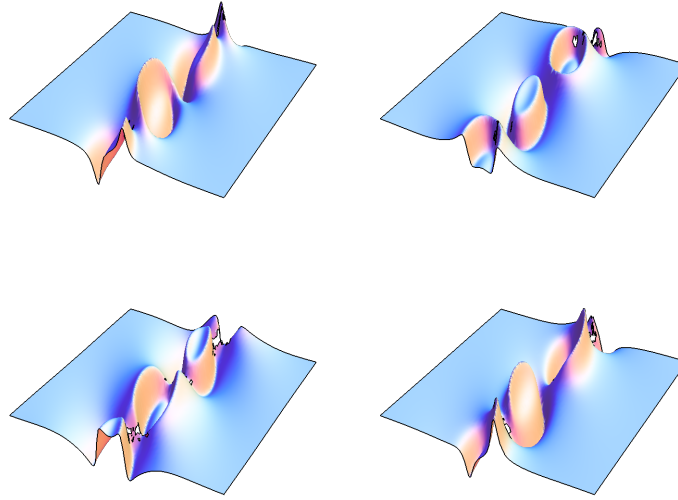


Fig. 6. The solution in Example 3b) is depicted for $-5 \leq x \leq 5$ and $-1 \leq t \leq 1$ with plot range between -5.5 and 5.5 .

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