ON THE VANISHING DISCOUNT PROBLEM FROM THE NEGATIVE DIRECTION

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Abstract. It has been proved in [10] that the unique viscosity solution of

$$\lambda u_\lambda + H(x, d_x u_\lambda) = c(H) \qquad \text{in } M, \tag{*}$$

uniformly converges, for $\lambda \to 0^+$, to a specific solution u_0 of the critical equation

$$H(x, d_x u) = c(H) \qquad \text{in } M,$$

where M is a closed and connected Riemannian manifold and $c(H)$ is the critical value. In this note, we consider the same problem for $\lambda \to 0^-$. In this case, viscosity solutions of equation (*) are not unique, in general, so we focus on the asymptotics of the minimal solution u_λ^- of (*). Under the assumption that constant functions are subsolutions of the critical equation, we prove that the u_λ^- also converges to u_0 as $\lambda \to 0^-$. Furthermore, we exhibit an example of H for which equation (*) admits a unique solution for $\lambda < 0$ as well.

1. Introduction and main results

Let M be a connected, closed smooth Riemannian manifold and $H : T^*M \to \mathbb{R}$ a C^3 Tonelli Hamiltonian, where Tonelli refers to the fact that H is strictly convex and superlinear with respect to p. We consider the Hamilton-Jacobi equation:

$$-\lambda u + H(x, d_x u) = c(H) \qquad \text{in } M, \tag{A$_\lambda$}$$

where λ is a positive parameter and $c(H)$ is the critical value, given by [9]

$$c(H) = \inf_{u \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x u).$$

We are interested in understanding the asymptotics of the solution(s) to (A$_\lambda$) as $\lambda \to 0^+$. The problem is well understood when $\lambda \to 0^-$, see [10]: when $\lambda < 0$, in fact, equation (A$_\lambda$) admits a unique viscosity solution and the latter converges, as $\lambda \to 0^-$, to a specific solution of the associated critical equation

$$H(x, d_x u) = c(H) \qquad \text{in } M. \tag{A$_0$}$$

The interest of the result relies on the fact that the critical equation (A$_0$) admits infinite solutions, even up to additive constants in general. We also refer the reader to [7, 10, 15, 16, 18, 19, 23, 30] for related results.

When $\lambda > 0$, the uniqueness of the viscosity solution to (A$_\lambda$) fails. For example, see [27, Example 1.1], the function $u_1 \equiv 0$ and the 1–periodic function u_2 satisfying $u_2(x) = x^2/2$ for $x \in [-1/2, 1/2]$ are both viscosity solutions of the equation

$$-u(x) + \frac{1}{2}|u'(x)|^2 = 0, \quad x \in \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}.$$

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Due to this nonuniqueness phenomenon, we will consider the vanishing discount problem for the minimal viscosity solutions of (A_λ). More precisely, let $\mathcal{S}_\lambda^-$ be the set of viscosity solutions of (A_λ) and denote by

$$u_\lambda^-(x) := \min_{v\in\mathcal{S}_\lambda^-} v(x).$$

The function u_λ^- is a Lipschitz continuous viscosity solution of (A_λ) as well, see [27, Theorem 1.2]. The asymptotic convergence is established under the assumption that constant functions are subsolution of the critical equation (A_0).

Theorem 1.1. *Let us assume that $H(x,0) \le c(H)$ for every $x \in M$. Then u_λ^- converges to u_0^- uniformly on M as $\lambda \to 0^+$, where u_0^- is the unique viscosity solution of* (A_0) *such that $u_0^- \equiv 0$ on the projected Aubry set $\mathcal{A}$ associated with* (A_0).

We point out that u_0^- is the same function that we obtain when we study the asymptotics of the solutions of (A_λ) for $\lambda \to 0^-$, see [10] or Theorem 2.8 below.

When (A_λ) has a unique viscosity solution for each $\lambda < 0$, Theorem 1.1 yields in particular that this solution converges to u_0^- uniformly on M as $\lambda \to 0^+$. It is worth mentioning that uniqueness of viscosity solutions of (A_λ) still holds under certain dynamical assumption. For example, the equation

$$-\lambda u + \frac{1}{2}|d_x u|^2 + U(x) = c, \quad x \in \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}, \quad \lambda > 0, \tag{E_D}$$

where $U : \mathbb{T}^1 \to \mathbb{R}$ is of class C^3 and has a unique maximum point x_0 with $U(x_0) = c$ and $U''(x_0) < 0$, has a unique viscosity solution, when $\lambda > 0$ is small enough, see Section 4 below.

By [27, Proposition 2.8], v is a backward (resp. forward) weak KAM solution of equation (A_λ) if and only if $-v$ is a forward (resp. backward) weak KAM solution of equation:

$$\lambda u + H(x, -d_x u) = c(H) \qquad \text{in } M, \tag{B_λ}$$

where backward weak KAM solutions and viscosity solutions are the same. The same holds for $\lambda = 0$ as well.

Let $\mathcal{S}_\lambda^+$ be the set of forward weak KAM solutions of (B_λ) and denote

$$u_\lambda^+(x) := \sup_{v\in\mathcal{S}_\lambda^+} v(x).$$

Based on the correspondence between viscosity solutions of (A_λ) and forward weak KAM solutions of (B_λ), Theorem 1.1 is equivalent to the following

Theorem 1.2. *Let us assume that $H(x,0) \le c(H)$ for every $x \in M$. Then u_λ^+ converges to u_0^+ uniformly on M as $\lambda \to 0^+$, where u_0^+ is the unique forward weak KAM solution of*

$$H(x, -d_x u) = c(H) \qquad \textit{in } M \tag{B_0}$$

such that $u_0^+ \equiv 0$ on the projected Aubry set $\check{\mathcal{A}}$ associated with (B_0).

We point out that $\check{\mathcal{A}} = \mathcal{A}$, as we will see in Section 2.

Our analysis is based on an extension of some aspects of Aubry-Mather theory to contact Hamiltonian systems, as developed in [25–28], for which discounted Hamilton-Jacobi equations serve as special models. Other output in this vein can be found in a series of papers including [4, 5, 21, 22, 29].

The condition that constant functions are subsolutions of the critical equation (A_0), under which the asymptotic convergence is established, is for instance satisfied whenever the Hamiltonian H is *reversible,* i.e. $H(x,p) = H(x,-p)$ for all $(x,p) \in T^*M$, see for instance Proposition 4.1 below. The model example is the mechanical Hamiltonian $H(x,p) = \frac{|p|_x^2}{2} + U(x)$.

Another example is provided by the Mañé's Hamiltonian $H(x,p) := \frac{1}{2}|p|_x^2 + \langle p, X\rangle_x$, where $X : M \to TM$ is a smooth vector field and $\langle \cdot,\cdot\rangle_x$ denotes the standard inner product in T_x^*M. Clearly, constant functions are solutions of the equation $H(x, d_x u) = 0$ in M.

Other examples of Hamiltonians for which constant functions are critical subsolutions can be obtained in the following way: according to [2], any C^3–Tonelli Hamiltonian H admits a $C^{1,1}$–critical subsolution φ. The new Hamiltonian $\tilde{H}(x,p) := H(x, d_x\varphi + p)$ satisfies $\tilde{H}(x,0) \le c(H) = c(\tilde{H})$ for every $x \in M$. In order to apply our results to $\tilde{H}$, we need the latter to be of class C^3, meaning we need the existence of a critical subsolution of class C^4 on M. This is true under proper dynamical assumptions, see [3], but it is not a general fact, see [2, Appendix A].

2. Generalities

In this paper, we assume the Hamiltonian $H : T^*M \to \mathbb{R}$ to be of class C^3 and to satisfy the following assumptions:

(H1) **(strict convexity)** $\dfrac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite as a quadratic form on T_x^*M, for every $x \in M$;

(H2) **(superlinearity)**

$$\inf_{x\in M} \frac{H(x,p)}{|p|_x} \to +\infty \qquad \text{as } |p|_x \to +\infty.$$

We will denote by $L : TM \to \mathbb{R}$ the Lagrangian associated with H via the Fenchel transform. The Lagrangian L is of class C^3 and satisfies assumptions analogous to (H1)-(H2). We point out that the C^3–regularity is only required in order to apply the results of [25–28].

Let H be a Hamiltonian satisfying the above assumptions and let us consider an Hamilton–Jacobi equation of the form

$$\lambda u + H(x, -d_x u) = a \qquad \text{in } M, \tag{2.1}$$

where $a \in \mathbb{R}$ and $\lambda \ge 0$. For the notion of viscosity (sub-, super-) solution of (2.1), we refer to [1]. Viscosity (sub-, super-) solutions will be always assumed continuous in the sequel, with no further specification. Set $\check{H}(x,p) := H(x,-p)$ and denote by $\check{L}$ the associated Lagrangian.

2.1. Subsolutions and the critical value.

Due to conditions (H1)-(H2), the following equivalence holds, see for instance [10, 11, 13] and references therein.

Proposition 2.1. *Let $\lambda \ge 0$ and $v \in \mathrm{C}(M,\mathbb{R})$. The following are equivalent facts:*

(i) *v is a viscosity subsolution of (2.1);*

(ii) *$v \in \mathrm{Lip}(M,\mathbb{R})$ and it is an almost everywhere subsolution of (2.1), i.e.*

$$\lambda v(x) + H(x, -d_x v) \le a \qquad \textit{for a.e. } x \in M;$$

(iii) *for every absolutely continuous curve* $\gamma : [t_1, t_2] \to M$ *we have*

$$e^{\lambda t_2} v(\gamma(t_2)) - e^{\lambda t_1} v(\gamma(t_1)) \le \int_{t_1}^{t_2} e^{\lambda s} \big(\check{L}(\gamma(s), \dot{\gamma}(s)) + a\big) ds. \tag{2.2}$$

Let $\gamma : [t_1, t_2] \to \mathbb{R}$ be a curve for which (2.2) holds with an equality for a subsolution v of (2.1). When v is differentiable at $x = \gamma(a)$, we have that γ is the projection on M of an integral curve of the discounted flow generated by

$$\begin{cases} \dot{x} = \frac{\partial \check{H}}{\partial p}(x, p), \\ \dot{p} = -\frac{\partial \check{H}}{\partial x}(x, p) - \lambda p. \end{cases} \tag{DH}$$

Namely, $\gamma(t) = \pi\big(\check{\Phi}_\lambda^{t-a}(x, d_x v)\big)$ for all $t \in [t_1, t_2]$, where $\pi : T^*M \to M$ denotes the standard projection and $\check{\Phi}_\lambda^t$ denotes the discounted flow generated by (DH).

When $\lambda > 0$, equation (2.1) admits a unique solution for every $a \in \mathbb{R}$. When $\lambda = 0$, on the other hand, there exists a a unique real constant $c(\check{H})$, hereafter called *critical*, for which the equation

$$H(x, -d_x u) = c(\check{H}) \qquad \text{in } M$$

admits viscosity solutions. Such a critical constant $c(\check{H})$ is also characterized as follows:

$$c(\check{H}) = \min\{a \in \mathbb{R} \mid \exists\, v \in \mathrm{Lip}(M, \mathbb{R}) \text{ such that } \check{H}(x, d_x v) \le a \text{ for a.e. } x \in M\}. \tag{2.3}$$

Since $v \in \mathrm{Lip}(M, \mathbb{R})$ is an a.e. subsolution of $\check{H}(x, d_x v) = a$ in M if and only if $-v$ is an a.e. subsolution of $H(x, d_x v) = a$ in M, it is clear from (2.3) that $c(\check{H}) = c(H)$.

2.2. **Forward weak KAM solutions and minimizing sets.** Let $\check{L} : TM \to \mathbb{R}$ denotes the Lagrangian associated with $\check{H}$ via the Fenchel transform. The following definition is an adaptation to the case of discounted equations of the notion of forward weak KAM solutions, first introduced by Fathi [12] for the non-discounted Hamilton-Jacobi equation and subsequently generalized in [27, Definition 2.2] for fairly general contact Hamiltonian systems.

Definition 2.2. *A function* $v \in C(M, \mathbb{R})$ *is called a forward weak KAM solution of* (B_λ) *if*

(i) *for each continuous piecewise* C^1 *curve* $\gamma : [t_1, t_2] \to M$*, we have*

$$e^{\lambda t_2} v(\gamma(t_2)) - e^{\lambda t_1} v(\gamma(t_1)) \le \int_{t_1}^{t_2} e^{\lambda s} \big(\check{L}(\gamma(s), \dot{\gamma}(s)) + c(H)\big)\, ds; \tag{2.4}$$

(ii) *for each* $x \in M$*, there exists a* C^1 *curve* $\gamma : [0, +\infty) \to M$ *with* $\gamma(0) = x$ *such that*

$$e^{\lambda t} v(\gamma(t)) - v(x) = \int_0^t e^{\lambda s} \big(\check{L}(\gamma(s), \dot{\gamma}(s)) + c(H)\big)\, ds \qquad \textit{for all } t > 0. \tag{2.5}$$

Let $\check{L}_\lambda(x, \dot{x}, u) := -\lambda u + \check{L}(x, \dot{x})$. Curves satisfying (2.5) are called $(v, \check{L}_\lambda, c(H))$-calibrated curves. We denote by $\mathcal{S}_\lambda^+$ the set of forward weak KAM solutions of (B_λ). Based on the forward Lax-Oleinik semigroup introduced in [28], we have

$$\check{T}_{t,\lambda}^+ \varphi(x) = \sup_{\gamma(0)=x} \left\{ e^{\lambda t} \varphi(\gamma(t)) - \int_0^t e^{\lambda s} \big(\check{L}(\gamma(s), \dot{\gamma}(s)) + c(H)\big) ds \right\}, \tag{2.6}$$

Let us denote by u_λ the unique viscosity solution of (B_λ). The following proposition will be employed to show uniqueness of viscosity solutions to(E_D).

Proposition 2.3 ([27]). *Let $v \in \mathcal{S}_\lambda^+$. The following holds:*

(i) *Denote by $\mathcal{I}_v^\lambda := \{x \in M \mid v(x) = u_\lambda(x)\}$. Then both v and u_λ are of class $C^{1,1}$ on $\mathcal{I}_v^\lambda$.*

(ii) *Denote by $\tilde{\mathcal{I}}_v^\lambda := \{(x,p) \in T^*M \mid v(x) = u_\lambda(x),\ p = d_x v = d_x u_\lambda\}$. Then $\tilde{\mathcal{I}}_v^\lambda$ is a non-empty and compact invariant set by the discounted flow $\check{\Phi}_\lambda^t$ generated by $\check{H}$. Furthermore, if we denote by*

$$(x(t), p(t)) := \check{\Phi}_\lambda^t(x_0, p_0) \qquad \text{for all } t \in \mathbb{R}$$

then for each $(x_0, p_0) \in \tilde{\mathcal{I}}_v^\lambda$, we have $p(t) = d_{x(t)} u_\lambda$ for each $t \in \mathbb{R}$.

(iii) *Given $x_0 \in M$, let $\gamma : [0, +\infty) \to M$ be a $(v, \check{L}_\lambda, c(H))$-calibrated curve with $\gamma(0) = x_0$. Let $p_0 := \frac{\partial \check{L}}{\partial \dot{x}}(x_0, \dot{\gamma}(0^+))$, where $\dot{\gamma}(0^+)$ denotes the right derivative of $\gamma(t)$ at $t = 0$. Then $d_{\gamma(t)} v$ exists for every $t > 0$ and*

$$(\gamma(t), d_{\gamma(t)} v) = \check{\Phi}_\lambda^t(x_0, p_0).$$

Furthermore $\omega(x_0, p_0) \subseteq \tilde{\mathcal{I}}_v^\lambda$, where $\omega(x_0, p_0)$ denotes the ω-limit set of (x_0, p_0) with respect to the discounted flow $\check{\Phi}_\lambda^t$.

Let w be a Lipschitz continuous function and set

$$G_w := \overline{\{(x,p) \in T^*M \mid w \text{ is differentiable at } x,\, p = d_x w\}}.$$

By [27, Theorem 1.1], G_{u_λ} and G_v for each $v \in \mathcal{S}_\lambda^+$ are backward and forward invariant by $\check{\Phi}_\lambda^t$, respectively. Define

$$\tilde{\mathcal{A}}^\lambda := \bigcap_{t \geq 0} \check{\Phi}_\lambda^{-t}(G_{u_\lambda}), \qquad \mathcal{A}^\lambda := \pi(\tilde{\mathcal{A}}^\lambda), \tag{2.7}$$

where $\pi : T^*M \to M$ denotes the standard projection. The sets $\tilde{\mathcal{A}}^\lambda$ and $\mathcal{A}^\lambda$ are called Aubry and projected Aubry set associated with $\check{H}_\lambda - c(H)$ where $\check{H}_\lambda(x,p,u) := \lambda u + \check{H}(x,p)$, respectively. According to [27, Theorem 1.1], the set $\tilde{\mathcal{A}}^\lambda$ is invariant under $(\check{\Phi}_\lambda^t)_{t \in \mathbb{R}}$. According to [27, Theorem 1.2], we have the following result.

Theorem 2.4. *The limit $\lim_{t \to +\infty} \check{T}_{t,\lambda}^+ u_\lambda$ exists. Let*

$$u_\lambda^+ := \lim_{t \to +\infty} \check{T}_{t,\lambda}^+ u_\lambda.$$

Then $u_\lambda^+ = \check{T}_{t,\lambda}^+ u_\lambda^+$ for each $t \geq 0$. Moreover, the following holds:

(i) *u_λ^+ is the maximal forward weak KAM solution of (B_λ);*

(ii) *$u_\lambda \geq u_\lambda^+$ in M and $u_\lambda = u_\lambda^+$ on $\mathcal{A}^\lambda$;*

(iii) *$\tilde{\mathcal{A}}^\lambda = \tilde{\mathcal{I}}_{u_\lambda^+}^\lambda$ and $\mathcal{A}^\lambda = \mathcal{I}_{u_\lambda^+}^\lambda$.*

Let $v \in \mathcal{S}_\lambda^+$. We will call *Mather measure* associated with v any Borel $\check{\Phi}_\lambda^t$-invariant probability measures supported in $\tilde{\mathcal{I}}_v^\lambda$. We shall denote by $\mathfrak{M}_v$ the set of such measures. Since $\tilde{\mathcal{I}}_v^\lambda$ is a non-empty and compact invariant set, Krylov-Bogoliubov's theorem, see [20, §II, Theorem I], guarantees that $\mathfrak{M}_v$ is non-empty. The Mather set associated with v is given by

$$\tilde{\mathcal{M}}_v^\lambda = \bigcup_{\mu \in \mathfrak{M}_v} \operatorname{supp}(\mu),$$

where $\mathrm{supp}(\mu)$ denotes the support of μ. A classical argument shows that there exists $\nu \in \mathfrak{M}_v$ such that $\tilde{\mathcal{M}}_v^\lambda = \mathrm{supp}(\nu)$, yielding that $\tilde{\mathcal{M}}_v^\lambda$ is also a non-empty and compact invariant set. Indeed, it suffices to define ν as a convex combination of a dense sequence of measures in $\mathfrak{M}_v$. Following [27], the Mather set of (DH) is defined as

$$\tilde{\mathcal{M}}^\lambda := \tilde{\mathcal{M}}^\lambda_{u_\lambda^+}, \tag{2.8}$$

where u_λ^+ denotes the maximal forward weak KAM solution of (E_I). Such a Mather set $\tilde{\mathcal{M}}^\lambda$ is maximal, in the sense that $\tilde{\mathcal{M}}_v^\lambda \subseteq \tilde{\mathcal{M}}^\lambda_{u_\lambda^+}$ and $v = u_\lambda$ on $\mathcal{M}_v^\lambda$ for each $v \in \mathcal{S}_\lambda^+$. This is a straightforward consequence of the fact that $\tilde{\mathcal{I}}_v^\lambda \subseteq \tilde{\mathcal{I}}^\lambda_{u_\lambda^+}$ and $v = u_\lambda$ on $\mathcal{I}_v^\lambda$ in view of Proposition 2.3 and Theorem 2.4.

The following holds.

Proposition 2.5. *Let $v \in \mathcal{S}_+$, $x_0 \in M$ and $\gamma : [0, +\infty) \to M$ be a $(v, \check{L}_\lambda, c(H))$-calibrated curve with $\gamma(0) = x_0$. Let $p_0 := \frac{\partial \check{L}}{\partial \dot{x}}(x_0, \dot{\gamma}(0^+))$, where $\dot{\gamma}(0^+)$ denotes the right derivative of $\gamma(t)$ at $t = 0$. Then*

$$\omega(x_0, p_0) \cap \tilde{\mathcal{M}}_v^\lambda \neq \emptyset,$$

where $\omega(x_0, p_0)$ denotes the ω-limit set of (x_0, p_0) with respect to the contact Hamiltonian flow $\check{\Phi}_\lambda^t$.

Proof. Let us assume by contradiction that $\omega(x_0, p_0) \cap \tilde{\mathcal{M}}_v^\lambda = \emptyset$. Since $\omega(x_0, p_0)$ is a non-empty, compact and invariant subset of $\tilde{\mathcal{I}}_v^\lambda$, see Proposition 2.3, by applying Krylov-Bogoliubov's theorem [20, §II, Theorem I] we would find a new Mather measure supported on $\omega(x_0, p_0)$, in contradiction with the very definition of $\tilde{\mathcal{M}}_v^\lambda$. □

As a consequence, we derive the following result.

Proposition 2.6. *Let v_1, v_2 be forward weak KAM solutions of (B_λ). If $v_1|_\mathcal{O} = v_2|_\mathcal{O}$, where $\mathcal{O}$ denotes a neighborhood of $\mathcal{M}_{v_1}$, then $v_1 \leq v_2$ in M.*

Proof. Pick $x_0 \in M$ and let $\gamma : [0, +\infty) \to M$ be a $(v_1, \check{L}_\lambda, c(H))$-calibrated curve with $\gamma(0) = x_0$. Let $p_0 := \frac{\partial \check{L}}{\partial \dot{x}}(x_0, \dot{\gamma}(0^+))$, where $\dot{\gamma}(0^+)$ denotes the right derivative of $\gamma(t)$ at $t = 0$.

Based on Proposition 2.5, there exists a $t > 0$ such that $\gamma(t) \in \mathcal{O}$. From the hypothesis that $v_1 = v_2$ on $\mathcal{O}$ we infer that $v_1(\gamma(t)) = v_2(\gamma(t))$. From the fact that $v_i = T^+_{t,\lambda} v_i$ on M for $i \in \{1, 2\}$ and (2.6) we get

$$e^{\lambda t} v_1(\gamma(t)) - v_1(x_0) = \int_0^t e^{\lambda s}\big(\check{L}(\gamma(s), \dot{\gamma}(s)) + c(H)\big) ds \geq e^{\lambda t} v_2(\gamma(t)) - v_2(x_0),$$

yielding $v_1(x_0) \leq v_2(x_0)$ since $v_1(\gamma(t)) = v_2(\gamma(t))$. The assertion follows since x_0 was arbitrarily chosen in M. □

2.3. **Viscosity solutions of** (B_0)**.** Viscosity solutions to equation (B_0) are not unique, even up to additive constants in general. A uniqueness set for equation (B_0) is given by the so-called *projected Aubry set* $\check{\mathcal{A}}$: it is a closed subset of M that can be characterized by the following property, see [13]:

$$y \in \check{\mathcal{A}} \qquad \text{iff} \qquad \text{any subsolution } v \text{ of } (\mathrm{B}_0) \text{ is differentiable at } y. \tag{$\mathcal{A}$}$$

Since v is a subsolution of (B_0) if and only if $-v$ is a subsolution of (A_0), we easily derive that $\check{\mathcal{A}} = \mathcal{A}$, where $\mathcal{A}$ is the projected Aubry set associated with equation

(A_0). In the sequel, we will always write $\mathcal{A}$ in place of $\check{\mathcal{A}}$. The following properties hold, see for instance [12–14]:

Proposition 2.7.

(i) *Let* $y \in \mathcal{A}$. *Then* $H(y, -d_y v) = c(H)$ *and* $d_y v = d_y w$ *for any pair* v, w *of subsolutions to* (B_0).

(ii) *Let* $y \in \mathcal{A}$. *Then there exists a unique curve* $\gamma : \mathbb{R} \to \mathcal{A} \subseteq M$ *with* $\gamma(0) = y$ *such that*

$$v(\gamma(b)) - v(\gamma(a)) = \int_a^b \big(\check{L}(\gamma(s), \dot{\gamma}(s)) + c(H)\big) ds \qquad \text{for every } a < b,$$

for any subsolution v *of* (B_0).

(iii) *Let* u, v *be viscosity solutions of* (B_0). *If* $u = v$ *on* $\mathcal{A}$, *then* $u = v$ *on* M.

We end this section by recalling the following result proved in [10], see Propositions 1.4 and 4.4 therein.

Theorem 2.8 ([10]). *For each* $\lambda > 0$, *let* u_λ *be the viscosity solution of* (B_λ). *Let us assume that* $H(x, 0) \leq c(H)$ *for every* $x \in M$. *Then* $u_\lambda \geq 0$ *in* M *for every* $\lambda > 0$ *and* $u_\lambda \nearrow u_0$ *uniformly on* M *as* $\lambda \searrow 0$, *where* u_0 *is the unique viscosity solution of* (B_0) *such that* $u_0 \equiv 0$ *on* $\mathcal{A}$.

As a corollary we infer

Corollary 2.9. *For each* $\lambda > 0$, *the function* u_λ *is a viscosity subsolution of* (B_0). *In particular*

(i) $u_\lambda \equiv 0$ *on* $\mathcal{A}$;

(ii) u_λ *is differentiable on* $\mathcal{A}$ *and* $d_x u_\lambda \equiv 0$ *on* $\mathcal{A}$.

Proof. From the fact that $u_\lambda \geq 0$ in M we get

$$H(x, -d_x u_\lambda) \leq \lambda u_\lambda + H(x, -d_x u_\lambda) = c(H) \qquad \text{in } M$$

in the viscosity sense, namely u_λ is a viscosity subsolution of (B_0). Item (i) follows from the inequality $0 \leq u_\lambda \leq u_0$ on M and the fact that $u_0 \equiv 0$ on $\mathcal{A}$, while item (ii) follows directly from Proposition 2.7 and the fact that any constant function is a subsolution of (B_0). □

3. Asymptotic convergence

This section is devoted to the proof of Theorem 1.2. We prove some preliminary results first.

Lemma 3.1. *The family* $\{u^+_\lambda\}_{\lambda \in (0,1]}$ *is equi-bounded and equi-Lipschitz continuous.*

Proof. First, we prove the equi-bounded character of $\{u^+_\lambda\}_{\lambda \in (0,1]}$. By Theorem 2.4, $u^+_\lambda \leq u_\lambda$ on M for each $\lambda \in [0, 1]$. By Theorem 2.8, u_λ converges uniformly to a function $u_0 \in \mathrm{C}(M, \mathbb{R})$, in particular there exists $C > 0$ such that $\|u_\lambda\|_\infty \leq C$ for each $\lambda \in [0, 1]$. It follows that the $\{u^+_\lambda\}_{\lambda \in (0,1]}$ are equi-bounded from above. Let us prove they are equi-bounded from below. By Proposition 2.3, we have $u^+_\lambda = \check{T}^+_{t,\lambda} u^+_\lambda$ for each $t \geq 0$. Take $x_0 \in \mathcal{I}_{u^+_\lambda} = \{x \in M \mid u_\lambda(x) = u^+_\lambda(x)\}$. Let $\alpha : [0, 1] \to M$ be a geodesic connecting x to x_0, parameterized by constant speed $|\dot\alpha(s)| := d(x, x_0) \leq \mathrm{diam}(M)$ for each $s \in [0,1]$. Let

$$C' := \max_{x\in M, |\dot x|\leq \mathrm{diam}(M)} \big(\check L(x,\dot x) + c(H)\big).$$

By (2.6),

$$u_\lambda^+(x) = \check T^+_{1,\lambda} u_\lambda^+(x) \geq e^\lambda u_\lambda(x_0) - \int_0^1 e^{\lambda s}\big(\check L(\alpha(s),\dot\alpha(s)) + c(H)\big)\,ds \geq e^\lambda\left(u_\lambda(x_0) - C'\right),$$

which implies u_λ^+ is bounded from below for each $\lambda \in (0,1]$. Thus, there exists $K > 0$ such that $\|u_\lambda^+\|_\infty \leq K$ for all $\lambda \in (0,1]$.

Next, we prove the equi-Lipschitz continuity. For each $x, y \in M$, let $\beta : [0, d(x,y)] \to M$ be a geodesic of length $d(x,y)$, parameterized by arclength and connecting x to y. Let

$$C'' := \sup\{\check L(x,\dot x) + c(H) \mid x \in M,\ \|\dot x\|_x = 1\}.$$

In view of (2.4), we have

$$e^{\lambda d(x,y)} v(x) - e^{\lambda d(x,y)} v(y) \leq v(y)(1 - e^{\lambda d(x,y)}) + \frac{C''}{\lambda}(e^{\lambda d(x,y)} - 1),$$

which yields from $0 < \lambda \leq 1$ and $1 - e^{-h} \leq h$ for all $h \in \mathbb{R}$,

$$\begin{aligned} v(x) - v(y) &\leq v(y)(e^{-\lambda d(x,y)} - 1) + \frac{C''}{\lambda}(1 - e^{-\lambda d(x,y)}) \\ &\leq (C'' + K)\frac{1 - e^{-\lambda d(x,y)}}{\lambda} \\ &\leq (C'' + K) d(x,y). \end{aligned}$$

We complete the proof by exchanging the roles of x and y. □

Next, we show the following result.

Proposition 3.2. *Let us assume that $H(x,0) \leq c(H)$ for every $x \in M$. Then projected Aubry set $\mathcal{A}$ associated with* (B$_0$) *is contained in $\mathcal{A}^\lambda$ for every $\lambda > 0$.*

Remark 3.3. *From the previous proposition and the fact that $d_x u_\lambda \equiv 0$ on $\mathcal{A}$ in view of Corollary 2.9, we get in particular that*

$$\{(y,0) \in T^*M \mid y \in \mathcal{A}\} \subseteq \tilde{\mathcal{A}}^\lambda.$$

Proof. Let us fix $\lambda > 0$ and pick $x \in \mathcal{A}$. According to Corollary 2.9, the function u_λ is differentiable at x, so $(x, d_x u_\lambda) \in G_{u_\lambda}$, and by Proposition 2.7 there exists a unique curve $\gamma : \mathbb{R} \to \mathcal{A} \subseteq M$ such that $\gamma(0) = x$ and

$$u_\lambda(\gamma(b)) - u_\lambda(\gamma(a)) = \int_a^b \big(\check L(\gamma(s),\dot\gamma(s)) + c(H)\big)\,ds \qquad \text{for all } a < b.$$

In particular,

$$\check L(\gamma(s),\dot\gamma(s)) + c(H) = \langle d_{\gamma(s)} u_\lambda, \dot\gamma(s)\rangle = \frac{d}{ds} u_\lambda(\gamma(s)) \qquad \text{for all } s \in \mathbb{R}.$$

By multiplying the above equality by $e^{\lambda s}$ and by integrating (by parts) in (a,b) we get

$$\int_a^b e^{\lambda s}\big(\check L(\gamma(s),\dot\gamma(s)) + c(H)\big)\,ds = e^{\lambda b} u_\lambda(\gamma(b)) - e^{\lambda a} u_\lambda(\gamma(a)) - \int_a^b \left(e^{\lambda s}\right)' u_\lambda(\gamma(s))\,ds,$$

hence, by taking into account that $\gamma(\mathbb{R}) \subseteq \mathcal{A}$ and $u_\lambda \equiv 0$ on $\mathcal{A}$, we get

$$e^{\lambda b} u_\lambda(\gamma(b)) - e^{\lambda a} u_\lambda(\gamma(a)) = \int_a^b e^{\lambda s}\big(\check{L}(\gamma(s), \dot\gamma(s)) + c(H)\big)\, ds.$$

This shows that γ is the projection on M of an integral curve of the flow $\check\Phi^t_\lambda$ generated by (DH), i.e. $\gamma(t) = \pi\big(\check\Phi^t_\lambda(x, d_x u_\lambda)\big)$ for all $t \in \mathbb{R}$. In particular, we get that $x \in \pi\big(\check\Phi^{-t}_\lambda(G_{u_\lambda})\big)$ for all $t \geq 0$, i.e. $x \in \mathcal{A}^\lambda$, as it was asserted. □

Proof of Theorem 1.2. In view of Lemma 3.1 and of Ascoli-Arzelá's theorem, it is enough to show that, if $u^+_{\lambda_n}$ converges to u_* uniformly on M as $\lambda_n \to 0^+$, then $u_* = u^+_0$ on M. In view of the correspondence between forward and backward, or viscosity, solutions, see [27, Proposition 2.8], we know that $-u^+_{\lambda_n}$ is a viscosity solution of (A_λ). By the stability of the notion of viscosity solution, $-u_*$ is a viscosity solution of (A_0), which means that u_* is a forward weak KAM solution of (B_0). Furthermore, by Corollary 2.9, Theorem 2.4 and Proposition 3.2, we know that $u^+_{\lambda_n} \equiv 0$ on $\mathcal{A}$, hence $u_* \equiv 0$ on $\mathcal{A}$. Hence, $-u_*$ and $-u^+_0$ are both viscosity solutions of (A_0) with $-u_* \equiv -u^+_0$ on $\mathcal{A}$. We conclude that $u_* \equiv u^+_0$ on M by Proposition 2.7. □

4. On the example (E_D)

By the recalled equivalence between viscosity, or backward weak KAM, solutions of (A_λ) and forward solutions of (B_λ), see [27, Proposition 2.8], it suffices to show the uniqueness of the forward weak KAM solution of

$$\lambda u + \frac{1}{2}|d_x u|^2 + U(x) = c, \quad x \in \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}, \quad \lambda > 0, \tag{E_I}$$

where $\mathbb{T}^1$ is a flat circle with the standard metric. For any two points $x, y \in M$, we use $|x - y|$ to denote the distance induced by the flat metric on $\mathbb{T}^1$. We recall that $U : \mathbb{T}^1 \to \mathbb{R}$ is of class C^3 and has a unique maximum point x_0 with $U(x_0) = c$, which is furthermore assumed to be non-degenerate, i.e. $U''(x_0) < 0$. When $U(x) = \cos(2\pi x)$, (E_I) corresponds to the dissipative pendulum.

To obtain this uniqueness result, we need some preliminary material that we will develop in the next section.

4.1. **Some preliminary facts.**

We start by the following known fact about reversible Hamiltonians.

Proposition 4.1. *Let us assume that H is reversible, i.e. $H(x,p) = H(x,-p)$ for all $(x,p) \in T^*M$. Then*

(i) *$H(x,p) > H(x,0)$ for every $x \in M$ and $|p| \neq 0$, moreover $c(H) = \max_{x\in M} H(x,0)$, in particular, $H(x,0) \leq c(H)$ for every $x \in M$;*
(ii) *the projected Aubry set $\mathcal{A}$ associated with (A_0) is given by*

$$\mathcal{A} = \{y \in M \mid H(y,0) = c(H)\};$$

Proof. Let us prove (i). Since H is reversible, we have $\frac{\partial H}{\partial p}(x,p) = -\frac{\partial H}{\partial p}(x,-p)$ for each $(x,p) \in T^*M$. In particular, $\frac{\partial H}{\partial p}(x,0) = 0$. Combining with $\frac{\partial^2 H}{\partial p^2}(x,p) > 0$, we have for each $x \in M$,

$$H(x,p) > H(x,0) = \min_{p \in T^*_x M} H(x,p) \qquad \text{for all } x \in M \text{ and } p \in T^*_x M \setminus \{0\}. \tag{4.1}$$

Let us set $c := \max_{x\in M} H(x,0)$. Since any constant function v on M is a subsolution of $H(x, d_x v) = c$ in M, we have $c(H) \leq c$ in view of (2.3). On the other hand, if v is subsolution of $H(x, d_x v) = c(H)$ in M, we have in particular

$$H(x,0) = \min_{p\in T^*_x M} H(x,p) \leq H(x, d_x v) \leq c(H) \qquad \text{for a.e. } x \in M,$$

yielding $c = \max_{x\in M} H(x,0) \leq c(H)$. This shows that $\max_{x\in M} H(x,0) = c(H)$.

(ii) Let us denote by $\mathcal{E}$ the set appearing at the right-hand side of the equality in (ii). The function $v_0 \equiv 0$ satisfies $H(x, d_x v_0) \leq c(H)$ for every $x \in M$, with a strict inequality holding when $x \notin \mathcal{E}$. This shows that $\mathcal{A} \subseteq \mathcal{E}$ in view of Proposition 2.7. Let now v be a subsolution of $H(x, d_x v) = c(H)$ in M. Then the Clarke generalized gradient $\partial^c v(x)$ of v at x satisfies $\partial^c v(x) \subseteq \{p \in T^*_x M \mid H(x,p) \leq c(H)\}$ for every $x \in M$, see for instance [24]. In particular, it is a singleton whenever $x \in \mathcal{E}$. This implies that v is (strictly) differentiable at any $x \in \mathcal{E}$, see [8, Proposition 2.2.4]. This show that $\mathcal{E} \subseteq \mathcal{A}$ by ($\mathcal{A}$). $\square$

We focus now on the properties enjoyed by the Mather set $\tilde{\mathcal{M}}^\lambda$ associated with equation (E_I). We start with a general fact for reversible Hamiltonians.

Proposition 4.2. *Let us assume that H is reversible and let μ be a $\check{\Phi}^t_\lambda$-invariant probability measure on T^*M. Then*

$$supp(\mu) \subseteq \{(x,0) \in T^*M \mid x \in M\}.$$

Proof. Fix $(x,p) \in T^*M$. For every $t \in \mathbb{R}$ we have

$$\frac{d}{dt} H\left(\check{\Phi}^t_\lambda(x,p)\right) = -\lambda \langle p(t), \frac{\partial \check{H}}{\partial p}\left(\check{\Phi}^t_\lambda(x,p)\right)\rangle \leq 0$$

by convexity, with equality holding if and only if $p(t) = 0$ in view of (4.1). If μ is $\check{\Phi}^t_\lambda$-invariant, we infer

$$0 = \int_{T^*M} H\left(\check{\Phi}^1_\lambda(x,p)\right)\, d\mu(x,p) - \int_{T^*M} H\left(\check{\Phi}^t_\lambda(x,p)\right)\, d\mu(x,p)$$
$$= \int_0^1 \left(\int_{T^*M} \frac{d}{ds} H\left(\check{\Phi}^s_\lambda(x,p)\right)\, d\mu(x,p)\right) ds = -\lambda \int_{T^*M} \langle p, \frac{\partial \check{H}}{\partial p}(x,p)\, d\mu(x,p),$$

yielding

$$\langle p, \frac{\partial \check{H}}{\partial p}(x,p)\rangle = 0 \qquad \text{for } \mu\text{–a.e. } (x,p) \in T^*M.$$

The assertion follows in view of (4.1). $\square$

We exploit the previous result to derive the following information.

Proposition 4.3. *Let $\tilde{\mathcal{M}}^\lambda$ be the Mather set associated with equation* (E_I)*, where $U : \mathbb{T}^1 \to \mathbb{R}$ is of class C^3 and has a unique maximum point x_0 with $U(x_0) = c$, which is furthermore assumed to be non-degenerate, i.e. $U''(x_0) < 0$. Then $(x_0, 0)$ is a hyperbolic fixed point for the discounted flow* (DH) *and, for $\lambda > 0$ small enough, $\tilde{\mathcal{M}}^\lambda = \{(x_0, 0)\}$.*

Proof. The fact that $(x_0, 0)$ a hyperbolic fixed point for the discounted flow (DH) is easily checked. For every fixed $\lambda > 0$, let us pick $(x_\lambda, 0) \in \tilde{\mathcal{M}}^\lambda$ and set $(x_\lambda(t), p_\lambda(t)) :=$

$\check{\Phi}^s_\lambda(x_\lambda, 0)$. By taking into account Proposition 4.2 together with $\tilde{\mathcal{M}}^\lambda \subseteq \tilde{\mathcal{I}}_{u_\lambda}$ and Proposition 2.3, we infer that

$$0 = p_\lambda(t) = d_{x_\lambda(t)} u_\lambda, \quad \dot{x}_\lambda(t) = p_\lambda(t) = 0, \quad 0 = \dot{p}_\lambda(t) = -\frac{\partial U}{\partial x}(x_\lambda(t), p_\lambda(t))$$

for all $t \in \mathbb{R}$, namely $x_\lambda(t) = x_\lambda$ for all $t \in \mathbb{R}$ and x_λ is a critical point for U. Furthermore, the static curve $x_\lambda(t) = x_\lambda$ for all $t \in \mathbb{R}$ is optimal for $u_\lambda(x_\lambda)$, i.e. for any $a \leq b$,

$$e^{\lambda b} u_\lambda(x_\lambda(b)) - e^{\lambda a} u_\lambda(x_\lambda(a)) = \int_a^b e^{\lambda s} \big(\check{L}(x_\lambda(s), \dot{x_\lambda}(s)) + c\big) ds.$$

By taking $b = 0$ and $a \to -\infty$ we get

$$u_\lambda(x_\lambda) = \int_{-\infty}^0 e^{\lambda s} \big(\check{L}(x_\lambda, 0) + c\big)\, ds = \frac{c - U(x_\lambda)}{\lambda} \geq 0.$$

Since u_λ is converging to u_0 as $\lambda \to 0^+$ by Theorem 2.8, we necessarily have that $x_\lambda \to x_0$ as $\lambda \to 0$. Since x_0 is an isolated critical point by the non-degeneracy condition, we infer that there exists $\lambda_0 > 0$ such that $x_\lambda = x_0$ for every $\lambda \leq \lambda_0$. □

Remark 4.4. *In the dissipative pendulum case, i.e. when* $U(x) = \cos(2\pi x)$*, the statement of Proposition 4.3 holds for any* $\lambda > 0$*. This follows as a consequence of [21, Example 2] together with Theorem 2.4-(iii). This is not true for a potential* U *of more general form. In fact, let* $w(x)$ *be a smooth function on* $\mathbb{T}^1$*. Let* x_0 *be its unique global minimum point with* $w(x_0) = 0$*,* $w''(x_0) > 0$ *and let* x_1 *be another critical point such that* $w'(x_1) = 0$ *and* $w(x_1) > 0$*. Set* $U(x) := -w(x) - \frac{1}{2}|d_x w|^2$*. Then* x_0 *is the unique global maximum point of* $U(x)$ *with* $U(x_0) = 0$*,* $U''(x_0) < 0$*. For* $\lambda_0 = 1$*,* w *is a smooth solution (hence both forward and backward weak KAM solution) of*

$$\lambda_0 u + \frac{1}{2}|d_x u|^2 + U(x) = 0.$$

However, one has $\{(x_0, 0), (x_1, 0)\} \subseteq \tilde{\mathcal{M}}^{\lambda_0}$*.*

4.2. **Uniqueness of the forward weak KAM solution.** Let us now come back to the analysis of equation (E_I). In view of Proposition 4.3, the uniqueness of the forward weak KAM solution to equation (E_I) is a consequence of the following more general result.

Proposition 4.5. *Let* $H : T^*\mathbb{T}^1 \to \mathbb{R}$ *be a* C^3*–Hamiltonian, satisfying hypotheses (H1)-(H2). Let us assume that the Mather set* $\tilde{\mathcal{M}}^\lambda$ *associated with the discounted Hamilton-Jacobi equation*

$$\lambda u + H(x, -d_x u) = c(H) \qquad \text{in } \mathbb{T}^1 \tag{4.2}$$

reduces to $\{(x_0, 0)\}$ *and that* $(x_0, 0)$ *is a hyperbolic fixed point for the discounted flow generated by* (DH)*. Then equation* (4.2) *admits a unique forward weak KAM solution.*

Proof. Let us denote by $\mathcal{S}^+_\lambda$ be the set of all forward weak KAM solutions of (4.2). For each $v \in \mathcal{S}^+_\lambda$, we have $\tilde{\mathcal{M}}^\lambda_v \subseteq \tilde{\mathcal{M}}^\lambda$. Since $\tilde{\mathcal{M}}^\lambda_v$ is nonempty, we necessarily have

$$\tilde{\mathcal{M}}^\lambda_v = \{(x_0, 0)\}.$$

This means $v(x_0) = 0$ and $d_{x_0} v = 0$.

We denote for simplicity, the stable submanifold of $(x_0, 0)$ with respect to $\check{\Phi}^t_\lambda$ by

$$W^s(x_0,0) := \{(x,p) \in T^*\mathbb{T}^1 \mid \lim_{t\to+\infty} d(\check{\Phi}^t_\lambda(x,p),(x_0,0)) = 0\},$$

where $d(\cdot,\cdot)$ is a Riemannian metric on $T^*\mathbb{T}^1$. Given $\varepsilon > 0$, we denote the local stable submanifold of $(x_0, 0)$ with respect to $\check{\Phi}^t_\lambda$ by

$$W^s_\varepsilon(x_0,0) := \{(x,p) \in W^s(x_0,0) \mid d(\check{\Phi}^t_\lambda(x,p),(x_0,0)) < \varepsilon,\ \forall t \geq 0\}.$$

Since $\check{H} \in C^3$, by the stable manifold theorem [17], there exist $\delta > 0$, $h \in C^2$ with $h(x_0) = 0$ such that

$$[x_0-\delta, x_0+\delta] \subseteq \pi\big(W^s_\varepsilon(x_0,0), \quad W^s_\varepsilon(x_0,0) = \{(x,h(x)) \mid x \in [x_0-\delta,x_0+\delta]\},$$

where $\pi : T^*\mathbb{T}^1 \to \mathbb{T}^1$ denotes the standard projection.

For each $v \in \mathcal{S}^+_\lambda$, v is Lipschitz continuous. Furthermore, it is semiconvex. Indeed, $-v$ is a solution to

$$H(x, d_x u) = c - \lambda v(x) \qquad \text{in } \mathbb{T}^1,$$

so it is semiconcave in view of the results in [6]. Denote $\mathcal{D}$ be the set of all differentiable points of v. Pick $\bar{x} \in \mathcal{D} \cap (x_0 - 1, x_0)$ and set $\bar{p} := d_{\bar{x}} v$. Let $(x(t), p(t)) := \check{\Phi}^t_\lambda(\bar{x}, \bar{p})$ for all $t \geq 0$. By Proposition 2.3, $d_{x(t)}v = p(t)$ for all $t \geq 0$. In view of Proposition 2.5, $(x_0, 0) \in \omega(\bar{x}, \bar{p})$. Thus, there exists a diverging sequence $(t_n)_n$ such that either $x(t_n) \to x_0$ or $x(t_n) \to x_0 - 1$ as $t_n \to +\infty$. We assert that the map $t \mapsto x(t)$ is monotone in $[0, +\infty)$. Let us assume for definiteness that $x(t_n) \to x_0$. We claim that $\dot{x}(t) \geq 0$ for all $t > 0$. In fact, let us assume by contradiction that $\dot{x}(t_0) < 0$ for some $t_0 > 0$. By the fact that $x(t_n) \to x_0$, one can find $t_1 > t_0$ such that $x(t_0) = x(t_1) =: \hat{x}$ and $\dot{x}(t_1) \geq 0$, a contradiction to the fact that

$$\dot{x}(t_0) = \frac{\partial \check{H}}{\partial p}(\hat{x}, d_{\hat{x}} v) = \dot{x}(t_1).$$

The monotonicity of $t \mapsto x(\cdot)$ implies that $x(t) \to x_0$. Furthermore, v is differentiable at x_0 and at $x(t)$ for every $t > 0$. By semiconvexity, we infer that $d_{x(t)}v \to d_{x_0}v = 0$ as $t \to +\infty$, i.e.

$$\check{\Phi}^t_\lambda(\bar{x}, \bar{p}) \to (x_0, 0) \qquad \text{as } t \to +\infty.$$

That implies $(\bar{x}, \bar{p}) \in W^s(x_0, 0)$. For each $\tilde{x} \in \mathcal{D} \cap [x_0 - \delta, x_0 + \delta]$, we have $(\tilde{x}, d_{\tilde{x}} v) \in W^s_\varepsilon(x_0, 0)$. Moreover, $d_x v = h$ on $\mathcal{D} \cap [x_0 - \delta, x_0 + \delta]$. Note that $\mathcal{D}$ has full Lebesgue measure on $\mathbb{T}^1$. It follows that, for each $x \in [x_0 - \delta, x_0 + \delta]$,

$$v(x) = \int_{x_0}^x d_y v(y)dy = \int_{x_0}^x h(y)dy.$$

This shows $d_x v = h$ on $[x_0 - \delta, x_0 + \delta]$.

For any $v_1, v_2 \in \mathcal{S}^+_\lambda$, we have $v_1(x_0) = v_2(x_0) = 0$ and $d_x v_1 = d_x v_2 = h$ on $[x_0 - \delta, x_0 + \delta]$. That yields $v_1 = v_2$ on $[x_0 - \delta, x_0 + \delta]$. By Proposition 2.6, $v_1 = v_2$ on $\mathbb{T}^1$, namely (4.2) has a unique forward weak KAM solution. □

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