On the limit configuration of four species strongly competing systems

Flavia Lanzara and Eugenio Montefusco

Abstract. We analysed some qualitative properties of the limit configuration of the solutions of a reaction-diffusion system of four competing species as the competition rate tends to infinity. Large interaction induces the spatial segregation of the species and only two limit configurations are possible: either there is a point where four species concur, a 4-point, or there are two points where only three species concur. We characterized, for a given datum, the possible 4-point configuration by means of the solution of a Dirichlet problem for the Laplace equation.

Mathematics Subject Classification (2010). 35J65; 35Bxx; 92D25.

Keywords. Spatial segregation, Competition-Diffusion system, Pattern formation.

1. Introduction and setting of the problem

In ecology, the competitive exclusion principle (also known as Gause's law) is the proposition that two (or more) species competing exactly for the same limiting resource cannot coexist at constant population values: if one specie has even the slightest advantage over another, the one with the advantage will dominate in the long term. This leads either to the extinction of the weaker competitors or to an evolutionary or behavioral shift toward a different ecological niche. The principle has been synthetized in the statement *complete competitors cannot coexist*. According to the competitive exclusion principle, many competing species cannot coexist under very strong competition, but when spatial movements are permitted more than one species can coexist thanks to the segregation of their habitats. For a theoretical discussion and some experimental results see [10, 12].

From a mathematical viewpoint the determination of the configuration of the habitat segregation for some populations is an interesting problem which can be modelled by an optimal (in a suitable sense) partition of a domain, for example in the papers [4, 5, 6, 7, 14] the problem is studied modelling the interspecies competition with a large interaction term in an elliptic system of partial differential equations inspired by classical models in populations dynamics. In [3, 8] the problem is modelled as a Cauchy problem for a parabolic system of semilinear partial differential equations describing the dynamics of the densities of different species. Starting from the quoted papers we want to tackle the segregation problem of 4 species in a planar region. Note that in the evolution works, in particular [8], it is proved that some populations can vanish under the competition of other species, moreover in [3] the authors are able to estimate the number of the long-term surviving populations.

Let $D \subseteq \mathbb{R}^n$ be an open bounded, simply connected domain with smooth boundary ∂D and consider the competition-diffusion system of k differential equations

$$\begin{cases} -\Delta u_i(x) = -\mu u_i(x) \sum_{j\neq i} u_j(x) & \text{in } D, \\ u_i(x) \geq 0 & \text{in } D, \\ u_i(x) = \phi_i(x) & \text{on } \partial D. \end{cases} \qquad i = 1,\dots,k \tag{1.1}$$

This system governs the steady states of k competing species coexisting in the same domain D. Any u_i represents a population density and the parameter μ determines the interaction strength between the populations.
The datum $\Phi = (\phi_1,\dots,\phi_k)$ is called **admissible** if any function $\phi_i \in H^{1/2}(\partial D)$ (for $i = 1,\dots,k$) is nonnegative, positive in a nonempty arc, and such that $\phi_i \cdot \phi_j = 0$ a.e. in ∂D for $i \neq j$.
Moreover, if $n = 2$, we will assume that $\phi_i \in W^{1,\infty}(\partial D)$, that the sets $\{\phi_i > 0\}$ are nonempty, open connected arcs and that the function $\sum_{i=1}^k \phi_i$ vanishes exactly in k points of ∂D (the endpoints of the ϕ_i's supports).
If Φ is admissible, the existence of positive solutions of (1.1) for any positive μ is proved in [5] using Leray-Schauder degree theory. The uniqueness is proved in [17], using the sub- and super-solution method.
Let define the class of segregated densities

$$\mathcal{U} = \left\{ \begin{array}{ll} U = (u_1,\dots,u_k) \in (H^1(D))^k : & u_i = \phi_i \text{ on } \partial D \\ & u_i \geq 0 \text{ in } D \\ & u_i \cdot u_j = 0 \text{ for } i \neq j \text{ a.e. in } D \end{array} \right\}$$

and the class

$$\mathcal{S} = \left\{ \begin{array}{ll} U = (u_1,\dots,u_k) \in \mathcal{U} : & -\Delta u_i \leq 0 \text{ in } D \\ & -\Delta\big(u_i - \sum_{j\neq i} u_j\big) \geq 0 \text{ in } D \end{array} \right\}.$$

Let $U^{(\mu)} = (u_{1,\mu},\dots,u_{k,\mu})$ be the solution of (1.1) for every $\mu > 0$. In [5] it is proved that there exists $\overline{U} = \{\bar{u}_1,\dots,\bar{u}_k\} \in \mathcal{U}$ such that, up to subsequences, $u_{i,\mu} \longrightarrow \bar{u}_i$ in $H^1(D)$ and $\mathcal{S}$ contains all the asymptotic limits of (1.1) that is $\overline{U} \in \mathcal{S}$.

The uniqueness of the limit solution of (1.1) as $\mu \to +\infty$ was proved in [5] in the case $k = 2$ and in [7] in the case of $k = 3$ and in dimension 2. Specifically, the authors prove that the class $\mathcal{S}$ consists of one element. In [17] it is proved that $\mathscr{S}$ consists of one element also in the case of arbitrary dimension and arbitrary number of species. A different proof of uniqueness of the limit configuration, based on the maximum principle and on the qualitative properties of the elements of $\mathscr{S}$, is given in [2].

Now consider the energy functional associated with k species

$$E(U) = \sum_{i=1}^{k} \int_D |\nabla u_i(x)|^2 dx$$

in the class of all possible segregated states subject to some boundary and positivity conditions. It is shown in [6] that the problem

$$\text{find} \quad U \in \mathscr{U} \quad \text{such that} \quad E(U) = \min_{V \in \mathscr{U}} E(V) \tag{1.2}$$

admits a unique solution, which belongs to $\mathscr{S} \subseteq \mathscr{U}$. It follows that the unique minimal energy configuration shares with the limit states of (1.1) the common property of belonging to $\mathscr{S}$. Specifically, the element of $\mathscr{S}$ is the critical point (in the weak sense) of the energy functional $E(U)$ among all segregated states, subject to the same boundary and positivity conditions.

The aim of this note is to study the case of four species in a planar smooth domain (i.e. the case $k = 4$ and $n = 2$ in (1.1)), in particular we are interested in the description and some qualitative properties of the limiting configuration.

2. Some known results

Suppose that D is a simply connected domain in $\mathbb{R}^2$. Due to the conformal invariance of the problem, with no loss of generality we can assume

$$D = B(O,1) = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$$

and consider the class

$$\mathscr{S} = \left\{ \begin{array}{l} U = (u_1, u_2, u_3, u_4) \in (H^1(D))^4 : u_i \geq 0 \text{ in } D,\ u_i = \phi_i \text{ on } \partial D \\ u_i \cdot u_j = 0 \text{ for } i \neq j, -\Delta u_i \leq 0, -\Delta\big(u_i - \sum_{j\neq i} u_j\big) \geq 0 \text{ in } D \end{array} \right\}.$$

The study of $\mathscr{S}$ provides the understanding of the segregated states of 4 species induced by strong competition. If $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is an admissible datum then $\phi_i, i = 1, \dots, 4$, are positive in their supports, the sets $\{\phi_i > 0\} \subset \partial D$ are open connected arcs and $\phi = \sum_{i=1}^4 \phi_i$ vanishes at exactly four points of ∂D, the endpoints p_1, p_2, p_3, p_4 in anticlockwise order.

In the following we will denote by U both the quadruple (u_1, u_2, u_3, u_4) and the function $\sum_{i=1}^4 u_i$. For any $U \in \mathscr{S}$ define the *nodal regions*

$$\omega_i = \{x \in D : u_i(x) > 0\} \qquad i = 1, \dots, 4,$$

the *multiplicity of a point* $x \in \overline{D}$ with respect to U:

$$m(x) = \#\{i : |\omega_i \cap B_r(x)| > 0 \quad \forall r > 0\}$$

where $B_r(x) = \{p \in \mathbb{R}^2 : |p - x| < r\}$ and the *interfaces* between two densities

$$\Gamma_{ij} = \partial\omega_i \cap \partial\omega_j \cap \{x \in D : m(x) = 2\}.$$

We say that ω_i and ω_j are adjacent if $\Gamma_{ij} \neq \emptyset$. Let us summarize the basic properties of the elements $U \in \mathscr{S}$:

(s1) $\sum_{i=1}^4 u_i \in W^{1,\infty}(\overline{D})$ ([6, Theorem 8.4]). It follows that $u_i \in C(\overline{D})$, ω_i is open and $x \in \omega_i$ implies $m(x) = 1$;

(s2) each ω_i is connected and each Γ_{ij} is either empty or a connected arc starting from the boundary ([7, Remark 2.1]);

(s3) u_i is harmonic in ω_i; $u_i - u_j$ is harmonic in $D \setminus \cup_{h\neq i,j}\overline{\omega}_h$ [6, Proposition 6.3];

(s4) if $x_0 \in D$ such that $m(x_0) = 2$ then ([6, Remark 6.4])

$$\lim_{\omega_i \ni y \to x_0} \nabla u_i(y) = -\lim_{\omega_j \ni y \to x_0} \nabla u_j(y);$$

(s5) if $x_0 \in D$ such that $m(x_0) = 2$ then $\nabla U(x_0) \neq 0$ and the set $\{x : m(x) = 2\}$ is locally a C^1-curve through x_0 ending either at points with higher multiplicity, or at the boundary ∂D [6, Lemma 9.4];

(s6) if $x_0 \in D$ such that $m(x_0) \geq 3$ then $|\nabla U(x)| \to 0$, as $x \to x_0$ [6, Theorem 9.3];

(s7) the set $\{x : m(x) \geq 3\}$ consists of a finite number of points [6, Lemma 9.11];

(s8) if $x_0 \in D$ with $m(x_0) = h \geq 3$ then there exists $\theta \in (-\pi, \pi]$ such that

$$U(r,\theta) = r^{h/2}\left|\cos\left(\frac{h}{2}(\theta+\theta_0)\right)\right| + o(r^{h/2}) \tag{2.1}$$

as $r \to 0$, where (r,θ) is a system of polar coordinates around x_0 [6, Theorem 9.6].

Remark 2.1. The asymptotic formula in (s8) describes the behavior of U in a neighborhood of a multiple point in D. As a consequence, at multiple point $U \in \mathscr{S}$ shares the angle in equal parts. This property does not hold true if $U \in \mathscr{S}$ has a multiple point p on the boundary ∂D. Consider the harmonic function

$$\Psi(x_1,x_2) = (x_2 - (x_1+1))\left(x_2 + (2+\sqrt{3})(x_1+1)\right)\left(x_2 + (2-\sqrt{3})(x_1+1)\right)$$

which nodal lines are represented in the following figure.

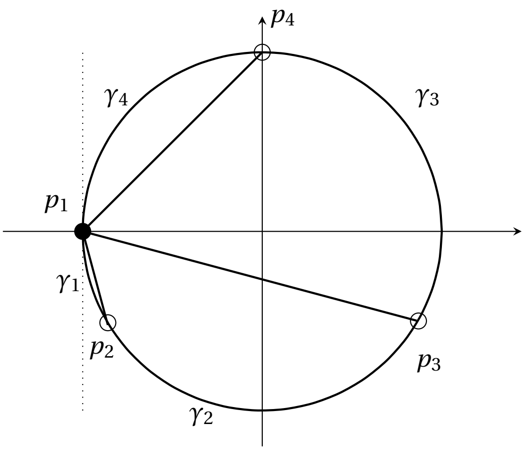

FIGURE 1. One 4-point on the boundary.

On the boundary of the unit disk Ψ vanishes in $p_1 = (-1,0)$, $p_{2,3} = (\mp\sqrt{3}/2, -1/2)$, $p_4 = (0,1)$. Denote by γ_i the arcs of positivity of $\Psi|_{\partial D}$ and by ω_i the sets of positivity of Ψ between γ_i and the lines where Ψ vanishes. If we assume $\phi_i = \Psi$ on γ_i, $i = 1,...,4$, then $\Phi = (\phi_1, \phi_2, \phi_2, \phi_4)$ is an admissible datum. Defining $u_i = |\Psi|$ in ω_i, then $U = (u_1, u_2, u_3, u_4)$ has in $p_1 \in \partial D$ a 4-point, that is a point with multiplicity 4, and it can be easily seen that the angles between the interfaces are different (see also figure 1).
On the contrary the function $U(x_1, x_2) = |x_1 x_2|$ belongs to $\mathscr{S}$ and has at the origin an interior point with multiplicity 4 with orthogonal nodal lines.
Note that in paper [15] it is considered a segregation model in which the competition between the species is not symmetric, in that case the authors are able to prove that the multiple points cannot occur on the boundary of the domain.

3. Main results and proofs

Define the set of points of multiplicity greater than or equal to $h \in \mathbb{N}$

$$\mathcal{Z}_h(U) = \{x \in \overline{D} : m(x) \geq h\}.$$

The set $\mathcal{Z}_h(U)$ consists of isolated points [6, Theorem 9.13].

Proposition 3.1. *Let $U \in \mathscr{S}$, then only one of the following statement is satisfied*
i. *$\mathcal{Z}_3(U)$ consists of one point $a_U \in \overline{D}$ such that $m(a_U) = 4$,*
ii. *$\mathcal{Z}_3(U)$ consists of two points $a_U, b_U \in \overline{D}$, $a_U \neq b_U$, with $m(a_U) = m(b_U) = 3$.*

Proof. The set $\mathcal{Z}_3(U)$ is nonempty. Indeed, if $\mathcal{Z}_3(U) = \emptyset$ then the interfaces between any two densities do not intersect. Since ϕ vanishes in exactly 4 points, there are only two interfaces which are nonempty, with endpoints on the boundary and do not intersect. This contradicts the fact that U defines four nodal regions.
Let $a_U \in \overline{B}$ with $m(a_U) = 4$. Then any neighborhood of a_U contains points of every ω_i, and every non empty Γ_{ij} is connected, starts from ∂B and satisfies $\overline{\Gamma}_{ij} \ni a_U$. Suppose that there exists $b_U \neq a_U$ with $m(b_U) \geq 3$. Then b_U belongs to an interface between two densities, say $b_U \in \Gamma_{12}$. It follows that there exists a neighborhood I of b_U such that $I \cap \omega_h \neq \emptyset$, $h = 1,2$, and I has positive distance from ω_k, $k = 3,4$. On the other hand, since $m(b_U) \geq 3$, $I \cap \omega_h \neq \emptyset$ for three different values of h. This contradicts the fact that every ω_i is connected.
If there aren't points with multiplicity 4, then there are at least two points $a_U \neq b_U$ of multiplicity 3. Suppose that there exists c_U with $m(c_U) = 3$, $c_U \neq a_U, b_U$. Then c_U belongs to an interface between two densities, for example $c_U \in \Gamma_{12}$. Since $m(c_U) = 3$ there exists a neighborhood I of c_U such that $I \cap \omega_h \neq \emptyset$ for three different values of h, for example $h = 1,2,3$. Then ω_3 is not connected, a contradiction. ∎

Remark 3.2. All the 4-partitions of the disk D generated by the admissible boundary datum $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ are topologically equivalent to one of the two configurations in figure 2 where ○ stands for an isolated zero of the boundary datum Φ, • is a 3-point and ● indicates a 4-point.

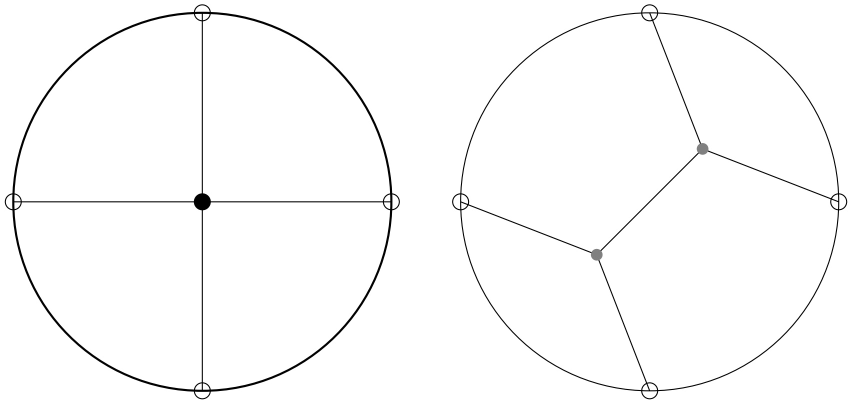

FIGURE 2. Configurations with one 4-point (on the left) and two 3–points (on the right).

In particular the multiple points can also lie on the boundary of D. In figures 1 and 4 we show, respectively, configurations with a 4-point on ∂D and with two 3-points on ∂D. The multiple points of U lie on $\overline{D}$, and it is possible to obtain either one 4-point or two 3-points.

Consider the boundary value problem

$$\begin{cases} -\Delta\psi = 0 & \text{in } D \\ \psi = \phi & \text{on } \partial D \end{cases} \tag{3.1}$$

Theorem 3.3. *Let* $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ *be an admissible datum. The harmonic function* ψ_a *which solves* (3.1) *with boundary datum* $\phi^a = \sum_{j=1}^{4}(-1)^j\phi_j$ *possesses at most one critical point* p *in* D *such that* $\psi_a(p) = 0$.

Proof. Since Φ is an admissible datum, the solution of (3.1) with boundary datum ϕ^a vanishes at exactly four points on ∂D, each arc $\Gamma_j = \{\phi_j > 0\} \subset \partial D$ is connected and ψ_a has different signs on adjacent arcs.

Since a harmonic function does not admit closed level lines, the set $\Gamma = \{x \in D : \psi_a(x) = 0\}$ has no closed loop. We infer that ψ_a has alternate positive or negative sign on 3 or 4 adjacent sets: the nodal components of ψ_a (see figure 3).

By standard theory of harmonic functions the zero set of ψ_a around a critical point at level 0 is made by (at least) 4 half-lines, meeting with equal angles. We infer that locally around each critical point at level 0 the function ψ_a defines 4 nodal components. Such components are exactly the 4 nodal regions of ψ_a, since we have already pointed out that Γ does not contain closed loop.

As a consequence, if the function ψ_a has 3 nodal components (figure 3a), there are no critical points at level 0. If the nodal components of ψ_a are 4, two different situations can occur.

1) The set Γ is made by 2 simple arcs with endpoints on the boundary, intersecting in a point $p \in D$ (figure 3b). Let q be a critical point at level 0, with $q \neq p$. Then one can construct a closed Jordan curve passing through p and q, belonging to

the positivity (or negativity) set of ψ_a, which disconnects ∂D from a negative (or positive) nodal components. This is impossible because Γ has no closed loop.
2) The set Γ is made by 4 simple arcs with endpoints on the boundary, intersecting in $p \in \partial D$ (figure 3c). If there exists a critical point q at level 0, with $q \neq p$ with the same arguments used in 1) we get a contradiction.

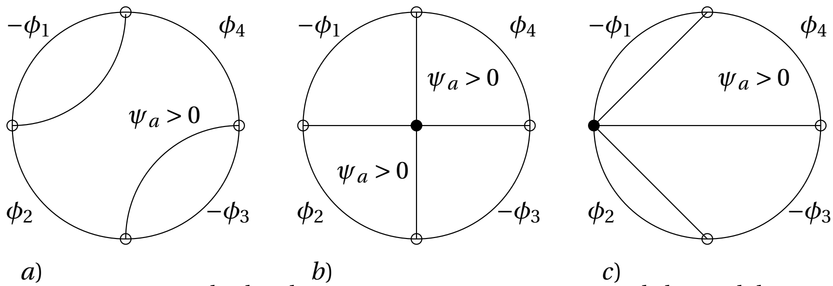

FIGURE 3. The level set $\Gamma = \{x \in D : \psi_a(x) = 0\}$ and the nodal components of ψ_a, where the function ψ_a solves (3.1) with boundary datum $\phi^a = \sum_{j=1}^4 (-1)^j \phi_j$. Three different situations can occur.

Remark 3.4. The existence of a critical point in theorem 3.3 is not guaranteed. Consider, for example, the harmonic polynomial

$$\psi_a(x_1, x_2) = 4(x_1^2 - x_2^2) + 10x_1 + 7$$

On the boundary of the unit disk

$$\psi_a\big|_{\partial D} = \Phi^a(\theta) = (4\cos(\theta) + 3)(2\cos(\theta) + 1)$$

Denoting by $\theta_0 \in (\pi/2, \pi) : \cos(\theta_0) = -3/4$, we have

$$\Phi^a(\theta) < 0 \quad \text{in} \quad (2\pi/3, \theta_0) \cup (-\theta_0, -2\pi/3)$$
$$\Phi^a(\theta) > 0 \quad \text{in} \quad (-2\pi/3, 2\pi/3) \cup (\theta_0, -\theta_0)$$

Hence $\Phi = |\Phi^a|$ is admissible and the unique critical point of ψ_a is $(-5/4, 0)$ which does not belong to D. Conditions which ensure the existence of critical points for solutions of two dimensional elliptic equation are given in [16] and [1].

Remark 3.5. Recall that all the critical points of ψ_a in D are always saddle points, since harmonic functions in D assume local maximum or local minimum only on the boundary of the ball. If x_0 is an internal critical point then, in a neighborhood of x_0, the level line $\{x \in D : \psi_a(x) = \psi_a(x^0)\}$ is made by 2 simple arcs intersecting in x_0 (see [1, Remark 1.2]).

Proposition 3.6. *Let Φ be an admissible datum. If the set $\mathscr{S}$ contains a function U with a 4-point a_U in $\overline{D}$ then $U = |\psi_a|$, where ψ_a is the harmonic function such that $\psi_a = \phi^a = \sum_{j=1}^4 (-1)^j \phi_j$ on ∂D.*

Proof. Let U be an element of $\mathscr{S}$ with a 4-point a_U, then there exist three or four arcs connecting a_U to any of the isolated zeros of the boundary datum (note that there are three arcs if and only if $a_U \in \partial D$). Then the function $\psi_a =$

$\sum_{j=1}^{4}(-1)^j u_j$ is harmonic in $D\setminus\{a_U\}$ (see [6, Proposition 6.3]), moreover ψ_a is bounded so, by Schwarz's removable singularity principle (see [13, Proposition 11.1]) ψ_a is harmonic in D and, by construction, $U=|\psi_a|$ in D and $U=|\Phi^a|=\Phi$ on ∂D. ∎

Proposition 3.7. *$\mathscr{S}$ contains only solutions of one type. In other words it is not possible that there exist $U,W\in\mathscr{S}$ such that U possesses one 4-point a_U and W possesses two 3-points a_W,b_W.*

Proof. This statement is an easy consequence of the uniqueness results contained in [2, 17], however we present here a different proof.
Suppose there exist $U,W\in\mathscr{S}$ such that $U=(u_1,\dots,u_4)$ possesses one 4-point a_U and $W=(w_1,\dots,w_4)$ possesses two 3-points a_W,b_W for an admissible boundary datum $\Phi=(\phi_1,\dots,\phi_4)$. Since W has two 3-points we can assume that there are two nodal set having positive distance, without loss of generality we can think that $\overline{\omega}_{W,1}\cap\overline{\omega}_{W,3}=\emptyset$, where $\omega_{W,1}=\{x\in D: w_1(x)>0\}$ and $\omega_{W,3}=\{x\in D: w_3>0\}$ (see, for example, figure 2). Consider the following functions

$$w^*=w_1-w_2+w_3-w_4\,,\qquad \psi_a=u_1-u_2+u_3-u_4\,.$$

By proposition 3.6 we know that ψ_a is an harmonic function in D such that $\psi_a=\phi^a$ on ∂D, moreover w^* is a superharmonic function, since is a gluing of harmonic functions which become superharmonic on the arc linking the a_W and b_W (see [6, Proposition 6.3], and also [13, Chapter 12]).
Then the function $z(x_1,x_2)=w^*(x_1,x_2)-\psi_a(x_1,x_2)$, with $(x_1,x_2)\in D$, is a superharmonic function such that $z(x_1,x_2)=0$ if $(x_1,x_2)\in\partial D$, and, by maximum principle, it follows $z(x_1,x_2)\ge 0$, that is

$$w^*(x_1,x_2)\ge\psi_a(x_1,x_2)\qquad\forall(x_1,x_2)\in D.\tag{3.2}$$

Inequality (3.2) implies that the positivity set of ψ_a is contained in the positivity set of w^*. Precisely it follows that $\omega_{U,1}\subseteq\omega_{W,1}$ and $\omega_{U,3}\subseteq\omega_{W,3}$ and the contradiction is reached since $\overline{\omega}_{U,1}\cap\overline{\omega}_{U,3}=a_U$ but $\overline{\omega}_{W,1}\cap\overline{\omega}_{W,3}=\emptyset$. ∎

Proposition 3.8. *Let $U,W\in\mathscr{S}$ be functions with only one 4-point $a_U,a_W\in\overline{D}$, then $U=W$.*

Proof. Assume there exist two different $U,W\in\mathscr{S}$ with a 4-point, then there exits three or four arcs connecting a_U to any of the isolated zeros of the boundary datum (there are three arcs if and only if $a_U\in\partial D$), and the same is true for a_W. In any case we can consider the functions $u_a=\sum_{j=1}^4(-1)^j u_j$ and $w_a=\sum_{j=1}^4(-1)^j w_j$, by proposition 6.3 in [6], both functions are solution to (3.1) with boundary datum $\phi^a=\sum_{j=1}^4(-1)^j\phi_j$, but it is well known that (3.1) possesses only one harmonic solution, that is $u_a\equiv w_a$, and this means that $U=W$.

∎

Let $U\in\mathscr{S}$ be given with trace $\Phi=(\phi_1,\dots,\phi_4)$ and 3-points on ∂D. The next proposition shows that, similarly to proposition 3.6, we can construct a harmonic function closely related to U, which has different signs on adjacent nodal regions.

Proposition 3.9. *Suppose that $U \in \mathcal{S}$ is a configuration with two 3–points on the boundary, that is there are $p, q \in \partial D$ such that $m(p) = m(q) = 3$. Then we can construct a harmonic function Ξ_a such that $U = |\Xi_a|$ in D.*

Proof. We denote by $\Sigma_j = \{x \in \partial D : \phi_j(x) > 0\}$, $j = 1, \ldots, 4$. Since Φ is admissible we have $\Sigma_i \cap \Sigma_j = \emptyset$, $i \neq j$, and $\partial D = \cup_{j=1}^{4} \overline{\Sigma}_j$. Two different configurations are possible (see figure 4) and, in both cases, we construct a function which is alternatively positive and negative on the nodal region of U and is harmonic in D.
a) There exists an index i such that p and q belong to $\overline{\Sigma}_i$ that is $\phi_i(p) = \phi_i(q) = 0$, say $i = 2$. Then there exists $j \neq i$ such that $p, q \notin \overline{\Sigma}_j$, say $j = 4$ (see figure 4, on the left). Then the function $\Xi_a = u_4 - (u_1 + u_2 + u_3)$ has different signs on adjacent nodal regions. By [6, Proposition 6.3], Ξ_a is harmonic in $D \setminus \Omega_{12}$, in $D \setminus \Omega_{13}$ and in $D \setminus \Omega_{23}$, where $\Omega_{hk} = \overline{\omega}_h \cup \overline{\omega}_k$. As $\Omega_{12} \cap \Omega_{23} \cap \Omega_{13} = \{p, q\}$ we deduce that Ξ_a is harmonic in D. By construction, $U = |\Xi_a|$ in D.
b) p, q do not belong to the same $\overline{\Sigma}_i$ for any i that is they are not consecutive. Suppose, e.g., that $p \in \overline{\Sigma}_1$, $q \in \overline{\Sigma}_3$, (see figure 4, on the right). Then the function $\Xi_a = u_1 + u_2 - u_3 - u_4$ has different signs on adjacent nodal regions and, by [6, Proposition 6.3], Ξ_a is harmonic in $D \setminus \Omega_{23}$, in $D \setminus \Omega_{13}$ and in $D \setminus \Omega_{14}$. As $\Omega_{23} \cap \Omega_{13} \cap \Omega_{14} = \{p, q\}$ we deduce that Ξ_a is harmonic in D. The function Ξ_a solves the problem (3.1) with boundary datum $\phi_1 + \phi_2 - \phi_3 - \phi_4$ and, by construction, $U = |\Xi_a|$ in D. ∎

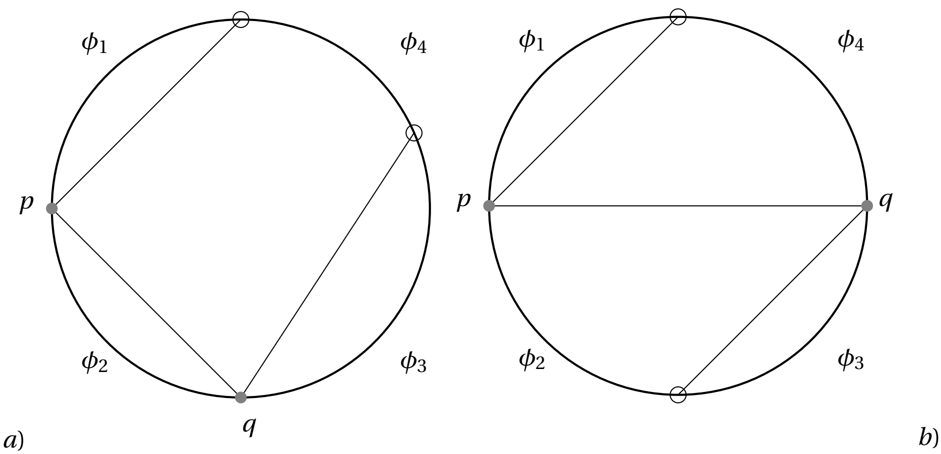

FIGURE 4. Configuration with 3-points on ∂D.

Remark 3.10. If $U \in \mathcal{S}$ produces a 3-points configuration with at least a 3-point $p \in D$ then proposition 3.7 does not hold true. Indeed, since $m(p) = 3$, there is a neighborhood I around p such that $I \cap \omega_h \neq \emptyset$ for three different values of h, and three interfaces with endpoint p are not empty. Then it is not possible to construct an harmonic function Ξ_a such that $U = |\Xi_a|$ with different signs on adjacent nodal regions around p.

Let us remark that, if $p \in D$ with $m(p) = 4$ then

$$U(r,\theta) = r^2 |\cos(2(\theta + \theta_0))| + o(r^2)$$

as $r \to 0$, where (r,θ) is a system of polar coordinates around p. Hence $U(p) = 0$ and $\nabla U(p) = 0$. Proposition 3.6 can be inverted if $p \in D$.

Proposition 3.11. *Let Φ be an admissible boundary datum and suppose that the related harmonic function ψ_a, solution to (3.1) with boundary datum ϕ^a, has a critical point $p \in D$ (a saddle) such that $\psi_a(p) = 0$, then p is a 4-point for the function $U = |\psi_a| \in \mathscr{S}$.*

Proof. We introduce the transformation

$$x = T_p(\zeta) = \frac{\zeta + p}{\overline{p}\zeta + 1}. \tag{3.3}$$

Here we identificate the complex numbers $x = x_1 + ix_2$ and $\zeta = \zeta_1 + i\zeta_2$ with the points (x_1, x_2) and $(\zeta_1, \zeta_2) \in \mathbb{R}^2$, respectively. T_p is a conformal map which maps the unit disk $\overline{D}$ into itself such that $T_p(\partial D) = \partial D$ and $T_p(0) = p$. Set

$$\widetilde{\Psi}(\zeta) = \Psi_a(T_p(\zeta)), \qquad \widetilde{\Phi}^a(\zeta) = \Phi^a(T_p(\zeta)). \tag{3.4}$$

Then $\widetilde{\Psi}$ solves the problem

$$\begin{cases} -\Delta\widetilde{\Psi} = 0 & \text{in } D, \\ \widetilde{\Psi} = \widetilde{\Phi}^a & \text{on } \partial D. \end{cases} \tag{3.5}$$

Denote by $u(\zeta) = \mathrm{Re}\,(T_p(\zeta))$ and $v(\zeta) = \mathrm{Im}\,(T_p(\zeta))$. By differentiation we get

$$\nabla_\zeta \widetilde{\Psi}(\zeta) = \begin{pmatrix} \partial_{\zeta_1} u(\zeta) & \partial_{\zeta_1} v(\zeta) \\ \partial_{\zeta_2} u(\zeta) & \partial_{\zeta_2} v(\zeta) \end{pmatrix} \nabla_x \Psi(T_p(\zeta)).$$

From the relations

$$\frac{\partial}{\partial \zeta_1} T_p(\zeta) = \frac{1 - |p|^2}{(1 + \overline{p}\zeta)^2} \qquad \frac{\partial}{\partial \zeta_2} T_p(\zeta) = i\frac{1 - |p|^2}{(1 + \overline{p}\zeta)^2}.$$

we obtain

$$\begin{pmatrix} \partial_{\zeta_1} u(\zeta) & \partial_{\zeta_1} v(\zeta) \\ \partial_{\zeta_2} u(\zeta) & \partial_{\zeta_2} v(\zeta) \end{pmatrix}\Bigg|_{\zeta=0} = \begin{pmatrix} (1 - |p|^2) & 0 \\ 0 & (1 - |p|^2) \end{pmatrix}.$$

Hence

$$\nabla_\zeta \widetilde{\Psi}(0) = \begin{pmatrix} (1 - |p|^2) & 0 \\ 0 & (1 - |p|^2) \end{pmatrix} \nabla_x \Psi_a(p). \tag{3.6}$$

We can write the Fourier expansion of $\widetilde{\Psi}$

$$\widetilde{\Psi}(r,\theta) = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos(k\theta) + B_k \sin(k\theta)) r^k, \qquad \zeta = (r,\theta).$$

The hypothesis $\widetilde{\Psi}(0) = \Psi(p) = 0$ implies that $A_0 = 0$ whereas the hypothesis $\nabla_x \Psi(p) = (0,0)$ gives $\nabla_\zeta \widetilde{\Psi}(0) = (0,0)$ and then $A_1 = B_1 = 0$. Moreover $(A_2, B_2) \neq (0,0)$ because $\widetilde{\Psi}$ has exactly four zeroes on the boundary. Therefore $U(x) = |\Psi_a(x)| = |\widetilde{\Psi}(T_p^{-1}(x))|$ is nonnegative, satisfies the boundary datum Φ^a and has exactly four nodal regions. This function generates an element of $\mathscr{S}$, with datum Φ and quadruple point p (see also [7, Lemma 3.2]). ∎

Proposition 3.12. *Suppose that* $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ *is an admissible datum. The solution* U *of the minimum problem* (1.2) *with admissible datum* Φ *has a 4-point in* $p \in D$ *if and only if*

$$\sum_{j=1}^{4} (-1)^j \int_{\partial D} \phi_j \left(\frac{\zeta + p}{\overline{p}\zeta + 1} \right) ds_\zeta \qquad = 0, \tag{3.7}$$

$$\sum_{j=1}^{4} (-1)^j \int_{\partial D} \phi_j \left(\frac{\zeta + p}{\overline{p}\zeta + 1} \right) \zeta_r \, ds_\zeta \qquad = 0, \qquad r = 1, 2. \tag{3.8}$$

Here we use the notation $\zeta = \zeta_1 + i\zeta_2$.

Proof. Suppose that U has a 4-point in $p \in D$. For proposition 3.6 we have $U = |\Psi_a|$ where Ψ_a solves (3.1) with boundary datum $\Phi^a = \sum_{j=1}^4 (-1)^j \phi_j$, and, by (2.1), $U(p) = 0$, $\nabla U(p) = (0,0)$. Hence $\nabla \Psi_a(p) = (0,0)$. We introduce the transformation (3.3) and define $\widetilde{\Psi}$ and $\widetilde{\Phi}^a$ according to (3.4). Then $\widetilde{\Psi}$ solves (3.5) and, by hypothesis, the origin is a 4-point for $\widetilde{U}(\zeta) = |\widetilde{\Psi}(\zeta)|$. By the Poisson integral formula

$$\widetilde{\Psi}(\zeta) = \frac{1 - |\zeta|^2}{2\pi} \int_{\partial D} \frac{\widetilde{\Phi}^a(\eta)}{|\zeta - \eta|^2} ds_\eta \tag{3.9}$$

and it belongs to $C^2(D) \cap C^0(\overline{D})$ (cf. [11, (2.27)]). We deduce that

$$0 = \widetilde{\Psi}(0) = \frac{1}{2\pi} \int_{\partial D} \frac{\widetilde{\Phi}^a(\eta)}{|\eta|^2} ds_\eta = \frac{1}{2\pi} \int_{\partial D} \widetilde{\Phi}^a(\eta) ds_\eta = \frac{1}{2\pi} \int_{\partial D} \Phi^a(T_p(\eta)) ds_\eta$$

which gives (3.7). By (3.6) we have $\nabla \widetilde{\Psi}(0) = (0,0)$. By direct differentation of the Poisson integral

$$\frac{\partial}{\partial \zeta_r} \widetilde{\Psi}(0) = \frac{1}{\pi} \int_{\partial D} \widetilde{\Phi}^a(\eta) \eta_r \, ds_\eta = \frac{1}{\pi} \int_{\partial D} \Phi^a(T_p(\eta)) \eta_r \, ds_\eta = 0, \quad r = 1, 2$$

which gives (3.8).

Conversely, suppose that (3.7) and (3.8) are valid. Let Ψ_a be the solution of (3.1) with boundary datum $\Phi^a = \sum_{j=1}^4 (-1)^j \phi_j$. Then the function $\widetilde{\Psi}(\zeta) = \Psi_a(T_p(\zeta))$ solves (3.5) and is given by (3.9). By (3.7) we have

$$\widetilde{\Psi}(0) = \frac{1}{2\pi} \int_{\partial D} \Phi^a(T_p(\eta)) ds_\eta = 0.$$

Hence $\Psi_a(p) = \Psi_a(T_p(0)) = \widetilde{\Psi}(0) = 0$.
Moreover, by (3.8),

$$\frac{\partial}{\partial \zeta_s} \widetilde{\Psi}(0) = \frac{1}{\pi} \int_{\partial D} \widetilde{\Phi}^a(\eta) \eta_s \, ds_\eta = \frac{1}{\pi} \int_{\partial D} \Phi^a(T_p(\eta)) \eta_r \, ds_\eta = 0, \quad r = 1, 2.$$

From (3.6) we get

$$\nabla_x \Psi_a(p) = \begin{pmatrix} (1 - |p|^2)^{-1} & 0 \\ 0 & (1 - |p|^2)^{-1} \end{pmatrix} \nabla_\zeta \widetilde{\Psi}(0) = (0,0).$$

For proposition 3.11 we conclude that the function $U = |\Psi_a|$ posses a 4-point and belongs to $\mathscr{S}$. ■

Remark 3.13. Conditions (3.7),(3.8) imply that an admissible datum generates a 4-point configuration if and only if 3 integral conditions are satisfied. These conditions identify three hyperplanes in the space $W^{1,\infty}(\partial D)$, then the boundary data which generate a 4-point configuration form (at most) a codimension 3 set in the space of admissible data.

If p is the origin, proposition 3.12 can be formulated as follows

Corollary 3.14. *Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be an admissible boundary datum. Then $U = |\psi_a|$, where ψ_a is the solution of (3.1) with boundary datum ϕ^a, has a 4-point in the origin if and only if*

$$\int_{\partial D} \phi^a(y)\, ds_y = 0 \qquad \text{and} \qquad \int_{\partial D} y_j \phi^a(y)\, ds_y = 0 \quad j = 1,2 \tag{3.10}$$

where $\phi^a = \sum_{j=1}^4 (-1)^j \phi_j$.

Remark 3.15. It is easy to verify that conditions (3.10) hold if the admissible datum $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ satisfies the following symmetry conditions

$$\phi_2(x_1, x_2) = \phi_1(-x_1, x_2) \quad \phi_3(x_1, x_2) = \phi_1(-x_1, -x_2) \quad \phi_4(x_1, x_2) = \phi_1(x_1, -x_2)$$

for any $x = (x_1, x_2) \in \partial D$.

Proposition 3.12 gives necessary and sufficient conditions on the datum such that the solution of the minimum problem (1.2) generates a 4-point configuration with the quadruple point $p \in D$ and suggests that the most probable configurations in nature are the one with 3–points. However a 4-point configuration can be obtained as limit of 3–points configurations (see Remark 3.13). This is a consequence of the continuously dependence of the minimum in (1.2) on the boundary data (cf. [6, Theorem 3.2]). In fact we expect that, generically, the limit configuration has two 3–points, with respect to a suitable measure in the space of the admissible boundary data.

Proposition 3.16. *Let $U = (u_1, u_2, u_3, u_4)$ be the solution of the minimum problem (1.2) with admissible datum $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, which generates a configuration with the 4-point $q \in D$, then there exists $\varepsilon_0 > 0$ and a continuum of functions $U^{(\varepsilon)} = (u_1^{(\varepsilon)}, u_2^{(\varepsilon)}, u_3^{(\varepsilon)}, u_4^{(\varepsilon)})$ belonging to a class of segregated states, with admissible boundary datum, such that*
i. $U^{(\varepsilon)}$ is a two 3–points configuration, for any $\varepsilon \in (0, \varepsilon_0)$,
ii. $u_j^{(\varepsilon)} \to u_j$ in $H^1(D)$, $j = 1, \ldots, 4$, as $\varepsilon \to 0$.

Proof. Let $\widetilde{\Phi}$ be an admissible datum which vanishes on the boundary ∂D in the same points of Φ and such that the solution of the minimum problem (1.2) with boundary condition $\widetilde{\Phi}$ generates a configuration with 3-points $t_1, t_2 \in D$. One among $\Phi \pm \varepsilon\widetilde{\Phi}$ is admissible for any $\varepsilon > 0$. Let us assume that $\Phi^{(\varepsilon)} = \Phi + \varepsilon\widetilde{\Phi}$ is admissibile. Let $\psi, \widetilde{\psi}$ and $\psi^{(\varepsilon)}$ be the solutions of (3.1) with boundary data $\Phi^a, \widetilde{\Phi}^a$ and $\Phi^a + \varepsilon\widetilde{\Phi}^a$, respectively. It is obvious that $\psi^{(\varepsilon)} = \psi + \varepsilon\widetilde{\psi}$.
If the solution $U^{(\varepsilon)}$ of (1.2) with boundary datum $\Phi^{(\varepsilon)}$ has a 4-point $Q \in D$ then, for proposition 3.6, $U^{(\varepsilon)} = |\psi^{(\varepsilon)}|$ and

$$\psi^{(\varepsilon)}(Q) = 0, \qquad \nabla\psi^{(\varepsilon)}(Q) = (0,0). \tag{3.11}$$

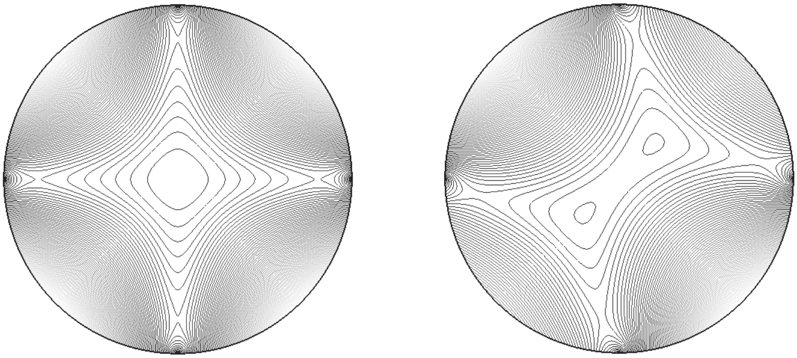

FIGURE 5. Level lines of the approximate solutions of (1.1) with 4 differential equations and for large μ, showing a 4-point configuration (on the left) and a two 3-points configuration (on the right).

If $Q=q$ then $\psi(Q)=0, \nabla\psi(Q)=(0,0)$ but at least one among $\widetilde{\psi}(Q)=0$ and $\nabla\widetilde{\psi}(Q)=(0,0)$ is not valid (because Q is not a 4-point for $\widetilde{U}=|\widetilde{\psi}|$). Hence (3.11) doesn't hold and we get a contradiction. It follows that $Q\neq q$.
If $\psi(Q)\neq 0$ and $\widetilde{\psi}(Q)=0$ then $\psi^{(\varepsilon)}(Q)=\psi(Q)\neq 0$. Hence Q cannot be a 4-point for $U^{(\varepsilon)}$. If $\psi(Q)=0$ then $\nabla\psi(Q)\neq(0,0)$ (because Q is not a 4-point for $|\psi|$). Since $\nabla\psi^{(\varepsilon)}(Q)=\nabla\psi(Q)+\varepsilon\nabla\widetilde{\psi}(Q)$ there exists $\varepsilon_0>0$ such that $\nabla\psi^{(\varepsilon)}(Q)\neq(0,0)$ for any $\varepsilon<\varepsilon_0$. It follows that $U^{(\varepsilon)}$ cannot generate a configuration with a 4-point if $\varepsilon<\varepsilon_0$.
For proposition 3.1 $U^{(\varepsilon)}$ has two 3-points.
We have $\Phi^{(\varepsilon)}-\Phi=\varepsilon\widetilde{\Phi}\to 0$ in $H^{1/2}(\partial D)$. For the continuous dependence of $U^{(\varepsilon)}$ on the boundary data ([6, Theorem 3.2]) the result follows. ∎

4. Some numerical examples

In [5] it is proved that the unique solution (see [17]) of the elliptic systems (1.1) converges to the solution of the segregation problem as $\mu\to+\infty$ and, at least for $k=2$ species, the rate of the convergence is $\mu^{-1/6}$. Using this result we provide numerical simulations of solutions of the system (1.1) with 4 differential equations and for large μ. Thanks to the FreeFEM++ software (a simple and powerful partial differential equation numerical solver, see [9]) we can obtain nodal partitions showing a 4-point solution (figure 5, on the left) and a two 3-points solution (figure 5, on the right).
The pictures 5, reflecting the model configurations in figure 2, show some level lines of the approximate solutions of (1.1) with 4 equations and for large μ. The function on the left side is the solution obtained with the boundary data $\phi_i(x_1,x_2)=15|x_1x_2|$ for $i=1,\ldots,4$ and $\mu=100$, the picture on the right is obtained choosing $\phi_i(x_1,x_2)=7|x_1x_2|$ for $i=1,3$, $\phi_i(x_1,x_2)=15|x_1x_2|$ for $i=2,4$ and $\mu=100$. For the interested reader we include the FreeFEM++ script which produces the two

3-points configuration above.

```
real R=1;
border gamma1(t=0,pi/2){x=R*cos(t);y=R*sin(t);};
border gamma2(t=pi/2,pi){x=R*cos(t);y=R*sin(t);};
border gamma3(t=pi,3*pi/2){x=R*cos(t);y=R*sin(t);};
border gamma4(t=3*pi/2,2*pi){x=R*cos(t);y=R*sin(t);};

mesh disk=buildmesh(gamma1(50)+gamma2(50)+gamma3(50)+gamma4(50));
plot(disk,wait=1);

fespace H(disk,P2);
H u1,u2,u3,u4,v1,v2,v3,v4,uold1=0,uold2=0,uold3=0,uold4=0,uT;
real mu=100;
real al=7,bt=15,ga=7,dt=15;
func f1=al*abs(x*y);
func f2=bt*abs(x*y);
func f3=ga*abs(x*y);
func f4=dt*abs(x*y);

problem segregazione(u1,u2,u3,u4,v1,v2,v3,v4)=
int2d(disk)(dx(u1)*dx(v1)+dy(u1)*dy(v1))
+int2d(disk)(dx(u2)*dx(v2)+dy(u2)*dy(v2))
+int2d(disk)(dx(u3)*dx(v3)+dy(u3)*dy(v3))
+int2d(disk)(dx(u4)*dx(v4)+dy(u4)*dy(v4))
+int2d(disk)(mu*u1*(uold2+uold3+uold4)*v1)
+int2d(disk)(mu*u2*(uold1+uold3+uold4)*v2)
+int2d(disk)(mu*u3*(uold2+uold1+uold4)*v3)
+int2d(disk)(mu*u4*(uold2+uold3+uold1)*v4)
+on(gamma1,u1=f1)+on(gamma2,u1=0)+on(gamma3,u1=0)+on(gamma4,u1=0)
+on(gamma1,u2=0)+on(gamma2,u2=f2)+on(gamma3,u2=0)+on(gamma4,u4=0)
+on(gamma1,u3=0)+on(gamma2,u3=0)+on(gamma3,u3=f3)+on(gamma4,u3=0)
+on(gamma1,u4=0)+on(gamma2,u4=0)+on(gamma3,u4=0)+on(gamma4,u4=f4);

real N=20;
for(real t=1;t<N;t+=1){
segregazione;
uold1=u1;
uold2=u2;
uold3=u3;
uold4=u4;
cout <<t<<endl;}

uT=u1+u2+u3+u4;
plot (uT,value=true,wait=true,fill=false,nbiso=80,grey=true);
```

5. Conclusions

The steady state of k competing species coexisting in the same area is governed by the competing-diffusion system (1.1). Large interaction induces the spatial segregation of the species in the limit configuration as μ tends to $+\infty$, that is different densities have disjoint support. We analysed the behaviour of the solutions in the case of 4 competing species as the competition rate tends to infinity and we proved that only two configurations are possible: or the interfaces between components meet in a 4-point that is a point where four components concur, or the interfaces between components meet in two 3-points, that is points where only three components concur (see figure 5).

We characterized, for a given datum, the possible 4-point configuration by means of the solution of a Dirichlet problem for the Laplace equation. We gave necessary and sufficient conditions on the datum which generates a 4-point which suggest that the most probable configurations in nature are those with 3-points.

Acknowledgment

The authors thank professor Andrea Dall'Aglio for many fruitful discussions.

References

[1] G. Alessandrini, *Critical points of solutions of elliptic equations in two variables*, Ann. S.N.S. **14**, 1987, 229-256.

[2] V. Arakelyan, F. Bozorgnia, *Uniqueness of Limiting Solution to a Strongly Competing System*, Elect. J. Diff. Eqs. **2017**, 2017.

[3] H. Berestycki, A. Zilio, *Predators-Prey models with Competition. Part I: Existence, Bifurcation and Qualitative Properites*, Comm. Contemp. Math. 2018.

[4] M. Conti, S. Terracini, G. Verzini, *Nehari's problem and competing species systems*, Ann. I.H. Poincaré A.N. **19**, 2002, 871-888.

[5] M. Conti, S. Terracini, G. Verzini, *Asymptotic estimates for the spatial segregation of competitive systems*, Adv. Math. **195**, 2005, 524-560.

[6] M. Conti, S. Terracini, G. Verzini, *A Variational Problem for the Spatial Segregation of Reaction-diffusion Systems*, Indiana Univ. Math. J. **54**, 2005, 779-815.

[7] M. Conti, S. Terracini, G. Verzini, *Uniqueness and least energy property for solutions to strongly competing systems*, Interf. Free Bound. **8**, 2006, 437-446.

[8] E.N. Dancer, K. Wang, Z. Zhang, *Dynamics of Strongly Competing Systems with many Species*, Trans. A.M.S. **364**, 2012, 961-1005.

[9] www.freefem.org

[10] G.F. Gause, *Experimental studies on the struggle for existence: 1. Mixed population of two species of yeast*, J. Exper. Biol **9**, 1932, 389-402.

[11] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag 2001.

[12] G. Hardin, *The Competive Exclusion Principle*, Science, New Series, **131**, 1960, 1292-1297.

[13] A.C. Ponce, *Elliptic PDEs, Measures and Capacities*, EMS Tracts in Mathematics **23**, European Mathematical Society, 2016.

[14] H.Tavares, S.Terracini, *Regularity of the nodal set of segregated critical configurations under weak reflection law*, Calc. Var.**45**, 2012, 273–317.

[15] S. Terracini, G. Verzini, A. Zilio, *Spiraling asymptotic profiles of competition-diffusion systems*, to appear in Comm. Pure Appl. Math.

[16] J. L. Walsh, *The location of Critical Points of Analytic and Harmonic Functions*, American Mathematical Society, 1950.

[17] K. Wang, Z. Zhang, *Some new results in competing systems with many species*, Ann. I.H. Poincaré A.N. **27**, 2010, 739-761.

Flavia Lanzara
Dipartimento di Matematica *G. Castelnuovo*
Sapienza Università di Roma, piazzale Aldo Moro 5, 00185 Roma
e-mail: `lanzara@mat.uniroma1.it`

Eugenio Montefusco
Dipartimento di Matematica *G. Castelnuovo*
Sapienza Università di Roma, piazzale Aldo Moro 5, 00185 Roma
e-mail: `montefus@mat.uniroma1.it`