SIGNED RADON MEASURE-VALUED SOLUTIONS OF FLUX SATURATED SCALAR CONSERVATION LAWS

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ABSTRACT. We prove existence and uniqueness for a class of signed Radon measure-valued entropy solutions of the Cauchy problem for a first order scalar hyperbolic conservation law in one space dimension. The initial data of the problem is a finite superposition of Dirac masses, whereas the flux is Lipschitz continuous and bounded. The solution class is determined by an additional condition which is needed to prove uniqueness.

1. INTRODUCTION

We study the Cauchy problem for the scalar conservation law:

$$(CL) \qquad \begin{cases} u_t + [H(u)]_x = 0 & \text{in } \mathbb{R}\times(0,T) =: S \\ u = u_0 & \text{in } \mathbb{R}\times\{0\}\,, \end{cases}$$

where

$$(A_0) \qquad H \in W^{1,\infty}(\mathbb{R})\,, \quad H(0) = 0$$

(obviously the condition $H(0) = 0$ is not restrictive). The initial condition u_0 is a signed Radon measure on $\mathbb{R}$. In most of the paper we shall assume that its singular part, u_{0s}, is a finite superposition of Dirac masses:

$$(A_1) \qquad u_{0s} = \sum_{j=1}^{p} c_j \delta_{x_j} \qquad (x_1 < x_2 < \cdots < x_p;\;\; c_j \in \mathbb{R}\setminus\{0\} \text{ for } 1 \le j \le p)\,.$$

In that case we denote the support of the singular measure u_{0s} by F:

$$F = \{x_1, x_2, \cdots, x_p\}.$$

In [3] we considered the case of *nonnegative* initial measures u_0. In the present paper we consider the case of *signed* measures (see [2, 5, 7, 9] for motivations and related remarks). A specific motivation is the link between measure-valued solutions of (CL) and *discontinuous* solutions of the Cauchy problem for the Hamilton-Jacobi equation

$$(HJ) \qquad \begin{cases} U_t + H(U_x) = 0 & \text{in } S \\ U = U_0 & \text{in } \mathbb{R}\times\{0\}\,, \end{cases}$$

where $U_0 \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $U_0' \in L^1_{loc}(\mathbb{R}\setminus F)$, and $U_0(x_j^+) \ne U_0(x_j^-)$ if $x_j \in F$. If (A_1) is satisfied, the distributional derivative U_0' is a Radon measure without

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singular continuous part, $U_0' = \sum_{j=1}^{p} \left[U_0(x_j^+) - U_0(x_j^-)\right] \delta_{x_j} + (U_0')_{ac}$, and problems (CL), (HJ) are formally related by the equality $u = U_x$. In a forthcoming paper [4], problem (HJ) will be studied in the context of viscosity solutions.

It is known ([2]) that (i) the singular part u_s of a suitably defined entropy solution may persist for some positive time (see [2, Theorem 3.5]) and (ii) entropy solutions are not always uniquely determined by the initial condition u_0 (see also Remark 3.2). To overcome the latter problem, we introduced in [3] a so-called compatibility condition at those points where $u_s(\cdot,t)$ is a Dirac mass, and used it as a uniqueness criterion for nonnegative measure-valued solutions.

The starting point of the present paper is the statement that, for general signed initial measures u_0, the singular part u_s of any local entropy solution u of (CL) (in the sense of Definition 3.2) satisfies a monotonicity result: both the positive and negative part of u_s, $[u(\cdot,t)]_s^{\pm}$, are nonincreasing with respect to t (see Theorem 3.1). For the class of initial measures satisfying (A_1) this implies that the support of the singular part u_s of an entropy solution of problem (CL) is a subset of $F \times [0,T]$ and, in addition, that the sign of $[u(\cdot,t)]_s$ is determined by that of u_{0s}. Having this in mind it is rather straightforward to adapt the concept of compatibility condition in [3] to *signed* measure-valued solutions (see Definition 3.3).

The main result of the paper is that if (A_0) and (A_1) are satisfied, then (CL) is well-posed in the class of entropy solutions which satisfy the compatibility condition at the p points $x_j \in F$.

Existence of a solution is proven by a constructive approach which can be outlined as follows. By (A_0)-(A_1) there exists a positive time τ until which all singularities persist (see [2, Theorem 3.5]), thus the real line is the disjoint union of $p+1$ intervals. In each interval we solve the initial-boundary value problem for the conservation law in (CL), the initial data being the restriction of u_{0r} to that interval, with "boundary conditions equal to infinity". Namely, we consider the *singular* Dirichlet initial-boundary value problems

$$
\begin{cases}
u_t + [H(u)]_x = 0 & \text{in } (x_{j-1}, x_j) \times (0,T) \\
u = \pm\infty & \text{in } \{x_{j-1}\} \times (0,T) \\
u = \pm\infty & \text{in } \{x_j\} \times (0,T) \\
u = u_{0r} & \text{in } (x_{j-1}, x_j) \times \{0\}
\end{cases}
\tag{1.1}
$$

with $j = 2, \dots, p$, and

$$
\begin{cases}
u_t + [H(u)]_x = 0 & \text{in } (-\infty, x_1) \times (0,T) \\
u = \pm\infty & \text{in } \{x_1\} \times (0,T) \\
u = u_{0r} & \text{in } (-\infty, x_1) \times \{0\},
\end{cases}
\tag{1.2}
$$

$$
\begin{cases}
u_t + [H(u)]_x = 0 & \text{in } (x_p, \infty) \times (0,T) \\
u = \pm\infty & \text{in } \{x_p\} \times (0,T) \\
u = u_{0r} & \text{in } (x_p, \infty) \times \{0\}.
\end{cases}
\tag{1.3}
$$

The choice between $u = \infty$ and $u = -\infty$ at x_j is determined by the sign of c_j: we choose ∞ if $c_j > 0$ and $-\infty$ if $c_j < 0$. Existence and uniqueness of an entropy solution to each problem (1.1)-(1.3) is proven in Sections 5-6. In particular, existence follows from an approximation procedure which makes use of BV initial and boundary data, avoiding the L^∞-theory of initial-boundary value problems developed in [11] (see Section 6).

The function determined by solutions of (1.1)-(1.3) in $\mathbb{R}\times(0,\tau)$ is, by definition, the regular part of a Radon measure, whose singular part is defined by observing that the variation of mass at each point x_j depends on the sweeping effect of the flux across x_j (see (7.2), (7.4), (7.6b) and Proposition 5.3). Then it is proven that this measure is the unique entropy solution of (CL) (in the sense of Definition 3.2) which satisfies the compatibility conditions at all $x_j \in F$ until the time $t=\tau$. Here we use that the required compatibility condition for the solution of the Cauchy problem (CL) at x_j is exactly the entropic formulation of the boundary conditions "$u = \pm\infty$" for the singular Dirichlet problems (see also Remark 5.3). If $\tau < T$ we iterate the procedure in $\mathbb{R}\times(\tau,T)$ with a smaller number a singularities, thus well-posedness of (CL) follows in a finite number of steps (see Section 7). We observe that the proof of uniqueness of entropy solutions to problem (CL) relies on a general comparison principle between entropy sub and super-solutions of (1.1)–(1.3) (see Definitions 5.2–5.5 and Theorem 5.2 below) which is independent of the above construction procedure. In this sense the comparison results are stronger than those in [3, Theorem 3.2].

The results in the paper can be esaily extended to the case that u_{0s} is a locally finite superposition of Dirac masses (namely, if the number of Dirac masses in every bounded interval is finite).

2. Preliminaries

Let χ_E denote the characteristic function of $E\subseteq\mathbb{R}$. For every $u\in\mathbb{R}$ we set

$$[u]_\pm := \max\{\pm u, 0\}, \quad \mathrm{sgn}_\pm(u) := \pm\chi_{\mathbb{R}_\pm}(u), \quad \mathrm{sgn}(u) := \mathrm{sgn}_-(u) + \mathrm{sgn}_+(u)\,.$$

For every real function f on $\mathbb{R}$ and $x_0\in\mathbb{R}$ we say that

$$\operatorname*{ess\,lim}_{x\to x_0^\pm} f(x) = l \in \mathbb{R}\,,$$

if there is a null set $E^*\subseteq\mathbb{R}$ such that $f(x_n)\to l$ if $\{x_n\}\subseteq\mathbb{R}\setminus(E^*\cup\{x_0\})$, $x_n\to x_0^\pm$.

For every open subset $\Omega\subseteq\mathbb{R}$ we denote by $C_c(\Omega)$ the space of continuous real functions with compact support in Ω and by $\mathcal{M}^+(\Omega)$ the cone of the nonnegative Radon measures on Ω. According to [6, Section 1.3], we say that ν is a (signed) Radon measure on Ω if there exist a (nonnegative) Radon measure $\mu\in\mathcal{M}^+(\Omega)$ and a locally μ-summable function $f:\Omega\to[-\infty,\infty]$ such that

$$\nu(K) = \int_K f\,d\mu$$

for all compact sets $K\subset\Omega$. The space of (signed) Radon measures on Ω will be denoted by $\mathcal{M}(\Omega)$.

If $\mu,\nu\in\mathcal{M}(\Omega)$, we say that $\mu\le\nu$ in $\mathcal{M}(\Omega)$ if $\nu-\mu\in\mathcal{M}^+(\Omega)$. We denote by $\langle\cdot,\cdot\rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and $C_c(\Omega)$. A sequence $\{\mu_n\}$ of Radon measures on $\mathbb{R}$ converges weakly* to a Radon measure μ, $\mu_n \overset{*}{\rightharpoonup} \mu$, if $\langle\mu_n,\rho\rangle_\mathbb{R}\to\langle\mu,\rho\rangle_\mathbb{R}$ for all $\rho\in C_c(\mathbb{R})$. For any compact $K\in\mathbb{R}$ the space $\mathcal{M}(K)$ is a Banach space with norm $\|\mu\|_{\mathcal{M}(K)} := |\mu|(K)$, where $|\mu|$ denotes the total variation of μ. A sequence $\{\mu_n\}$ converges strongly to μ in $\mathcal{M}(K)$ if $\|\mu_n-\mu\|_{\mathcal{M}(K)}\to 0$ as $n\to\infty$. Similar definitions are used for Radon measures on any subset of $S := \mathbb{R}\times(0,T)$.

Every $\mu\in\mathcal{M}(\mathbb{R})$ has a unique decomposition $\mu = \mu_{ac}+\mu_s$, with $\mu_{ac}\in\mathcal{M}(\mathbb{R})$ absolutely continuous and $\mu_s\in\mathcal{M}(\mathbb{R})$ singular with respect to the Lebesgue measure. We denote by $\mu_r\in L^1_{loc}(\mathbb{R})$ the density of μ_{ac}. Every function $f\in L^1_{loc}(\mathbb{R})$ can be identified to an absolutely continuous Radon measure on $\mathbb{R}$; we shall denote this measure by the same symbol f used for the function.

The restriction $\mu \llcorner E$ of $\mu \in \mathcal{M}(\mathbb{R})$ to a Borel set $E \subseteq \mathbb{R}$ is defined by $(\mu \llcorner E)(A) := \mu(E \cap A)$ for any Borel set $A \subseteq \mathbb{R}$. Similar notations are used for $\mathcal{M}(S)$.

For every open subset $\Omega \subseteq \mathbb{R}$ we denote by $BV(\Omega)$ the Banach space of functions of bounded variation in Ω:

$$BV(\Omega) := \left\{ z \in L^1(\Omega) \,|\, z' \in \mathcal{M}(\Omega), \|z'\|_{\mathcal{M}(\Omega)} < \infty \right\}, \quad \|z\|_{BV(\Omega)} := \|z\|_{L^1(\Omega)} + \|z'\|_{\mathcal{M}(\Omega)},$$

where z' is the first order distributional derivative. The total variation in Ω of z is $TV(z;\Omega) := \|z'\|_{\mathcal{M}(\Omega)}$. We say that $z \in BV_{loc}(\mathbb{R})$ if $z \in BV(\Omega)$ for every open subset $\Omega \subset\subset \mathbb{R}$.

In the remainder of this section Ω denotes an open subset of $\mathbb{R}$, and $Q_T = \Omega \times (0,T)$. By $C([0,T];\mathcal{M}(\Omega))$ we denote the subset of strongly continuous mappings from $[0,T]$ into $\mathcal{M}(\Omega)$ - namely, $u \in C([0,T];\mathcal{M}(\Omega))$ if for all $t_0 \in [0,T]$ and for every compact $K \in \Omega$ there holds $\|u(\cdot,t) - u(\cdot,t_0)\|_{\mathcal{M}(K)} \to 0$ as $t \to t_0$.

Definition 2.1. We denote by $L^\infty(0,T;\mathcal{M}^+(\Omega))$ the set of nonnegative Radon measures $u \in \mathcal{M}^+(Q_T)$ such that for a.e. $t \in (0,T)$ there is a measure $u(\cdot,t) \in \mathcal{M}^+(\Omega)$ with the following properties:
(i) if $\zeta \in C([0,T];C_c(\Omega))$ the map $t \mapsto \langle u(\cdot,t), \zeta(\cdot,t)\rangle_\Omega$ belongs to $L^1(0,T)$ and

$$\langle u, \zeta \rangle_{Q_T} = \int_0^T \langle u(\cdot,t), \zeta(\cdot,t)\rangle_\Omega \, dt\,; \tag{2.1}$$

(ii) the map $t \mapsto \|u(\cdot,t)\|_{\mathcal{M}(K)}$ belongs to $L^\infty(0,T)$ for every compact $K \subset \Omega$.

Remark 2.1. Definition 2.1 implies that for all $\rho \in C_c(\Omega)$ the map $t \mapsto \langle u(\cdot,t), \rho\rangle_\Omega$ is measurable, thus the map $u : (0,T) \to \mathcal{M}^+(\Omega)$ is weakly* measurable. For simplicity we prefer the notation $L^\infty(0,T;\mathcal{M}^+(\Omega))$ to the more correct one $L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$. Moreover, as a consequence of Definition 2.1-(i), it can be seen that for every Borel set $E \subseteq Q_T$ the map $t \to u(\cdot,t)\big(E^t\big)$ is Lebesgue measurable and there holds

$$u(E) = \int_0^T u(\cdot,t)\big(E^t\big)\, dt \qquad (E^t = \{x \in \Omega : \ (x,t) \in E\}). \tag{2.2}$$

If $u \in L^\infty(0,T;\mathcal{M}^+(\Omega))$, then $u_{ac}, u_s \in L^\infty(0,T;\mathcal{M}^+(\Omega))$ as well, and $u_r \in L^\infty(0,T;L^1_{\text{loc}}(\Omega))$. Moreover, equality (2.1) implies

$$\langle u_{ac}, \zeta\rangle_{Q_T} = \iint_{Q_T} u_r \, \zeta \, dxdt \quad \text{and} \quad \langle u_s, \zeta\rangle_{Q_T} = \int_0^T \langle u_s(\cdot,t), \zeta(\cdot,t)\rangle_\Omega \, dt.$$

Denoting by $[u(\cdot,t)]_{ac}$, $[u(\cdot,t)]_s \in \mathcal{M}^+(\Omega)$ the absolutely continuous and the singular part of the measure $u(\cdot,t) \in \mathcal{M}^+(\Omega)$, a routine proof shows that for *a.e.* $t \in (0,T)$

$$u_s(\cdot,t) = [u(\cdot,t)]_s\,, \quad u_{ac}(\cdot,t) = [u(\cdot,t)]_{ac}\,, \quad u_r(\cdot,t) = [u(\cdot,t)]_r\,, \tag{2.3}$$

where $[u(\cdot,t)]_r$ denotes the density of the measure $[u(\cdot,t)]_{ac}$. In view of (2.3), we shall always identify the quantities which appear on either side of equalities (2.3).

We say that a (signed) Radon measure $u \in \mathcal{M}(Q_T)$ belongs to $L^\infty(0,T;\mathcal{M}(\Omega))$ if both u^+ and u^- belong to $L^\infty(0,T;\mathcal{M}^+(\Omega))$. In particular, this implies that:
(α) the total variation $|u|$ of the measure u belongs to $L^\infty(0,T;\mathcal{M}^+(\Omega))$;
(β) conditions (i) and (ii) of Definition 2.1 hold with $u(\cdot,t) := u^+(\cdot,t) - u^-(\cdot,t)$ for *a.e.* $t \in (0,T)$.

Moreover, since u^+ and u^- are mutually singular, it follows that for *a.e.* t the nonnegative measures $u^+(\cdot,t)$ and $u^-(\cdot,t)$ are mutually singular, whence

$$u^{\pm}(\cdot,t) = [u(\cdot,t)]^{\pm}\,, \qquad |u(\cdot,t)| = |u|(\cdot,t) \quad \text{for } a.e.\ t\in(0,T)\,, \tag{2.4}$$

and

$$u_s^{\pm}(\cdot,t) = [u(\cdot,t)]_s^{\pm}, \quad |u_s|(\cdot,t) = |[u(\cdot,t)]_s| \quad \text{for } a.e.\ t\in(0,T)\,. \tag{2.5}$$

3. Results

For any $\tau\in(0,T]$ and open subset $\Omega\subseteq\mathbb{R}$ set $Q_\tau := \Omega\times(0,\tau]$, $Q_T \equiv Q$; set also $S_\tau := \mathbb{R}\times(0,\tau]$, $S_T\equiv S$. Solutions of problem (CL) are meant in the following sense.

Definition 3.1. Let u_0 be a signed Radon measure on Ω and let (A_0) be satisfied. A measure $u\in L^\infty(0,T;\mathcal{M}(\Omega))$ is a *solution* of problem (CL) in Q_τ if for all $\zeta\in C^1([0,\tau];C^1_c(\Omega))$, $\zeta(\cdot,\tau)=0$ in Ω there holds

$$\iint_{Q_\tau}\left[u_r\zeta_t + H(u_r)\,\zeta_x\right]dxdt + \int_0^\tau \langle u_s(\cdot,t),\zeta_t(\cdot,t)\rangle_\Omega\,dt = -\langle u_0,\zeta(\cdot,0)\rangle_\Omega\,. \tag{3.1}$$

Solutions of (CL) in S are simply referred to as "solutions of (CL)".

Definition 3.2. Let u_0 be a signed Radon measure on Ω and let (A_0) be satisfied. A solution of (CL) in Q_τ is called an *entropy solution in* Q_τ if it satisfies the *entropy inequality*

$$\begin{aligned}\iint_{Q_\tau}\{|u_r-k|\,\zeta_t + \operatorname{sgn}(u_r-k)\,[H(u_r)-H(k)]\,\zeta_x\}\,dxdt\ +\\ +\int_0^\tau\langle|u_s(\cdot,t)|,\zeta_t(\cdot,t)\rangle_\Omega\,dt \ge -\int_\Omega|u_{0r}(x)-k|\,\zeta(x,0)\,dx - \langle|u_{0s}|,\zeta(\cdot,0)\rangle_\Omega\end{aligned} \tag{3.2}$$

for all $\zeta\in C^1([0,\tau];C^1_c(\Omega))$, $\zeta\ge0$, $\zeta(\cdot,\tau)=0$ in Ω, and for all $k\in\mathbb{R}$;

If $Q_\tau\ne Q_T$, an (entropy) solution in Q_τ can be considered as a *local* (entropy) solution of (CL). For general initial measures, local entropy solutions satisfy the following monotonicity result.

Theorem 3.1. *Let (A_0) be satisfied, let u_0 be a signed Radon measure on Ω and let u be an entropy solution u of problem (CL) in Q_T. Then, for a.e. $0<t_1<t_2<T$, there holds*

$$[u(\cdot,t_2)]_s^{\pm} \le [u(\cdot,t_1)]_s^{\pm} \le u_{0s}^{\pm} \quad \text{in } \mathcal{M}(\Omega)\,. \tag{3.3}$$

Now we consider the case that u_{0s} is the sum of a finite number of Dirac masses with support F.

Remark 3.1. Let $(A_0)-(A_1)$ be satisfied and let u be an entropy solution of problem (CL) in Q_T. Arguing as in the proof of Proposition 3.20 in [2], it follows that $u\in C((0,T];\mathcal{M}(\Omega))$.

Corollary 3.2. *Let $(A_0)-(A_1)$ be satisfied and let u be an entropy solution u of problem (CL) in Q_T. Then $u_s\in C([0,T];\mathcal{M}(\Omega))$, $[u(\cdot,0)]_s = u_{0s}$, (3.3) holds for any $0\le t_1\le t_2\le T$ and for every $x_j\in F\cap\Omega$ there exists $t_j\in(0,T]$ such that*

$$\begin{cases} u_s(\cdot,t)(\{x_j\})\ne0 & \text{if } t\in[0,t_j),\\ u_s(\cdot,t)(\{x_j\})=0 & \text{if } t\in(t_j,T].\end{cases} \tag{3.4}$$

We observe that the proof of Corollary 3.2 provides an explicit lower bound for t_j.

If $x_j \in F$ and $t_j \in (0,T]$ as in Corollary 3.2, Theorem 3.1 implies that the support of the singular part of any entropy solution is a subset of $F \times [0,T]$ and that the Delta mass at $x_j \in F$ does not change sign in the interval $[0,t_j)$. Therefore we may formulate a compatibility condition at x_j which depends on the sign of c_j, i.e. on the sign of the initial Delta mass at x_j:

Definition 3.3. Let $(A_0) - (A_1)$ be satisfied. An entropy solution u of (CL) in Q_T is said to satisfy the **compatibility condition** at $x_j \in F \cap \Omega$ if

$$\text{(3.5a)}\quad \operatorname{ess}\lim_{x\to x_j^+}\int_0^{t_j}\operatorname{sgn}_\pm(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \le 0 \quad \text{if } \pm c_j < 0,$$

$$\text{(3.5b)}\quad \operatorname{ess}\lim_{x\to x_j^-}\int_0^{t_j}\operatorname{sgn}_\pm(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \ge 0 \quad \text{if } \pm c_j < 0$$

for all $\beta \in C_c^1(0,t_j)$, $\beta \ge 0$ and $k \in \mathbb{R}$, where $t_j \in (0,T]$ is defined by Corollary 3.2.

We shall prove below (see Remark 5.4) that, if (A_0)-(A_1) hold, for every entropy solution u of (CL) the limits

$$\text{(3.6)}\quad \operatorname{ess}\lim_{x\to x_j^\pm}\int_0^{T}\operatorname{sgn}_\pm(u_r(x,t)-k)[H(u_r(x,t))-H(k)]\,\beta(t)\,dt \qquad (j=1,\dots,p)\,,$$

with β and k as above, exist and are finite. Hence Definition 3.3 is well-posed.

The main result of the paper is the well-posedness of problem (CL), if u_0 satisfies (A_1), in the class of entropy solutions in $C([0,T];\mathcal{M}(\mathbb{R}))$ which satisfy the compatibility condition in $\operatorname{supp} u_0$.

Theorem 3.3. *Let (A_0)-(A_1) be satisfied. Then there exists a unique entropy solution of problem (CL) which belongs to $C([0,T];\mathcal{M}(\mathbb{R}))$ and satisfies the compatibility condition at all $x_j \in \operatorname{supp} u_0$.*

Remark 3.2. It was already observed in [2] that in general measure-valued entropy solutions are not unique. This is essentially a consequence of the elementary observation that there exists a unique entropy solution for which $[u_s(t)] = u_{0s}$ for a.e. $t \in (0,T)$ (it is enough to set $u = u_0 + \tilde{u}$, where $\tilde{u}$ is the entropy solution with initial data u_{0r}). But if u_0 satisfies (A_1) and $F \ne \emptyset$, one easily checks that if the function H, satisfying (A_0), is not constant in intervals of the type (a,∞) and $(-\infty,b)$, then such solution does not satisfy the compatibility condition at $x_j \in F$. In particular, it does not coincide with the solution defined by Theorem 3.3.

4. Monotonicity of u_s.

In this section we prove Theorem 3.1 and Corollary 3.2.

Proof of Theorem 3.1. By (3.1), for every $k \in \mathbb{R}$ we get

$$\begin{aligned}
&\iint_S (u_r-k)\beta'(t)\rho(x)\,dxdt + \int_0^T \langle u_s(\cdot,t),\rho\rangle_{\mathbb{R}}\,\beta'(t)\,dt + \\
&+ \iint_S \left[H(u_r)-H(k)\right]\rho'(x)\beta(t)\,dxdt = \\
&= -\beta(0)\left\{\int_{\mathbb{R}}(u_{0r}-k)\rho(x)\,dx + \langle u_{0s}\,,\,\rho\rangle_{\mathbb{R}}\right\}
\end{aligned}$$

for all $\rho \in C^1_c(\Omega)$ and $\beta \in C^1_c([0,T))$. By summing and subtracting the above equality from the entropy inequality (3.2), for every nonnegative ρ and β as above we obtain

$$\iint_S [u_r - k]_\pm \, \beta'(t)\rho(x)\,dxdt + \int_0^T \langle [u(\cdot,t)]^\pm_s, \rho\rangle_{\mathbb{R}} \, \beta'(t)\,dt + \\ + \iint_S \operatorname{sgn}_\pm(u_r - k)\,[H(u_r) - H(k)]\,\rho'(x)\beta(t)\,dxdt \geq \\ \geq -\beta(0)\left\{\int_{\mathbb{R}} [u_{0r} - k]_\pm \, \rho(x)\,dx + \langle u^\pm_{0s}\,,\,\rho\rangle_{\mathbb{R}}\right\},$$

Letting $k \to \infty$ with "+" and $k \to -\infty$ with "-", we obtain that

$$\int_0^T \langle [u(\cdot,t)]^\pm_s, \rho\rangle_{\mathbb{R}} \, \beta'(t)\,dt \geq \langle u^\pm_{0s}\,,\,\rho\rangle_{\mathbb{R}}\,.$$

Let $0 < t_1 < t_2 \leq T$. By standard approximation arguments we can choose

$$\beta(t) = \beta_n(t) = n(t - t_1)\chi_{[t_1,t_1+1/n]}(t) + \chi_{(t_1+1/n,t_2-1/n)}(t) + n(t_2 - t)\chi_{[t_2-1/n,t_2]}\,.$$

Arguing as in the proof of Proposition 3.8(i) in [2], there exists a null set $N \in (0,T)$ which does not depend on the function ρ such that, letting $n \to \infty$,

$$\langle [u(\cdot,t_2)]^\pm_s\,,\,\rho\rangle_{\mathbb{R}} \leq \langle [u(\cdot,t_1)]^\pm_s\,,\,\rho\rangle_{\mathbb{R}} \quad \text{if } t_1, t_2 \notin N\,. \tag{4.7}$$

Hence the first inequality in (3.3) follows from the arbitrariness of ρ.

The second inequality in (3.3) can be proved in a similar way, replacing β_n by

$$\beta_n(t) = \chi_{(0,t_1-1/n)}(t) + n(t_1 - t)\chi_{[t_1-1/n,t_1]}\,.$$

□

Proof of Corollary 3.2. Arguing as in the proof of Theorem 3.1, for every $\rho \in C^1_c(\Omega)$ and $t \in (0,T]$ from (3.1) we get

$$\langle u(\cdot,t)\,,\,\rho\rangle_{\mathbb{R}} - \langle u_0\,,\,\rho\rangle_{\mathbb{R}} = \int_0^t \int_{\mathbb{R}} H(u_r)\,\rho'(x)\,dxds\,. \tag{4.8}$$

Fix any $x_j \in F \cap \Omega$. By standard approximation arguments we can choose in (4.8)

$$\rho(x) = \rho_n(x) = n(x - x_j + 1/n)\chi_{[x_j-1/n,x_j]} + n(x_j + 1/n - x)\chi_{(x_j,x_j+1/n]}\,.$$

Then letting $n \to \infty$, and observing that

$$\left|\int_0^t \int_{\mathbb{R}} H(u_r)\,\rho'_n(x)\,dxds\right| \leq \|H\|_\infty t \int_{\mathbb{R}} |\rho'_n(x)|\,dx \leq 2\,\|H\|_\infty t\,,$$

we obtain

$$|u_s(\cdot,t)\,(\{x_j\}) - u_{0s}\,(\{x_j\})| \leq 2\,\|H\|_\infty t\,. \tag{4.9}$$

Since, by Theorem 3.1, $u_s(\cdot,t) = u_s(\cdot,t) \llcorner F$ and F contains p points, we obtain that $\|u_s(\cdot,t) - u_{0s}\| \leq 2p\|H\|_\infty t$. Hence $u_s \in C([0,T];\mathcal{M}(\Omega))$ and $[u(\cdot,0)]_s = u_{0s}$ (see also Remark 3.1).

If $u_{0s} \llcorner \{x_j\} = \pm u^\pm_{0s} \llcorner \{x_j\}$, it follows from (3.3) that $[u(\cdot,t)]^\mp_s \llcorner \{x_j\} = 0$ for any $t \in [0,T]$. Then inequality (4.9) gives

$$[u_s(\cdot,t)]^\pm\,(\{x_j\}) \geq u^\pm_{0s}\,(\{x_j\}) - 2t\|H\|_\infty > 0$$

for any $t \in [0,t_j)$, with $t_j := \frac{u^\pm_{0s}(\{x_j\})}{2\|H\|_\infty}$. Then by the monotonicity of the mappings $t \mapsto u^\pm_s(\cdot,t)$ (see (3.3)) the conclusion follows. □

5. Problem (D): comparison and uniqueness

As already said, to address (CL) we need results concerning *singular* Dirichlet initial-boundary value problems for the scalar conservation law:

$$
(D) \qquad \begin{cases} u_t + [H(u)]_x = 0 & \text{in } \Omega \times (0,T) =: Q \\ u = m_1 & \text{in } \{a\} \times (0,T) \\ u = m_2 & \text{in } \{b\} \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}
$$

where $\Omega = (a,b)$ is a bounded interval, $m_1 = \pm\infty$, $m_2 = \pm\infty$, and $u_0 : \Omega \mapsto \mathbb{R}$. Similar problems will be considered also for half-lines, either $\Omega \equiv (a,\infty)$, or $\Omega \equiv (-\infty,b)$; obviously, the above condition at $\{b\} \times (0,T)$ is omitted when $\Omega \equiv (a,\infty)$, and that at $\{a\} \times (0,T)$ is omitted when $\Omega \equiv (-\infty,b)$.

We shall denote problem (D) by (D_S) when $m_1 = \pm\infty$, $m_2 = \pm\infty$, or by (D_R) when both m_1 and m_2 are finite. When $\Omega = (a,b)$ problem (D_S) stands for four different initial-boundary value problems, which we denote by (D^+_+), (D^-_-), (D^-_+) and (D^+_-) according to the four choices $m_1 = m_2 = \infty$, $m_1 = m_2 = -\infty$, $m_1 = \infty, m_2 = -\infty$ and $m_1 = -\infty, m_2 = \infty$. In the case of half-lines problem (D_S) consists only of two cases, namely

$$
(D_\pm) \qquad \begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = \pm\infty & \text{in } \{a\} \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}
$$

if $\Omega = (a,\infty)$, and

$$
(D^\pm) \qquad \begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = \pm\infty & \text{in } \{b\} \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}
$$

if $\Omega = (-\infty,b)$. We shall write that a statement holds for problem (D_S), if it collectively holds for all problems $(D^\pm_\pm)$.

The following definition concerns problem (D_R) (see [13]).

Definition 5.1. Let $\Omega = (a,b)$, $u_0 \in BV(\Omega)$.
(i) An *entropy subsolution* of (D_R) is any $\underline{u} \in BV(Q)$ such that:
(a) for every $k \in \mathbb{R}$ and for all $\zeta \in C^1([0,T]; C^1_c(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω, $\zeta \ge 0$ in Q,

$$
\iint_Q \{[u-k]_+\zeta_t + \operatorname{sgn}_+(u-k)[H(u)-H(k)]\zeta_x\}\,dxdt \ge -\int_\Omega [u_0-k]_+\zeta(x,0)\,dx; \tag{5.1}
$$

(b) for a.e. $t \in (0,T)$ there holds

$$
\operatorname{sgn}_+(\underline{u}(a^+,t)-k)\,[H(\underline{u}(a^+,t))-H(k)] \le 0 \quad \text{if } k > m_1\,, \tag{5.2}
$$

$$
\operatorname{sgn}_+(\underline{u}(b^-,t)-k)\,[H(\underline{u}(b^-,t))-H(k)] \ge 0 \quad \text{if } k > m_2\,. \tag{5.3}
$$

(ii) An *entropy supersolution* of (D_R) is any $\overline{u} \in BV(Q)$ such that:
(a') for every $k \in \mathbb{R}$ and if $\zeta \in C^1([0,T]; C^1_c(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω, $\zeta \ge 0$ in Q,

$$
\iint_Q \{[u-k]_-\zeta_t + \operatorname{sgn}_-(u-k)[H(u)-H(k)]\zeta_x\}\,dxdt \ge -\int_\Omega [u_0-k]_-\zeta(x,0)\,dx; \tag{5.4}
$$

(b') for a.e. $t \in (0,T)$ there holds

$$
\operatorname{sgn}_-(\overline{u}(a^+,t)-k)\,[H(\overline{u}(a^+,t))-H(k)] \le 0 \quad \text{if } k < m_1\,, \tag{5.5}
$$

$$\text{sgn}_-(\overline{u}(b^-,t)-k)\,[H(\overline{u}(b^-,t))-H(k)] \ge 0 \quad \text{if } k < m_2 \,. \tag{5.6}$$

(iii) A function $\overline{u} \in BV(Q) \cap C([0,T];L^1(\Omega))$ is an *entropy solution* of (D_R) if it is both an entropy subsolution and an entropy supersolution.

When $\Omega = (a,\infty)$ entropy sub- and supersolutions of (D_R) are defined as above, only dropping conditions (5.3) and (5.6); similarly, conditions (5.2) and (5.5) are omitted if $\Omega = (-\infty,b)$. Moreover, in these cases we require that $\underline{u},\overline{u}$ belong to $BV_{loc}(Q) \cap L^\infty(Q)$.

Remark 5.1. If $\underline{u},\overline{u} \in BV(Q)$, the traces $\underline{u}(a^+,t) := \text{ess}\lim_{\xi\to a^+} \underline{u}(\xi,t)$, $\underline{u}(b^-,t) := \text{ess}\lim_{\eta\to b^-} \underline{u}(\eta,t)$ exist for a.e. $t \in (0,T)$, and similarly for $\overline{u}$. Hence the above definitions are well-posed. By the same token, conditions (5.2)-(5.3) and (5.5)-(5.6) can be reformulated as follows: for every $\beta \in C^1_c(0,T)$, $\beta \ge 0$,

$$\text{ess}\lim_{\xi\to a^+} \int_0^T \text{sgn}_+(\underline{u}(\xi,t)-k)\left[H(\underline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \le 0 \ \text{ if } k > m_1 \,, \tag{5.7a}$$

$$\text{ess}\lim_{\eta\to b^-} \int_0^T \text{sgn}_+(\underline{u}(\eta,t)-k)\left[H(\underline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \ge 0 \ \text{ if } k > m_2 \,, \tag{5.7b}$$

$$\text{ess}\lim_{\xi\to a^+} \int_0^T \text{sgn}_-(\overline{u}(\xi,t)-k)\left[H(\overline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \le 0 \ \text{ if } k < m_1 \,, \tag{5.7c}$$

$$\text{ess}\lim_{\eta\to b^-} \int_0^T \text{sgn}_-(\overline{u}(\eta,t)-k)\left[H(\overline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \ge 0 \ \text{ if } k < m_2 \,. \tag{5.7d}$$

The following definitions for problem (D_S) are formulated for a wider class of initial data.

Definition 5.2. Let $\Omega = (a,b)$, $u_0 \in L^1(\Omega)$.
(i) An *entropy subsolution* of (D^+_+) is any $\underline{u} \in C([0,T];L^1(\Omega))$ such that:
(a) for every $k \in \mathbb{R}$ and $\zeta \in C^1_c(Q)$, $\zeta \ge 0$ in Q

$$\iint_Q \{[\underline{u}-k]_+\zeta_t \,+\, \text{sgn}_+(\underline{u}-k)\,[H(\underline{u})-H(k)]\zeta_x\}\; dxdt \ge 0 \,, \tag{5.8}$$

and for any interval $I \subseteq \Omega$

$$\lim_{t\to 0^+} \int_I [\underline{u}(x,t)-u_0(x)]_+\, dx = 0 \,. \tag{5.9}$$

(ii) An *entropy subsolution* of (D^-_-) is any $\underline{u} \in C([0,T];L^1(\Omega))$ such that (a) holds, and for every $k \in \mathbb{R}$, $\beta \in C^1_c(0,T)$, $\beta \ge 0$,

$$\text{ess}\lim_{\xi\to a^+} \int_0^T \text{sgn}_+(\underline{u}(\xi,t)-k)\left[H(\underline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \le 0 \,, \tag{5.10a}$$

$$\text{ess}\lim_{\eta\to b^-} \int_0^T \text{sgn}_+(\underline{u}(\eta,t)-k)\left[H(\underline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \ge 0 \,. \tag{5.10b}$$

(iii) An *entropy subsolution* of (D^-_+) is any $\underline{u} \in C([0,T];L^1(\Omega))$ such that (a) holds, and for every $k \in \mathbb{R}$, $\beta \in C^1_c(0,T)$, $\beta \ge 0$ inequality (5.10b) holds.
(iv) An *entropy subsolution* of (D^+_-) is any $\underline{u} \in C([0,T];L^1(\Omega))$ such that (a) holds, and for every $k \in \mathbb{R}$, $\beta \in C^1_c(0,T)$, $\beta \ge 0$ inequality (5.10a) holds.

Definition 5.3. Let $\Omega = (a,b)$, $u_0 \in L^1(\Omega)$.
(i) An *entropy supersolution* of (D_+^+) is any $\overline{u} \in C([0,T];L^1(\Omega))$ such that:
(a') for every $k \in \mathbb{R}$ and $\zeta \in C_c^1(Q)$, $\zeta \geq 0$ in Q

$$\iint_Q \{[\overline{u}-k]_-\zeta_t + \operatorname{sgn}_-(\overline{u}-k)\,[H(\overline{u})-H(k)]\zeta_x\}\,dxdt \geq 0\,, \tag{5.11}$$

and for any interval $I \subseteq \Omega$

$$\lim_{t\to 0^+}\int_I [\overline{u}(x,t)-u_0(x)]_-\,dx = 0\,; \tag{5.12}$$

(b') for every $k \in \mathbb{R}$ and $\beta \in C_c^1(0,T)$, $\beta \geq 0$,

$$\operatorname*{ess\,lim}_{\xi\to a^+}\int_0^T \operatorname{sgn}_-(\overline{u}(\xi,t)-k)\left[H(\overline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \leq 0\,, \tag{5.13a}$$

$$\operatorname*{ess\,lim}_{\eta\to b^-}\int_0^T \operatorname{sgn}_-(\overline{u}(\eta,t)-k)\left[H(\overline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \geq 0\,. \tag{5.13b}$$

(ii) An *entropy supersolution* of (D_-^-) is any $\overline{u} \in C([0,T];L^1(\Omega))$ such that (a') holds.
(iii) An *entropy supersolution* of (D_+^-) is any $\overline{u} \in C([0,T];L^1(\Omega))$ such that (a') holds, and for every $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$, inequality (5.13a) holds.
(iv) An *entropy supersolution* of (D_-^+) is any $\overline{u} \in C([0,T];L^1(\Omega))$ such that (a') holds, and for every $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$, inequality (5.13b) holds.

Definition 5.4. A function $u \in C([0,T];L^1(\Omega))$ is called an *entropy solution* of (D_S) if it is both an entropy subsolution and an entropy supersolution of (D_S).

Observe that (5.13a)-(5.13b) can be regarded as limiting cases of (5.7c)-(5.7d), since for every $k \in \mathbb{R}$ there holds $\operatorname{sgn}_-(m_i - k) \to 0$ as $m_i \to \infty$ $(i = 1,2)$. Similarly, (5.10a)-(5.10b) can be regarded as limiting cases of (5.7a)-(5.7b) as $m_i \to -\infty$.

Remark 5.2. Let us prove that every entropy solution of (D_S) satisfies the weak formulation

$$\iint_Q \{u\zeta_t + H(u)\zeta_x\}\,dxdt = 0 \tag{5.14}$$

for every $\zeta \in C_c^1(Q)$.

To this aim, we fix any sequence $k_j \to -\infty$. By (5.8), for all $\zeta \in C_c^1(Q)$, $\zeta \geq 0$, there holds

$$\iint_Q \{[u-k_j]_+\zeta_t + \operatorname{sgn}_+(u-k_j)\,[H(u)-H(k_j)]\zeta_x\}\,dxdt \geq 0\,. \tag{5.15}$$

Let us take the limit as $j \to \infty$ in (5.15). Since $u \in L^1(Q)$ and H is bounded, we have

$$\int_Q \operatorname{sgn}_+(u-k_j)\,[H(u)-H(k_j)]\zeta_x\,dxdt = \iint_{\{u>k_j\}} H(u)\zeta_x\,dxdt \tag{5.16}$$
$$- \underbrace{\iint_Q H(k_j)\,\zeta_x\,dxdt}_{=0} + \iint_{\{u\leq k_j\}} H(k_j)\zeta_x\,dxdt \to \iint_Q H(u)\zeta_x\,dxdt\,,$$

and

$$\iint_Q [u-k_j]_+ \zeta_t \, dxdt = \iint_{\{u>k_j\}} u\zeta_t \, dxdt - \overbrace{\iint_Q k_j \zeta_t \, dxdt}^{=0} + \iint_{\{u\le k_j\}} k_j \zeta_t \, dxdt \to \iint_Q u\zeta_t \, dxdt \,, \tag{5.17}$$

(here we have used that $\left|\iint_{\{u\le k_j\}} k_j \zeta_t \, dxdt\right| \le \iint_{\{u\le k_j\}} |u| \, |\zeta_t| \, dxdt \to 0$, as $k_j \to -\infty$). In view of (5.16)–(5.17), letting $j \to \infty$ in (5.15) gives

$$\iint_Q \{u\zeta_t + H(u)\zeta_x\} \, dxdt \ge 0 \tag{5.18}$$

for every $\zeta \in C^1_c(Q)$, $\zeta \ge 0$. Analogously, letting $k_j \to \infty$ in

$$\iint_Q \{[u-k_j]_- \zeta_t + \mathrm{sgn}_-(u-k_j)\,[H(u) - H(k_j)]\zeta_x\} \, dxdt \ge 0 \tag{5.19}$$

(see (5.11)) gives, for every ζ as above,

$$\iint_Q \{u\zeta_t + H(u)\zeta_x\} \, dxdt \le 0\,. \tag{5.20}$$

Therefore the conclusion follows combining (5.18) and (5.20).

Remark 5.3. The conditions (5.10a-5.10b) and (5.13a-5.13b) are entropy boundary conditions for singular Dirichlet problems and give a meaning, in a hyperbolic sense, to the boundary conditions "$u = -\infty$" and "$u = \infty$". As already mentioned in the Introduction, they coincide with the compatibility conditions (3.5a) and (3.5b) for entropy solutions of (CL) at points x_j where a signed Dirac mass is concentrated.

Remark 5.4. Let u denote either $\underline{u}$ in (5.8), or $\overline{u}$ in (5.11). Choosing $\zeta(x,t) = \alpha(x)\beta(t)$ with $\alpha \in C^1_c(\Omega)$, $\beta \in C^1_c(0,T)$, $\alpha, \beta \ge 0$, gives

$$\iint_Q \{[u(x,t)-k]_\pm \, \alpha(x)\beta'(t) + \mathrm{sgn}_\pm(u(x,t)-k)[H(u(x,t))-H(k)]\alpha'(x)\beta(t)\} dxdt \ge 0 \tag{5.21}$$

for any $k \in \mathbb{R}$. Since $0 \le [u_r - k]_\pm \le [u]_\pm + |k|$, from the above inequality we get

$$\begin{aligned} -\int_\Omega dx\, \alpha'(x) \left\{ \int_0^T \mathrm{sgn}_\pm(u(x,t)-k)[H(u(x,t))-H(k)]\beta(t)\,dt \right\} \le \\ \le \|\beta'\|_\infty \int_\Omega dx\, \alpha(x) \left\{ \int_0^T \left([u]_\pm(x,t) + |k|\right) dt \right\} = \\ = -\|\beta'\|_\infty \int_\Omega dx\, \alpha'(x) \left\{ \int_0^T \int_c^x \left([u]_\pm(x,t) + |k|\right) dt \right\}. \end{aligned}$$

for every $c \in \overline{\Omega}$. Hence the distributional derivative of the function

$$x \mapsto \int_0^T \mathrm{sgn}_\pm(u(x,t)-k)\,[H(u(x,t)) - H(k)]\,\beta(t)\,dt - \|\beta'\|_\infty \int_0^T \int_c^x \left([u]_\pm(x,t) + |k|\right) dydt$$

is nonpositive. Therefore, the limits

$$\operatorname*{ess\,lim}_{x\to a^+} \int_0^T \mathrm{sgn}_\pm(u(x,t)-k)\,[H(u(x,t)) - H(k)]\,\beta(t)\,dt\,, \tag{5.22a}$$

$$\operatorname*{ess\,lim}_{x\to b^-} \int_0^T \mathrm{sgn}_\pm(u(x,t)-k)\,[H(u(x,t)) - H(k)]\,\beta(t)\,dt \tag{5.22b}$$

exist and are finite, thus the above definitions are well-posed.

The same statement can be applied to entropy solutions of (CL), since they satisfy inequalities (5.21) in every domain $Q_j = (x_j, x_{j+1}) \times (0,T)$ $(j = 1,\dots,p-1)$, or $Q^- = (-\infty, x_1) \times (0,T)$, $Q^+ = (x_p, \infty) \times (0,T)$ (recall that by Theorem 3.1 and assumption (A_1) the singular part of an entropy solution of (CL) is not supported in these domains).

Remark 5.5. Conditions (5.10a)-(5.10b) for subsolutions of (D_-^-) can be equivalently rewritten as follows: for all $k \in \mathbb{R}$ and β as above and for a.e. $\xi, \eta \in (a,b)$

(5.23a)
$$\int_0^T \int_a^\xi [\underline{u}(x,t) - k]_+ \beta'(t)\, dxdt \geq \int_0^T \operatorname{sgn}_+(\underline{u}(\xi,t) - k) \left[H(\underline{u}(\xi,t)) - H(k)\right] \beta(t)\, dt$$

(5.23b)
$$\int_0^T \int_\eta^b [\underline{u}(x,t) - k]_+ \beta'(t)\, dxdt \geq -\int_0^T \operatorname{sgn}_+(\underline{u}(\eta,t) - k) \left[H(\underline{u}(\eta,t)) - H(k)\right] \beta(t)\, dt.$$

Similarly, conditions (5.13a)-(5.13b) for supersolutions of (D_+^+) equivalently read: for all $k \in \mathbb{R}$ and for a.e. $\xi, \eta \in (a,b)$

(5.24a)
$$\int_0^T \int_a^\xi [\overline{u}(x,t) - k]_- \beta'(t)\, dxdt \geq \int_0^T \operatorname{sgn}_-(\overline{u}(\xi,t) - k) \left[H(\overline{u}(\xi,t)) - H(k)\right] \beta(t)\, dt$$

(5.24b)
$$\int_0^T \int_\eta^b [\overline{u}(x,t) - k]_- \beta'(t)\, dxdt \geq -\int_0^T \operatorname{sgn}_-(\overline{u}(\eta,t) - k) \left[H(\overline{u}(\eta,t)) - H(k)\right] \beta(t)\, dt.$$

When $\Omega = (a,\infty)$ we have the following definition (we omit the formulation for the case $\Omega = (-\infty, b)$).

Definition 5.5. Let $\Omega = (a,\infty)$, $u_0 \in L^1_{loc}(\Omega)$.
(i) An *entropy subsolution* of (D_+) is any $\underline{u} \in C([0,T]; L^1_{loc}(\Omega))$ such that (a) of Definition 5.2 holds.
(ii) An *entropy subsolution* of (D_-) is any $\underline{u} \in C([0,T]; L^1_{loc}(\Omega))$ such that (a) of Definition 5.2 holds, and for every $k \in \mathbb{R}$, $\beta \in C^1_c(0,T)$, $\beta \geq 0$ inequality (5.10a) holds.
(iii) An *entropy supersolution* of (D_+) is any $\overline{u} \in C([0,T]; L^1_{loc}(\Omega))$ such that (a') of Definition 5.3 holds, and for every $k \in \mathbb{R}$, $\beta \in C^1_c(0,T)$, $\beta \geq 0$, inequality (5.13a) holds.
(iv) An *entropy supersolution* of (D_-) is any $\overline{u} \in C([0,T]; L^1_{loc}(\Omega))$ such that (a') of Definition 5.3 holds.
(v) A function $u \in C([0,T]; L^1_{loc}(\Omega))$ is called an *entropy solution* of (D_S) if it is both an entropy subsolution and an entropy supersolution of (D_S).

Comparison and uniqueness results for problem (D_R) are given by the following theorem (see [13, Theorem 1.1]).

Theorem 5.1. *Let $\Omega = (a,b)$. Let $u_0, v_0 \in BV(\Omega)$, and $m_1, m_2, n_1, n_2 \in \mathbb{R}$. Let $\underline{u}$ be an entropy subsolution of (D_R), and $\overline{v}$ be an entropy supersolution of (D_R) with u_0, m_1 and m_2 replaced by v_0, n_1 and n_2. Then for a.e. $t \in (0,T)$*

(5.25)
$$\int_\Omega [\underline{u}(x,t) - \overline{v}(x,t)]_+\, dx \leq \int_\Omega [u_0(x) - v_0(x)]_+\, dx + \left([m_1 - n_1]_+ + [m_2 - n_2]_+\right) \|H'\|_\infty\, t\, .$$

Similar results hold for $\Omega = (a,\infty)$ *and* $\Omega = (-\infty,b)$ *if* $u_0, v_0 \in BV_{loc}(\Omega)\cap L^\infty(\Omega)$. *In these cases for a.e.* $t\in(0,T)$ *there holds*

(5.26)
$$\int_a^R [\underline{u}(x,t)-\overline{v}(x,t)]_+\,dx \le \int_a^{R+\|H'\|_\infty t}[u_0(x)-v_0(x)]_+\,dx + [m_1-n_1]_+\|H'\|_\infty\, t$$

for every $R>a$ *if* $\Omega=(a,\infty)$, *respectively*

$$\int_R^b [\underline{u}(x,t)-\overline{v}(x,t)]_+\,dx \le \int_{R-\|H'\|_\infty t}^b [u_0(x)-v_0(x)]_+\,dx + [m_2-n_2]_+\|H'\|_\infty\, t$$

for every $R<b$ *if* $\Omega=(-\infty,b)$. *Therefore, in all cases there exists at most one solution of* (D_R).

As for problem (D_S), the following holds.

Theorem 5.2. *Let* (A_0) *hold. Let* $\underline{u}, \overline{u}$ *be an entropy sub- and supersolution of* (D_S) *with the same boundary conditions. Then* $\underline{u}\le\overline{u}$ *a.e. in* Q. *In particular, there exists at most one entropy solution of* (D_S).

Proof. We only give the proof for (D_+^-), as in the other cases of (D_S) it is similar. We use the Kružkov doubling method adapted to boundary valued problems (see [3, 10, 11, 12]). Let ρ_ϵ ($\epsilon>0$) be a symmetric mollifier in $\mathbb{R}$, and set

$$\zeta(x,t,y,s) := \rho_{\epsilon_1}(x-y)\,\rho_{\epsilon_2}(t-s)\,\sigma_1\Big(\frac{x+y}{2}\Big)\sigma_2\Big(\frac{t+s}{2}\Big) \qquad ((x,t),\,(y,s)\in Q)\,,$$

with $\sigma_1\in C_c^1(\Omega)$, $\sigma_2\in C_c^1(0,T)$, $\sigma_1\ge0$, $\sigma_2\ge0$. From (5.11) and (5.8) we get

$$\iint_Q \big\{\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(y,s))[H(\overline{u}(x,t))-H(\underline{u}(y,s))]\zeta_x(x,t,y,s)\, + \\ +[\overline{u}(x,t)-\underline{u}(y,s)]_-\,\zeta_t(x,t,y,s)\big\}\,dxdt \ge 0 \quad \text{for all } (y,s)\in Q\,,$$

$$\iint_Q \big\{\mathrm{sgn}_+(\underline{u}(y,s)-\overline{u}(x,t))[H(\underline{u}(y,s))-H(\overline{u}(x,t))]\zeta_y(x,t,y,s)\, + \\ +[\underline{u}(y,s)-\overline{u}(x,t)]_+\,\zeta_s(x,t,y,s)\big\}\,dyds \ge 0 \quad \text{for all } (x,t)\in Q\,.$$

Recalling that $[u]_+ = [-u]_-$ and $\mathrm{sgn}_+(u) = -\mathrm{sgn}_-(-u)$ ($u\in\mathbb{R}$), we sum the above inequalities integrated over Q:

(5.27)
$$\iiiint_{Q\times Q}\rho_{\epsilon_1}(x-y)\,\rho_{\epsilon_2}(t-s)\Big\{[\overline{u}(x,t)-\underline{u}(y,s)]_-\,\sigma_1\Big(\frac{x+y}{2}\Big)\sigma_2'\Big(\frac{t+s}{2}\Big)+ \\ +\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(y,s))[H(\overline{u}(x,t))-H(\underline{u}(y,s))]\sigma_1'\Big(\frac{x+y}{2}\Big)\sigma_2\Big(\frac{t+s}{2}\Big)\Big\}\,dxdtdyds \ge 0.$$

Set

$$I_1 := \iiiint_{Q\times Q}\rho_{\epsilon_1}(x-y)\,\rho_{\epsilon_2}(t-s)[\overline{u}(x,t)-\underline{u}(y,s)]_-\,\sigma_1(y)\,\sigma_2'\Big(\frac{t+s}{2}\Big)\,dxdtdyds\,,$$

$$I_2 := \iiiint_{Q\times Q}\rho_{\epsilon_1}(x-y)\,\rho_{\epsilon_2}(t-s)\,\times \\ \times\ \mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(y,s))[H(\overline{u}(x,t))-H(\underline{u}(y,s))]\,\sigma_1'(y)\,\sigma_2\Big(\frac{t+s}{2}\Big)\,dxdtdyds\,.$$

Observe that the difference between I_1 and the first term in (5.27) vanishes as $\epsilon_1\to0^+$; the same holds for the difference between I_2 and the second term in (5.27).

Let $a<\xi<\eta<b$ be fixed. By standard approximation arguments we can choose

$$\sigma_1(y)\equiv\sigma_{1,n}(y) = n(y-\xi)\chi_{[\xi,\xi+1/n]}(y)+\chi_{(\xi+1/n,\eta-1/n)}(y)-n(y-\eta)\chi_{[\eta-1/n,\eta]}(y),$$

where $n \in \mathbb{N}$ and $y \in \Omega$, thus

$$\sigma_1'(y) = n\chi_{[\xi,\xi+1/n]}(y) - n\chi_{[\eta-1/n,\eta]}(y)\,.$$

With this choice of σ_1, I_2 reads

$$\begin{aligned}(5.28)\qquad I_2 = {} & n\int_0^T ds\int_\xi^{\xi+1/n} dy\int_a^b dx\,\rho_{\epsilon_1}(x-y)\,\times\\ & \times\int_0^T dt\,\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(y,s))[H(\overline{u}(x,t))-H(\underline{u}(y,s))]\,\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big) -\\ & - n\int_0^T ds\int_{\eta-1/n}^{\eta} dy\int_a^b dx\,\rho_{\epsilon_1}(x-y)\,\times\\ & \times\int_0^T dt\,\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(y,s))[H(\overline{u}(x,t))-H(\underline{u}(y,s))]\,\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big).\end{aligned}$$

By (5.24a) and (5.23b), from (5.28) we obtain for a.e. $\xi,\eta \in (a,b)$

$$\begin{aligned}\lim_{n\to\infty} I_2 = {} & \int_0^T ds\int_a^b dx\,\rho_{\epsilon_1}(x-\xi)\,\times\\ & \times\int_0^T dt\,\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(\xi,s))[H(\overline{u}(x,t))-H(\underline{u}(\xi,s))]\,\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big) -\\ & - \int_0^T ds\int_a^b dx\,\rho_{\epsilon_1}(x-\eta)\,\times\\ & \times\int_0^T dt\,\mathrm{sgn}_-(\overline{u}(x,t)-\underline{u}(\eta,s))[H(\overline{u}(x,t))-H(\underline{u}(\eta,s))]\,\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big) \le\\ & \le \int_0^T ds\int_a^b dx\,\rho_{\epsilon_1}(x-\xi)\int_0^T dt\int_a^x dz\,[\overline{u}(z,t)-\underline{u}(\xi,s)]_-\Big[\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big)\Big]_t +\\ & + \int_0^T dt\int_a^b dx\,\rho_{\epsilon_1}(x-\eta)\int_0^T ds\int_\eta^b dz\,[\underline{u}(z,s)-\overline{u}(x,t)]_+\Big[\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big)\Big]_s =: S\,.\end{aligned}$$

As $\epsilon_1 \to 0^+$ we obtain that

$$\begin{aligned}(5.29)\qquad \lim_{\epsilon_1\to 0^+} S = {} & \int_0^T ds\int_0^T dt\int_a^\xi dz\,[\overline{u}(z,t)-\underline{u}(\xi,s)]_-\Big[\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big)\Big]_t +\\ & + \int_0^T dt\int_0^T ds\int_\eta^b dz\,[\underline{u}(z,s)-\overline{u}(\eta,t)]_+\Big[\rho_{\epsilon_2}(t-s)\,\sigma_2\Big(\frac{t+s}{2}\Big)\Big]_s \le\\ & \le C_{\epsilon_2}T\Big\{\int_0^T dt\int_a^\xi dz\,|\overline{u}(z,t)| + (\xi-a)\int_0^T |\underline{u}(\xi,s)|\,ds\,+\\ & + \int_0^T ds\int_\eta^b dz\,|\underline{u}(z,s)| + (b-\eta)\int_0^T |\overline{u}(\eta,t)|\,dt\Big\}.\end{aligned}$$

Clearly, there holds

$$\lim_{\xi\to a^+}\int_0^T dt\int_a^\xi dz\,|\overline{u}(z,t)| = \lim_{\eta\to b^-}\int_0^T ds\int_\eta^b dz\,|\underline{u}(z,s)| = 0\,.$$

On the other hand, since $\int_0^T |\underline{u}(\xi,s)|\,ds \ge 0$ and the map $\xi \to \int_0^T |\underline{u}(\xi,s)|\,ds$ belongs to $L^1(\Omega)$, there holds

$$\operatorname{ess}\liminf_{\xi\to a^+}\,(\xi-a)\int_0^T |\underline{u}(\xi,s)|\,ds = 0$$

- for, otherwise there would exist $c,\delta > 0$ such that $(\xi-a)\int_0^T |\underline{u}(\xi,s)|\,ds \ge c$ for a.e. $\xi \in (a,a+\delta)$, thus $\xi \to \int_0^T |\underline{u}(\xi,s)|\,ds \notin L^1(\Omega)$. Therefore, for every $n \in \mathbb{N}$ there exist

$\delta_n > 0$ and $E_n \subseteq (a, a+\delta_n)$, $|E_n| > 0$, such that $(\xi - a)\int_0^T |\underline{u}(\xi,s)|\,ds < 1/n$ for a.e. $\xi \in E_n$. It follows that a sequence $\{\xi_n\} \subseteq \Omega$ exists, such that:

(i) ξ_n is a Lebesgue point of $\int_0^T |\underline{u}(\xi,s)|\,ds$,

(ii) $\xi_n \to a^+$ as $n \to \infty$, and $(\xi_n - a)\int_0^T |\underline{u}(\xi_n,s)|\,ds < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Similarly, there holds

$$\operatorname{ess}\liminf_{\eta \to b^-} (b-\eta)\int_0^T |\underline{u}(\eta,t)|\,dt = 0\,,$$

hence there exists $\{\eta_n\} \subseteq \Omega$, $\eta_n \to b^-$ as $n \to \infty$, with properties analogous to (i)-(ii) above. Then writing (5.29) with $\xi = \xi_n$, $\eta = \eta_n$ and letting $n \to \infty$ we obtain that the right-hand side of (5.29) goes to zero.

To sum up, following the above procedure and letting $\epsilon_2 \to 0^+$ from (5.27), we get for any $\sigma_2 \in C^1_c(0,T)$, $\sigma_2 \geq 0$,

$$\iint_Q [\overline{u}(x,t) - \underline{u}(x,t)]_- \,\sigma_2'(t)\,dxdt \geq 0\,. \tag{5.30}$$

Let $0 < t_1 < t_2 \leq T$ be fixed. By standard approximation arguments we can choose

$$\sigma_2(t) \equiv \sigma_{2,n}(t) = n(t-t_1)\chi_{[t_1,t_1+1/n]} + \chi_{(t_1+1/n,t_2-1/n)} - n(t-t_2)\chi_{[t_2-1/n,t_2]} \qquad (n \in \mathbb{N})\,.$$

Then from (5.30) we get for all n

$$n\int_{t_2-1/n}^{t_2}\int_\Omega [\overline{u}(x,t) - \underline{u}(x,t)]_-\,dxdt \;\leq\; n\int_{t_1}^{t_1+1/n}\int_\Omega [\overline{u}(x,t) - \underline{u}(x,t)]_-\,dt\,,$$

whence as $n \to \infty$

$$\int_\Omega [\overline{u}(x,t_2) - \underline{u}(x,t_2)]_-\,dx \leq \int_\Omega [\overline{u}(x,t_1) - \underline{u}(x,t_1)]_-\,dx\,.$$

Since $\underline{u}, \overline{u} \in C([0,T];L^1(\Omega))$, as $t_1 \to 0^+$ by (5.9) and (5.12) there holds

$$\int_\Omega [\overline{u}(x,t_2) - \underline{u}(x,t_2)]_-\,dx = 0 \quad \text{for all } t_2 \in (0,T]\,.$$

This proves the result. □

For future reference we prove the following generalization of [3, Lemma 4.4].

Proposition 5.3. (i) *Let u be an entropy solution either of (D^+_+), or of (D^-_+). Then there exists $f^+_a \in L^\infty(0,T)$ such that*

$$\operatorname{ess}\lim_{x \to a^+}\int_0^T H(u(x,t))\,\beta(t)\,dt = \int_0^T f^+_a(t)\,\beta(t)\,dt \tag{5.31}$$

for every $\beta \in C^1_c(0,T)$, and

$$\limsup_{u \to \infty} H(u) \leq f^+_a(t) \leq \sup_{u \in \mathbb{R}} H(u) \quad \text{for a.e. } t \in (0,T)\,. \tag{5.32}$$

(ii) *Let u be an entropy solution either of (D^+_-), or of (D^-_-). Then there exists $f^-_a \in L^\infty(0,T)$ such that*

$$\operatorname{ess}\lim_{x \to a^+}\int_0^T H(u(x,t))\,\beta(t)\,dt = \int_0^T f^-_a(t)\,\beta(t)\,dt \tag{5.33}$$

for every $\beta \in C^1_c(0,T)$, and

$$\inf_{u \in \mathbb{R}} H(u) \leq f^-_a(t) \leq \liminf_{u \to -\infty} H(u) \quad \text{for a.e. } t \in (0,T)\,. \tag{5.34}$$

(iii) Let u be an entropy solution either of (D^+_-), or of (D^+_+). Then there exists $f^+_b \in L^\infty(0,T)$ such that

$$\operatorname{ess}\lim_{x\to b^-}\int_0^T H(u(x,t))\,\beta(t)\,dt = \int_0^T f^+_b(t)\,\beta(t)\,dt \tag{5.35}$$

for every $\beta \in C^1_c(0,T)$, and

$$\inf_{u\in\mathbb{R}} H(u) \le f^+_b(t) \le \liminf_{u\to\infty} H(u) \quad \textit{for a.e. } t\in(0,T)\,. \tag{5.36}$$

(iv) Let u be an entropy solution either of (D^-_+), or of (D^-_-). Then there exists $f^-_b \in L^\infty(0,T)$ such that

$$\operatorname{ess}\lim_{x\to b^-}\int_0^T H(u(x,t))\,\beta(t)\,dt = \int_0^T f^-_b(t)\,\beta(t)\,dt \tag{5.37}$$

for every $\beta \in C^1_c(0,T)$, and

$$\limsup_{u\to-\infty} H(u) \le f^-_b(t) \le \sup_{u\in\mathbb{R}} H(u) \quad \textit{for a.e. } t\in(0,T)\,. \tag{5.38}$$

Proof. The existence of the limits in the left-hand side of (5.31), (5.33), (5.35) and (5.37) follows from (5.22a)-(5.22b), since for *a.e.* $x\in\Omega$ there holds

$$x \mapsto \int_0^T H(u(x,t))\beta(t)\,dt = \int_0^T [\operatorname{sgn}_+(u(x,t)) - \operatorname{sgn}_-(u(x,t))]H(u(x,t))\beta(t)\,dt$$

(recall that $H(0)=0$). On the other hand, for every sequence $\{x_n\}$, $x_n \to a^+$, the sequence $\{H(u(x_n,\cdot))\}$ is bounded in $L^\infty(0,T)$. Hence there exist a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and a function $f^+_a \in L^\infty(0,T)$ (independent of $\{x_{n_k}\}$) such that $H(u(x_{n_k},\cdot)) \overset{*}{\rightharpoonup} f^+_a$ in $L^\infty(0,T)$, thus (5.31) follows. Equalities (5.33), (5.35) and (5.37) are similarly proven.

Let us prove (5.32). Clearly, there holds $f^+_a(t) \le \sup_{u\in\mathbb{R}} H(u)$ for a.e. $t\in(0,T)$. To prove the first inequality, let us choose in (5.8) $\zeta(x,t) = \rho(x)\beta(t)$ with $\rho \in C^1_c([a,b))$, $\rho\ge 0$, $\beta\in C^1_c(0,T)$, $\beta\ge 0$. By standard arguments we can also choose $\rho = \alpha\sigma_\epsilon$ with $\alpha \in C^1_c([a,b))$, $\alpha\ge 0$, and

$$\sigma_\epsilon(x) := \frac{2(x-a)-\epsilon}{\epsilon}\chi_{[a+\epsilon/2,a+\epsilon]}(x) + \chi_{(a+\epsilon,b]}(x) \qquad (x\in\Omega)\,. \tag{5.39}$$

Then for every $k\in\mathbb{R}$ we obtain that

$$\begin{aligned}
&\iint_Q \Big\{[u(x,t)-k]_+\alpha(x)\sigma_\epsilon(x)\beta'(t)\,+ \\
&\quad +\operatorname{sgn}_+(u(x,t)-k)[H(u(x,t))-H(k)]\,\alpha'(x)\sigma_\epsilon(x)\beta(t)\Big\}\,dxdt \ge \\
&\quad \ge -\frac{2}{\epsilon}\int_0^T dt\,\beta(t)\int_{a+\epsilon/2}^{a+\epsilon}\operatorname{sgn}_+(u(x,t)-k)[H(u(x,t))-H(k)]\,\alpha(x)\,dx\,.
\end{aligned}$$

Letting $\epsilon \to 0^+$ and using (5.13a) and (5.31), we get that for every $k \in \mathbb{R}$

$$\iint_Q \{[u(x,t)-k]_+\alpha(x)\beta'(t) + \mathrm{sgn}_+(u(x,t)-k)[H(u(x,t))-H(k)]\alpha'(x)\beta(t)\}\,dxdt \geq$$

$$\geq -\alpha(a)\,\mathrm{ess}\lim_{x\to a^+}\int_0^T \mathrm{sgn}_+(u(x,t)-k)[H(u(x,t))-H(k)]\beta(t)\,dt =$$

$$= -\alpha(a)\Big\{\mathrm{ess}\lim_{x\to a^+}\int_0^T [H(u(x,t))-H(k)]\,\beta(t)\,dt +$$

$$\underbrace{+\,\mathrm{ess}\lim_{x\to a^+}\int_0^T \mathrm{sgn}_-(u(x,t)-k)[H(u(x,t))-H(k)]\,\beta(t)\,dt}_{\leq 0}\Big\} \geq$$

$$\geq -\alpha(a)\int_0^T [f_a^+(t)-H(k)]\,\beta(t)\,dt\,.$$

Letting $k \to \infty$ in the above inequality gives

$$0 \leq \int_0^T [f_a^+(t) - \limsup_{k\to\infty} H(k)]\,\beta(t)\,dt\,,$$

whence by the arbitrariness of β inequality (5.32) follows.

To prove (5.34) we argue as for (5.32), using inequality (5.10a), (5.11) and (5.33) instead of (5.8), (5.13a) and (5.31). Then we get for every $k \in \mathbb{R}$

$$\iint_Q \{[u(x,t)-k]_-\alpha(x)\beta'(t) + \mathrm{sgn}_-(u(x,t)-k)[H(u(x,t))-H(k)]\,\alpha'(x)\beta(t)\}\,dxdt \geq$$

$$\geq -\alpha(a)\,\mathrm{ess}\lim_{x\to a^+}\int_0^T \mathrm{sgn}_-(u(x,t)-k)[H(u(x,t))-H(k)]\beta(t)\,dt =$$

$$= -\alpha(a)\Big\{-\,\mathrm{ess}\lim_{x\to a^+}\int_0^T [H(u(x,t))-H(k)]\,\beta(t)\,dt +$$

$$\underbrace{+\,\mathrm{ess}\lim_{x\to a^+}\int_0^T \mathrm{sgn}_+(u(x,t)-k)[H(u(x,t))-H(k)]\,\beta(t)\,dt}_{\leq 0}\Big\} \geq$$

$$\geq \alpha(a)\int_0^T [f_a^-(t)-H(k)]\,\beta(t)\,dt\,.$$

As $k \to -\infty$ in the above inequality, by the arbitrariness of β we obtain

$$f_a^-(t) \leq \liminf_{k\to-\infty} H(k) \quad \text{for } a.e.\ t \in (0,T)\,,$$

thus (5.34) follows. The proof of (5.36) and (5.38) is similar to that of (5.32) and (5.34), using

(5.40) $$\sigma_\epsilon(x) := \chi_{[a,b-\epsilon)}(x) - \frac{2(x-b)+\epsilon}{\epsilon}\chi_{[b-\epsilon,b-\epsilon/2]}(x) \qquad (x \in \Omega)$$

instead of (5.39); we leave the details to the reader. □

Finally we prove the following result.

Lemma 5.4. *Let u be an entropy solution of (D_R). Then for every $t \in (0,T]$*

(5.41) $$\|u(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + 2\,\|H\|_\infty t\,.$$

Proof. By (5.1) and (5.4) there holds

$$\iint_Q \{|u-k|\zeta_s + \mathrm{sgn}(u-k)\,[H(u)-H(k)]\zeta_x\}\,dxds \geq -\int_\Omega |u_0-k|\,\zeta(x,0)\,dx$$

for every $k \in \mathbb{R}$ and ζ as above. By standard arguments we can choose $\zeta(x,s) = \alpha_p(x)\beta_q(s)$ with

$$\alpha_p(x) = p(x-a)\chi_{[a,a+1/p)}(x) + \chi_{[a+1/p,b-1/p)}(x) - p(x-b)\chi_{[b-1/p,b]}(x)\,,$$
$$\beta_q(s) = \chi_{[0,t-1/q)}(s) - q(s-t)\chi_{[t-1/q,t)}(s)$$

for any fixed $t \in (0,T]$ and $p,q \in \mathbb{N}$ sufficiently large. Then for $k=0$ as $q \to \infty$ we get

$$\int_\Omega |u(x,t)|\,\alpha_p(x)\,dx - \int_\Omega |u_0(x)|\,\alpha_p(x)\,dx \le 2\,\|H\|_\infty t\,,$$

whence as $p \to \infty$ (5.41) follows. □

6. Problem (D): existence

Let us recall the following result (see [1, 13]).

Theorem 6.1. *Let $\Omega = (a,b)$, and let $u_0 \in BV(\Omega)$. Then there exists a unique entropy solution $u \in BV(Q) \cap C([0,T];L^1(\Omega))$ of problem (D_R). Moreover,*

$$\|u\|_{L^\infty(Q)} \le \max\left\{|m_1|, |m_2|, \|u_0\|_{L^\infty(\Omega)}\right\}\,. \tag{6.1}$$

The same holds for $\Omega = (a,\infty)$ and $\Omega = (-\infty,b)$ with $u_0 \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$, $\operatorname{supp} u_0$ compact. In these cases there holds $u \in BV_{loc}(Q) \cap L^\infty(Q) \cap C([0,T];L^1_{loc}(\Omega))$, and inequality (6.1) *is replaced by*

$$\|u\|_{L^\infty(Q)} \le \max\left\{|m_1|, \|u_0\|_{L^\infty(\Omega)}\right\} \tag{6.2}$$

if $\Omega = (a,\infty)$, respectively by $\|u\|_{L^\infty(Q)} \le \max\left\{|m_2|, \|u_0\|_{L^\infty(\Omega)}\right\}$ if $\Omega = (-\infty,b)$.

The above uniqueness claim follows from Theorem 5.1. Let us outline the proof of the existence part; we limit ourselves to the case $\Omega = (a,\infty)$, the proof being the same for $\Omega = (-\infty,b)$ and easier for $\Omega = (a,b)$.

Let $f_{1,\epsilon}, f_{2,\epsilon} \in C^\infty(\mathbb{R})$ $(0<\epsilon<1)$ be a partition of unity:

$$\begin{cases} f_{1,\epsilon} = 1 \quad \text{in } (-\infty, a+2\sqrt{\epsilon}\,], \quad \operatorname{supp} f_{1,\epsilon} \subseteq (-\infty, a+3\sqrt{\epsilon}\,] \\ f_{2,\epsilon} = 1 \quad \text{in } [a+3\sqrt{\epsilon},\infty)\,, \quad \operatorname{supp} f_{2,\epsilon} \subseteq [a+2\sqrt{\epsilon},\infty)\,, \\ 0 \le f_{i,\epsilon} \le 1\,,\ \sum_{i=1}^2 f_{i,\epsilon} = 1 \quad \text{in } \mathbb{R}\,, \end{cases}$$

such that, for $i = 1,2$,

$$\sup_{\epsilon\in(0,1)} \|f'_{i,\epsilon}\|_{L^1(\mathbb{R})} < \infty\,, \qquad \sup_{\epsilon\in(0,1)} \sqrt{\epsilon}\,\|f'_{i,\epsilon}\|_{L^\infty(\mathbb{R})} < \infty\,, \qquad \sup_{\epsilon\in(0,1)} \sqrt{\epsilon}\|f''_{i,\epsilon}\|_{L^1(\mathbb{R})} < \infty.$$

Let $u_0 \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$ have compact support. Set

$$u_{0\epsilon} := m_1 f_{1,\epsilon} + [\sigma_\epsilon * u_0] f_{2,\epsilon}\,, \tag{6.3}$$

where $\{\sigma_\epsilon\}$ is a family of standard mollifiers with $\operatorname{supp}\sigma_\epsilon \subseteq [-\sqrt{\epsilon},\sqrt{\epsilon}]$. Then there holds $u_{0\epsilon} \in C^\infty(\mathbb{R})$, $u_{0\epsilon} = m_1$ in $[a, a+\sqrt{\epsilon}]$, $\operatorname{supp} u_{0\epsilon}$ compact. Moreover,

$$\sup_{\epsilon\in(0,1)} \|u_{0\epsilon}\|_{L^\infty(\Omega)} \le \max\left\{|m_1|, \|u_0\|_{L^\infty(\Omega)}\right\}\,, \tag{6.4}$$

$$\sup_{\epsilon\in(0,1)} \|u'_{0\epsilon}\|_{L^1(\Omega)} < \infty\,, \qquad \sup_{\epsilon\in(0,1)} \epsilon\,\|u''_{0\epsilon}\|_{L^1(\Omega)} < \infty \tag{6.5}$$

$$u_{0\epsilon} \to u_0 \quad \text{in } L^p(\Omega) \quad \text{for every } p \in [1,\infty)\,, \qquad u_{0\epsilon} \overset{*}{\rightharpoonup} u_0 \quad \text{in } L^\infty(\Omega)\,. \tag{6.6}$$

Let $H \in W^{1,\infty}(\mathbb{R})$, $H(0) = 0$. Set

$$H_\epsilon(u) := g_\epsilon(u)\left([\sigma_\epsilon * H](u) - [\sigma_\epsilon * H](0)\right) \qquad (u \in \mathbb{R})\,,$$

where the family $\{g_\epsilon\} \in C_c^\infty(\mathbb{R})$ satisfies $g_\epsilon = 1$ in $(-1/\epsilon, 1/\epsilon)$, $0 \le g_\epsilon(x) \le 1$ in $\mathbb{R}$, $\operatorname{supp} g_\epsilon \subseteq (-2/\epsilon, 2/\epsilon)$. It is easily seen that

$$\begin{cases} H_\epsilon(0) = 0\,, \quad \|H_\epsilon\|_{W^{1,\infty}(\mathbb{R})} \le \|H\|_{W^{1,\infty}(\mathbb{R})}\,, \\ H_\epsilon \to H \quad \text{uniformly on the compact subsets of } \mathbb{R}\,. \end{cases} \tag{6.7}$$

Let $u_\epsilon \in C^{2,1}(\overline{Q})$ be the unique classical solution of the parabolic problem

$$\begin{cases} u_{\epsilon t} + [H_\epsilon(u_\epsilon)]_x = \epsilon u_{\epsilon xx} & \text{in } Q \\ u_\epsilon = m_1 & \text{in } \{a\} \times (0,T) \\ u_\epsilon = u_{0\epsilon} & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{D_ϵ}$$

with $m_1 \in \mathbb{R}$, $u_{0\epsilon}$ and H_ϵ as above (*e.g.*, see [8]).

Lemma 6.2. *There holds*

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(Q)} \le \max\left\{|m_1|, \|u_0\|_{L^\infty(\Omega)}\right\}\,, \tag{6.8}$$

and there exists $c > 0$ only depending on m_1, $TV(u_0;\Omega)$, and $\|H\|_{W^{1,\infty}(\mathbb{R})}$) such that

$$\sup_{\epsilon\in(0,1)} \|u_{\epsilon x}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \tag{6.9}$$

$$\sup_{\epsilon\in(0,1)} \|u_{\epsilon t}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \tag{6.10}$$

$$\sup_{\epsilon\in(0,1)} \left(\epsilon \|u_{\epsilon x}\|_{L^\infty(Q)}\right) \le c\,. \tag{6.11}$$

Proof. Inequality (6.8) follows by the maximum principle and (6.4). Arguing as in the proof of [13, Proposition 3.1] (see also [1]) and using (6.5) gives (6.9)-(6.10). As for (6.11), integrating the first equation of (D_ϵ) over (a, x) gives

$$\epsilon u_{\epsilon x}(x,t) - \epsilon u_{\epsilon x}(a,t) = \int_a^x u_{\epsilon t}(y,t)\,dy + H_\epsilon(u_\epsilon(x,t)) - H_\epsilon(m_1)\,, \tag{6.12}$$

whence

$$\epsilon\,|u_{\epsilon x}(a,t)| \le \int_a^x |u_{\epsilon t}(y,t)|\,dy + 2\|H\|_\infty + |u_{\epsilon x}(x,t)|\,.$$

Integrating the above inequality over $(a, a+1)$ and using (6.9)-(6.10) we get

$$\epsilon\,|u_{\epsilon x}(a,t)| \le 2\|H\|_\infty + \tilde{c} \tag{6.13}$$

for some $\tilde{c} > 0$ independent of ϵ. Then by (6.9)-(6.10) and (6.12)-(6.13) the estimate in (6.11) follows. □

Proof of Theorem 6.1. By estimates (6.8)-(6.10) the family $\{u_\epsilon\}$ is bounded in $L^\infty(Q)$, and there exists $M > 0$ (only depending on m_1, $TV(u_0;\Omega)$, $\|H\|_{W^{1,\infty}(\mathbb{R})}$) such that

$$\sup_{\epsilon\in(0,1)} \|u_{\epsilon x}\|_{L^\infty(0,T;L^1(\Omega))} + \sup_{\epsilon\in(0,1)} \|u_{\epsilon t}\|_{L^\infty(0,T;L^1(\Omega))} \le M\,.$$

Then by embedding theorems there exist a sequence $\{u_{\epsilon_n}\} \subseteq \{u_\epsilon\}$ and a function $u \in BV(Q) \cap C([0,T];L^1(\Omega))$ such that

$$u_{\epsilon_n} \to u \quad \text{in } C([0,T];L^1(\Omega)) \quad \text{as } n \to \infty\,. \tag{6.14}$$

Arguing as in [1] shows that u is an entropy solution of problem (D_R), In fact, let $E, F_\epsilon : \mathbb{R} \to \mathbb{R}$, $E \in C^2(\mathbb{R})$, $F_\epsilon \in C^1(\mathbb{R})$ and $F'_\epsilon = E'H'_\epsilon$. Multiplying the first equation in (D_ϵ) by $E'(u_\epsilon)\,\zeta$ gives for any $\zeta \in C^1([0,T]; C^1_c(\overline{\Omega}))$, $\zeta(\cdot,T) = 0$ in Ω,

$$(6.15)\qquad -\int_\Omega E(u_{0\epsilon})(x)\zeta(x,0)\,dx + \epsilon\iint_Q \left\{E''(u_\epsilon)u^2_{\epsilon x}\zeta \,+\, [E(u_\epsilon)]_x\zeta_x\right\}dxdt =$$
$$= \iint_Q \{E(u_\epsilon)\zeta_t + F_\epsilon(u_\epsilon)\zeta_x\}\,dxdt + \int_0^T \{F_\epsilon(m_1) - \epsilon E'(m_1)u_{\epsilon x}(a,t)\}\zeta(a,t)\,dt\,.$$

By standard regularization arguments we can choose in (6.15) $E(u_\epsilon) = [u_\epsilon - k]_\pm$, thus obtaining for all $k \in \mathbb{R}$ and ζ as above, $\zeta \ge 0$,

$$(6.16)\qquad \iint_Q \{[u_\epsilon - k]_\pm\zeta_t + \mathrm{sgn}_\pm(u_\epsilon - k)\,[H_\epsilon(u_\epsilon) - H_\epsilon(k)]\zeta_x\}\,dxdt \ge$$
$$\ge \epsilon\iint_Q \mathrm{sgn}_\pm(u_\epsilon - k)\,u_{\epsilon x}\,\zeta_x\,dxdt - \int_\Omega [u_{0\epsilon} - k]_\pm\,\zeta(x,0)\,dx\, -$$
$$-\mathrm{sgn}_\pm(m_1 - k)\int_0^T [H_\epsilon(m_1) - H_\epsilon(k) - \epsilon u_{\epsilon x}(a,t)]\zeta(a,t)\,dt\,.$$

If $\zeta(\cdot,t) \in C^1_c(\Omega)$ for all $t \in (0,T)$, from (6.16) we obtain

$$(6.17)\qquad \iint_Q \{[u_\epsilon - k]_\pm\zeta_t + \mathrm{sgn}_\pm(u_\epsilon - k)\,[H_\epsilon(u_\epsilon) - H_\epsilon(k)]\zeta_x\}\,dxdt \ge$$
$$\ge \epsilon\iint_Q \mathrm{sgn}_\pm(u_\epsilon - k)\,u_{\epsilon x}\,\zeta_x\,dxdt - \int_\Omega [u_{0\epsilon} - k]_\pm\,\zeta(x,0)\,dx\,.$$

On the other hand, choosing in (6.16) $\zeta(x,t) = \chi_{[a,\,\xi+1/n)}(x)\beta(t)$ $(\xi \in \Omega, n \in \mathbb{N})$ with $\beta \in C^1_c(0,T)$, $\beta \ge 0$, and letting $n \to \infty$ plainly gives for every $k \in \mathbb{R}$

$$(6.18)\qquad \int_0^T\!\!\int_a^\xi [u_\epsilon(x,t) - k]_\pm\beta'(t)\,dxdt\, -$$
$$-\int_0^T \mathrm{sgn}_\pm(u_\epsilon(\xi,t) - k)\left[H_\epsilon(u_\epsilon(\xi,t)) - H_\epsilon(k)\right]\beta(t)\,dt \ge$$
$$\ge \epsilon\int_0^T \mathrm{sgn}_\pm(u_\epsilon(\xi,t) - k)\,u_{\epsilon x}(\xi,t)\,\beta(t)dt\, -$$
$$-\,\mathrm{sgn}_\pm(m_1 - k)\int_0^T [H_\epsilon(m_1) - H_\epsilon(k) - \epsilon u_{\epsilon x}(a,t)]\beta(t)\,dt\,.$$

Multiplying the first equation of (D_ϵ) by $\zeta(x,t) = \chi_{[a,\,\xi+1/n)}(x)\beta(t)$ and letting $n \to \infty$, one easily sees that

$$(6.19)\; -\epsilon\int_0^T u_{\epsilon x}(a,t)\beta(t)\,dt = -\,\epsilon\int_0^T u_{\epsilon x}(\xi,t)\beta(t)\,dt\, -$$
$$-\int_0^T\!\!\int_a^\xi [u_\epsilon(x,t) - k]\beta'(t)\,dxdt \,+ \int_0^T \left[H_\epsilon(u_\epsilon(\xi,t)) - H_\epsilon(m_1)\right]\beta(t)\,dt\,.$$

From (6.18)-(6.19) we get

$$(6.20)\quad \int_0^T\!\!\int_a^\xi \{[u_\epsilon(x,t) - k]_\pm - \mathrm{sgn}_\pm(m_1 - k)[u_\epsilon(x,t) - k]\}\,\beta'(t)\,dxdt \ge$$
$$\ge \int_0^T [\mathrm{sgn}_\pm(u_\epsilon(\xi,t) - k) - \mathrm{sgn}_\pm(m_1 - k)]\,[H_\epsilon(u_\epsilon(\xi,t)) - H_\epsilon(k)]\,\beta(t)\,dt\, +$$
$$+\,\epsilon\int_0^T [\mathrm{sgn}_\pm(u_\epsilon(\xi,t) - k)\mathrm{sgn}_\pm(m_1 - k)]\;u_{\epsilon x}(\xi,t)\,\beta(t)dt\,.$$

By (6.7), (6.9) and (6.14) we can take the limit as $\epsilon_n \to 0^+$ in (6.17) and (6.20) (written with $\epsilon = \epsilon_n$). It follows that the function u in (6.14) satisfies the following inequalities:
- for every $k \in \mathbb{R}$ and for all $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω, $\zeta \geq 0$ in Q,

$$\iint_Q \{[u-k]_\pm \zeta_t + \operatorname{sgn}_\pm(u-k)\,[H(u) - H(k)]\zeta_x\}\, dxdt \geq -\int_\Omega [u_0 - k]_\pm\, \zeta(x,0)\, dx\,;$$

- for every $k \in \mathbb{R}$ and $\beta \in C_c^1(0,T)$, $\beta \geq 0$ and for a.e. $\xi \in \Omega$,

$$\int_0^T\!\!\int_a^\xi \{[u(x,t)-k]_\pm - \operatorname{sgn}_\pm(m_1 - k)[u(x,t)-k]\}\,\beta'(t)\, dxdt \geq$$
$$\geq \int_0^T [\operatorname{sgn}_\pm(u(\xi,t)-k) - \operatorname{sgn}_\pm(m_1-k)]\,\big[H(u(\xi,t)) - H(k)\big]\beta(t)\, dt\,.$$

Letting $\xi \to a^+$ in the latter inequality and using Remark 5.1 we conclude that u is an entropy solution of (D_R). Hence the result follows. □

Remark 6.1. In the proof of Theorem 6.1 when $\Omega = (a,b)$ one uses the family of solutions of the problem

$$(D'_\epsilon) \qquad \begin{cases} u_{\epsilon t} + [H_\epsilon(u_\epsilon)]_x = \epsilon u_{\epsilon xx} & \text{in } Q \\ u_\epsilon = m_1 & \text{in } \{a\} \times (0,T) \\ u_\epsilon = m_2 & \text{in } \{b\} \times (0,T) \\ u_\epsilon = u_{0\epsilon} & \text{in } \Omega \times \{0\}\,, \end{cases}$$

with $m_1, m_2 \in \mathbb{R}$, H_ϵ as above and $u_{0\epsilon}$ defined by a suitable partition of unity; we leave the details to the reader.

Concerning (D_S) the following holds.

Theorem 6.3. *Let (A_0) hold. When $\Omega = (a,b)$ for any $u_0 \in L^1(\Omega)$ there exists an entropy solution of (D_S). The same holds for any $u_0 \in L^1_{loc}(\Omega)$ if $\Omega = (a,\infty)$, or $\Omega = (-\infty,b)$.*

Proof. Let $\Omega = (a,b)$. Let us prove the result for (D_+^-), the proof being the same for (D_+^+), (D_-^-) and (D_-^+). Let $u_0 \in BV(\Omega)$. By Theorem 6.1, for all $n,p \in \mathbb{N}$ there exists an entropy solution $u_{n,-p} \in BV(Q) \cap C([0,T]; L^1(\Omega))$ of problem (D_R) with $m_1 = n$, $m_2 = -p$. In particular, there holds:

(a) by (5.8)-(5.9) and (5.11)-(5.12), for every $k \in \mathbb{R}$ and for all $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω, $\zeta \geq 0$ in Q,

$$\iint_Q \{[u_{n,-p} - k]_\pm \zeta_t + \operatorname{sgn}_\pm(u_{n,-p} - k)\,[H(u_{n,-p}) - H(k)]\zeta_x\}\, dxdt \geq \tag{6.21}$$
$$\geq -\int_\Omega [u_0 - k]_\pm\, \zeta(x,0)\, dx\,;$$

(b) by (5.24a), for every $\beta \in C_c^1(0,T)$, $\beta \geq 0$, for *a.e.* $\xi \in \Omega$ and for all $k < n$ and $p \in \mathbb{R}$,

$$\int_0^T\!\!\int_a^\xi [u_{n,-p}(x,t) - k]_- \beta'(t)\, dxdt \geq \tag{6.22a}$$
$$\geq \int_0^T \operatorname{sgn}_-(u_{n,-p}(\xi,t) - k)\,\big[H(u_{n,-p}(\xi,t)) - H(k)\big]\beta(t)\, dt;$$

(c) by (5.23b), for every $\beta \in C^1_c(0,T)$, $\beta \geq 0$, for $a.e.$ $\eta \in \Omega$ and for all $n \in \mathbb{R}$ and $k > -p$,

$$\int_0^T \int_\eta^b [u_{n,-p}(x,t) - k]_+ \beta'(t)\, dxdt \geq \tag{6.22b}$$
$$\geq - \int_0^T \operatorname{sgn}_+(u_{n,-p}(\eta,t) - k) \left[H(u_{n,-p}(\eta,t)) - H(k)\right] \beta(t)\, dt;$$

(d) by (5.41), for every $t \in (0,T)$

$$\|u_{n,-p}(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + 2\,\|H\|_\infty t\,. \tag{6.23}$$

Moreover, by inequality (5.25), for all $n,p \in \mathbb{N}$ there holds a.e. in Q

$$u_{n,-p} \leq u_{n+1,-p}\,, \tag{6.24a}$$

$$u_{n,-p} \geq u_{n,-p-1}\,. \tag{6.24b}$$

Let $p \in \mathbb{R}$ be fixed. By (6.23) and (6.24a) there exists $u_{\infty,-p} \in L^\infty(0,T;L^1(\Omega))$ such that

$$u_{n,-p} \to u_{\infty,-p} \quad \text{in } L^1(Q) \text{ as } n \to \infty. \tag{6.25}$$

Then letting $n \to \infty$ in (6.21) gives

$$\iint_Q \{[u_{\infty,-p} - k]_\pm \zeta_t + \operatorname{sgn}_\pm(u_{\infty,-p} - k)\,[H(u_{\infty,-p}) - H(k)]\zeta_x\}\, dxdt \geq \tag{6.26}$$
$$\geq - \int_\Omega [u_0 - k]_\pm\, \zeta(x,0)\, dx\,,$$

whereas from (6.22a)-(6.22b) we get, for every $\beta \in C^1_c(0,T)$, $\beta \geq 0$, for $a.e.$ $\xi, \eta \in \Omega$ and for all $k,p \in \mathbb{R}$:

$$\int_0^T \int_a^\xi [u_{\infty,-p}(x,t) - k]_- \beta'(t)\, dxdt \geq \tag{6.27a}$$
$$\geq \int_0^T \operatorname{sgn}_-(u_{\infty,-p}(\xi,t) - k) \left[H(u_{\infty,-p}(\xi,t)) - H(k)\right] \beta(t)\, dt\,,$$

$$\int_0^T \int_\eta^b [u_{\infty,-p}(x,t) - k]_+ \beta'(t)\, dxdt \geq \tag{6.27b}$$
$$\geq - \int_0^T \operatorname{sgn}_+(u_{\infty,-p}(\eta,t) - k)\,[H(u_{\infty,-p}(\eta,t)) - H(k)]\,\beta(t)\, dt \quad (k > -p)\,.$$

Moreover, from (6.23) and (6.24b) we obtain

$$\|u_{\infty,-p}(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + \|H\|_\infty t \quad \text{for every } t \in (0,T)\,, \tag{6.28}$$

$$u_{\infty,-p} \geq u_{\infty,-p-1} \quad \text{a.e. in } Q\,. \tag{6.29}$$

By (6.28)-(6.29) there exists $u_{\infty,-\infty} \in L^\infty(0,T;L^1(\Omega))$ such that

$$u_{\infty,-p} \to u_{\infty,-\infty} \quad \text{in } L^1(Q) \text{ as } p \to \infty. \tag{6.30}$$

Then letting $p \to \infty$ in (6.26) shows that $u_{\infty,-\infty}$ satisfies (5.8) and (5.11). In addition, letting $p \to \infty$ in (6.27a)-(6.27b) proves that $u_{\infty,-\infty}$ satisfies (5.10b) and (5.13a) for every $k \in \mathbb{R}$ (see Remark 5.5). By Remark 3.1 and arguing as in the proof of [2, Proposition 3.20], it can be checked that $u_{\infty,-\infty} \in C([0,T];L^1(\Omega))$ and, by construction, $u_{\infty,-\infty}(\cdot,0) = u_0$ in $\mathcal{M}(\Omega)$. Therefore (5.9) and (5.12) follow as well, and $u_{\infty,-\infty}$ is an entropy solution of (D^-_+).

It remains to remove the assumption $u_0 \in BV(\Omega)$. To this purpose, let $v_0 \in BV(\Omega)$, and let $v_{\infty,-\infty}$ be the entropy solution of (D^-_+) with initial data v_0 constructed by the above procedure (and with the same boundary conditions considered in the construction of $u_{\infty,-\infty}$). Then by (5.25) there holds

$$\|u_{\infty,-\infty}(\cdot,t) - v_{\infty,-\infty}(\cdot,t)\|_{L^1(\Omega)} \le \|u_0 - v_0\|_{L^1(\Omega)} \quad \text{for every } t \in (0,T)\,. \tag{6.31}$$

Let $u_0 \in L^1(\Omega)$, let $\{u_{0j}\} \subseteq BV(\Omega)$ be any sequence such that $u_{0j} \to u_0$ in $L^1(\Omega)$. Let $\{u_j\} \equiv \{(u_{\infty,-\infty})_j\}$ be the sequence of entropy solutions to problem (D^-_+) constructed as above, with initial data u_{0j}. Then for all $j \in \mathbb{N}$:

(a) for every $k \in \mathbb{R}$ and for all $\zeta \in C^1([0,T];C^1_c(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω, $\zeta \ge 0$ in Q,

$$\begin{aligned}\iint_Q \{[u_j - k]_\pm \zeta_t + \operatorname{sgn}_\pm(u_j - k)\,[H(u_j) - H(k)]\zeta_x\}\, dxdt \ge \\ \ge -\int_\Omega [u_{0j} - k]_\pm\, \zeta(x,0)\, dx\,;\end{aligned} \tag{6.32}$$

(b) for all $\beta \in C^1_c(0,T)$, $\beta \ge 0$, for all $k \in \mathbb{R}$ and for $a.e.\ \xi,\eta \in \Omega$:

$$\begin{aligned}\int_0^T\int_a^\xi [u_j(x,t) - k]_- \beta'(t)\, dxdt \ge \\ \ge \int_0^T \operatorname{sgn}_-(u_j(\xi,t) - k)\left[H(u_j(\xi,t)) - H(k)\right]\beta(t)\, dt\,,\end{aligned} \tag{6.33a}$$

$$\begin{aligned}\int_0^T\int_\eta^b [u_j(x,t) - k]_+ \beta'(t)\, dxdt \ge \\ \ge -\int_0^T \operatorname{sgn}_+(u_j(\eta,t) - k)\left[H(u_j(\eta,t)) - H(k)\right]\beta(t)\, dt\,.\end{aligned} \tag{6.33b}$$

By (6.31) there holds

$$\|u_i - u_j\|_{L^1(Q)} \le T\|u_{0i} - u_{0j}\|_{L^1(\Omega)} \quad \text{for all } i,j \in \mathbb{N}\,,$$

thus there exists $u \in L^1(Q)$ such that $u_j \to u$ in $L^1(Q)$ as $j \to \infty$. As before, there holds $u \in C([0,T];L^1(\Omega))$. Then letting $j \to \infty$ in (6.32) and (6.33a)-(6.33b) the result for (D^-_+) follows. The other cases of (D_S) can be dealt with similarly, hence the conclusion follows if $\Omega = (a,b)$.

The above arguments easily extend to the case of half-lines. For instance, let $\Omega = (a,\infty)$, $m_1 = \infty$ and $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$. Then by Theorem 6.1 and inequality (5.26) there exists a sequence $\{u_n\}$ of entropy solutions of (D_R), such that for every $t \in (0,T)$ $\|u_n(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)} + 2\,\|H\|_\infty t$, and $u_n \le u_{n+1}$ a.e. in Q for all $n \in \mathbb{N}$. Then letting $n \to \infty$ we obtain an entropy solution u_∞ of (D_+) in this case. Moreover, if $v_0 \in BV(\Omega) \cap L^\infty(\Omega)$ and v_∞ is the corresponding entropy solution of (D_+) with initial data v_0 constructed as before, by (5.26) there holds

$$\|u_\infty(\cdot,t) - v_\infty(\cdot,t)\|_{L^1(a,R)} \le \|u_0 - v_0\|_{L^1(a,R+\|H'\|_\infty t)} \quad \text{for every } t \in (0,T)\,. \tag{6.34}$$

Now let $u_0 \in L^1_{loc}(\Omega)$, and let $\{u_{0j}\} \subseteq BV(\Omega) \cap L^\infty(\Omega)$, $\operatorname{supp} u_{0j}$ compact, $u_{0j} \to u_0$ in $L^1_{loc}(\Omega)$. Let $\{u_j\} \equiv \{(u_\infty)_j\}$ be the sequence of entropy solutions of (D_+) constructed as above, with initial data u_{0j}. By (6.34) $\{u_j\}$ is a Cauchy sequence in $L^\infty(0,T;L^1(K))$ for every compact subset $K \subset \Omega$. Then by a diagonal argument the conclusion easily follows. □

7. Well-posedness of problem (CL)

In this section we prove Theorem 3.3.

We first prove the existence claim. Rewrite (A_1) as follows:

$$u_{0s} = \sum_{j=1}^{r} c_j^+ \delta_{x_j'} - \sum_{j=1}^{s} c_j^- \delta_{x_j''} \qquad (c_j^\pm > 0,\ r+s=p)\,. \tag{7.1}$$

For every $j = 1,\dots,p$ such that $c_j > 0$ we set

$$C_j^+(t) := \Big[c_j - \int_0^t \Big[f^+_{x_j^+}(s) - f^+_{x_j^-}(s) \Big] ds \Big]_+ \qquad (t \in [0,T])\,, \tag{7.2}$$

with $f^+_{x_j^+}$ satisfying (5.31) (written with x_j^+ instead of a) and $f^+_{x_j^-}$ satisfying (5.35) (written with x_j^- instead of b); observe that by (5.32) and (5.36) there holds

$$f^+_{x_j^+}(s) - f^+_{x_j^-}(s) \ge 0 \quad \text{for a.e. } s \in (0,T)\,. \tag{7.3}$$

Similarly, for every $j = 1,\dots,p$ such that $c_j < 0$ we set

$$C_j^-(t) := \Big[c_j - \int_0^t \Big[f^-_{x_j^+}(s) - f^-_{x_j^-}(s) \Big] ds \Big]_- \qquad (t \in [0,T])\,, \tag{7.4}$$

with $f^-_{x_j^+}$ satisfying (5.33) (written with x_j^+ instead of a) and $f^-_{x_j^-}$ satisfying (5.37) (written with x_j^- instead of b); observe that by (5.34) and (5.38) there holds

$$f^-_{x_j^+}(s) - f^-_{x_j^-}(s) \le 0 \quad \text{for a.e. } s \in (0,T)\,. \tag{7.5}$$

Let $\bar t_j := \sup\{t \in [0,T] \,|\, C_j^\pm(t) > 0\} > 0$ $(j = 1,\dots,p)$. Then $\bar t_j > 0$ since $C_j^\pm(0) = \pm c_j > 0$. By (7.3) and (7.5) $C_j^\pm$ is nonincreasing in $(0,T)$, whence $C_j^\pm > 0$ in $[0,\bar t_j)$ and, if $\bar t_j < T$, there holds $C_j^\pm = 0$ in $[\bar t_j, T]$. Let $\tau_1 := \min\{\bar t_1,\dots,\bar t_p\}$, and define $u \in C([0,\tau_1];\ \mathcal{M}(\mathbb{R}))$ as follows:

(7.6a)

$$\begin{cases} \text{in } Q_{1,\tau_1}\ u_r \text{ is the entropy solution of } (D^+) \text{ if } c_1 > 0, \text{ of } (D^-) \text{ if } c_1 < 0; \\ \text{in } Q_{j,\tau_1}\ (j = 2,\dots,p)\ u_r \text{ is the entropy solution of } (D^+_+) \text{ if } \min\{c_{j-1},c_j\} > 0, \\ \text{of } (D^-_-) \text{ if } \max\{c_{j-1},c_j\} < 0, \text{ of } (D^-_+) \text{ if } c_{j-1} > 0 > c_j, \text{ of } (D^+_-) \text{ if } c_{j-1} < 0 < c_j; \\ \text{in } Q_{p+1,\tau_1}\ u_r \text{ is the entropy solution of } (D_+) \text{ if } c_p > 0, \text{ of } (D_-) \text{ if } c_p < 0\,. \end{cases}$$

$$u_s(\cdot,t) := \sum_{j=1}^{r} C_j^+(t) \delta_{x_j'} - \sum_{j=1}^{s} C_j^-(t) \delta_{x_j''}\,. \tag{7.6b}$$

By Definitions 3.2 and 5.2-5.4 u is an entropy solution of (CL) in Q_{j,τ_1} for $j = 1,\dots,p+1$. Hence u is an entropy solution of (CL) in S_{τ_1}, if we prove (3.1)-(3.2) with $\Omega = \mathbb{R}$, $\tau = \tau_1$ for all $\zeta \in C^1([0,\tau_1];C^1_c(\mathbb{R}))$, $\zeta \ge 0$, $\zeta(\cdot,\tau_1) = 0$ in $\mathbb{R}$, such that

$$\operatorname{supp}\zeta \cap \big(\{x_j\} \times (0,\tau_1)\big) \ne \varnothing \quad \text{for some } j = 1,\dots,p\,.$$

We only give the proof when $\zeta(x,t) = \alpha(x)\beta(t)$ with $\alpha \in C^1_c(\mathbb{R})$, $\alpha \ge 0$, $\alpha(x_j) > 0$ for a unique $j \in \{1,\dots,p\}$, and $\beta \in C^1([0,\tau_1])$, $\beta \ge 0$, $\beta(\tau_1) = 0$ (the general case can be dealt with similarly). We also assume $c_j > 0$, since the proof is similar for $c_j < 0$. Let us first prove (3.1) in this case, namely

$$\int_0^{\tau_1}\!\!\int_{I_j\cup I_{j+1}} \big[u_r\zeta_t + H(u_r)\,\zeta_x\big]\,dxdt + \int_{I_j\cup I_{j+1}} u_{0r}(x)\zeta(x,0)\,dx = \\ -\int_0^{\tau} \langle u_s(\cdot,t),\zeta_t(\cdot,t)\rangle_{(x_{j-1},x_{j+1})}\,dt - \langle u_{0s},\zeta(\cdot,0)\rangle_{(x_{j-1},x_{j+1})} \tag{7.7}$$

for all ζ as above. From (7.2) we obtain

$$\text{(7.8)}\quad \int_0^{\tau_1} \langle u_s(\cdot,t), \zeta_t(\cdot,t)\rangle_{(x_{j-1},x_{j+1})}\, dt + \langle u_{0s}, \zeta(\cdot,0)\rangle_{(x_{j-1},x_{j+1})} = \\ = \alpha(x_j)\left(\int_0^{\tau_1} \beta'(t) C_j^+(t)\, dt + c_j\beta(0)\right) = \alpha(x_j)\int_0^{\tau_1}\left[f^+_{x_j^+}(t) - f^+_{x_j^-}(t)\right]\beta(t)\, dt.$$

On the other hand, since u is a solution of (CL) in Q_{j+1,τ_1}, by (3.1) there holds

$$\iint_{Q_{j+1,\tau_1}} \left\{(u_r - k)\xi_t + [H(u_r) - H(k)]\,\xi_x\right\} dxdt = -\int_{I_{j+1}} [u_{0r}(x) - k]\,\xi(x,0)\, dx$$

for all $k \in \mathbb{R}$ and $\xi \in C^1([0,\tau_1]; C_c^1(I_{j+1}))$, $\xi(\cdot,\tau_1) = 0$ in I_{j+1}. Let η_q be defined by

$$\eta_q(x) = [2q(x - x_j) - 1]\,\chi_{\left[x_j + \frac{1}{2q}, x_j + \frac{1}{q}\right]}(x) + \chi_{\left(x_j + \frac{1}{q}, x_{j+1}\right]}(x),$$

and let $\zeta \in C^1([0,\tau_1]; C_c^1([x_j, x_{j+1})))$, $\zeta(\cdot,\tau_1) = 0$ in I_{j+1} (here $x_{j+1} = \infty$ if $j = p$). By standard arguments we can choose $\xi = \zeta\eta_q$ in the above equality. Then we get

$$\iint_{Q_{j+1,\tau_1}} \left\{(u_r - k)\zeta_t\eta_q + [H(u_r) - H(k)]\,\zeta_x\eta_q\right\} dxdt + \\ + \int_{I_{j+1}} [u_{0r}(x) - k]\,\zeta(x,0)\eta_q(x) dx = -2q\int_0^{\tau_1}\int_{x_j+1/2q}^{x_j+1/q} [H(u_r) - H(k)]\,\zeta\, dxdt\,.$$

Letting $q \to \infty$ in the above equality plainly gives (see (5.31) and (7.6a)):

$$\text{(7.9)}\quad \iint_{Q_{j+1,\tau_1}} \{(u_r - k)\zeta_t + [H(u_r) - H(k)]\zeta_x\}\, dxdt + \int_{I_{j+1}} [u_{0r}(x) - k]\zeta(x,0)\, dx = \\ = -\operatorname{ess}\lim_{x\to x_j^+} \int_0^{\tau_1} [H(u_r(x,t)) - H(k)]\,\zeta(x,t)\, dt = \\ = -\int_0^{\tau_1} [f^+_{x_j^+}(t) - H(k)]\,\zeta(x_j,t)\, dt\,.$$

Since u is an entropy solution of (CL) in Q_{j+1,τ_1}, arguing as before we obtain

$$\iint_{Q_{j+1,\tau_1}} \{|u_r - k|\zeta_t + \operatorname{sgn}(u_r - k)\,[H(u_r) - H(k)]\,\zeta_x\} dxdt + \int_{I_{j+1}} |u_{0r}(x) - k|\zeta(x,0) dx \geq \\ \geq -\operatorname{ess}\lim_{x\to x_j^+} \int_0^{\tau_1} \operatorname{sgn}(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\, dt$$

for all ζ as above, $\zeta \geq 0$. Choosing $\zeta(x,t) = \alpha(x)\beta(t)$ with $\alpha \in C_c^1([x_j, x_{j+1}))$, $\beta \in C^1([0,\tau_1])$, $\alpha \geq 0$, $\beta \geq 0$ and $\beta(\tau_1) = 0$, by (5.13a) from the above inequality we obtain

$$\text{(7.10)}\quad \iint_{Q_{j+1,\tau_1}} \{|u_r - k|\,\zeta_t + \operatorname{sgn}(u_r - k)\,[H(u_r) - H(k)]\,\zeta_x\}\, dxdt + \\ + \int_{I_{j+1}} |u_{0r}(x) - k|\zeta(x,0)\, dx + \operatorname{ess}\lim_{x\to x_j^+} \int_0^{\tau_1} [H(u_r(x,t)) - H(k)]\,\zeta(x,t)\, dt \geq \\ \geq -2\operatorname{ess}\lim_{x\to x_j^+} \int_0^{\tau_1} \operatorname{sgn}_-(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\, dt = \\ = -2\alpha(x_j)\operatorname{ess}\lim_{x\to x_j^+} \int_0^{\tau_1} \operatorname{sgn}_-(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\beta(t)\, dt \geq 0,$$

since $\operatorname{sgn}(u) = 1 + 2\operatorname{sgn}_-(u)$.

Replacing Q_{j+1,τ_1} by Q_{j,τ_1} we obtain, similarly to (7.9)-(7.10),

$$\iint_{Q_{j,\tau_1}} \left\{(u_r-k)\zeta_t + [H(u_r)-H(k)]\,\zeta_x\right\} dxdt + \int_{I_j} [u_{0r}(x)-k]\,\zeta(x,0)\,dx = \tag{7.11}$$
$$= \operatorname*{ess\,lim}_{x\to x_j^-} \int_0^{\tau_1} [H(u_r(x,t))-H(k)]\zeta(x,t)\,dt = \int_0^{\tau_1} [f^+_{x_j^-}(t)-H(k)]\zeta(x_j,t)\,dt,$$

$$\iint_{Q_{j,\tau_1}} \{|u_r-k|\,\zeta_t + \operatorname{sgn}(u_r-k)\,[H(u_r)-H(k)]\,\zeta_x\}\,dxdt + \tag{7.12}$$
$$+\int_{I_j} |u_{0r}(x)-k|\zeta(x,0)\,dx - \operatorname*{ess\,lim}_{x\to x_j^-} \int_0^{\tau_1} [H(u_r(x,t))-H(k)]\,\zeta(x,t)\,dt \geq 0,$$

Summing (7.9) and (7.11) gives

$$\int_0^{\tau_1}\int_{I_j\cup I_{j+1}} \left[u_r\zeta_t + H(u_r)\,\zeta_x\right] dxdt + \int_{I_j\cup I_{j+1}} u_{0r}(x)\zeta(x,0)\,dx = \tag{7.13}$$
$$= -\,\alpha(x_j)\int_0^{\tau_1} \left[f^+_{x_j^+}(t) - f^+_{x_j^-}(t)\right]\beta(t)\,dt\,.$$

Then equality (7.7) follows from (7.8) and (7.13).

Next we prove (3.2) for all ζ as above, namely

$$\int_0^{\tau_1}\int_{I_j\cup I_{j+1}} \{|u_r-k|\,\zeta_t + \operatorname{sgn}(u_r-k)\,[H(u_r)-H(k)]\,\zeta_x\}\,dxdt + \tag{7.14}$$
$$+\int_0^{\tau_1} \langle |u_s(\cdot,t)|,\zeta_t(\cdot,t)\rangle_{(x_{j-1},x_{j+1})}\,dt + \langle |u_{0s}|,\zeta(\cdot,0)\rangle_{(x_{j-1},x_{j+1})} \geq$$
$$\geq -\int_{I_j\cup I_{j+1}} |u_{0r}(x)-k|\,\zeta(x,0)\,dx$$

Since u is an entropy solution of (CL) in Q_{j,τ_1} and Q_{j+1,τ_1}, from (5.31), (5.35), (7.10) and (7.12) it follows that

$$\int_0^{\tau_1}\int_{I_j\cup I_{j+1}} \{|u_r-k|\;\;\zeta_t + \operatorname{sgn}(u_r-k)\,[H(u_r)-H(k)]\,\zeta_x\}\,dxdt +$$
$$+\int_{I_j\cup I_{j+1}} |u_{0r}(x)-k|\,\zeta(x,0)\,dx \;\geq\; -\,\alpha(x_j)\int_0^{\tau_1} \left[f^+_{x_j^+}(t) - f^+_{x_j^-}(t)\right]\beta(t)\,dt\,.$$

Since $|u_{0s}| \llcorner (x_{j-1},x_{j+1}) = u_{0s} \llcorner (x_{j-1},x_{j+1})$ (recall that $c_j > 0$ by assumption) and by (7.6b) $|u_s(\cdot,t)| \llcorner (x_{j-1},x_{j+1}) = u_s(\cdot,t) \llcorner (x_{j-1},x_{j+1})$ for all $t\in[0,T]$, the above inequality together with (7.8) implies (7.14). Therefore, the measure u defined by (7.6) is an entropy solution of (CL) in S_{τ_1}.

If $\tau_1 < T$, either $u_s(\cdot,\tau_1) = 0$, or $u_s(\cdot,\tau_1) \neq 0$. If $u_s(\cdot,\tau_1) = 0$, there holds $C_j^{\pm}(\tau_1) = 0$ for all $j = 1,\dots,p$, thus $u_s(\cdot,t) = 0$ for all $t\in[\tau_1,T]$. Then, by the standard theory of scalar conservation laws, we can continue the solution of (CL) in $(\tau_1,T]$ with initial data $u_r(\cdot,\tau_1)$. On the other hand, if $u_s(\cdot,\tau_1)\neq 0$, then $C_j^{\pm}(\tau_1)\neq 0$ for some $j = 1,\dots,p$ and, arguing as before, we can continue the solution of (CL) in $(\tau_1,\tau_2]$, with initial data $u(\cdot,\tau_1)$, for some $\tau_2\in(\tau_1,T]$. Iterating the procedure q times with $2\leq q\leq p$, we obtain that either $\tau_q = T$, or $u_s(\cdot,\tau_q) = 0$.

Let us now address uniqueness. Let $u,v\in C([0,T];\mathcal{M}(\mathbb{R}))$ be entropy solutions of (CL), and let $\tau := \min\{t_u,t_v\}$, where

$$\begin{cases} t_u := \sup\{t\in[0,T)\,|\,\operatorname{supp} u_s(\cdot,t) = \operatorname{supp} u_{0s}\} \\ t_v := \sup\{t\in[0,T)\,|\,\operatorname{supp} v_s(\cdot,t) = \operatorname{supp} u_{0s}\}\,. \end{cases}$$

Arguing as at the end of the existence proof, it is enough to show that $u = v$ in $\mathcal{M}(S_\tau)$. We claim that this follows, if we prove that

$$u_r = v_r \quad \text{a.e. in } S_\tau \,. \tag{7.15}$$

In fact, equalities (3.1) and (7.15) imply that, for all $\zeta \in C^1([0,\tau];C^1_c(\mathbb{R}))$, $\zeta(\cdot,\tau) = 0$ in $\mathbb{R}$,

$$\int_0^\tau \langle u_s(\cdot,t) - v_s(\cdot,t), \zeta_t(\cdot,t)\rangle_{\mathbb{R}}\, dt = \iint_{S_\tau} \{(u_r - v_r)\,\zeta_t + [H(u_r) - H(u_r)]\zeta_x\}\, dxdt = 0\,.$$

Hence $\langle u_s(\cdot,t) - v_s(\cdot,t), \alpha\rangle_{\mathbb{R}} = 0$ for a.e. $t \in (0,\tau)$, for all $\alpha \in C^1_c(\mathbb{R})$. Therefore $u_s = v_s$ in $L^\infty(0,\tau;\mathcal{M}(\mathbb{R}))$, thus (7.15) implies $u = v$ in $\mathcal{M}(S_\tau)$.

It remains to prove (7.15), which is equivalent to showing that $u_r = v_r$ a.e. in $Q_{j,\tau}$ for all $j = 1,\dots,p+1$. However, this follows from the uniqueness results provided by Theorem 5.2.. Then the result follows. □

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