# Free groups with involution satisfying a $*$-group identity*

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**Abstract**

Let $G$ be a nonabelian free group with involution $*$. In the present note, we show that $G$ satisfies a $*$-group identity if and only if $*$ is the classical involution, given by $g^* = g^{-1}$ for all $g \in G$.

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## 1 Introduction

Let $G$ be a group. We say that $G$ satisfies a group identity if there exists a nontrivial word $w(x_1, \dots, x_n)$ in the free group $\langle x_1, x_2, \dots \rangle$ such that $w(g_1, \dots, g_n) = 1$ for all $g_i \in G$. For instance, an abelian group satisfies the identity $x_1^{-1}x_2^{-1}x_1x_2 = 1$, whereas a nonabelian free group cannot satisfy a group identity.

Recall that an involution on $G$ is a function $* : G \to G$ satisfying $(gh)^* = h^*g^*$ and $(g^*)^* = g$ for all $g, h \in G$. One example that is always present is the classical involution, given by $g^* = g^{-1}$ for all $g \in G$. The free group $\langle x_1, x_2, \dots \rangle$ has an involution given by $x_1^* = x_2$, $x_3^* = x_4$, and so on. Renumbering, we obtain the free group with involution, $\langle x_1, x_1^*, x_2, x_2^*, \dots \rangle$. We say that a group $G$ with involution satisfies a $*$-group identity (or $*$-GI)

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if there exists a nontrivial word $w(x_1, x_1^*, \ldots, x_n, x_n^*) \in \langle x_1, x_1^*, \ldots \rangle$ such that $w(g_1, g_1^*, \ldots, g_n, g_n^*) = 1$ for all $g_i \in G$.

Inspired by the work of Amitsur [1], where it was shown that if a ring with involution satisfies a $*$-polynomial identity, then it satisfies a polynomial identity, Giambruno, Polcino Milies and Sehgal [3] looked at the group ring $FG$ of a torsion group with involution $G$ over an infinite field $F$. They demonstrated that if the unit group of $FG$ satisfies a $*$-GI (under the involution linearly extended from the one on $G$), then the symmetric units of $FG$ satisfy a group identity. Here, an element $g$ is said to be symmetric if $g^* = g$. This does not, in general, imply that the entire unit group satisfies a group identity.

Let us now consider the case of a nonabelian free group $G$ with involution. As the composition of two involutions is an automorphism, and every automorphism of $G$ commutes with the classical involution, we see that the involutions of $G$ are found by composing the classical involution with an automorphism of order at most 2. For example, if $G = \langle x_1, x_2, x_3, x_4 \rangle$, then we can obtain an automorphism $\sigma$ of order 2 by letting $\sigma(x_1) = x_1$, $\sigma(x_2) = x_2^{-1}$, $\sigma(x_3) = x_4$ and $\sigma(x_4) = x_3$. Composing with the classical involution, we get $x_1^* = x_1^{-1}$, $x_2^* = x_2$, $x_3^* = x_4^{-1}$ and $x_4^* = x_3^{-1}$. Then, for instance, $(x_1x_3)^* = x_4^{-1}x_1^{-1}$. In fact, Dyer and Scott [2, Theorem 3] have classified the automorphisms of prime order of a free group.

If the involution on $G$ is classical, then it is immediately obvious that $G$ satisfies the $*$-GI $x_1x_1^*$. Our purpose here is to demonstrate that the classical involution is rather special in that sense, for if the involution is anything else, then $G$ will not satisfy any $*$-GI at all. In analogy to the main theorem of [3], we obtain the following result.

**Theorem.** *Let $G$ be a nonabelian free group having an involution $*$. Then the following are equivalent:*

1. *$G$ satisfies a $*$-GI,*
2. *the symmetric elements of $G$ satisfy a group identity, and*
3. *$*$ is the classical involution.*

# 2 Proof of the theorem

Throughout, let $G$ be a nonabelian free group with involution $*$.

**Lemma 1.** *If $G$ has noncommuting symmetric elements $a$ and $b$, then $G$ does not satisfy a $*$-GI.*

*Proof.* We may as well let $G = \langle a, b \rangle$, which is a free group of rank 2. If $G$ satisfies a $*$-GI, then by [3, Lemma 1], it satisfies one of the form $w(x, x^*) = 1$. Notice that $(ab)^* = ba$, $(ab)^{-1} = b^{-1}a^{-1}$ and $((ab)^*)^{-1} = a^{-1}b^{-1}$. Letting $x = ab$, we note that no cancellation is possible, since $x$ and $x^{-1}$ do not occur together in $w$, and neither do $x^*$ and $(x^*)^{-1}$. Therefore, $w(ab, (ab)^*) \neq 1$. □

**Lemma 2.** *If $G$ has noncommuting elements $a$ and $b$ satisfying $a^* = a$, $b^* = b^{-1}$, then $G$ does not satisfy a $*$-GI.*

*Proof.* We can assume that $G = \langle a, b \rangle$, a free group of rank 2. Then we note that $a$ and $bab^{-1}$ are symmetric and do not commute. Apply Lemma 1. □

**Lemma 3.** *Suppose that $*$ is not the classical involution. If $x$ is in any free generating set of $G$, and $x^* = x^{-1}$, then $G$ does not satisfy a $*$-GI.*

*Proof.* As $*$ cannot act as the classical involution upon the entire free generating set, let $y$ be another generator such that $y^* \neq y^{-1}$. If $x$ and $yy^*$ do not commute, then Lemma 2 finishes the proof. Therefore, assume that they do commute. Then, since $x$ is part of the generating set, $yy^* = x^i$, for some integer $i$. However, $*$ fixes $yy^*$ and inverts $x^i$. Therefore, $yy^* = 1$, which is impossible. □

Let us now prove the main result.

*Proof of Theorem.* If $*$ is the classical involution, then since $G$ is torsion-free, its only symmetric element is the identity. Therefore, in this case, the symmetric elements of $G$ satisfy every group identity, so (3) implies (2). To show that (2) implies (1), we note that if the symmetric elements satisfy $w(x_1, \ldots, x_n) = 1$, then $G$ satisfies $w(x_1x_1^*, \ldots, x_nx_n^*) = 1$. Thus, it remains only to show that (1) implies (3). To this end, suppose that $G$ satisfies a $*$-GI, but $*$ is not the classical involution.

Taking any element $x$ of a free generating set, Lemma 3 tells us that $x^* \neq x^{-1}$. Let $y$ be any element of $G$ that does not centralize $xx^*$. If $xx^*$ and $yy^*$ do not commute, then Lemma 1 finishes the proof, so assume that they do commute. Then there exists $z \in G$ such that $xx^* = z^m$ and $yy^* = z^n$, for some integers $m$ and $n$. By choosing $z$ judiciously, we may assume that $m$ and $n$ are relatively prime. Then, since $z^m$ and $z^n$ are symmetric, so is

$z$. Furthermore, $y^* = y^{-1}z^n$. Therefore, the group $\langle y, z\rangle$ is $*$-invariant. If $y$ commutes with $z$, then it commutes with $z^m = xx^*$, giving us a contradiction. Recalling that $z$ is symmetric, we may therefore assume that $G = \langle y, z\rangle$, a free group of rank 2.

If $z$ and $y^2(y^2)^*$ do not commute, then by Lemma 1, we are done, so assume that they commute. As $z$ is part of the free generating set, $y^2(y^2)^* = z^i$, for some integer $i$. But also,

$$y^2(y^*)^2 = y(yy^*)y^{-1}(yy^*) = yz^ny^{-1}z^n.$$

Therefore, $yz^ny^{-1}z^{n-i} = 1$. As $y$ and $z$ are free generators, this can only happen if $n = 0$. But in this case, $yy^* = z^0 = 1$, hence $y^* = y^{-1}$. Lemma 3 completes the proof. □


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