A CHARACTERIZATION OF MINIMAL VARIETIES OF $\mathbb{Z}_p$-GRADED PI ALGEBRAS

Onofrio Mario DI VINCENZO - Viviane Ribeiro Tomaz da SILVA - Ernesto SPINELLI [1]

Abstract. Let F be a field of characteristic zero and p a prime. In the present paper it is proved that a variety of $\mathbb{Z}_p$-graded associative PI F-algebras of finite basic rank is minimal of fixed $\mathbb{Z}_p$-exponent d if, and only if, it is generated by an upper block triangular matrix algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$ equipped with a suitable elementary $\mathbb{Z}_p$-grading, whose diagonal blocks are isomorphic to $\mathbb{Z}_p$-graded simple algebras $A_1, \ldots, A_m$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

Classification MSC 2010: 16R10, 16R50, 16W50.
Keywords: Graded Algebras, Graded Polynomial Identities, G-exponent, Minimal Varieties, Upper Block Triangular Matrix Algebras.

1. Introduction

Let F be a field of characteristic zero and A an associative F-algebra satisfying a polynomial identity (in the sequel we shall refer to these algebras as PI algebras) graded by a finite group G. In the last two decades there has been an extensive investigation of graded polynomial identities satisfied by A. Apart from their own interesting features, the main motivations for this type of work come from the significant information on quite general questions that the additional graded structure and related objects may provide. This is the case, for instance, for the solution of the Specht problem due to Kemer (see [19]) in which $\mathbb{Z}_2$-gradings play a key role. Along this line, we are going to focus on the growth of *graded codimensions*.

More specifically, let $F\langle X_G\rangle$ be the free associative G-graded algebra on a countable set over F and $\mathrm{Id}_G(A)$ the ideal of G-graded identities of a G-graded F-algebra A. Then one considers the relatively free G-graded algebra $F\langle X_G\rangle/\,\mathrm{Id}_G(A)$ and the dimension of the subspace of its multilinear elements in n free generators, denoted by $c_n^G(A)$. The sequence $\{c_n^G(A)\}_{n\geq 1}$ is called the sequence of the *G-graded codimensions* of A. It was introduced by Regev in the ungraded case (which actually corresponds to when $G = \langle 1_G\rangle$) in the seminal paper [21], where it was proved that in presence of a non-zero polynomial identity it is exponentially bounded. As a consequence of this

[1]The first named author was partially supported by RIL of Università della Basilicata. The second named author was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) - grant 306534/2016-9, Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG) - grant APQ-01149-18, and "Para mulheres na Ciência"(L'OREAL-ABC-UNESCO).

fact, in [13] it was stated that $c_n^G(A)$ is exponentially bounded if, and only, A is a PI algebra. In such an event, Aljadeff and Giambruno [1] have recently captured the exponential growth of this sequence proving that

$$\exp_G(A) := \lim_{m\to+\infty} \sqrt[m]{c_m^G(A)}$$

exists and is a non-negative integer, which is called the *G-graded exponent* or simply the *G-exponent* of A (for the sake of completeness, we have to mention that this was established in [12] when G is finite and abelian and, previously, in [2] under the extra assumption that A is finite-dimensional). This extends the deep contribution of Giambruno and Zaicev ([14] and [15]) on the existence of the exponent of a PI algebra.

These results are the most striking culminating points of quantitative investigations of (non-necessarily) graded polynomial identities satisfied by a PI algebra: in fact, the existence of the graded exponent allows us to classify varieties on an integral scale whose steps are the *minimal varieties* of given G-exponent d, namely those varieties of G-exponent d such that every proper subvariety has G-exponent strictly less than d.

In this direction, Giambruno and Zaicev [17] proved that in the ungraded case a variety of exponential growth is minimal if, and only if, it is generated by the Grassmann envelope of a so called *minimal superalgebra*. Previously the same authors in [16] stated that a variety of PI algebras of finite basic rank (that is generated by a finitely generated PI algebra) is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(m_1, \dots, m_n)$. Motivated by all these results, in [10] varieties of $\mathbb{Z}_2$-graded PI algebras or *supervarieties* of *finite basic rank* (that is, generated by a finitely generated superalgebra satisfying an ordinary polynomial identity) which are minimal of fixed $\mathbb{Z}_2$-graded exponent have been investigated. In particular, it has been established that any such supervariety is generated by one of the above mentioned minimal superalgebras. But despite some partial results in [8] and [9], until this moment their complete characterization is still unknown.

In the present paper we deal with varieties of $\mathbb{Z}_p$-graded PI algebras of finite basic rank where p is a prime and our goal is the classification of those which are minimal of a given $\mathbb{Z}_p$-exponent (obviously this solves the problem for the above discussed case $p = 2$ as well). In more detail, given a m-tuple of $\mathbb{Z}_p$-graded simple algebras $(A_1, \dots, A_m)$ we construct an upper block triangular matrix algebra $UT_{\mathbb{Z}_p}(A_1, \dots, A_m)$ equipped with a suitable elementary $\mathbb{Z}_p$-grading, whose i-th diagonal block is isomorphic to the graded algebra A_i. The main result we prove is that a variety of $\mathbb{Z}_p$-graded PI-algebras of finite basic rank is minimal of $\mathbb{Z}_p$-exponent d if, and only if, it is generated by a $\mathbb{Z}_p$-graded algebra $UT_{\mathbb{Z}_p}(A_1, \dots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

2. Preliminaries and minimal G-graded algebras

Throughout the rest of the paper, unless otherwise stated, G is a finite group, F is a field of characteristic zero and all the algebras are assumed to be associative and to have the same ground field F. For any ring R and pair of positive integers s and t the symbol $M_{s\times t}(R)$ means the space of all matrices with s rows and t columns over R and set $M_s(R) := M_{s\times s}(R)$. In the case in which $R = F$, we simply write $M_{s\times t}$ and M_s instead of $M_{s\times t}(F)$ and $M_s(F)$, respectively. Finally, if $m_1, \ldots, m_n$ is a sequence of positive integers, let $UT(m_1, \ldots, m_n)$ be the upper block triangular matrix algebra of size $m_1, \ldots, m_n$ over F.

An algebra A is *G-graded* if it has a vector space decomposition $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. The elements of A_g are called *homogeneous of degree g*. An element w of A is *homogeneous* if there exists $g \in G$ such that $w \in A_g$ (and in this case denote its degree by $|w|_A$), whereas a subalgebra or an ideal $V \subseteq A$ is *homogeneous* if $V = \oplus_{g\in G}(V \cap A_g)$.

Let $X_G := \cup X_g$ be the union of disjoint countable sets $X_g := \{x_1^g, x_2^g, \ldots\}$, where $g \in G$. The free algebra $F\langle X_G\rangle$ freely generated by X_G is equipped in a natural way with a G-graded algebra structure if we require that the variables from X_g have degree g (and then extend this grading to the monomials on X_G). This algebra is said to be the *free G-graded algebra* over F. An element $f(x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}})$ of $F\langle X_G\rangle$ is a *G-graded polynomial identity* for the G-graded algebra $A = \oplus_{g\in G} A_g$ if $f(a_1, \ldots, a_n) = 0_A$ for every $a_1 \in A_{g_{i_1}}, \ldots, a_n \in A_{g_{i_n}}$. We use the symbol $\mathrm{Id}_G(A)$ to indicate the set of all the G-graded polynomial identities satisfied by A, which is easily seen to be a *T_G-ideal* of $F\langle X_G\rangle$, namely a two-sided ideal of the free G-graded algebra stable under every endomorphism of $F\langle X_G\rangle$ preserving the grading.

Given a T_G-ideal I of $F\langle X_G\rangle$, the *variety of G-graded algebras* $\mathcal{V}^G$ associated to I is the class of all the G-graded algebras A such that I is contained in $\mathrm{Id}_G(A)$. The T_G-ideal I is denoted by $\mathrm{Id}_G(\mathcal{V}^G)$. The variety $\mathcal{V}^G$ is generated by the G-graded algebra A if $\mathrm{Id}_G(\mathcal{V}^G) = \mathrm{Id}_G(A)$, and in this case we write $\mathcal{V}^G = \mathrm{var}_G(A)$ and set $c_n^G(\mathcal{V}^G) := c_n^G(A)$, the n-th *G-graded codimension* of the variety $\mathcal{V}^G$. It is defined as the dimension of $\frac{P_n^G}{P_n^G \cap \mathrm{Id}_G(A)}$, where P_n^G is the space of multilinear polynomials of degree n of $F\langle X_G\rangle$ in the variables x_i^g for $1 \le i \le n$ and $g \in G$. Since F has characteristic zero, $\mathrm{Id}_G(\mathcal{V}^G)$ is completely determined by the multilinear polynomials it contains. Thus the behaviour of the sequence of its graded codimensions in some sense measures the rate of growth of the G-graded polynomial identities of $\mathcal{V}^G$.

It is worth recalling that if G has more than one element and a G-graded algebra satisfies an ordinary polynomial identity, obviously it satisfies a G-graded polynomial identity, but the converse is in general not true: for instance, it is enough to consider the free algebra W generated by two indeterminates with the trivial G-grading. Anyway we shall assume throughout that all the G-graded algebras A we deal with are PI algebras because, if this happens, the sequence of the graded codimensions $\{c_n^G(\mathrm{var}_G(A))\}_{n\ge 1}$ is exponentially bounded and, as reported in the Introduction,

$$\exp_G(\mathrm{var}_G(A)) := \lim_{m\to+\infty} \sqrt[m]{c_m^G(\mathrm{var}_G(A))}$$

exists and is a non-negative integer, which is called the *G-graded exponent* or simply the *G-exponent* of the variety $\mathrm{var}_G(A)$.

In the present paper we actually consider finite-dimensional algebras: in fact, we analyze the behaviour of varieties of G-graded PI algebras of finite basic rank, i.e. generated by a finitely generated G-graded PI algebra W. But, in this event, Aljadeff and Kanel-Belov [4] have proved that there exist a field extension K of F and a finite-dimensional G-graded algebra A over K such that $\mathrm{Id}_G(W) = \mathrm{Id}_G(A)$. The fact that A is a K-algebra has no effect on our final aims because, since F has characteristic zero, $\mathrm{Id}_G(W) = \mathrm{Id}_G(W \otimes_F L)$ (in $F\langle X_G\rangle$) for any field extension L of F. Consequently also the G-graded codimensions of W do not change upon extension of the base field. Hence for the proofs of our main results we can assume that F (as well as its extensions) is algebraically closed. This is convenient because over algebraically closed fields it is easier to describe special G-graded algebras. In particular, a result of Bahturin, Sehgal and Zaicev, which completely determines the structure of finite-dimensional *G-graded simple* or *G-simple* algebras, will be very useful. We recall that a G-graded algebra A is said to be G-simple if the multiplication is non-trivial and it has no non-trivial homogeneous ideals.

Theorem 2.1 ([6]). *Let F be an algebraically closed field and A be a finite-dimensional G-simple algebra. Then there exists a subgroup H of G, a 2-cocycle $f : H \times H \longrightarrow F \setminus \{0_F\}$ where the action of H on F is trivial, an integer k and a k-tuple $(g_1 := 1_G, g_2, \ldots, g_k) \in G^k$ such that A is G-graded isomorphic to $C := F^f H \otimes M_k$, where $C_g = \mathrm{span}_F\{b_h \otimes E_{ij} \mid g = g_i^{-1} h g_j\}$. Here $b_h \in F^f H$ is a representative of $h \in H$ and $E_{ij} \in M_k$ is its (i,j)-matrix unit.*

Assume now that A is a finite-dimensional G-graded algebra over the algebraically closed field F. Hence, by the generalization of the Wedderburn-Malcev Theorem we can write $A = A_{ss} + J(A)$, where A_{ss} is a maximal semisimple subalgebra of A homogeneous in the G-grading and $J(A)$ is its Jacobson radical (which is homogeneous as well). Also A_{ss} can be written as the direct sum of G-graded simple algebras. In analogy with the definition given by Giambruno and Zaicev in [17] in the case in which $G = \mathbb{Z}_2$, we introduce that of a *minimal* G-graded algebra.

Definition 2.2. *Let F be an algebraically closed field. A G-graded algebra A is called* minimal *if it is finite-dimensional and either A is a G-simple algebra or $A = A_{ss} + J(A)$ where*

(i) $A_{ss} = A_1 \oplus \cdots \oplus A_n$, *with* $A_1, \ldots, A_n$ *G-simple algebras and* $n \geq 2$;

(ii) *there exist homogeneous elements* $w_{12}, \ldots, w_{n-1,n} \in J(A)$ *and minimal homogeneous idempotents* $e_1 \in A_1, \ldots, e_n \in A_n$ *such that*

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \qquad 1 \leq i \leq n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;$$

(iii) $w_{12}, \ldots, w_{n-1,n}$ *generate* $J(A)$ *as a two-sided ideal of* A.

We notice that the order of the components $A_1, \ldots, A_n$ of the semisimple part A_{ss} of a minimal G-graded algebra A is important. For this reason, in the sequel we shall tacitly agree that if $A_{ss} = A_1 \oplus \ldots \oplus A_n$, then $A_1 J(A) A_2 J(A) \cdots J(A) A_n \neq 0_A$. According to the main result of [1], $\exp_G(A) = \dim_F A_{ss}$.

It is easily seen that the minimal G-graded algebra A has the following vector space decomposition

$$A = \bigoplus_{1 \leq i \leq j \leq n} A_{ij},$$

with

$$A_{ij} := \begin{cases} A_i & \text{if } i = j, \\ A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < j. \end{cases}$$

Moreover $J(A) = \oplus_{i<j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where δ_{jk} is the Kronecker delta. Finally, for every $1 \leq r \leq s \leq n$, set $[r, s] := \{r, r+1, \ldots, s\}$,

$$A^{[r,s]} := \bigoplus_{r \leq i \leq j \leq s} A_{ij} \tag{1}$$

and, for each $1 < k < n$,

$$A^{(\check{k})} := \bigoplus_{\substack{1 \leq i \leq j \leq n; \\ i \neq k \neq j}} A'_{ij}, \tag{2}$$

where

$$A'_{ij} := \begin{cases} A_i w_{i,i+1} A_{i+1} \cdots A_{k-1} w_{k-1,k} w_{k,k+1} A_{k+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < k < j, \\ A_{ij} & \text{otherwise.} \end{cases}$$

We remark that both $A^{[r,s]}$ and $A^{(\check{k})}$ are minimal G-graded algebras of G-exponent $\dim_F(A_r \oplus \cdots \oplus A_s)$ and $\dim_F(A_1 \oplus \cdots \oplus A_{k-1} \oplus A_{k+1} \oplus \cdots \oplus A_n)$, respectively.

The aim of the present paper is to contribute to the classification of minimal varieties of G-graded PI algebras of fixed G-exponent. We recall the definition.

Definition 2.3. *A variety $\mathcal{V}^G$ of G-graded PI algebras is said to be* minimal *of G-exponent d if $\exp_G(\mathcal{V}^G) = d$ and $\exp_G(\mathcal{U}^G) < d$ for every proper subvariety $\mathcal{U}^G$ of $\mathcal{V}^G$.*

The first technical result along the way is the following

Lemma 2.4. *Let A be a finite-dimensional G-graded algebra over an algebraically closed field. Then there exists a minimal G-graded algebra $A' \subseteq A$ such that $\exp_G(A) = \exp_G(A')$.*

Proof. It is sufficient to use exactly the same arguments of the proof of Lemma 8.1.4 of [18] (which deals with the case $G = \mathbb{Z}_2$). $\square$

The next statement represents the core of this section and establishes an important connection among minimal varieties and minimal G-graded algebras.

Theorem 2.5. *Let $\mathcal{V}^G$ be a variety of G-graded PI-algebras of finite basic rank. If $\mathcal{V}^G$ is minimal of G-exponent d, then there exists a minimal G-graded algebra A such that $\mathcal{V}^G = \mathrm{var}_G(A)$.*

Proof. Since $\mathcal{V}^G$ is of finite basic rank, according to Theorem 1.1 of [4] and the deductions before Theorem 2.1 there exists a finite-dimensional G-graded algebra B over an algebraically closed field extension K of F such that $\mathcal{V}^G = \mathrm{var}_G(B)$ and $\exp_G(B) = d$. By Lemma 2.4, B contains a minimal G-graded algebra A of G-exponent d. In particular, the algebra A belongs to $\mathcal{V}^G$ and, by the minimality of $\mathcal{V}^G$, we get that $\mathcal{V}^G = \mathrm{var}_G(A)$. $\square$

3. The $\mathbb{Z}_p$-graded algebra $UT_{\mathbb{Z}_p}(A_1, \dots, A_m)$ related to a minimal $\mathbb{Z}_p$-graded algebra

We are interested to see in more detail the special case in which p is a prime and G is isomorphic to the cyclic group of order p, denoted by $\mathbb{Z}_p$. To this end, some preliminary considerations are in order. A G-grading on the complete matrix algebra M_m is called *elementary* if there exists an m-tuple $(g_1, \dots, g_m) \in G^m$ such that the matrix units E_{ij} of M_m are homogeneous and $E_{ij} \in (M_m)_g$ if, and only if, $g = g_i^{-1} g_j$. In an equivalent manner, we can define a map $\alpha : \{1, \dots, m\} \longrightarrow G$ inducing a grading on M_m setting the degree of E_{ij} equal to $\alpha(i)^{-1}\alpha(j)$. Obviously, the algebra of upper block triangular matrices admits an elementary grading: in fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra. To denote that the grading on M_m (as well as on one of its homogeneous subalgebras) is induced by α we write (M_m, α).

Now, let ϵ be a primitive p-th root of unity in F and assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$. Consider M_p endowed with the elementary grading induced by the p-tuple $(1_G, \epsilon, \epsilon^2, \dots, \epsilon^{p-1})$. Given the permutation $\sigma := (1\ 2 \cdots p)$ of the symmetric group S_p, let us denote by D the homogeneous subalgebra of M_p generated, as a vector space, by the elements

$$c_{\epsilon^{k-1}} := \sum_{i=0}^{p-1} E_{\sigma^i(1)\sigma^i(k)} \qquad \text{with } k \in [1, p]. \tag{3}$$

That is,

$$D = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ a_p & a_1 & \ddots & & a_{p-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_3 & & \ddots & \ddots & a_2 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix}, \text{ where } a_1, a_2, \dots, a_p \in F,$$

with its natural grading induced by the p-tuple $(1_G, \epsilon, \epsilon^2, \dots, \epsilon^{p-1})$. Directly from Theorem 2.1 one has the following characterization.

Proposition 3.1. *Let F be an algebraically closed field and $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ a group of prime order p. If A is a finite-dimensional G-simple algebra, then it is isomorphic to one of the following G-graded algebras:*

(i) M_n with an elementary grading;

(ii) $D \otimes M_r$ with the grading induced by the trivial grading on M_r and the natural one on D. In other words, in this case, A is isomorphic to the homogeneous subalgebra $M_r(D)$ of M_{pr} with the grading induced by the (pr)-tuple $(\underbrace{\underbrace{1_G, \epsilon, \epsilon^2, \dots, \epsilon^{p-1}}, \dots, \underbrace{1_G, \epsilon, \epsilon^2, \dots, \epsilon^{p-1}}}_{r \text{ times}})$.

Proof. From the deductions of Section 3.1 of [7] it follows that, when A is non-simple graded simple, then it is isomorphic as a graded algebra to $M_r(D)$, where r is such that $r^2 p = \dim_F A$.

In the case in which A is simple graded simple the result was established in Theorems 5 and 6 of [5]. $\square$

We are going to strengthen the conclusion of Theorem 2.5 proving that among the minimal $\mathbb{Z}_p$-graded algebras generating a minimal variety of finite basic rank we can always choose a special upper block triangular matrix algebra. To this end, we need to fix some notation.

For the rest of this section assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and F is algebraically closed. Given an m-tuple $(A_1, \dots, A_m)$ of G-simple algebras, let

$$\Gamma_0 := \{\, i \mid i \in [1, m],\ A_i \text{ is simple graded simple}\} \quad \text{and} \quad \Gamma_1 := [1, m] \setminus \Gamma_0.$$

For every $k \in [1, m]$, let us denote the *size* of A_k by

$$s_k := \begin{cases} n_k & \text{if } k \in \Gamma_0 \text{ and } A_k \cong M_{n_k}, \\ pn_k & \text{if } k \in \Gamma_1 \text{ and } A_k \cong M_{n_k}(D) \subseteq M_{pn_k} \end{cases}$$

and, set $\eta_0 := 0$, let $\eta_k := \sum_{i=1}^{k} s_i$ and $\mathrm{Bl}_k := [\eta_{k-1} + 1, \eta_k]$. Finally, let us define

- $\mathbf{U} := UT(s_1, \dots, s_m) := \{(a_{ij})_{i,j \in [1,m]} \mid a_{ij} \in M_{s_i \times s_j} \text{ if } 1 \le i \le j \le m \text{ and } a_{ij} = 0_{M_{s_i \times s_j}} \text{ otherwise}\}$;
- $\mathbf{R} := UT(A_1, \dots, A_m) := \{(a_{ij}) \in \mathbf{U} \mid a_{kk} \in M_{n_k}(D)\ \forall k \in \Gamma_1\}$;

and, for all $1 \le r \le t \le m$,

- $\mathbf{U}_{r,t} := \{(a_{ij}) \in \mathbf{U} \mid a_{ij} = 0_{M_{s_i \times s_j}}\ \ \forall (i,j) \ne (r,t)\}$;
- $\mathbf{R}_{r,t} := \mathbf{R} \cap \mathbf{U}_{r,t}$.

A canonical basis for $UT(A_1, \dots, A_m)$. For any $i \in [1, s_r]$ and $j \in [1, s_t]$, set

$$E_{ij}^{(r,t)} := E_{\eta_{r-1}+i, \eta_{t-1}+j},$$

where $E_{\eta_{r-1}+i,\eta_{t-1}+j}$ is the $(\eta_{r-1}+i, \eta_{t-1}+j)$-matrix unit of M_{η_m}. These elements form a basis of $\mathbf{U}_{r,t}$. Consequently they form also a basis of $\mathbf{R}_{r,t}$ in all the cases except those in which $r = t \in \Gamma_1$, which we call *canonical.*

When $k \in \Gamma_1$, if $a := (a_{ij}) \in \mathbf{R}_{k,k}$, then a_{kk} is a $(n_k \times n_k)$-matrix with entries in D and $a_{ij} = 0_{M_{s_i \times s_j}}$ for any pair $(i,j) \neq (k,k)$. Hence we can take as a basis of $\mathbf{R}_{k,k}$ the elements $(c_{\epsilon^h} \otimes e_{ij})^{(k,k)}$, where $h \in [0, p-1]$, $i,j \in [1, n_k]$, the c_{ϵ^h}'s are defined as in (3) and the e_{ij}'s are the matrix units of M_{n_k}. We refer to this basis of $\mathbf{R}_{k,k}$ as *canonical.*

Definition 3.2. *The basis $\mathcal{B}$ of $\boldsymbol{R}$ consisting of the union of the canonical basis of $\boldsymbol{R}_{r,t}$ for every $1 \leq r \leq t \leq m$ is said to be the* canonical basis *of $\boldsymbol{R}$.*

Clearly the element $b := (c_{\epsilon^h} \otimes e_{ij})^{(k,k)}$ of $\mathbf{R}_{k,k}$ is a sum of p distinct matrices in $\mathbf{U}_{k,k}$, write

$$b = E^{(k,k)}_{i_1 j_1} + E^{(k,k)}_{i_2 j_2} + \cdots + E^{(k,k)}_{i_p j_p}. \tag{4}$$

In this case, we say that $E^{(k,k)}_{i_t j_t}$ appears in b and, when it is convenient, set $b = \overline{E}^{(k,k)}_{i_1 j_1} = \ldots = \overline{E}^{(k,k)}_{i_p j_p}$. Moreover one observes that, for every $i, j \in [1, s_k]$, there exists a unique canonical basis element b of $\mathbf{R}_{k,k}$ such that $E^{(k,k)}_{ij}$ appears in b (and so $b = \overline{E}^{(k,k)}_{ij}$).

Finally, for each $k \in \Gamma_1$, the left (and right) indices of the matrix units $E^{(k,k)}_{rs}$ appearing in the decomposition (4) of $b = (c_{\epsilon^h} \otimes e_{ij})^{(k,k)}$ are pairwise distinct. Therefore, using the fact that $E^{(r,t)}_{ij} E^{(r',t')}_{i'j'} = \delta_{t,r'} \delta_{j,i'} E^{(r,t')}_{ij'}$, one has that $\mathcal{B}$ is a multiplicative basis of $\mathbf{R}$, that is, for every $b_1, b_2 \in \mathcal{B}$, if $b_1 b_2 \neq 0_{\mathbf{R}}$ then $b_1 b_2 \in \mathcal{B}$. As a consequence of this fact one has the following useful result.

Lemma 3.3. *Let $b_1, \ldots, b_s \in \mathcal{B}$ and assume that $b := b_1 \cdots b_s \neq 0_{\boldsymbol{R}}$.*

- (i) *If $b \in A_k$, then $b_i \in A_k$ for every $i \in [1, s]$ and*
  - (a) *if $k \in \Gamma_0$, then there exist $i, j \in [1, s_k]$ such that $b = E^{(k,k)}_{ij}$. Furthermore, for every $\pi \in S_s$ such that $b^\pi := b_{\pi(1)} \cdots b_{\pi(s)} \neq 0_{\boldsymbol{R}}$, one has that $b^\pi = E^{(k,k)}_{ij}$ when $i \neq j$, whereas $b^\pi = E^{(k,k)}_{\ell\ell}$ for some $\ell \in [1, s_k]$ otherwise;*
  - (b) *if $k \in \Gamma_1$, then $A_k \cong M_{n_k}(D)$ and $b = (c_{\epsilon^h} \otimes e_{ij})^{(k,k)}$ for some $h \in [0, p-1]$ and $i, j \in [1, n_k]$. Moreover, if $b^\pi \neq 0_{\boldsymbol{R}}$ for some $\pi \in S_s$, then $b^\pi = (c_{\epsilon^h} \otimes e_{ij})^{(k,k)}$ when $i \neq j$, whereas $b^\pi = (c_{\epsilon^h} \otimes e_{\ell\ell})^{(k,k)}$ for some $\ell \in [1, n_k]$ otherwise.*
- (ii) *If $b \in J(\boldsymbol{R})$, then there exist $1 \leq r < t \leq m$, $i \in [1, s_r]$ and $j \in [1, s_t]$ such that $b = E^{(r,t)}_{ij}$. Moreover if $b^\pi := b_{\pi(1)} \cdots b_{\pi(s)} \neq 0_{\boldsymbol{R}}$ for some $\pi \in S_s$, then $b^\pi = E^{(r,t)}_{ij}$.*

Proof. Part (i) directly follows from Lemma 1 of [20] taking into account that D is commutative.

The statement (ii) is a consequence of (i) and the fact that the basis $\mathcal{B}$ is multiplicative. $\square$

A grading on $UT(A_1,\dots,A_m)$. Assume that, in the case in which $k\in\Gamma_0$, A_k is isomorphic to (M_{n_k},α_k) whereas, when $k\in\Gamma_1$, A_k is isomorphic to $(M_{n_k}(D),\alpha_k)$, where $\alpha_k:[1,s_k]\longrightarrow G$ is the map inducing the elementary grading on A_k. In particular, if $k\in\Gamma_1$, as seen in Proposition 3.1

$$(\alpha_k(1),\dots,\alpha_k(s_k)) := (\underbrace{\underbrace{1_G,\epsilon,\dots,\epsilon^{p-1}},\dots,\underbrace{1_G,\epsilon,\dots,\epsilon^{p-1}}}_{n_k \text{ times}}).$$

Let us define the maps

$$\alpha:[1,\eta_m]\longrightarrow G,\qquad i\longmapsto \alpha_k(i-\eta_{k-1})$$

and, for any m-tuple $\tilde{g}:=(g_1,\dots,g_m)\in G^m$,

$$\alpha_{\tilde{g}}:[1,\eta_m]\longrightarrow G,\qquad i\longmapsto g_k\alpha(i),$$

where $k\in[1,m]$ is the (unique) integer such that $i\in \mathrm{Bl}_k$. A very simple, but crucial fact is the following

Proposition 3.4. *The G-graded algebra $(UT(A_1,\dots,A_m),\alpha_{\tilde{g}})$ is a minimal G-graded algebra whose k-th graded simple component of its maximal semisimple homogeneous subalgebra is isomorphic to (M_{n_k},α_k), if $k\in\Gamma_0$, and to $(M_{n_k}(D),\alpha_k)$ otherwise. Furthermore, we can require that in these isomorphisms the minimal homogeneous idempotent e_k of A_k corresponds to $E_{11}^{(k,k)}$ if $k\in\Gamma_0$ and to $(c_{1_G}\otimes e_{11})^{(k,k)}=E_{11}^{(k,k)}+E_{22}^{(k,k)}+\cdots+E_{pp}^{(k,k)}=\overline{E}_{11}^{(k,k)}$ when $k\in\Gamma_1$.*

When convenient, any such G-graded algebra will be simply denoted by $UT_G(A_1,\dots,A_m)$.

Now, let $A=A_{ss}+J(A)$ be a minimal G-graded algebra such that $A_{ss}=A_1\oplus\ldots\oplus A_m$. Using the same notation for the homogeneous radical elements defining A which appear in Definition 2.2, set

$$g_1:=1_G,\qquad g_k:=|w_{12}w_{23}\cdots w_{k-1,k}|_A\quad \forall k\in[2,m]$$

and $\tilde{g}:=(g_1,g_2,\dots,g_m)$. Let us denote the minimal G-graded algebra $(UT(A_1,\dots,A_m),\alpha_{\tilde{g}})$ by $(UT(A_1,\dots,A_m),\widetilde{\alpha}_A)$ in order to stress that the grading on $UT(A_1,\dots,A_m)$ depends upon that on A.

Definition 3.5. *The G-graded algebra $(UT(A_1,\dots,A_m),\widetilde{\alpha}_A)$ is said to be* the upper block triangular matrix algebra related to the minimal G-graded algebra *A.*

We are in a position to state the main result of this section.

Proposition 3.6. *Let $G=\langle\epsilon\rangle\cong\mathbb{Z}_p$ be a group of prime order p and $A=A_{ss}+J(A)$ a minimal G-graded algebra such that $A_{ss}=A_1\oplus\cdots\oplus A_m$. Then $(UT(A_1,\dots,A_m),\widetilde{\alpha}_A)$ belongs to $\mathrm{var}_G(A)$. In particular, if $\mathrm{var}_G(A)$ is minimal, then $\mathrm{var}_G(A)=\mathrm{var}_G(UT(A_1,\dots,A_m),\widetilde{\alpha}_A)$.*

Proof. For convenience of notation, let us denote by $\mathbf{R}$ the G-graded algebra $(UT(A_1,\dots,A_m),\widetilde{\alpha}_A))$. In order to prove that $\mathbf{R}\in\mathrm{var}_G(A)$ we proceed by induction on m.

If $m=1$, then $\mathbf{R}=UT(A_1)\cong A_1=A$ and we are done.

Thus, suppose that $m \geq 2$ and let $f = f(x_1^{h_1}, \dots, x_n^{h_n}) \in F\langle X_G \rangle \backslash \mathrm{Id}_G(\mathbf{R})$. We aim to show that f is not a graded polynomial identity for A. To this end, since the characteristic of the ground field is zero, we can assume that f is multilinear. Hence there exist $b_1, \dots, b_n$ in the canonical basis $\mathcal{B}$ of $\mathbf{R}$, with $|b_i|_{\mathbf{R}} = h_i$ for each $i \in [1, n]$, such that $f(b_1, \dots, b_n) \neq 0_{\mathbf{R}}$.

Let s be the number of the b_k's which are in $J(\mathbf{R})$. Since $J(\mathbf{R})$ is nilpotent of index $m-1$, one has that $s \leq m-1$. Assume first that $s < m-1$. Hence there exists $i \in [1, m-1]$ such that, for every $j \in [i+1, m]$, none of the elements $b_1, \dots, b_n$ is in $\mathbf{R}_{i,j}$. At this stage, we have two cases to distinguish. If there exists $\ell \in [1, n]$ such that $b_\ell \in \mathbf{R}_{r,i}$ for some $r \in [1, i]$, then all the b_k's are in

$$\bigoplus_{1 \leq r \leq s \leq i} \mathbf{R}_{r,s} \;\cong\; UT(A_1, \dots, A_i)$$

with the induced G-grading and $f \notin \mathrm{Id}_G(UT(A_1, \dots, A_i))$. Otherwise the basis elements $b_1, \dots, b_n$ are in

$$\bigoplus_{\substack{1 \leq r \leq s \leq m; \\ r \neq i \neq s}} \mathbf{R}_{r,s} \;\cong\; UT(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m)$$

with the induced G-grading and, hence, $f \notin \mathrm{Id}_G(UT(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m))$. But the G-graded algebras $UT(A_1, \dots, A_i)$ and $UT(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m)$ are the upper block triangular matrix algebras related to the homogeneous subalgebras $A^{[1,i]}$ and $A^{(\check{i})}$ of A defined in (1) and (2), respectively. Therefore, by the induction assumption, we conclude that $f \notin \mathrm{Id}_G(A^{[1,i]})$ in the former case and $f \notin \mathrm{Id}_G(A^{(\check{i})})$ in the latter one. As a consequence, in any event f does not belong to $\mathrm{Id}_G(A)$.

Thus suppose that $s = m-1$. Hence there exist $t_1, \dots, t_{m-1} \in [1, n]$ such that

$$b_{t_1} = E_{i_1 j_2}^{(1,2)}, \;\dots,\; b_{t_{m-1}} = E_{i_{m-1} j_m}^{(m-1,m)},$$

where $i_k \in [1, s_k]$ and $j_{k+1} \in [1, s_{k+1}]$ for every $k \in [1, m-1]$. Moreover all the remaining elements of the set $\{b_1, \dots, b_n\}$ are in the diagonal blocks of $\mathbf{R}$. According to Lemma 3.3,

$$f(b_1, \dots, b_n) = \beta E_{ij}^{(1,m)}$$

for some $\beta \in F \setminus \{0_F\}$, $i \in [1, s_1]$ and $j \in [1, s_m]$. Without loss of generality, we can assume that $b := b_1 \cdots b_n = E_{ij}^{(1,m)}$. Therefore $\beta = \sum_\pi \beta_\pi$, where π runs over the elements of the symmetric group S_n such that $b_{\pi(1)} \cdots b_{\pi(n)}$ is not zero in $\mathbf{R}$ and β_π is the coefficient of the monomial $x_{\pi(1)}^{h_{\pi(1)}} \cdots x_{\pi(n)}^{h_{\pi(n)}}$ in f. Set $j_1 := i$ and $i_m := j$, let

$$b_0 := \overline{E}_{1 j_1}^{(1,1)} \qquad \text{and} \qquad b_{n+1} := \overline{E}_{i_m 1}^{(m,m)},$$

where, in order to use the same notation, $\overline{E}_{rs}^{(k,k)} := E_{rs}^{(k,k)}$ for every $r, s \in [1, s_k]$ whenever $k \in \Gamma_0$. Then

$$b_0 f(b_1, \dots, b_n) b_{n+1} = \overline{E}_{1i}^{(1,1)} (\beta E_{ij}^{(1,m)}) \overline{E}_{j1}^{(m,m)} = \beta E_{11}^{(1,m)}.$$

Now, for each $k \in [1, m-1]$, let $v_k \in A_k$ and $z_{k+1} \in A_{k+1}$ be the elements corresponding to $\overline{E}_{i_k 1}^{(k,k)}$ and $\overline{E}_{1 j_{k+1}}^{(k+1,k+1)}$ in the isomorphisms $A_k \cong \mathbf{R}_{k,k}$ and $A_{k+1} \cong \mathbf{R}_{k+1,k+1}$ of G-graded algebras of Proposition 3.4, respectively. Denoting by a_{t_k} the element $v_k w_{k,k+1} z_{k+1}$ of A (recall that $w_{k,k+1}$ is the k-th homogeneous radical element defining the minimal G-graded algebra A), from the equality

$$E_{i_k j_{k+1}}^{(k,k+1)} = \overline{E}_{i_k 1}^{(k,k)} E_{11}^{(k,k+1)} \overline{E}_{1 j_{k+1}}^{(k+1,k+1)}$$

it easily follows that

$$\begin{aligned} |a_{t_k}|_A &= |v_k|_A \, |w_{k,k+1}|_A \, |z_{k+1}|_A \\ &= |\overline{E}_{i_k 1}^{(k,k)}|_{\mathbf{R}} \, |E_{11}^{(k,k+1)}|_{\mathbf{R}} \, |\overline{E}_{1 j_{k+1}}^{(k+1,k+1)}|_{\mathbf{R}} = |E_{i_k j_{k+1}}^{(k,k+1)}|_{\mathbf{R}} = |b_{t_k}|_{\mathbf{R}}. \end{aligned}$$

Furthermore, setting $t_0 := 0$ and $t_m := n+1$, we notice that, for every $i \in [t_{k-1}+1, t_k - 1]$, $b_i \in \mathbf{R}_{k,k}$. Let $a_i \in A_k$ be the element corresponding to b_i, $z_1 := a_0$ that corresponding to b_0 in A_1 and $v_m := a_{n+1}$ that corresponding to b_{n+1} in A_m (in the isomorphisms of Proposition 3.4).

We claim that, for every $\pi \in S_n$, one has that $a_0 a_{\pi(1)} \cdots a_{\pi(n)} a_{n+1} \neq 0_A$ if, and only if, $b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_{\mathbf{R}}$. In fact, assume first that $a_0 a_{\pi(1)} \cdots a_{\pi(n)} a_{n+1}$ is a non-zero element of A. Then $\pi(t_k) = t_k$ for all $k \in [1, m-1]$. Moreover, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \pi(l) < t_q$ since $a_{\pi(l)} \in A_q$. Thus

$$\begin{aligned} 0_A \neq & z_1 a_{\pi(1)} \cdots a_{\pi(n)} v_m = z_1 a_{\pi(1)} \cdots a_{\pi(t_1-1)} v_1 w_{12} z_2 a_{\pi(t_1+1)} \cdots \\ & \cdots a_{\pi(t_2-1)} v_2 w_{23} z_3 \cdots v_{m-1} w_{m-1,m} z_m a_{\pi(t_{m-1}+1)} \cdots a_{\pi(n)} v_m, \end{aligned}$$

which yields the following equivalent statements:

$$z_k a_{\pi(t_{k-1}+1)} \cdots a_{\pi(t_k-1)} v_k \neq 0_A \ \text{ for all } k \in [1, m],$$

$$\overline{E}_{1 j_k}^{(k,k)} b_{\pi(t_{k-1}+1)} \cdots b_{\pi(t_k-1)} \overline{E}_{i_k 1}^{(k,k)} \neq 0_{\mathbf{R}} \ \text{ for all } k \in [1, m],$$

$$b_{\pi(t_{k-1}+1)} \cdots b_{\pi(t_k-1)} = \overline{E}_{j_k i_k}^{(k,k)} \ \text{ for all } k \in [1, m],$$

$$\overline{E}_{1 j_1}^{(1,1)} b_{\pi(1)} \cdots b_{\pi(t_1-1)} E_{i_1 j_2}^{(1,2)} b_{\pi(t_1+1)} \cdots b_{\pi(t_2-1)} E_{i_2 j_3}^{(2,3)} \cdots E_{i_{m-1} j_m}^{(m-1,m)} b_{\pi(t_{m-1}+1)} \cdots b_{\pi(n)} \overline{E}_{i_m 1}^{(m,m)} = E_{11}^{(1,m)}$$

and

$$b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_{\mathbf{R}}.$$

Conversely, if $b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_{\mathbf{R}}$, according to what was established above one has that $z_k a_{\pi(t_{k-1}+1)} \cdots a_{\pi(t_k-1)} v_k \neq 0_A$ for all $k \in [1, m]$. Furthermore this product coincides with the minimal homogeneous idempotent e_k of A_k (since e_k corresponds to $\overline{E}_{11}^{(k,k)}$) and, consequently,

$$z_1 a_{\pi(1)} \cdots a_{\pi(n)} v_m = e_1 w_{12} e_2 w_{23} \cdots e_{m-1} w_{m-1,m} e_m = w_{12} \cdots w_{m-1,m} \neq 0_A.$$

The final outcome of these deductions is that

$$a_0 f(a_1, \ldots, a_n) a_{n+1} = \beta w_{12} \cdots w_{m-1,m} \neq 0_A.$$

Hence f is not a graded polynomial identity for A, and this is enough to conclude that $(UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$ is in $\mathrm{var}_G(A)$.

The final part of the statement is an immediate consequence of the fact that $\exp_G(A) = \exp_G((UT(A_1, \ldots, A_m), \widetilde{\alpha}_A))$. □

4. Kemer polynomials for $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$

Throughout this section assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$, F is algebraically closed, $(A_1, \ldots, A_m)$ is an m-tuple of G-simple algebras and $A := UT_G(A_1, \ldots, A_m)$. The aim is to construct a family of G-graded polynomials which are not G-graded identities for the algebra A but, as we shall see in the sequel, they are in $\mathrm{Id}_G(UT_G(B_1, \ldots, B_n))$ for any sequence of G-simple algebras $(B_1, \ldots, B_n)$ such that $\exp_G(A) = \exp_G(UT_G(B_1, \ldots, B_n))$ whenever either $n \neq m$ or $n = m$ and $\dim_F(A_k)_g \neq \dim_F(B_k)_g$ for some $k \in [1, m]$ and $g \in G$. This is a key point in the proof of the main result. More specifically, these graded polynomials are suitable *Kemer polynomials* for A (for the definition and more details we refer the reader to [4]).

Suppose first that $m = 1$, which means that A is G-simple. If $A = M_n$, then one obtains its matrix unit E_{11} as a product of the n^2 different canonical basis elements of A. Let us fix such a product and refer to it as the *standard total product* (of basis elements) of A.

If $A = M_n(D)$, considering the standard total product of the matrix units e_{ij} of M_n, for each $h \in G$ one has a product of all the n^2 different basis elements of A of the form $c_h \otimes e_{ij}$ giving $c_h^{n^2} \otimes e_{11}$. Hence, repeating this process for every $h \in G$, we produce the following product of all the pn^2 different canonical basis elements of A

$$\Pi_{h\in G}(c_h^{n^2} \otimes e_{11}) = (\Pi_{h\in G} c_h)^{n^2} \otimes e_{11} = \begin{cases} c_\epsilon \otimes e_{11} & \text{if } n \text{ is odd and } p = 2, \\ c_{1_G} \otimes e_{11} & \text{otherwise.} \end{cases}$$

We shall still refer to it as the *standard total product* (of basis elements) of A. Notice that, due to the embedding $M_n(D) \subseteq M_{pn}$, we can write $c_{1_G} \otimes e_{11} = \overline{E}_{11}$ and $c_\epsilon \otimes e_{11} = \overline{E}_{12}$.

At this stage, we construct a monomial $m_A \in F\langle X_G \rangle$ (in pairwise different variables) replacing each element appearing in the standard total product of A with a variable of X_G of the same degree. Clearly, m_A has n^2 variables if $A = M_n$ and pn^2 when $A = M_n(D)$. An evaluation of m_A (by basis elements of A) is said to be the *standard total* if each variable of m_A is evaluated by the corresponding canonical basis element of A appearing in the standard total product of A (used to define m_A).

Assume now that m is an arbitrary positive integer. For each $k \in [1, m]$ and $g \in G$, set

$$d_k^A := \dim_F A_k, \quad d_{k,g}^A := \dim_F(A_k)_g \quad \text{and} \quad d_{ss,g}^A := \dim_F(A_{ss})_g = \sum_{k\in[1,m]} d_{k,g}^A.$$

For any $\nu \geq m$ and $k \in [1, m]$, consider ν copies of m_{A_k} in pairwise disjoint sets of graded variables. For each $i \in [1, \nu]$, let us denote by $m_{A_k}^{(i)}$ the i-th copy of m_{A_k} and by $S(k, i)$ the set of variables of $m_{A_k}^{(i)}$, respectively. Clearly we can write $S(k, i) = \cup_{g\in G} S(k, i, g)$, where $S(k, i, g)$ is the set of variables of degree g in $S(k, i)$. Hence

$$|S(k, i)| = d_k^A \qquad \text{and} \qquad |S(k, i, g)| = d_{k,g}^A.$$

Furthermore, for every $i \in [1, \nu]$ and $g \in G$, let $T(i, g) := \cup_{k\in[1,m]} S(k, i, g)$. Clearly,

$$(5) \qquad |T(i, g)| = \sum_{k\in[1,m]} d^A_{k,g} = d^A_{ss,g}.$$

We notice that the standard total evaluations of each monomial $m^{(i)}_{A_k}$ give a graded evaluation of $m^{(1)}_{A_k} \cdots m^{(\nu)}_{A_k}$ equal to $\overline{E}^{(k,k)}_{11}$ for almost all $k \in [1, m]$. More precisely, this does not occur only when $A_k = M_{n_k}(D)$, n_k and ν are both odd and $p = 2$. In this case the same evaluation coincides with $\overline{E}^{(k,k)}_{12}$ and we shall say that (k, ν, p) *is an exception*.

Now, for each $j \in [1, m-1]$, consider the homogeneous radical element $E^{(j,j+1)}_{11}$ of A if (j, ν, p) is not an exception, and $E^{(j,j+1)}_{21}$ otherwise. Let us call g_j its degree and choose a variable z_j of degree g_j such that all the elements of $(\cup_{i\in[1,\nu],k\in[1,m]} S(k, i)) \cup (\cup_{j\in[1,m-1]}\{z_j\})$ are pairwise different. Setting $Z_j := T(j, g_j) \cup \{z_j\}$, one has that

$$(6) \qquad |Z_j| = d^A_{ss,g_j} + 1.$$

Moreover there exists a graded evaluation of the product

$$\pi_{A,\nu} := m^{(1)}_{A_1} \cdots m^{(\nu)}_{A_1} \, z_1 \, m^{(1)}_{A_2} \cdots m^{(\nu)}_{A_2} \, z_2 \cdots \, z_{m-1} \, m^{(1)}_{A_m} \cdots m^{(\nu)}_{A_m},$$

by canonical basis elements of A, equal to $E^{(1,m)}_{11}$ if (m, ν, p) is not an exception, and to $E^{(1,m)}_{12}$ otherwise.

We extend the monomial $\pi_{A,\nu}$ by inserting $\nu(\dim_F A_{ss}) + m$ pairwise different variables of degree 1_G and not involved in $\pi_{A,\nu}$, bordering each variable appearing in $\pi_{A,\nu}$. We denote the monomial obtained in this way by $\widetilde{\pi}_{A,\nu}$. For each $k \in [1, m]$ and $i \in [1, \nu]$, let $Y(k, i)$ be the set of all the variables inserted on the left of the variables of the set $S(k, i)$. Still, for each $k \in [1, m]$, let $\widetilde{y}_k$ be the variable inserted on the right of the monomial $m^{(\nu)}_{A_k}$ (and therefore, for each $k \in [1, m-1]$, on the left of the variable z_k) and set $Y_k := \cup_{i\in[1,\nu]} Y(k, i) \cup \{\widetilde{y}_k\}$.

For each $j \in [1, m-1]$ we alternate in $\widetilde{\pi}_{A,\nu}$ the variables of the set Z_j and, for each $i \in [m, \nu]$ and $g \in G$, those of $T(i, g)$, respectively. We denote the graded polynomial obtained in this way (which clearly depends upon ν) by $f_{A,\nu}$. We prove the following

Lemma 4.1. *For every $\nu \geq m$ the polynomial $f_{A,\nu}$ is not a G-graded polynomial identity for the G-graded algebra $A := UT_G(A_1, \dots, A_m)$.*

Proof. For every $k \in [1, m]$ and $i \in [1, \nu]$, consider the standard total evaluation $\overline{S(k, i)}$ of the monomial $m^{(i)}_{A_k}$ in A. In particular, if $k \in \Gamma_0$, for each variable $v^{(i)}_t \in S(k, i)$ one has that $\overline{v}^{(i)}_t = E^{(k,k)}_{rs}$ for some $r, s \in [1, s_k]$. In this case, evaluate the variable $y^{(i)}_t \in Y(k, i)$ appearing on the left of $v^{(i)}_t$ by $E^{(k,k)}_{rr}$. Since $\overline{y}^{(i)}_1 \overline{v}^{(i)}_1 \cdots \overline{y}^{(i)}_{d^A_k} \overline{v}^{(i)}_{d^A_k} = E^{(k,k)}_{11}$, evaluate the variable $\widetilde{y}_k$ by $E^{(k,k)}_{11}$. Similarly, if $k \in \Gamma_1$, for each $v^{(i)}_t \in S(k, i)$ it holds that $\overline{v}^{(i)}_t = (c_h \otimes e_{rs})^{(k,k)}$ for some $c_h \in D$ and $r, s \in [1, n_k]$, and then evaluate the variable $y^{(i)}_t \in Y(k, i)$

appearing on the left of $v_t^{(i)}$ by $(c_{1_G}\otimes e_{rr})^{(k,k)}$. As $\overline{y}_1^{(i)}\overline{v}_1^{(i)}\cdots\overline{y}_{d_k^A}^{(i)}\overline{v}_{d_k^A}^{(i)}$ is equal either to $(c_{1_G}\otimes e_{11})^{(k,k)}=\overline{E}_{11}^{(k,k)}$ or to $(c_\epsilon\otimes e_{11})^{(k,k)}=\overline{E}_{12}^{(k,k)}$ (when n_k is odd and $p=2$), evaluate the variable $\widetilde{y}_k$ by $(c_{1_G}\otimes e_{11})^{(k,k)}=\overline{E}_{11}^{(k,k)}=\overline{E}_{22}^{(k,k)}$. Finally, for every $j\in[1,m-1]$, set $\overline{z}_j:=E_{11}^{(j,j+1)}$ if (j,ν,p) is not an exception, and $\overline{z}_j:=E_{21}^{(j,j+1)}$ otherwise. In this way we have obtained a graded evaluation of the monomial $\widetilde{\pi}_{A,\nu}$ in A equal to $E_{11}^{(1,m)}$ if (m,ν,p) is not an exception, and to $E_{12}^{(1,m)}$ otherwise. For simplicity, let us denote such an evaluation by $\overline{S}_A$.

Now, fixed $i\in[m,\nu]$ and $g\in G$, take a permutation σ of the variables of $\widetilde{\pi}_{A,\nu}$ which possibly moves only the elements of $T(i,g)$ and assume that the evaluation of the monomial $\sigma(\widetilde{\pi}_{A,\nu})$ in A by $\overline{S}_A$ is non-zero. We claim that σ must be the identity permutation. In fact, since $z_1,\dots,z_{m-1}$ are not moved by σ and $aa'=0_A$ for any $a\in A_r$ and $a'\in A_s$ when $r\neq s$, then σ permutes the variables of $S(k,i,g)$ for each $k\in[1,m]$. Moreover in each monomial of $f_{A,\nu}$ the variables in Y_k appear in the same order. Therefore, by the choice of the elements in $\overline{Y}_k$, we conclude that, in order to not get zero, the evaluation of each variable $v_t^{(i)}$ is uniquely determined by those of the bordering variables and by the degree of $v_t^{(i)}$. Since we do not alternate variables of different degrees, the claim is confirmed.

Finally, for each $j\in[1,m-1]$, applying the same arguments presented above one has that, for any non-trivial permutation τ of the variables of the set Z_j in $\widetilde{\pi}_{A,\nu}$, the evaluation of the monomial $\tau(\widetilde{\pi}_{A,\nu})$ by $\overline{S}_A$ is zero.

The outcome of all these deductions is that $\widetilde{\pi}_{A,\nu}$ is the unique monomial of $f_{A,\nu}$ which does not vanish under the evaluation by $\overline{S}_A$, and this concludes the proof. □

We shall refer to the graded polynomial $f_{A,\nu}$ as a *Kemer polynomial* for A.

Remark 4.2. The polynomial $f_{A,\nu}$ is constructed starting, for each G-simple algebra A_k, from a choice of a product of all the canonical basis elements whose result is a fixed canonical basis element. In particular, if A_1 and A_m are simple graded simple, namely $A_1=M_{n_1}$ and $A_m=M_{n_m}$, taking $i\in[1,n_1]$ and $j\in[1,n_m]$, we may fix new standard total products (of basis elements) of A_1 and A_m equal to $E_{ii}^{(1,1)}$ and $E_{jj}^{(m,m)}$, respectively. Making the necessary changes in the degrees of the variables involved in $\pi_{A,\nu}$ we obtain a new Kemer polynomial, denoted by $f_{A,\nu}^{ij}$, for which there exists a suitable graded evaluation in A equal to $E_{ij}^{(1,m)}$. Clearly $f_{A,\nu}$ is one of the polynomials $f_{A,\nu}^{11}$.

5. Proof of the main result

Throughout this section assume that $G=\langle\epsilon\rangle\cong\mathbb{Z}_p$. According to Theorem 2.5 and Proposition 3.6 one has that any minimal variety of G-graded PI algebras of finite basic rank of fixed G-exponent d is generated by a suitable G-graded upper block triangular matrix algebra $UT_G(A_1, \dots, A_m)$ satisfying $\dim_F(A_1 \oplus \dots \oplus A_m) = d$ (we shall freely use in the sequel that $\exp_G(UT_G(A_1, \dots, A_m)) = \dim_F(A_1 \oplus \dots \oplus A_m)$). Our final goal is to provide a characterization of these varieties proving that any G-graded algebra of the form $UT_G(A_1, \dots, A_m)$ generates a minimal variety. To this end, the crucial step consists of relating, for any pair of sequences $(A_1, \dots, A_m)$ and $(B_1, \dots, B_n)$ of G-simple algebras, the G-graded polynomial identities of the G-graded algebras $A := UT_G(A_1, \dots, A_m)$ and $B := UT_G(B_1, \dots, B_n)$ satisfying $\exp_G(A) = \exp_G(B)$ with the existence (or not) of a graded isomorphism between them.

The first result in this direction concerns the algebraic structures of A and B under a condition a bit weaker than we need, namely $\exp_G(A) = d_{ss}^A \geq d_{ss}^B = \exp_G(B)$. A key ingredient is the Kemer polynomial $f_{A,\nu}$ presented in the previous section. The notation we shall use is introduced there.

Lemma 5.1. *Let $A := UT_G(A_1, \dots, A_m)$ and $B := UT_G(B_1, \dots, B_n)$ be G-graded algebras satisfying $d_{ss}^A \geq d_{ss}^B$ and $\nu := m + n - 1$. If $f_{A,\nu}$ is not a G-graded polynomial identity for B, then the following conditions hold:*

- *(i)* $d_{ss,g}^B = d_{ss,g}^A$ *for every* $g \in G$;
- *(ii)* $n = m$;
- *(iii)* $d_{k,g}^B = d_{k,g}^A$ *for every* $k \in [1, m]$ *and* $g \in G$.

Proof. Since $f_{A,\nu}$ is a multilinear graded polynomial and $f_{A,\nu} \notin \mathrm{Id}_G(B)$, there exists a non-zero graded evaluation $\overline{S}_B$ of $f_{A,\nu}$ by canonical basis elements of B. Moreover we may assume, without loss of generality, that $\overline{S}_B$ is a non-zero graded evaluation of the monomial $\widetilde{\pi}_{A,\nu}$ of $f_{A,\nu}$. As $J(B)$ is nilpotent of index n and $|[m, m+n-1]| = n$, one has that there exists $m \leq \ell \leq m+n-1$ such that all the variables of $\cup_{g \in G} T(\ell, g) = \cup_{k \in [1,m]} S(k, \ell)$ as well as all those of $\cup_{k \in [1,m]} Y(k, \ell)$ must be necessarily evaluated only by semisimple elements in $\overline{S}_B$.

The fact that, for every $g \in G$, $f_{A,\nu}$ alternates in the set $T(\ell, g)$ forces $|T(\ell, g)| \leq d_{ss,g}^B$, and thus (5) yields

$$d_{ss,g}^A = |T(\ell, g)| \leq d_{ss,g}^B \qquad \forall\, g \in G.$$

Combining the above inequality with the original assumption we obtain

$$d_{ss}^A = \sum_{g \in G} d_{ss,g}^A \leq \sum_{g \in G} d_{ss,g}^B = d_{ss}^B \leq d_{ss}^A$$

and, consequently, that $d_{ss,g}^A = d_{ss,g}^B$ for every $g \in G$, which concludes the proof of item (i).

Now, $\cup_{g \in G} \overline{T(\ell, g)} = \cup_{k \in [1,m]} \overline{S(k, \ell)}$ is a total evaluation of the product $m_{A_1}^{(\ell)} \cdots m_{A_m}^{(\ell)}$ by canonical basis elements of B_{ss}, i.e. an evaluation involving all (and only) the canonical basis elements of B_{ss} (and, hence, each one exactly once). Furthermore, as $bb' = 0_B$ for any $b \in B_i$ and $b' \in B_j$ whenever $i \neq j$, for each $k \in [1, m]$ the monomial $m_{A_k}^{(\ell)}$ must be evaluated in a unique block of B_{ss}. At this stage assume, if possible, that $m < n$. Then there exist $h \in [1, m]$ and at least two elements from different B_t's appearing in the same $\overline{S(h,\ell)}$, which is a contradiction. Hence $n \leq m$.

On the other hand, for every $j \in [1, m-1]$, $f_{A,\nu}$ alternates in the set Z_j, which, according to (6), has cardinality

$$|Z_j| = d^A_{ss,g_j} + 1 = d^B_{ss,g_j} + 1.$$

Therefore we must have at least $m-1$ canonical basis elements of $J(B)$ in $\overline{S}_B$. But this implies that $m-1$ is smaller than the nilpotent index of $J(B)$. The combination with the above disequality gives item (ii).

Finally, as we have seen above, one has that, for every $k \in [1, m]$ the monomial $m^{(\ell)}_{A_k}$ must be evaluated in B_k. Since $\overline{S(k,\ell)}$ is a total evaluation of $m^{(\ell)}_{A_k}$ by canonical basis elements of B_k, item (iii) follows looking at the number of variables of $m^{(\ell)}_{A_k}$ of degree g. $\square$

The next and crucial step is to show that, if A and B have the same G-exponent, the condition $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$ is sufficient to guarantee that they are isomorphic. It is, in some sense, a generalization of Theorem 3.3 of [11] when G is a group of prime order p (and it was actually proved there for the particular case in which all the algebras $A_1, \ldots, A_m$ are simple graded simple). To this aim we shall use as a main tool the concept of *invariance subgroup* of a graded simple algebra still introduced in [11]. We recall the definition and fix some notation.

Given the G-graded algebra $A := UT_G(A_1, \ldots, A_m)$, assume that $\alpha_A : \cup_{k=1}^m \mathrm{Bl}_k \longrightarrow G$ is the map inducing the grading on A. For any $1 \leq r \leq s \leq m$ let us denote by $\alpha_A^{[r,s]}$ the restriction of α_A to $\cup_{k\in[r,s]} \mathrm{Bl}_k$ and set $\alpha_A^{(r)} := \alpha_A^{[r,r]}$. Thus $A^{[r,s]} \cong (UT(A_r, \ldots, A_s), \alpha_A^{[r,s]})$ and $A^{[r,r]} \cong (A_r, \alpha_A^{(r)})$. In the last case, when convenient, we shall simply write A_r. For each $k \in [1, m]$ set

$$w_A^{(k)}(g) := |\{s \mid s \in \mathrm{Bl}_k,\ \alpha_A^{(k)}(s) = g\}| \quad \forall\, g \in G.$$

The subgroup

$$H_A^{(k)} := \{h \,|\, h \in G,\, w_A^{(k)}(hg) = w_A^{(k)}(g) \text{ for all } g \in G\}$$

of G is said to be the *invariance subgroup* of A_k. It is immediate that, since G has prime order, either $H_A^{(k)} = \langle 1_G \rangle$ or $H_A^{(k)} = G$.

Proposition 5.2. *Let $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ be G-graded algebras satisfying $d^A_{ss} = d^B_{ss}$. If $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, then A and B are isomorphic as G-graded algebras.*

Proof. Since $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, Lemma 4.1 yields that $f_{A,\nu} \notin \mathrm{Id}_G(B)$ for all $\nu \geq m$. In particular, taking $\nu := m + n - 1$, from Lemma 5.1 it follows that $n = m$ and $d^B_{k,g} = d^A_{k,g}$ for every $k \in [1, m]$ and $g \in G$. Consequently,

$$d^A_k = d^B_k \qquad \forall\, k \in [1, m]. \tag{7}$$

The fact that $\dim_F M_{n_k}(D) = pn_k^2$ whereas $\dim_F M_{n_k} = n_k^2$ allows us to conclude that A_k is non-simple graded simple if, and only if, so is B_k.

Assume first that $A_k \cong M_{n_k}(D) \subseteq M_{pn_k}$. From Proposition 3.1 and its proof it follows that A_k must be isomorphic to B_k (and hence we are done when $m = 1$ and $A_1 = A$ is non-simple graded simple). Furthermore, as seen in Section 3,

$$(\alpha_A^{(k)}(\eta_{k-1}+1), \dots, \alpha_A^{(k)}(\eta_{k-1}+pn_k)) = (h_k, h_k\epsilon, \dots, h_k\epsilon^{p-1}, \dots, h_k, h_k\epsilon, \dots, h_k\epsilon^{p-1})$$

for some $h_k \in G$ (and the same is true for $\alpha_B^{(k)}$ for a suitable $h'_k \in G$). This implies that

$$w_A^{(k)}(g) = w_A^{(k)}(gh) = n_k = w_B^{(k)}(g) = w_B^{(k)}(gh) \qquad \forall\, g, h \in G,$$

and hence $H_A^{(k)} = H_B^{(k)} = G$.

If A_k is simple graded simple, from the previously presented arguments we get that there exists an isomorphism of (ordinary) algebras between A_k and B_k, which is not necessarily a G-graded isomorphism. More precisely, we may conclude that A_k and B_k are isomorphic to the same matrix algebra M_{n_k}, but endowed with some not necessarily isomorphic elementary G-gradings.

In the case in which $m = k = 1$, from Corollary 2.3 of [11] one has that $A_1 = A$ is isomorphic to $B_1 = B$ as a graded algebra, and the desired conclusion holds.

Therefore assume that $m \geq 2$ and let us consider the G-graded algebras $A^{[1,m-1]}$ and $B^{[1,m-1]}$. We claim that $\mathrm{Id}_G(B^{[1,m-1]}) \subseteq \mathrm{Id}_G(A^{[1,m-1]})$. In fact, suppose, if possible, that the inclusion does not hold. Hence there exists a polynomial $f_1 \in \mathrm{Id}_G(B^{[1,m-1]}) \setminus \mathrm{Id}_G(A^{[1,m-1]})$. By virtue of Lemma 4.1, the Kemer polynomial $f_{A^{[m-1,m]},2}$ (whose variables can be assumed to be pairwise different from those involved in f_1) is not a graded identity for $A^{[m-1,m]}$. But, from (7), one has that

$$d_{ss}^{A^{[m-1,m]}} = d_{m-1}^A + d_m^A > d_m^B = d_{ss}^{B_m}$$

and, applying Lemma 5.1, obtains that $f_{A^{[m-1,m]},2} \in \mathrm{Id}_G(B_m)$. Let $u := \sum_{\ell \in G} x^\ell$, where the x^ℓ's are pairwise different variables of degree l of $F\langle X_G\rangle$ involved neither in f_1 nor in $f_{A^{[m-1,m]},2}$. Then $f_1 u f_{A^{[m-1,m]},2}$ is not a G-graded polynomial identity for the algebra A. But $f_1 u f_{A^{[m-1,m]},2} \in \mathrm{Id}_G(B^{[1,m-1]}) \cdot \mathrm{Id}_G(B_m) \subseteq \mathrm{Id}_G(B)$, which contradicts the original assumption that $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$.

Arguing in the same manner we obtain that $\mathrm{Id}_G(B^{[2,m]}) \subseteq \mathrm{Id}_G(A^{[2,m]})$ and, continuing in a similar fashion, that

$$\mathrm{Id}_G(B^{[k,k']}) \subseteq \mathrm{Id}_G(A^{[k,k']}) \qquad \forall\, 1 \leq k \leq k' \leq m \tag{8}$$

and hence

$$\mathrm{Id}_G(B_k) \subseteq \mathrm{Id}_G(A_k) \qquad \forall\, k \in [1, m].$$

This inclusion allows us to directly apply Corollary 2.3 of [11].

In particular, the combination of the last mentioned result with what we proved when A_k and B_k are non-simple graded simple guarantees for each $k \in [1, m]$ the existence of an element $g_k \in G$ such that

$$w_A^{(k)}(g_k x) = w_B^{(k)}(x) \qquad \forall\, x \in G,$$

that A_k is isomorphic to B_k as a G-graded algebra and $H_A^{(k)} = H_B^{(k)}$. For this reason, in the rest of the proof let us denote $H_A^{(k)}$ simply by $H^{(k)}$.

In order to prove that A and B are isomorphic as G-graded algebras, it is enough to show that there exists $g \in G$ such that

$$w_A^{(k)}(gx) = w_B^{(k)}(x) \qquad \text{for all } k \in [1,m] \text{ and } x \in G. \tag{9}$$

In fact, if the equality occurs, one can find a permutation $\sigma \in S_{\eta_m}$, satisfying $\sigma(\mathrm{Bl}_k) = \mathrm{Bl}_k$ for each $k \in [1,m]$ (and preserving the structure of A_k when $k \in \Gamma_1$) such that the restriction of the endomorphism τ of M_{η_m} induced by σ (namely, that defined on the matrix units E_{ij} of M_{η_m} by $\tau(E_{ij}) := E_{\sigma(i),\sigma(j)}$) to A is an isomorphism of graded algebras from A into B.

To this end we observe that if the integer $k \in [1,m]$ is such that $H^{(k)} = G$, then, for every $g, x \in G$,

$$w_A^{(k)}(gx) = w_A^{(k)}(x) = w_A^{(k)}(g_k x) = w_B^{(k)}(x).$$

Consequently, if $H^{(k)} = G$ for every $k \in [1,m]$, the equality (9) is satisfied for any choice of $g \in G$. On the other hand, if there exists a unique integer $\ell \in [1,m]$ such that $H^{(\ell)} = \langle 1_G \rangle$, it is sufficient to set $g := g_\ell$ to obtain (9).

Therefore we are left with the case in which there exist at least two different integers $k_1, k_2 \in [1,m]$ such that $H^{(k_1)} = H^{(k_2)} = \langle 1_G \rangle$. As $H^{(k)} = G$ for every $k \in \Gamma_1$, one has that $|\Gamma_0| \geq 2$. Now, for any pair of elements $r < s$ in Γ_0, we replace the polynomial Ψ_{ij}^A appearing in the proof of Theorem 3.3 of [11] with the Kemer polynomial $f^{ij}_{A^{[r,s]},\nu}$ introduced in Remark 4.2. Applying the same arguments used there (replacing the inclusion (1) appearing in that proof with (8)), we conclude that $g_r^{-1} g_s \in H^{(r)} H^{(s)}$. Invoking again the fact that $H^{(k)} = G$ when A_k is non-simple graded simple, we get that

$$g_r^{-1} g_s \in H^{(r)} H^{(s)} \qquad \forall\, 1 \leq r < s \leq m.$$

Since there exists $d \in [1,m]$ such that $H^{(d)} = \langle 1_G \rangle$, following again the same reasonings of the final part of the proof of Theorem 3.3 in [11], equation (9) occurs setting $g := g_d$. $\square$

An immediate consequence of the previous result is the following statement, which is in the same spirit of what was proved in [3] for G-simple algebras.

Corollary 5.3. *Let $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ be G-graded algebras satisfying $d_{ss}^A = d_{ss}^B$. Then A and B are isomorphic as G-graded algebras if, and only if, $\mathrm{Id}_G(A) = \mathrm{Id}_G(B)$.*

We are now in a position to state the main result of the paper, namely a characterization of minimal varieties of G-graded PI algebras of finite basic rank.

Theorem 5.4. *Let F be a field of characteristic zero and G a group of prime order p. A variety $\mathcal{V}^G$ of G-graded PI-algebras of finite basic rank is minimal of G-exponent d if, and only if, it is generated by a G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.*

Proof. If $\mathcal{V}^G$ is minimal of G-exponent d, from Theorem 2.5 and Proposition 3.6 it follows that $\mathcal{V}^G$ is generated by a suitable G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

Conversely, set $A := UT_G(A_1, \ldots, A_m)$ and consider a subvariety $\mathcal{U}^G \subseteq \mathrm{var}_G(A)$ such that $\exp_G(\mathcal{U}^G) = \exp_G(\mathrm{var}_G(A))$. Since $\mathrm{var}_G(A)$ satisfies some Capelli identities, according to what was established in Section 7.1 of [4] we conclude that $\mathcal{U}^G$ has finite basic rank. Hence, by virtue of Theorem 1.1 of [4], $\mathcal{U}^G$ is generated by a finite-dimensional G-graded algebra $\hat{B}$ (we recall that, as remarked in Section 2, we can extend the basis field and assume that it is algebraically closed and this has no effect on the codimensions). Lemma 2.4 yields that there exists a minimal G-graded algebra B' such that $\mathrm{Id}_G(\hat{B}) \subseteq \mathrm{Id}_G(B')$ and $\exp_G(\hat{B}) = \exp_G(B')$. In particular, Proposition 3.6 guarantees the existence of a G-graded algebra $B := UT_G(B_1, \ldots, B_n)$ such that $\mathrm{Id}_G(B') \subseteq \mathrm{Id}_G(B)$ and $\exp_G(B') = \exp_G(B)$. Therefore $\mathrm{Id}_G(A) \subseteq \mathrm{Id}_G(B)$ and $\exp_G(A) = \exp_G(B)$. At this stage, we can directly apply Proposition 5.2 and conclude that A is isomorphic to B as a G-graded algebra. Consequently $\mathrm{Id}_G(A) = \mathrm{Id}_G(B)$, and this completes the proof. $\square$

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ONOFRIO MARIO DI VINCENZO
Dipartimento di Matematica, Informatica ed Economia
Università degli Studi della Basilicata
Via dell'Ateneo Lucano, 10
85100-POTENZA (Italy)
e-mail: onofrio.divincenzo@unibas.it

VIVIANE RIBEIRO TOMAZ DA SILVA
Departamento de Matemática, Instituto de Ciências Exatas
Universidade Federal de Minas Gerais
30161-970 BELO HORIZONTE (Brazil)
e-mail: viviane@mat.ufmg.br

ERNESTO SPINELLI
Dipartimento di Matematica "G. Castelnuovo"
Università degli Studi di Roma "La Sapienza"
Piazzale Aldo Moro 5
00185-ROMA (Italy)
e-mail: spinelli@mat.uniroma1.it