1 Introduction

The EOQ mathematical models usually deal with the problem of a wholesaler who has to manage a goods restocking policy, settling his best amount of goods to be procured. *Best* means capable of minimizing all the costs concerning the trade of the stored goods. The relevant seminal contributions are due to Harris, [19] and [20] and Wilson, [36], where an easy scenario is analyzed with a certain demand uniform all over the time so that its instantaneous change rate is fixed, with stocking charges not dependent on time. In subsequent years the subject attracted the auctors' continuous efforts to improve the assumptions of the ingenuous early models about the stored goods' demand, the charges due to goods stocking and to their perishability, if any.

The model main features considered by us concern the store blow-down which will depend on the products demand and on perishability, as for food or medicines or vaporizing liquids. The theoretical models presented hereinafter provide single mathematical representations of the blowdown and of charges. E.g. [3] and [31] considering the case of a perishable good stored in two different warehouses, get blowdown dynamics ruled by two different time laws. In [2] and [29] time changes of production / demand rates not due to perishability are taken into account, while [5] analyzes a frame where the store level decrement is a function of its own level. Anyway the effort of providing a full overview of the main contributions is out of our purpose for being giant the relevant literature. Very often some Journals publish review articles on the subject like [18, 25, 27, 28]; alternatively monographs are available as [37]. The theoretical treatments reviewed throughout this article are concerning a stocks blow-down dynamics depending on their level itself. For an extended overview see [33].

In such a field our own contribution consists of establishing sufficient conditions ensuring the well posedness to the problem of minimum cost and relationships providing either closed form solutions or, alternatively, quadrature formulae- without *ex ante* approximations- allowing a numerical solution to the transcendental (or algebraical of high degree) equation providing the most economical batch. Let us introduce the unified notation used throughout all the paper:

$q(t)$ store level at time t

$f(t,q)$ demand level ruled by time t and store level q

$\hat{h}(t) > 0$ holding cost, assumed as a positive function of t

$\hat{k}(q)$ a factor affecting the holding cost as a increasing function of q.

$A > 0$ costs for delivery

Let the stored goods blow-down according to:

$$\begin{cases} \dot{q}(t) = -f(t,q(t)) \\ q(0) = Q > 0 \end{cases} \tag{1}$$

where the function $f : [0,\infty[\times[0,\infty[\to \mathbb{R}$ is assumed positive, so that the solution to (1) fulfills $q(t) \le Q$ for each $t \ge 0$. We call *reordering time* generated by the batch Q the real positive value $T(Q)$ solution of $q(t) = 0$ where $q(t)$ solves (1). If $A > 0$ is the delivery cost, $\hat{h}(t) > 0$ models the holding cost at time t as a *continuous* function, so that $\hat{h}(0) > 0$, if $\hat{k}(q)$ denotes a continuous and positive function of q so that $\hat{k}(q) \to \infty$ for $q \to \infty$ and that $\hat{k}(0) = 0$, then the total cost for reordering an amount $Q > 0$ of goods is:

$$C(Q) = \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^{T(Q)} \hat{h}(t)\,\hat{k}(q(t))\,\mathrm{d}t. \tag{2}$$

The Wilson originary treatment, [36] follows putting $f(t,q) = \delta > 0$, $\hat{h}(t) = h$, $\hat{k}(q) = q$. Several literature models: [12, 15, 17, 35] are all particular cases of what above, being there $f(t,q) = aq + bq^{\beta}$, $\hat{h}(t) = ht^{\alpha}$. For several models the function $f(t,q)$ is piecewise defined, e.g. [2, 21, 30, 11, 9]. In [4] is treated the case $f(t,q) = \delta(t)$ where $\delta(t)$ is a given positive and continuous function of time, $\hat{h}(t) = h$, $\hat{k}(q) = q$.

The statement of the problem is quite clear: find Q^* such that

$$C(Q^*) = \inf_{Q>0} C(Q) \tag{3}$$

The general problem (3) can be solved explicitly when:

(a) one succeeds in solving the differential equation (1) finding $q(t)$

(b) one succeeds in solving explicitly the equation $q(T) = 0$

(c) one succeeds in solving explicitly the critical point equation

$$C'(Q) = \frac{T'(Q)}{T^2(Q)} \left\{ T(Q)\hat{h}(T(Q))\hat{k}(q(T(Q))) - \int_0^{T(Q)} \hat{h}(t)\hat{k}(q(t))\, \mathrm{d}t \right\} \tag{4}$$

In the Wilson model the blow-down law will be: $q(t) = Q - \delta\, t$ and $T(Q) = Q/\delta$ and the cost

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(Q)} q(t)\mathrm{d}t = \frac{\delta\, A}{Q} + \frac{h}{2}\, Q.$$

In more elaborate models one shall solve, either exactly or numerically, the equation (4), but a previous knowledge is needed whether problem (3) is well posed-existence of solution-or not; so that a numerical treatment for solving equation (4)has a meaning. When possible, some uniqueness conditions for the solution will be provided. Let us notice that in [4] an example is provided of a not-unique solution taking $f(t, q) = t^2 - (9/2)t + 13/2$, $\hat{h} = 1$, $A = 1$, $\hat{k}(q) = q$.

We will provide existence-uniqueness conditions following different demand good dynamics. We will follow [4, 23, 24, 13], as far as it concerns the store costs given by functions $\hat{h}(t)$ e $\hat{k}(q)$. For each theoretical case we will provide applications leading- even if not always- either to closed form solutions by means of Special Functions (e.g. Gauss hypergeometric function, Lambert W function) or to quadrature formulae allowing a direct settlement of the best batch Q^*. Furthermore the problem of backordering will be embodied: it has been recently tackled by several authors, but always under a constant rate of store level change, [6, 7, 8, 32]. They try to detect the optimal batch backordering levels without calculus, but founding upon classic inequalities such that they are between the arithmetic and geometric means powered by the methods in [14, 26]. Anyway in our very general frame where the stock inventory level is ruled by a nonlinear dynamics, the classic approach through the infinitesimal calculus is compulsory.

2 Demand depending on the stock level only

Let us start with (1) when $f(t, q) = f(q)$,and let the stored goods blow-down change according to law:

$$\begin{cases} \dot{q}(t) = -f(q(t)) \\ q(0) = Q > 0 \end{cases} \tag{5}$$

The autonomous structure of (5) allows a closed form solution: defining

$$F(q) := \int_q^Q \frac{1}{f(u)}\, \mathrm{d}u = t \tag{6}$$

then, inverting $F(q)$ we find that $q(t) = F^{-1}(t)$ solves (5).

The *reordering time* generated by the batch Q is the positive value $T(Q)$ solution of $q(t) = 0$:

$$T(Q) = F(0) = \int_0^Q \frac{1}{f(u)}\, \mathrm{d}u.$$

The total cost for reordering an amount $Q > 0$ of goods is here:

$$C(Q) = \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^{T(Q)} \hat{h}(t)\, \hat{k}(q(t))\, \mathrm{d}t. \tag{7}$$

The Wilson early treatment, [36] follows putting $f(u) = \delta$, $\hat{h}(t) = h$, $\hat{k}(q) = q$. Notice that several literature models: [15, 17, 35] are nothing else but particular cases of what above, being there $f(u) = au + bh^\beta$, $\hat{k}(q) = q$, $\hat{h}(t) = ht^\alpha$. In [16] $f(q)$ is defined as $f(q) = -\theta q - \alpha q^\beta$ for $0 \le t \le t_1$ and $f(q) = -\theta q - D$ for $\le t_1 \le t \le T$. In [23] is treated as the case for arbitrary $f(u)$.

Theorem 2.1. *Suppose that function f in (1) is such that*

$$\lim_{v\to\infty}\int_0^v \frac{\mathrm{d}u}{f(u)} = \infty \tag{8}$$

Moreover we assume that if $f(0)=0$, the integrability in $u=0$ of both functions:

$$\frac{1}{f(u)}, \quad \frac{\hat{k}(u)}{f(u)}. \tag{9}$$

Then the cost function of (7) attains its absolute minimum at $Q^*>0$, which is unique.

Proof. In the integral at the right hand side of (7) we do the change $t=F(u)$ minding that $t=0 \Rightarrow u=Q$, $t=T(Q)\Rightarrow u=0$, and that $\mathrm{d}t=-(1/f(u))\mathrm{d}u$, and $q(t)=F^{-1}(t)$, we get:

$$\begin{aligned} C(Q) &= \frac{A}{T(Q)} + \frac{1}{T(Q)}\int_0^Q \hat{h}(F(u))\,\hat{k}(F^{-1}(F(u)))\,\frac{\mathrm{d}u}{f(u)} \\ &= \frac{A}{T(Q)} + \frac{1}{T(Q)}\int_0^Q \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\mathrm{d}u \end{aligned} \tag{10}$$

The good position of (10) follows from (9). The structure of (10) implies that $Q\mapsto C(Q)$, $Q>0$ has exactly one minimizer. First observe that:

$$\lim_{Q\to 0^+} C(Q) = \infty.$$

Then from (8) we see that the cost function (10) diverges when $Q\to\infty$, as immediately checked through De l'Hospital rule:

$$\lim_{Q\to\infty} C(Q) = \lim_{Q\to\infty} \frac{\dfrac{\hat{h}(F(Q))\,\hat{k}(Q)}{f(Q)}}{\dfrac{1}{f(Q)}} = \lim_{Q\to\infty} \hat{h}(0)\,\hat{k}(Q) = \infty$$

Thus $C(Q)$ is bounded from below: so it has at least one stationary value. The extremum will be attained at only one value since the first derivative of $C(Q)$ vanishes if and only if the batch Q solves the equation:

$$\hat{h}(0)\,\hat{k}(Q)\int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{A + \int_0^Q \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,\mathrm{d}u\right\} = 0. \tag{11}$$

But the function

$$\mathcal{N}(Q) := \hat{h}(0)\,\hat{k}(Q)\int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{A + \int_0^Q \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,\mathrm{d}u\right\}$$

is the difference of two increasing functions; thus this minimizing batch is unique.

□

Through a similar way it is possible to prove that thesis of Theorem 2.1 holds with slightly different assumptions on f.

Corollary 2.1.1. *The same conclusion of Theorem 2.1 holds if:*

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^\infty \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,\mathrm{d}u = \infty$$

and

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^\infty \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,\mathrm{d}u \in \mathbb{R}$$

2.1 Applications to known models

For the model [17], where $f(q) = \delta q^{\beta}$, $\hat{h}(t) = h$, $\hat{k}(q) = q$ the optimum condition (4) gives:

$$hQ^{2-\beta} - A\left(\beta^2 - 3\beta + 2\right)\delta = 0.$$

Finally, in the model [15], being there $f(q) = \theta q + \delta q^{\beta}$, $\hat{h}(t) = h$, $\hat{k}(q) = q$, the detection of the optimum batch, firs order condition (4) leads at the Q-equation:

$$\frac{hQ}{(1-\beta)\vartheta}\ln\left(1+\frac{\vartheta}{\delta}Q^{1-\beta}\right) - A - \frac{hQ}{\vartheta}\left(1 - {}_2F_1\left(1, \frac{1}{1-\beta}; \frac{2-\beta}{1-\beta}; -\frac{\vartheta}{\delta}Q^{1-\beta}\right)\right) = 0.$$

Anyway the above formula involving the Gauss hypergeometric function ${}_2F_1$is not present in the article [15]; it is founded upon the integral identities:

$$\int_0^Q \frac{\mathrm{d}u}{\vartheta\, u + \delta\, u^{\beta}} = \frac{1}{\vartheta(1-\beta)}\ln\left(1+\frac{\vartheta}{\beta}Q^{1-\beta}\right),$$
$$\int_0^Q \frac{u\,\mathrm{d}u}{\vartheta\, u + \delta\, u^{\beta}} = \frac{1}{\vartheta}\left[Q - Q\,{}_2F_1\left(1, \frac{1}{1-\beta}; \frac{2-\beta}{1-\beta}; -\frac{\vartheta}{\delta}Q^{1-\beta}\right)\right].$$

We limit here to recall that ${}_2F_1$ is the Gauss hypergeometric function defined as a $|x| < 1$ power series:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$. ${}_2F_1$ is analytically continued by the integral representation theorem:

$${}_2F_1\,(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b}\,\mathrm{d}t,$$

whose validity ranges are: $\operatorname{Re} c > \operatorname{Re} a > 0$, $|x| < 1$. It provides the way for extending the region where the (complex) hypergeometric function is defined, namely for its analytical continuation to the (almost) whole complex plane excluding the half-line $]1,\infty[$.

Let us now introduce some $f(q)$ not considered up to this time. Notice that $f(q)$ could be known as experimental data set to be fitted in some reliable analytical expression: this explains the theoretical laws we are going to study.

2.2 More applications

2.2.1 Affine demand

The demand function which provides the most immediate generalization to the old one (Wilson and Harris), consists of modelling the inventory blow-down through an affine function of the stock level q, namely $f(q) = \delta + \varepsilon\, q$ with δ, $\varepsilon > 0$. The optimum condition (4) in such a case will lead to the transcendental Q-equation

$$\frac{h}{\varepsilon^2}\left[(\delta+\varepsilon Q)\ln\left(\frac{\delta+\varepsilon Q}{\delta}\right) - \varepsilon Q\right] - A = 0. \tag{12}$$

Equation (12) can be solved solely by a numerical approach.

2.2.2 Rational

By rational demand functions we find algebraic first order conditions, in fact, if the inventory blow-down is rational,

$$f(q) = \frac{a}{b+q},$$

a, $b > 0$, then the optimum condition leads (4) to a cubic Q-equation:

$$hQ^3 + 3bhQ^2 - 6aA = 0.$$

If:

$$f(q) = \frac{a}{b^2 + q^2}$$

from (4) we get:

$$hQ^4 + 6b^2hQ^2 - 12aA = 0.$$

2.2.3 Quadratic demand

Let the instantaneous inventory stock level be ruled by (1) with $0 < f(u) = (u-a)(u-b)$, $a, b < 0$. In such a way the optimum condition (11) will specialize in:

$$\frac{h\,Q}{a-b}\ln\left[\frac{b(Q-a)}{a(Q-b)}\right] - \frac{h}{a-b}\left[a\ln\left(\frac{a-Q}{a}\right) - b\ln\left(\frac{b-Q}{b}\right)\right] - A = 0, \tag{13}$$

being (13) to be solved to Q, the only possible approach is numerical. E. g. the left hand side of (13) as a function of Q is plotted below, showing the unique optimal solution $Q \simeq 5.28169$.

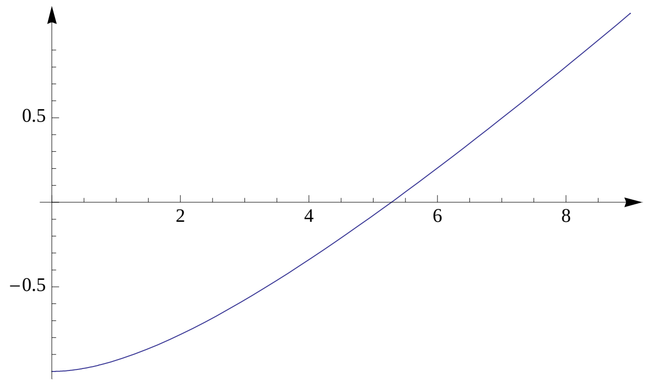

Figure 1: The solution to quadratic demand with $a = -3$, $b = -2$, $h = 1$, $A = 3$.

2.3 Exponential

The inventory manager is faced with aperiodic demand which either is always increasing or decreasing: for instance $f(q) = a\,e^q$, or, $f(q) = a\,e^{-q}$. Even if the integrals in (4) are all elementary for the exponential situation, the relevant Q-equations:

$$hQe^Q - (aA + h)\,e^Q + h = 0, \tag{14}$$
$$-hQe^{-Q} - (aA + h)\,e^{-Q} + h = 0, \tag{15}$$

are transcendental yet. Nevertheless they can be solved through a special function, not being compulsory a numerical solution any more.

The Lambert function(s) $W(y)$ can be achieved starting from $\ell(x) = xe^x$ and after taking its inverse. Of course $\ell(x) = xe^x$ is a not monotonic function of $x \in \mathbb{R}$, and then its inverse is multivalued. So that, we do not have *one* Lambert W- function but *two* Lambert functions on the real line, both coming from the relationship $W(y)e^{W(y)} = y$ where the discriminating point, in order to decide the branch, is $x = -e^{-1}$, $W = -1$. Some special values are $W(0) = 0$; $W(-1) = -e^{-1}$; $W(e) = 1$; $W(1) = \Omega = 0,67143\ldots$ In such a way, looking at the figure, four behaviors are possible:

- if $y \geq 0$, we move on the *principal* branch, say $W_0(y)$, or simply $W(y)$, when no ambiguity can occur;
- if $-e^{-1} \leq y \leq 0$, we move on the principal branch again if $W(u) \geq -1$;
- if $-e^{-1} \leq y \leq 0$, but $W(y) < -1$, we move on the *secondary* branch, say $W_{-1}(y)$;
- if $y < -e^{-1}$, we do not have at all real values of W any more.

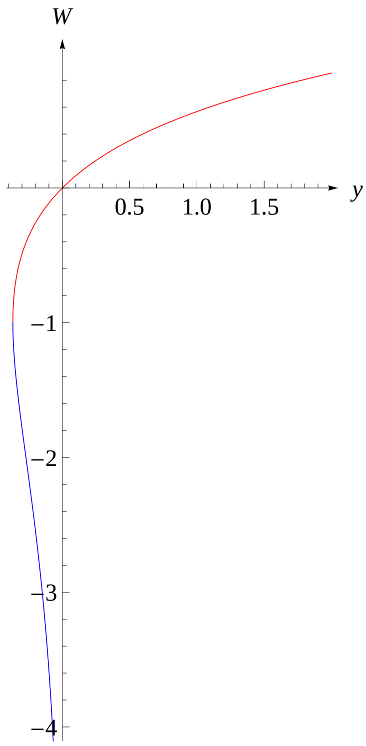

Figure 2: The couple of real branches of the Lambert W-function: $W_0(y)$ and $W_{-1}(y)$.

Anyway, there is no possibility of expressing $W(y)$ in terms of elementary functions. A method for computing $W(y)$ for each y could be: to develop $\ell(x) = xe^x$ in a power series, what we know has a sum equal to y; and to revert such a series by the Lagrange inversion theorem. In such a way one obtains W expanded in ascending powers of y:

$$W(y) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} y^n$$

whose convergence radius is e^{-1}. As far as we are concerned, the first appearance of W function in an economics context was in [22], a paper where we generalized a Goodwin microeconomic model, while the first use of this function in the EOQ contest is due to [34]. An almost exhaustive survey on Lambert functions can be read at [10]. After this short synopsis, let us go back to our equations (14) and (15) and solve them by means of W. Let us begin writing (14) as: $e^Q (Q - b) = -1$ with the obvious meaning of $b > 1$. We change variable putting $Q - b = R$ obtaining:

$$R\, e^R = -e^{-b}. \tag{16}$$

It is worth noting that (16) is well posed, i.e. has two real roots because $b > 1$: they are $W_{-1}(-e^{-b})$ and $W_0(-e^{-b})$. Only $W_0(-e^{-b})$ has economic meaning, in fact recalling that, $-1 < W_0(-e^{-b}) < 0$ and going back to the original Q we find:

$$Q_+^* = b + W_0(-e^{-b}),$$

where the index $+$ reminds we started from a positive exponential. To solve (15), observe that it can be written as:

$$e^{-Q-b} (-Q - b) = -e^{-b}, \tag{17}$$

where the meaning of b does not change. The solution, of economic interest, i.e. positive, is then:

$$Q_-^* = -b - W_{-1}(-e^{-b}),$$

since $-\infty < W_{-1}(-e^{-b}) < -1$.

The W Lambert function is available by several computer algebra packages like Mathematica®, for automatic computing. In addition the $f(q)$ exponential nature is not an analytical oddness, but has a deep market meaning.

3 Backordering

In order to analyze the backordering we present our recent contribution [13]. Assume $f(t,q) = f(q)$, $\hat{h} = \text{const.}$ and $\hat{k}(q) = q$. The quantity Q ordered at each cycle undergoes two different uses: a first share $Q - R$ covers the demand of the previous cycle, and then does not enter the inventory; while R is the residual share which enters the store so that the outstanding amount is again $Q - R$, and so on. As a consequence, the reordering time becomes:

$$T(Q) = F(R-Q) = \int_{R-Q}^{R} \frac{1}{f(u)}\,\mathrm{d}u,$$

where the function $f : [R-Q, \infty[\to \mathbb{R}$ is assumed positive, and the total cost is:

$$C(R,Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(R)} q(t)\,\mathrm{d}t - \frac{b}{T(Q)} \int_{T(R)}^{T(Q)} q(t)\,\mathrm{d}t. \tag{18}$$

It is possible to get easier (18), by the following Lemma.

Lemma 3.1. *Let $f(q)$ be the law describing the the q-blowdown dynamics: then the total cost is given by:*

$$C(R,Q) = \frac{A + h\displaystyle\int_0^R \frac{u}{f(u)}\mathrm{du} - b\int_{R-Q}^0 \frac{u}{f(u)}\mathrm{d}u}{\displaystyle\int_{R-Q}^R \frac{\mathrm{d}u}{f(u)}}, \tag{19}$$

where, if $f(0) = 0$ we assume the integrability of both functions:

$$\frac{1}{f(u)}, \quad \frac{u}{f(u)}.$$

Proof. Putting in (18) $t = F(u)$, notice that $t = 0 \Rightarrow u = R$, $t = T(R) \Rightarrow u = 0$, $t = T(Q) \Rightarrow u = R - Q$, and $\mathrm{d}t = -(1/f(u))\mathrm{d}u$, so that, minding that $q(t) = F^{-1}(t)$ one finds:

$$\begin{aligned} C(R,Q) &= \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_{F(R)}^{F(0)} F^{-1}(t)\,\mathrm{d}t - \frac{b}{T(Q)} \int_{F(0)}^{F(R-Q)} F^{-1}(t)\,\mathrm{d}t \\ &= \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^R F(q)\,\mathrm{d}t - \frac{b}{T(Q)} \left((R-Q)F(R-Q) + \int_{R-Q}^0 F(q)\,\mathrm{d}t \right) \end{aligned}$$

writing F in terms of f we find out:

$$C(R,Q) = \frac{1}{T(Q)} \left(A + h \int_0^R \left(\int_q^R \frac{\mathrm{du}}{f(u)} \right) \mathrm{d}q - b \left((R-Q) \int_{R-Q}^R \frac{\mathrm{du}}{f(u)} + \int_{R-Q}^0 \left(\int_q^R \frac{\mathrm{du}}{f(u)} \right) \mathrm{d}q \right) \right)$$

exchanging the integrations order and computing the inner one

$$\begin{aligned} C(R,Q) &= \frac{1}{T(Q)} \left[A + h \int_0^R \frac{u}{f(u)}\mathrm{d}u \right. \\ &\left. -b \left((R-Q) \int_{R-Q}^R \frac{\mathrm{d}u}{f(u)} + \int_{R-Q}^0 \frac{u-(R-Q)}{f(u)}\,\mathrm{d}u + \int_0^R \frac{Q-R}{f(u)}\,\mathrm{d}u \right) \right] \\ &= \frac{A + h\displaystyle\int_0^R \frac{u}{f(u)}\mathrm{d}u + b \left((Q-R) \int_{R-Q}^R \frac{\mathrm{d}u}{f(u)} - \int_{R-Q}^0 \frac{u-(R-Q)}{f(u)}\mathrm{d}u - \int_0^R \frac{Q-R}{f(u)}\mathrm{d}u \right)}{\displaystyle\int_{R-Q}^R \frac{\mathrm{d}u}{f(u)}} \end{aligned}$$

A numerator straightforward reduction completes the proof. □

Theorem 3.1. *The cost function introduced in* (19) *attains its absolute minimum at proper positive values* (Q^*, R^*). *Such a minimizing batch is unique.*

Proof. Recall that

$$\lim_{(R,Q)\to(0,0)} C(R,Q) = \infty$$

and that if $R \to Q$, we go back to the originary model, furthermore, by De l'Hospital rule one finds that:

$$\lim_{Q\to+\infty} C(R,Q) = +\infty$$

Let us change variables passing from $C(R,Q)$ to $C(R,Q-R)$: accordingly, the total cost $C(R,Q-R)$ is:

$$\frac{A + h\displaystyle\int_0^R \frac{u}{f(u)}\mathrm{d}u - b\int_{R-Q}^0 \frac{u}{f(u)}\mathrm{d}u}{\displaystyle\int_{R-Q}^R \frac{\mathrm{d}u}{f(u)}} \tag{20}$$

Partial derivatives with respect to R and Q provide:

$$\frac{\partial C}{\partial Q} = \frac{-A + b\,(Q-R)\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u}{f(R-Q)\left(\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u\right)^2}$$

$$\frac{\partial C}{\partial R} = \frac{-A + h\,R\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u}{f(R)\left(\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u\right)^2} - \frac{\partial C}{\partial Q}$$

Imposing partial derivatives to vanish:

$$\frac{\partial C}{\partial Q} = \frac{-A + b\,(Q-R)\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u}{f(R-Q)\left(\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u\right)^2} = 0$$

$$\frac{\partial C}{\partial R} = \frac{-A + h\,R\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u}{f(R)\left(\displaystyle\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u\right)^2} = 0$$

We assume $f > 0$ for each u, then both the denominators are strictly positive; setting the numerators to be zero, first order conditions will provide the critical point system:

$$\begin{aligned} g(R,Q-R) &= -A + h\,R\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u = 0 \\ m(R,Q-R) &= -A + b\,(Q-R)\int_{R-Q}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{R-Q}^0 \frac{u}{f(u)}\,\mathrm{d}u = 0. \end{aligned} \tag{21}$$

To solve (21), subtracting side by side, one finds:

$$\begin{aligned} Q &= \frac{hR}{b} + R \\ m(R,\frac{hR}{b}) &= -A + hR\int_{\frac{hR}{b}}^R \frac{1}{f(u)}\,\mathrm{d}u - h\int_0^R \frac{u}{f(u)}\,\mathrm{d}u + b\int_{\frac{hR}{b}}^0 \frac{u}{f(u)}\,\mathrm{d}u = 0 \end{aligned}$$

$m(R, hR/b)$ is an increasing function being:

$$\frac{\mathrm{d}}{\mathrm{d}R} m(R, hR/b) = h \int_{\frac{hR}{b}}^{R} \frac{1}{f(u)}\,\mathrm{d}u > 0,$$

so observing that $m(0,0) < 0$, then $m(R, hR/b)$ has a unique real root, and we have one and only one critical point for the cost function (19). Let us show it is a minimum. The Hessian determinant at the critical point is:

$$H = \frac{bh}{f(R) f(-\frac{hR}{b}) \left(\int_{-\frac{hR}{b}}^{R} \frac{1}{f(u)}\,\mathrm{d}u \right)^2} \tag{22}$$

In fact being:

$$H = \frac{\partial^2}{\partial R^2} C(R,Q) \frac{\partial^2}{\partial Q^2} C(R,Q) - \left(\frac{\partial^2}{\partial R \partial Q} C(R,Q) \right)^2$$

minding that $g(R, Q-R) = m(R, Q-R) = 0$ we have

$$\frac{\partial^2 C}{\partial Q^2} = \frac{\frac{\partial k}{\partial Q}}{Z^2 f(R-Q)}, \quad \frac{\partial^2 C}{\partial R^2} = \frac{\frac{\partial g}{\partial R}}{Z^2 f(R)} - \frac{\frac{\partial k}{\partial R}}{Z^2 f(R-Q)}, \quad \frac{\partial^2 C}{\partial Q \partial R} = \frac{\frac{\partial k}{\partial R}}{Z^2 f(R-Q)}$$

where we put:

$$Z = \int_{R-Q}^{R} \frac{1}{f(u)}\,\mathrm{d}u.$$

Eventually, recalling that

$$Q = \frac{hR}{b} + R, \quad \frac{\partial m}{\partial R} = \frac{\partial g}{\partial Q} = 0$$

we find (22) proving the stationary point to be a minimum.

□

4 Sample problems

We provide now some applications of above to known models of the literature extended to backorders, getting in any case a transcendental (or algebraic) R- resolvent equation. The following conditions are assumed:

$$h > 0,\ Q > 0,\ A > 0\ R > 0,\ 0 < p < 1,\ b > 0,\ \delta > 0$$

Wilson model

$$f(u) = \delta \Rightarrow C(R,Q) = \frac{\delta}{Q} \left(A + \frac{b\,(Q-R)^2}{2\,\delta} + \frac{h\,R^2}{2\,\delta} \right)$$

Such a case has a theoretical interest due to its final (not transcendental and) exactly solvable resolvent: the minimizing batch is found to be:

$$Q^* = \sqrt{\frac{2A\delta(b+h)}{bh}}; \qquad R^* \sqrt{\frac{2Ab\delta}{h(b+h)}}$$

In such a way the minimized cost will be:

$$C^* = b \left(\sqrt{\frac{b+h}{b}} - \sqrt{\frac{b}{b+h}} \right) \sqrt{2A\delta h}$$

Goh's model $p = 1/2$

$$f(u) = \sqrt{|u|} \Rightarrow C(R,Q) = \frac{3\,A + 2\,h\,R^{\frac{3}{2}} + 2\,b\,(Q-R)^{\frac{3}{2}}}{6\,\left(\sqrt{R} + \sqrt{Q-R}\right)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q-R) = 0$ we get:

$$-3\,A + 4\,h\,\left(1 + \sqrt{\frac{h}{b}}\right)\,R^{\frac{3}{2}} = 0$$

Goh's model p general

If $f(u) = |u|^p$ then

$$C(R,Q) = \frac{(1-p)\,\left(b\,R^p\,(Q-R)^2 + h\,R^2\,(Q-R)^p + A\,d\,(2-p)\,(R\,(Q-R))^p\right)}{(2-p)\,(R^p(Q-R) + R(Q-R)^p)}$$

By $Q - R = \frac{hR}{b}$ e $g(R, Q-R) = 0$ we get:

$$\frac{\left(\frac{R^2}{b}\right)^p\,\left\{h\,\left[b^p\,h - b\,h^p\,(p-3)\right]\,R^2 - A\,b\,(p-2)\,(h\,R)^p\right\}}{b} = 0$$

collecting R^p

$$R^p\,\left\{-\left[A\,b\,h^p\,(-2+p)\right] + h\,\left[b^p\,h - b\,h^p\,(-3+p)\right]\,R^{2-p}\right\} = 0$$

the solution is straightforward solving to R^{2-p}.

Exponentials

$$f(u) = e^{-u} \Rightarrow C(R,Q) = \frac{e^{Q-R}\,\left\{A + h\,\left[1 + e^R\,(R-1)\right]\right\} + b\,\left(e^{Q-R} - Q + R - 1\right)}{e^Q - 1}$$

If $Q - R = \frac{hR}{b}$ e $g(R, Q-R) = 0$ we get the transcendental R-equation:

$$b - e^{\frac{h\,R}{b}}\,\left(A + b + h - e^R\,h\right) = 0$$

It is then provided a simulation for$A = 1$, $b = 1/3$, $h = 1/4$. Figure 3 shows the iso-cost curves, highlighting the minimizer (Q^*, R^*) numerically detected.

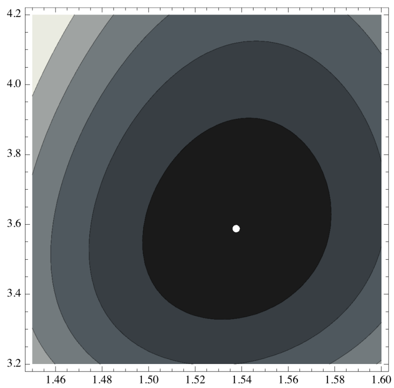

Figure 3: Level curves relevant to the cost function

Figure 4 shows the crossing of the loci of roots of the single first partial derivatives.

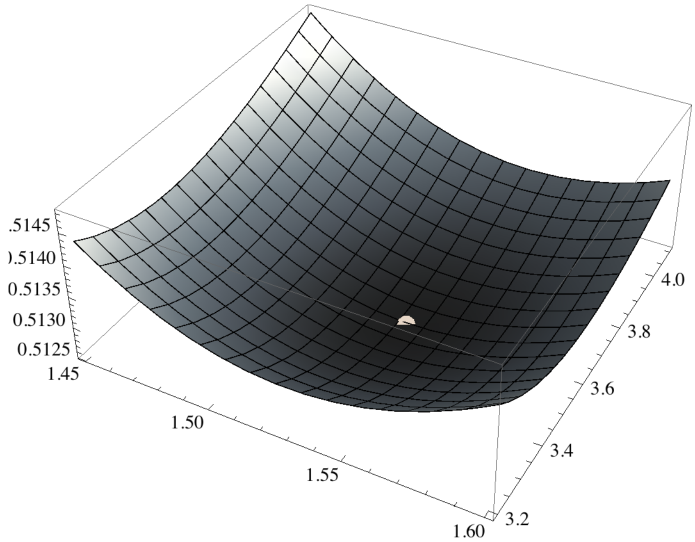

Figure 5: a 3-D plot of $C(Q,R)$ for $f(u) = e^{-u}$

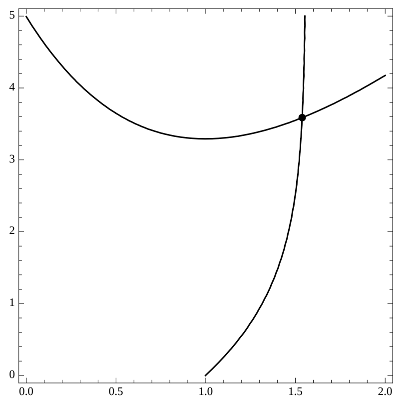

Figure 4: Crossing of the loci of roots of the single partial derivatives

And finally in Figure 5, a 3-D plot of the global cost function. We have a similar behavior for $f(u) = e^{u}$.

Rational (first)

$$f(u) = \frac{k}{n+u} \Rightarrow C(R,Q) = \frac{6\,A\,k + h\,R^2\;(3\,n + 2\,R) + b\;(3\,n - 2\,Q - R)\;(Q-R)^2}{3Q\;(2\,n + 2R - Q)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q-R) = 0$ one obtains:

$$\frac{-6\,A\,b^2\,k + h\;(b+h)\;R^2\;(-(h\,R) + b\;(3\,n + R))}{b^2} = 0$$

which is providing promptly the batch.

Rational (second)

$$f(u) = \frac{k}{n+u} \Rightarrow C(R,Q) = \frac{3\;\left(4\,A\,k + h\;\left(2\,n\,R^2 + R^4\right) + b\,(Q-R)^2\;\left(2\,n + (Q-R)^2\right)\right)}{4\;\left(R^3 + (Q-R)^3 + 3\,n\,Q\right)}$$

By $Q-R=\frac{hR}{b}$ and $g(R,Q-R)=0$ one finds:

$$-12\,A\,k+\frac{6\,h\,\left(b+h\right)\,n\,R^2}{b}+\frac{h\,\left(b^3+h^3\right)\,R^4}{b^3}=0$$

biquadratic equation.

5 Conclusions

We proved two existence-uniqueness theorem 2.1 and 3.1 about a minimum cost batch for a class of EOQ models with perishable inventory and nonlinear cost and with sole backordering, leading to a set of sufficient conditions which require to check the convergence of some improper integrals, and form the article's main theoretical effort. As application, several cases have been treated of demand $f(q)$ as a continuous function of the stock level q. Being one of the sufficient conditions met in any case, the economic order quantity is unique, and the relevant computations lead to transcendental equations. In some cases the plot of the global cost function is provided, and, even if the optimality condition can be written in closed (but transcendental) form, whose solution shall mostly be faced numerically. Mind that the re-ordering time, the global cost function and the minimum cost (optimum) condition are here detected without any previous approximation, being a numerical treatment required-if any- only at the end, in order to solve the (often) transcendental equation for the economic batch.

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