A vertical analysis of difficulties in mathematics by secondary school to level: Some evidence stems from standardized assessment

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A longitudinal analysis of the recurring mistakes at different school levels in national standardized assessment tests is presented. The analysis of the outcomes highlights some difficulties common across different school grades. Subsequently, we extend our research to university students: we investigate the results of tasks solved by students at the end of high school and at the beginning of university in an e-learning environment called AlmaMathematica. We examine whether there are commonalities between errors that lead to wrong answers at school level and university level. Results show that university students share the same difficulties of high school students when faced with similar tasks.

Keywords: Mathematics test, summative evaluation, semiotics.

Introduction

Our research is carried out selecting a set of tasks sharing common features among the national standardized assessment tests INVALSI (National Assessment Institute for the School System) and the results of the AlmaMathematica tests administered to students at the end of high secondary school and at the beginning of university. As we can see in the next section, the INVALSI gives back the results at national level to each item of each test. Thus, we have, each year, results for grade 2 and 5 (in Italy, second and fifth years of Primary School), grade 8 (third year of low secondary school) and grade 10 (second year of high secondary school). Most of the research based on these data concerns vertical and common features arising at different test level outcomes (Branchetti et al, 2015). In this research perspective, we investigate common difficulties that emerged from the vertical analysis of INVALSI secondary schools test results. In the second phase of the project we further develop the research expanding to university level students. In order to do this, we analyzed the results of mathematics tests in the AlmaMathematica project, an e-learning environment in which many students of the fifth (and last) year of high secondary school and university freshmen perform mathematics tasks. Hence, we asked the following research question: Is it possible to identify some common student behavior when facing mathematics tests from secondary school to university? For this we analyse two tasks of INVALSI mathematics tests of grade 8 and grade 10 giving an example of analysis of linked tasks in which different approaches are implemented. The comparison between the two tasks allows us to interpret some difficulties encountered by the students. The analysis of a related task in AlmaMathematica suggests some possible ways to interpret students' behaviors at university level. All three tasks have the same structure and the same mathematical content, but the likely difficulty is represented by the switch from a representation register to another one. The same skills are required for all three tasks and the purpose of our research is to investigate if there are common aspects in the task solutions.

National Evaluation Service and AlmaMathematica Project

INVALSI is the Italian national institute for the educational assessment of instruction and every year, through National Evaluation Service (SNV), it carries out periodic and systematic national assessment to check student knowledge and skills in Mathematics and Italian language. Every year, INVALSI performs assessment tests in a census to school grades 2 and 5 (Primary School), 8 (low secondary school), and 10 (high secondary school) and it returns the results of the sample for each item of all tests administered. As part of our research we consider in detail the results of two tasks in mathematics standardized tests carried out by grade 8 students in a.y. 2010/11 and by grade 10 students in a.y. 2011/12. To conduct this research, it has been crucial the use of a research tool of INVALSI tests, the *GestInv Database* that we will describe in detail later. To investigate difficulties at university level, we examine the results of the performance of students using AlmaMathematica. AlmaMathematica (almamathematica.unibo.it) is a project of the University of Bologna designed specifically to create links between Secondary School and University. It is aimed at students who wish to enroll in undergraduate courses at the University of Bologna and it provides basic courses in mathematics, statistics and probability. The environment that has been created refers to the tasks of the Entrance Tests to a restricted number of Curriculum courses (TOLC), and to the evaluation of basic knowledge tests. Access to the statistical data allows us to investigate the percentage of correct answers for each question.

Theoretical lenses

Our main hypothesis is that a longitudinal analysis involving a large number of students can give relevant information about difficulties existing at secondary school level and can allow us to infer whether these difficulties remain at the beginning of university and in what fashion. Our research stems from evidence arising from the analysis of Large Scale Assessment (LSA) tests. We do not consider LSA merely as a way to provide a ranking or scores for benchmarks or as a search for correlations between variables of context, but we assume that its results can provide information on the teaching/learning process. In accordance with many researches (i.e. Looney, 2006) we consider the analysis of the results of standardized assessment through the lens of formative assessment. The information given back by LSA contains not only global scores (measured by statistical models), but it also highlights the specific phenomenon observed individually. Among the results of standardized assessments many significant macro-phenomena are visible that can be explored and interpreted through some of the lenses of mathematics education as the most frequent difficulties described in literature are also reflected in the students' responses. We conjecture that this kind of longitudinal analysis carried out through the comparison between the data sets from different years and levels could be useful to better interpret the difficulties that arise in secondary school and remain until university. For this reason, we need some criteria to link tasks from different grades: we have chosen the tasks that had the lowest correct response rate in the standardized tests and for which the topic is present both at secondary school and university. In particular, our research focuses on powers and manipulation of exponents and the difficulties with these topics have been widely reported in the literature (Pitta-Pantazi et al., 2007, Cangelosi et al., 2013). Indeed some studies have already reported common difficulties with management of exponentials between university students and high school students (Cangelosi et al., 2013). This led us to think that some misconceptions regarding exponential expressions are persistent over time. The students' mental constructions and the way in which they develop a meaningful understanding of exponentials has been the subject of other studies (i.e. Pitta-Pattanzi et al., 2007). In this is a vertical analysis made among secondary school students the authors found that, independently from the age of the students, there is an issue in the treatment of the exponentials that led them to provide wrong arguments in comparing powers with the same base or with the same exponents. Starting from these evidence, we conducted our research concerning expressions with exponentials, in particular the manipulation of different representations of exponentials. We are interested in understanding if the phenomenon showed from the quantitative analysis also implies that the difficulties encountered by students at different school levels are the same. To investigate this phenomenon, we need a further qualitative analysis, which is on-going, and its results will not be included in this paper. As we can see below, we detect that the main common difficulties are related to the semiotic representations management and, in order to interpret it, we use the semiotic approach proposed by Duval (1993, 2006). According to Duval, for each object there is more than one possible semiotic representation and one of the highest processes of mathematics is precisely the management of different representations of the same object. Our analysis will show that recurrent errors made by students, in all investigated levels, can be reduced to the difficulties concerning the management of different semiotic representation of the same object and the transformation of representations within different registers. We can then identify the main difficulties in the conversion (Duval, 1993). Duval (2006) suggests that the switch from natural language (verbal register) to algebra (symbolic register) requires a high level of complexity. Furthermore, according to Duval, it is possible to classify the different representations of a mathematical object in different registers, which are a set of signs and rules that can be manipulated. Such registers may themselves be classified as discursive (natural language written or spoken, mathematical symbols) or non-discursive (diagrams and figures). Still, it is possible to distinguish within each category those that are multifunctional registers, i.e. suitable to explain processes that cannot be put in algorithmic form, from those mono-functional, i.e. especially dedicated to algorithmic processes. In the first category there is natural language, in the second arithmetic and algebraic symbols. In the mono-functional register treatments can take the form of an algorithm, while in multi-functional ones, this is not possible. This fact will be crucial in the analysis of the behaviour of the students in our research.

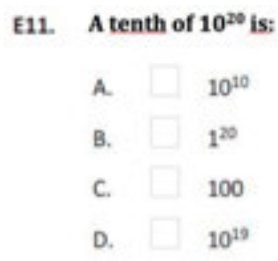
Methodology and data analysis

Our research is a mixed method sequential research (Johnson & Onwuegbuzie, 2004), with design QUAN → QUAL → QUAN. The first quantitative phase consists in an analysis of statistical results of the standardized items. Then, among the selected standardized items, we search for the ones with a topic common between secondary school and university levels. Subsequently, we look for the ones that highlight the same educational phenomena. Finally, we search on AlmaMathematica for the ones with the same features. We conduct a research looking at tasks at low secondary school level, high secondary school level and the initial stage of university undergraduate level. We needed a common topic to start with and we chose powers. In Italy, this is a topic used in the final national examination at the end of low secondary school, which is then elaborated in the second year of high secondary school and it is considered an “entry requirement” (and therefore investigated) for all university courses that require mathematics knowledge and skills. To search for tasks concerning powers among all the ones of the standardized assessment INVALSI test from 2008, which has had a low rate of correct answers, the INVALSI Database Gestinv is used. This database is an online tool of research (www.gestinv.it) that contains more than 1,400 items administered in the Italian national standardized tests and it is used in professional development programs implemented by schools and in research in mathematics education. Inside the database, there is a PDF of all tests administered in Italy from 2008, in which each item of these tests is accompanied by detailed results, statistical classifications, and data split into different categories. In respect of each item there is the image of the question, the goal of the content, the process, the reference to the National Guidelines for Curricula, some keywords characterizing the content, the text of the question, the correct answer or the image of the correct answer, the percentage of national response, the characteristic curves, and the item information. The Gestinv database can be used in many ways: when entering the section of Mathematics it is possible to search by National Guidelines for Curricula, Keywords, Full Text, and to do a Guided Search: a cross-search - with connectors and/or - of all parameters in respect of each item and all its features, such as the percentage of national response. Through the tool Guided Search we searched for all of the secondary school tasks of INVALSI Tests of Mathematics, referring to the keyword “powers” which had percentages of correct national responses below 50%. The research displayed about ten tasks with these features, and we looked for those whose analyses represent the same didactic phenomenon. As we see in the next section, we studied two tasks that had “common errors” displayed at national level: one by low secondary student (13 years old) and one by high secondary school students (15 years old).

Once such tasks are identified, we check if the analysis of the tasks about powers in AlmaMathematica Project shows the same type of errors. Students who performed exercises and problems in this e-learning environment are 18-20 years old; this allows us to investigate whether the same phenomenon persists with students of different age and how it occurs in various levels.

The analysis of tasks

The following task was administered in a census at grade 8 Italian students in a.y. 2010/11.



E11. **A tenth of 10^{20} is:**

A. ☐ 10^{10}

B. ☐ 1^{20}

C. ☐ 100

D. ☐ 10^{19}

Figure 1: Question D11, Grade 08, INVALSI Test, a.y. 2010/11

Figure 2 shows the national percentage of correct, mistaken, missed, and invalid answers and the percentage of choice for each option.

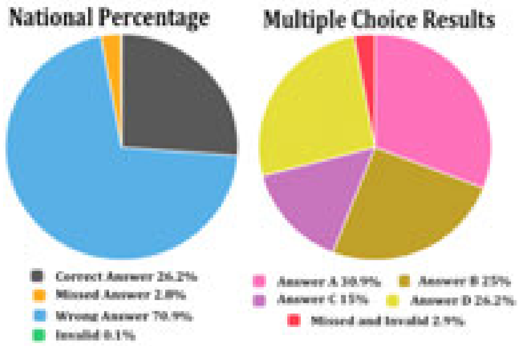


Figure 2: Data of Question D10, Grade 08, INVALSI Test a.y. 2010/11

The question was administered in a.y. 2010/11 to a population of approximately 600,000 grade 8 students, the sample (from which the statistical data are calculated) was composed by about 27,000 students. To give the correct answer a correct management from natural language to algebra is necessary. As we notice, the percentage of correct answers is low (the correct answer is option D, and it has been chosen by 26.2% of students). Option A and B have been chosen almost by the same percentage of students. Students that answered A worked incorrectly on the exponents (they probably halved the exponents or subtracted ten from the exponents). Students who choose option B had divided the base by ten. A similar situation appears also in the following question, administered at grade 10 students in the INVALSI N of the a.y. 2011/12.

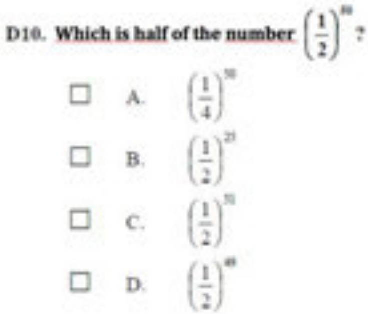


Figure 3: Question D10, Grade 10, INVALSI Test, a.y. 2011/12

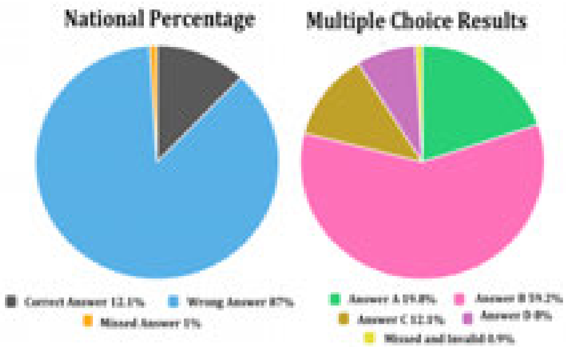


Figure 4: Data of Question D10, Grade 10, INVALSI Test, a.y. 2011/12

As shown in Figure 4 the correct answer was chosen only by 12.1% of students. This question was administered in a census to 530,000 grade 10 students, and about 42,000 students composed the sample. The most common option is B: students who have chosen this option have halved the exponents. This situation has some features in common with the previous one: the task structure is the same and the construction of the distractor is similar. In the 8th grade task, the numbers involved are integers and in the 10th grade task are fractions, but the solution of both tasks require the manipulation of powers, and the conversion from different registers. Another similar situation occurred in the analysis of the results of the exercise referring to powers in the AlmaMathematica Project. Entering the e-learning environment AlmaMathematica there are 5 sections and one of them is about algebra. Inside this section there are 7 subsections including "Powers and Roots". Students enter the online environment and perform the exercises; each student can perform the exercises more than once. When a student makes a mistake, it is only reported that the given answer is incorrect but the display does not show the right one. The percentages of answers, shown in Table 1, are related to the first attempts given by students. When we extrapolated the data, there were 1625 registered and 773 students attempted the exercise. By analysing the data we can see that one of the tasks has some characteristics in common with the previous ones; the task is the following one (Figure 5).

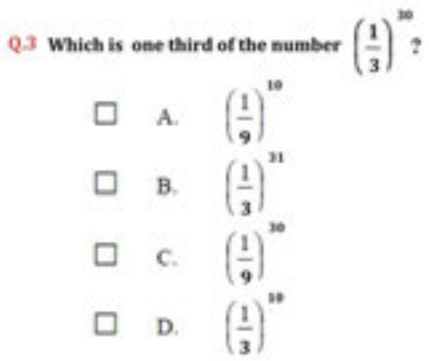


Figure 5: Question Q.3, Section "Powers and Roots", AlmaMathematica Project

Answer A	Answer B	Answer C	Answer D
13.7%	25.9%	12.3 %	48.1%

Table 1: Data of Question Q.3, Section "Powers and Roots", AlmaMathematica Project

Also in this case we notice that the structure of the task has several elements in common with the previous two. Indeed, the situation is similar to the situations that occurred in the INVALSI tests: the solution of the task requires interpreting a verbal delivery and employing working with powers. Also the distractors are similar to the distractors in the INVALSI task grade 10th. Specifically, the number presented in option C, in which the exponent is divided by a third, is obtained by an incorrect manipulation of the exponent exactly as the number present in option B in the previous task. As we can see in Table 1, the percentage of correct answer is slightly higher and almost half of the students chose option D, just like the majority of students of grade 10 chose option B. The exercises in AlmaMathematica were performed by students at the end of the secondary school and at the beginning of university but almost all users are university students. Thus, we can observe that among all analysed levels (from low secondary school to university) the conversion from natural language to symbolic representation about power manipulation is an issue. Specifically, observing the results obtained in high secondary school and university tests, we show that the students who made the same mistake: they manipulated in the wrong way the exponent of the powers leading back to the exponent the "verbal indication" provided in the stimulus. Indeed, in both tasks they chose the option in which there is an incorrect manipulation of the exponent.

Conclusions and further directions

Our main hypothesis is that a longitudinal analysis, performed with many students, can give relevant information on the directions to link (and, then interpret) longitudinal shared difficulties from low and high secondary school to university. We study students' behaviour when solving mathematics exercises in which the management of representation of powers in different registers is required. As shown by Cangelosi et al. (2013) certain errors when working with exponential expressions persist as the students progress through their mathematical studies. Many students memorize algebraic rules with little or no conceptual understanding of their meaning because the rules of algebra and its terminology seem distant from their way of thinking. It follows that these students have trouble keeping track and applying the rules appropriately (Kieran, 2007). The description of the difficulties of students in algebra, particularly in the interpretation of mathematical symbols, was also addressed by Carraher and Schliemann (2007). Kieran (2007) noted furthermore that a main issue is the ways in which students work with variables and algebraic expressions, discussing in depth the development of algebraic thinking in middle and high school. In our case the problem is the management of verbal representation of power and algebraic representation. The processes put in place to manage these different representations are well framed in Duval (2006). In all tasks analysed, a switch from natural language (verbal register) to the algebra (symbolic register) is necessary to give the correct answer, and this presents a high level of complexity (Duval, 2006). Indeed, we studied the difficulties of students in conversion from two different registers, from natural language to symbolic representation. Particularly, we analysed the difficulties to convert from one multi-functional register to one mono-functional register, and despite this represents a difficulty, it is impossible to avoid this situation in the teaching/learning processes. Results show that students make common errors in managing different representations of an object. For a better interpretation of the phenomena that we observed, we shall need a further qualitative analysis and for this reason we are conducting some interviews with school and university students. Research in mathematics education regarding the transition from secondary to tertiary education highlights that students' difficulties are related to a multiplicity of factors – cognitive and meta-cognitive – and it is still more problematic when accessing university education (Gueudet, 2008). These difficulties highlight that one of the causes is the gap in the prerequisite knowledge, specifically in the manipulation of different objects representations. In conclusion, information acquired by LSA and by the e-learning environment has brought to light some recurrent mistakes. The analysis of this data allows us to interpret a didactic phenomenon, and it is also in this perspective that we consider standardized assessment as a tool for formative assessment.

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