Bounded Engel and solvable unitary units in group rings*

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Abstract

Let FG be the group ring of a group G over a field F. We consider the group of unitary units of FG with respect to the classical involution. Under suitable restrictions upon F, we show that if the unitary units of FG are both bounded Engel and solvable, then the entire unit group of FG is nilpotent. This extends a result of Fisher, Parmenter and Sehgal.

1 Introduction

Let G be a group and F a field of characteristic $p \neq 2$. It is a natural problem to try to determine the structure of the unit group $\mathcal{U}(FG)$ of the group ring FG. In particular, we would like to know the conditions under which $\mathcal{U}(FG)$ satisfies various group identities. This topic has been studied extensively, and we refer to [8] for an overview.

In particular, it is known when $\mathcal{U}(FG)$ is nilpotent. For modular group rings, Khripta presented the answer in [7]. The case where FG is not modular was handled independently in Fisher-Parmenter-Sehgal [2] and Khripta [6].

Certain subsets of $\mathcal{U}(FG)$ are also interesting. Consider the classical involution on FG, given by $(\sum_{g\in G} \alpha_g g)^* = \sum_{g\in G} \alpha_g g^{-1}$. The identities satisfied by the set of symmetric units (namely, those satisfying $\alpha^* = \alpha$), have also received a good deal of attention and again, we refer to [8] for a discussion.

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However, the unitary units are also worthy of study. If R is any ring with involution $*$, then we let $Un(R) = \{r \in \mathcal{U}(R) : rr^* = 1\}$. It is easy to see that $Un(R)$ is a subgroup of $\mathcal{U}(R)$. When $R = FG$, we will always let $*$ be the classical involution. As such, we note that G is a subgroup of $Un(FG)$.

Only a handful of papers have examined identities satisfied by the unitary units. Giambruno-Polcino Milies [3] and Broche-Dooms-Ruiz [1] looked at general results concerning group identities. Gonçalves-Passman [4] discussed when $Un(FG)$ contains a nonabelian free group. Recently, the authors in [10] considered group rings whose unitary units form a nilpotent group; as it turns out, this is usually enough to imply that $\mathcal{U}(FG)$ is nilpotent, but there are exceptions.

However, the Fisher-Parmenter-Sehgal paper did more. Let us establish some notation. On any group G, let $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$ and, recursively,

$$(g_1, \dots, g_{n+1}) = ((g_1, \dots, g_n), g_{n+1}).$$

Then, of course, G is nilpotent if, for some n, we have $(g_1, \dots, g_n) = 1$ for all $g_i \in G$. We say that G is n-Engel if, for every $g, h \in G$, we have

$$(g, \underbrace{h, \dots, h}_{n \text{ times}}) = 1$$

and bounded Engel if it is n-Engel for some n. Also, we let $(g_1, g_2)^o = (g_1, g_2)$ and, recursively,

$$(g_1, \dots, g_{2^{n+1}})^o = ((g_1, \dots, g_{2^n})^o, (g_{2^n+1}, \dots, g_{2^{n+1}})^o).$$

Then G is solvable if, for some n, we have $(g_1, \dots, g_{2^n})^o = 1$ for all $g_i \in G$. Of course, every nilpotent group is both bounded Engel and solvable, but even the bounded Engel property and solvability together are not enough to guarantee nilpotence. However, it is shown in [2] that if FG is not modular, then whenever $\mathcal{U}(FG)$ is both bounded Engel and solvable, it is also nilpotent.

Inspired by this result, we ask if it is sufficient to assume that the unitary units are both bounded Engel and solvable, in order to prove that the entire unit group is nilpotent. We show that, under certain restrictions upon the field, this is the case.

Recall that FG is said to be modular if char $F = p > 0$ and G has an element of order p. Also, G is p-abelian if G' is a finite p-group, and 0-abelian means abelian. Our main theorems are the following.

Theorem 1. *Let F be an infinite field of characteristic $p > 2$ and G a group, such that FG is modular. Then the following are equivalent:*

1. *$Un(FG)$ is bounded Engel and solvable,*

2. *$\mathcal{U}(FG)$ is nilpotent, and*

3. *G is nilpotent and p-abelian.*

When FG is not modular, we record

Theorem 2. *Let F be an algebraically closed field of characteristic different from 2 and G a group, such that FG is not modular. Then the following are equivalent:*

1. *$Un(FG)$ is bounded Engel and solvable,*

2. *$\mathcal{U}(FG)$ is nilpotent, and*

3. *G is nilpotent and the torsion elements of G are central.*

Note that if G is torsion and has no 2-elements, then just as in [10, Theorem 2], we need not insist that F be algebraically closed in Theorem 2. However, in general, some restriction upon the field is needed. Indeed, in [10, Proposition], we pointed out that if F is the field of 5 elements and G is the dihedral group of order 8, then $Un(FG)$ is nilpotent; however, $\mathcal{U}(FG)$ is neither bounded Engel nor solvable (see [8, Theorems 5.2.1 and 6.2.2]).

2 Some necessary lemmas

Let us gather a few known results. We begin with some group theory. Write G for a group and T for its set of torsion elements. The following crucial result is due to Gruenberg (see [5] and [15, Theorem 7.36]).

Lemma 1. *If G is both bounded Engel and solvable, then G is locally nilpotent, T is a subgroup and G/T is nilpotent.*

We will also need

Lemma 2. *Suppose that G has an abelian normal subgroup of finite index and, for some prime p, an infinite normal p-subgroup of bounded exponent. Then G contains an infinite direct product $A_1 \times A_2 \times \cdots$, where each A_i is a finite, nontrivial, abelian normal p-subgroup.*

Proof. See [12, Lemma 7]. □

Next, we need to know the conditions under which $\mathcal{U}(FG)$ is nilpotent. Let F be a field and $p \geq 0$ its characteristic.

Lemma 3. *If FG is modular, then $\mathcal{U}(FG)$ is nilpotent if and only if G is nilpotent and p-abelian.*

Proof. See [8, Theorem 4.2.1]. □

Lemma 4. *If F is infinite and FG is not modular, then $\mathcal{U}(FG)$ is nilpotent if and only if G is nilpotent and its torsion elements are central.*

Proof. See [8, Theorem 4.2.9]. □

We require the famous result of Isaacs and Passman concerning group rings satisfying a polynomial identity. Recall that if R is an F-algebra, then R is said to satisfy a polynomial identity if there is a nonzero polynomial $f(x_1, \ldots, x_n)$ in the free algebra on noncommuting indeterminates $F\{x_1, x_2, \ldots\}$ such that $f(r_1, \ldots, r_n) = 0$ for all $r_i \in R$.

Lemma 5. *The group ring FG satisfies a polynomial identity if and only if G has a p-abelian normal subgroup of finite index.*

Proof. See [14, Corollaries 5.3.8 and 5.3.10]. □

In order to obtain a polynomial identity, we need a result on group identities. We say that a group G satisfies a group identity if there is a nontrivial word $w(x_1, \ldots, x_n)$ in the free group $\langle x_1, x_2, \ldots \rangle$ such that $w(g_1, \ldots, g_n) = 1$ for all $g_i \in G$. For instance, an n-Engel group satisfies the group identity

$$(x_1, \underbrace{x_2, \ldots, x_2}_{n \text{ times}}) = 1.$$

We have the following.

Lemma 6. *Let F be an infinite field with $p > 2$ and R an F-algebra with involution $*$. Suppose that $Un(R)$ satisfies a group identity. Then every $*$-invariant nil ideal of R satisfies a polynomial identity.*

Proof. See [1, Theorem 2.3]. □

Another useful reduction concerning group identities comes from

Lemma 7. *Let R be an F-algebra with involution $*$ and I a $*$-invariant nil ideal of R. If $p \neq 2$ and $Un(R)$ satisfies a group identity, then $Un(R/I)$ satisfies the same group identity.*

Proof. See [4, Lemma 1.1]. □

Finally, we need to know something about a particular polynomial identity. On any ring R, let

$$[x_1, x_2]^o = [x_1, x_2] = x_1x_2 - x_2x_1$$

and, recursively,

$$[x_1, \dots, x_{2^{n+1}}]^o = [[x_1, \dots, x_{2^n}]^o, [x_{2^n+1}, \dots, x_{2^{n+1}}]^o].$$

We say that a subset S of R is Lie solvable if there exists an n such that $[s_1, \dots, s_{2^n}]^o = 0$ for all $s_i \in S$. Also, if R is a ring with involution $*$, write R^- for the set of skew elements; that is, $R^- = \{r \in R : r^* = -r\}$. We have

Lemma 8. *Suppose that $p \neq 2$ and G is a group without 2-elements. If $(FG)^-$ is Lie solvable, then G is p-abelian.*

Proof. If G is torsion, then by Zalesskiĭ-Smirnov [18, Theorem 3.4], FG is Lie solvable. If G is not torsion, then the authors proved the same thing in [9, Proposition 3.5]. (See [11, Section 5] for a discussion.) Thus, by Passi-Passman-Sehgal [13], G is p-abelian. □

3 Proofs of the theorems

We can now prove our results. A portion of the proof of Theorem 1 is similar to part of the proof of [10, Theorem 1], but for the sake of clarity, we shall include it in full. Let us introduce some notation. Write ζ for the centre of G. Also, if H is a finite subgroup of G, let $\hat{H} = \sum_{h \in H} h$, and if $H = \langle h \rangle$, then write $\hat{h} = \hat{H}$. Finally, if N is a normal subgroup of G, write $\Delta(G, N)$ for the kernel of the natural homomorphism $FG \to F(G/N)$.

Proof of Theorem 1. In view of Lemma 3, we only need to prove that (1) implies (3). Suppose that $Un(FG)$ is bounded Engel and solvable. Take any $a, b \in G$. We claim that if $Un(FG)$ is p^n-Engel, then $a^{p^{n+1}}b = ba^{p^{n+1}}$. To prove this, it is sufficient to work in $H = \langle a, b, c \rangle$, where c is any element of order p in G. Note that H is a finitely generated subgroup of $Un(FG)$; thus, by Lemma 1, H is nilpotent. Therefore, as H contains an element of order p, it contains a central element z of order p. Let $\eta = \hat{z}$.

Take any $x, y \in H$. Then notice that

$$(1+\eta(x-x^{-1}))^{-1} = 1-\eta(x-x^{-1}) = (1+\eta(x-x^{-1}))^*.$$

Thus, $1+\eta(x-x^{-1}), y \in Un(FH)$. Therefore,

$$1 = (1+\eta(x-x^{-1}), \underbrace{y,\ldots,y}_{p^n \text{ times}}) = 1+\eta(y^{-p^n}(x-x^{-1})y^{p^n} - (x-x^{-1})).$$

(See [8, Lemma 4.1.1].) Thus, by [17, Proposition III.4.18],

$$y^{-p^n}(x-x^{-1})y^{p^n} - (x-x^{-1}) \in \Delta(H, \langle z\rangle).$$

Let $\bar{H} = H/\langle z\rangle$. Then working in $F\bar{H}$,we have

$$(\bar{y})^{-p^n}(\bar{x}-(\bar{x})^{-1})(\bar{y})^{p^n} - (\bar{x}-(\bar{x})^{-1}) = \bar{0}.$$

This leaves two possibilities. If $(\bar{y})^{-p^n}\bar{x}(\bar{y})^{p^n} = (\bar{y})^{-p^n}(\bar{x})^{-1}(\bar{y})^{p^n}$, then $(\bar{x})^2 = \bar{1}$. Otherwise, $(\bar{y})^{-p^n}\bar{x}(\bar{y})^{p^n} = \bar{x}$, in which case $\bar{x}$ and $(\bar{y})^{p^n}$ commute. Let us assume that $(\bar{x})^2 = \bar{1}$, and prove that the same conclusion is reached.

Assuming that $\bar{H}$ is not the trivial group (for otherwise, there is nothing to do), we note that $\bar{H}$ is still nilpotent and, therefore, has a nontrivial central element. Suppose there is a central element $\bar{v}$ with $(\bar{v})^2 \neq \bar{1}$. Then replacing x with xv, we see that $(\bar{x}\bar{v})^2 \neq \bar{1}$, and therefore $(\bar{y})^{p^n}$ commutes with $\bar{x}\bar{v}$, and hence with $\bar{x}$. Thus, we may assume that the centre of $\bar{H}$ has exponent 2. By [16, 5.2.22], $\bar{H}$ is a 2-group. If $\bar{u} \in \bar{H}$ has order greater than 2, then by our earlier discussion, $\bar{u}$ commutes with $(\bar{t})^{p^n}$, for every $\bar{t} \in \bar{H}$. But as $\bar{H}$ is a 2-group, this means that $\bar{u}$ is central, and we can let $v = u$. If no such $\bar{u}$ exists, then $\bar{H}$ has exponent 2, in which case it is abelian.

Therefore, in any case, we have $(\bar{y})^{-p^n}\bar{x}(\bar{y})^{p^n} = \bar{x}$. In other words, $y^{-p^n}xy^{p^n} = xz^i$, for some integer i. Conjugating by y^{p^n} an additional $p-1$ times, we get $y^{-p^{n+1}}xy^{p^{n+1}} = xz^{ip} = x$, proving our claim.

As n was independent of the choice of a and b, we now know that G/ζ is a p-group of bounded exponent. By a theorem of Schur (see [17, Corollary I.4.3]), we see that G' is a p-group of bounded exponent.

We claim that G' is, in fact, finite. Suppose otherwise. Let P be the set of p-elements in G. As $G' \subseteq P$, we know that P is a subgroup of

G. Notice that $\Delta(G,P)$ is a nil ideal. Indeed, any element of the ideal is surely in $\Delta(K, K \cap P)$, where K is some finitely generated subgroup of G. But by Lemma 1, K is nilpotent, hence $K \cap P$ is finitely generated and, therefore, finite. By [8, Lemma 1.1.1], $\Delta(K, K \cap P)$ is a nilpotent ideal. Thus, $\Delta(G,P)$ is nil. It is also invariant under the classical involution. Thus, by Lemma 6, $\Delta(G,P)$ satisfies a polynomial identity, $f(x_1,\ldots,x_n)=0$. But as $G' \leq P$, we note that $FG/\Delta(G,P)$ is commutative. Therefore, FG satisfies the polynomial identity $f([x_1,x_2],\ldots,[x_{2n-1},x_{2n}])=0$. By Lemma 5, G has a p-abelian normal subgroup A of finite index.

As we have just observed, $\Delta(G,A')$ is nilpotent. Thus, by Lemma 7, the group $Un(F(G/A'))$ is bounded Engel and solvable. Also, to prove our claim, it is sufficient to show that $(G/A')' = G'/A'$ is finite. Therefore, let us assume that A is abelian. Lemma 2 then presents us with an infinite direct product $A_1 \times A_2 \times \cdots$ of nontrivial finite normal p-subgroups in G. As $Un(FG)$ is solvable, let us say that it satisfies $(x_1,\ldots,x_{2^n})^o = 1$. Then letting $\eta_i = \hat{A}_i$ and taking any $g_i \in G$, we observe that $1+\eta_i(g_i - g_i^{-1}) \in Un(FG)$. Thus,

$$\begin{aligned} 1 &= (1+\eta_1(g_1-g_1^{-1}),\ldots,1+\eta_{2^n}(g_{2^n}-g_{2^n}^{-1}))^o \\ &= 1+\eta_1\cdots\eta_{2^n}[g_1-g_1^{-1},\ldots,g_{2^n}-g_{2^n}^{-1}]^o \end{aligned}$$

(see [8, Lemma 6.2.4]).

Now, letting $B = A_1 \times \cdots \times A_{2^n}$, we have $\eta_1 \cdots \eta_{2^n} = \hat{B}$. Thus, $[g_1 - g_1^{-1},\ldots,g_{2^n}-g_{2^n}^{-1}]^o$ annihilates $\hat{B}$ and, as we have seen before, this means that $[g_1-g_1^{-1},\ldots,g_{2^n}-g_{2^n}^{-1}]^o \in \Delta(G,B)$. Letting $\bar{G} = G/B$, we have

$$[\bar{g}_1-\bar{g}_1^{-1},\ldots,\bar{g}_{2^n}-\bar{g}_{2^n}^{-1}]^o = \bar{0}$$

for all $\bar{g}_i \in \bar{G}$. As $(F\bar{G})^-$ consists of the linear combinations of the terms $\bar{g}-\bar{g}^{-1}$, we conclude that $(F\bar{G})^-$ is Lie solvable.

If we can show that $(G/B)' = G'B/B$ is finite, then as B is finite, we will know that G' is finite as well. Therefore, we will replace G with $\bar{G}$ and assume that FG is Lie solvable. Now, as G is locally nilpotent, its 2-elements form a normal subgroup Q. Also, $(F(G/Q))^-$ must be Lie solvable. Thus, by Lemma 8, $(G/Q)'$ is a finite p-group. But $(G/Q)' = G'Q/Q \simeq G'/(G' \cap Q)$. However, as G' is a p-group, $G' \cap Q = 1$. Thus, G' is a finite p-group, proving our claim.

We have now deduced that G is locally nilpotent and p-abelian, say $|G'| = p^m$. If L is any finitely generated subgroup of G, then L is nilpotent and $L' \leq G'$. In particular, there is a fixed upper bound upon the nilpotency class of any such L, as the terms in the lower central series must get strictly smaller until they reach 1. So let us say that any such L must satisfy

$$(\underbrace{L, L, \ldots, L}_{r \text{ times}}) = 1.$$

If G does not satisfy this same identity, then choose $g_1, \ldots, g_r \in G$ such that $(g_1, \ldots, g_r) \neq 1$. Then letting $L = \langle g_1, \ldots, g_r \rangle$, we observe that L does not satisfy the identity either. This gives us a contradiction. Therefore, G is indeed nilpotent and p-abelian. The proof is complete. □

When FG is not modular, we can use essentially the same proof as in [10].

Proof of Theorem 2. It again suffices to show that (1) implies (3). But here, nearly all of the work was done in [10]. Indeed, we notice that Lemmas 6, 7 and 8 in [10] all work just as well if each instance of "nilpotent" in the statement is replaced with "locally nilpotent". Furthermore, [10, Lemma 9] holds for any group identity. Now, we follow the proof of [10, Theorem 3] and deduce that the torsion elements of G form a central subgroup T. But Lemma 1 tells us that G/T is nilpotent. Thus, G is nilpotent, and we are finished. □

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