http://www.aimspress.com/journal/mine

Mathematics in Engineering, 3(3): 1–15.
DOI:10.3934/mine.2021022
Received: 22 June 2019
Accepted: 12 October 2019
Published: 15 July 2020

Research article

Existence of nonradial positive and nodal solutions to a critical Neumann problem in a cone†

Mónica Clapp[1,*] **and Filomena Pacella**[2]

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 Coyoacán, CDMX, Mexico

[2] Dipartimento di Matematica, Sapienza Università di Roma, P.le. Aldo Moro 2, 00185 Roma, Italy

† **This contribution is part of the Special Issue:** Critical values in nonlinear pdes – Special Issue dedicated to Italo Capuzzo Dolcetta

Guest Editor: Fabiana Leoni

Link: `www.aimspress.com/mine/article/5754/special-articles`

* **Correspondence:** Email: monica.clapp@im.unam.mx.

Abstract: We study the critical Neumann problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Sigma_\omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma_\omega, \end{cases}$$

in the unbounded cone $\Sigma_\omega := \{tx : x \in \omega \text{ and } t > 0\}$, where ω is an open connected subset of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ with smooth boundary, $N \geq 3$ and $2^* := \frac{2N}{N-2}$. We assume that some local convexity condition at the boundary of the cone is satisfied. If ω is symmetric with respect to the north pole of $\mathbb{S}^{N-1}$, we establish the existence of a nonradial sign-changing solution. On the other hand, if the volume of the unitary bounded cone $\Sigma_\omega \cap B_1(0)$ is large enough (but possibly smaller than half the volume of the unit ball $B_1(0)$ in $\mathbb{R}^N$), we establish the existence of a positive nonradial solution.

Keywords: semilinear elliptic equation; critical nonlinearity; conical domain; Neumann boundary condition; nonradial solution

1. Introduction

We consider the Neumann problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Sigma_\omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma_\omega, \end{cases} \tag{1.1}$$

in the unbounded cone $\Sigma_\omega := \{tx : x \in \omega \text{ and } t > 0\}$, where ω is an open connected subset of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ with smooth boundary, $N \geq 3$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

It is well known that, if $\omega = \mathbb{S}^{N-1}$, i.e., if Σ_ω is the whole space $\mathbb{R}^N$, then the only positive solutions to the critical problem

$$-\Delta w = |w|^{2^*-2}w, \qquad w \in D^{1,2}(\mathbb{R}^N), \tag{1.2}$$

are the rescalings and translations of the standard bubble U defined in (2.3). Moreover, they are the only nontrivial radial solutions to (1.2), up to sign. It is immediately deduced that, up to sign, the restriction of the bubbles (3.1) to Σ_ω are the only nontrivial radial solutions of (1.1) in any cone; see Proposition 3.4. In addition, if the cone Σ_ω is convex, it was shown in [8, Theorem 2.4] that these are the only positive solutions to (1.1). The convexity property of the cone is crucial in the proof of this result, and it is strongly related to a relative isoperimetric inequality obtained in [7].

The aim of this paper is to establish the existence of nonradial solutions to (1.1), both positive and sign-changing. As mentioned above, the positive ones can only exist in nonconvex cones. On the other hand, nodal radial solutions to (1.1) do not exist, as this would imply the existence of a nontrivial solution to problem (2.5) in the bounded cone $\Lambda_\omega := \{tx : x \in \omega \text{ and } t \in (0, 1)\}$, which is impossible because of the Pohozhaev identity (2.6) and the unique continuation principle.

For the problem (1.2) in $\mathbb{R}^N$ various types of sign-changing solutions are known to exist; see [2–5]. In particular, a family of entire nodal solutions, which are invariant under certain groups of linear isometries of $\mathbb{R}^N$, were exhibited in [2]. These solutions arise as blow-up profiles of symmetric minimizing sequences for the critical equation in a ball, and are obtained through a fine analysis of the concentration behavior of such sequences.

Here we use some ideas from [2] to produce sign-changing solutions to (1.1), but we exploit a different kind of symmetry. Our main result shows that, if ω is symmetric with respect to the north pole of $\mathbb{S}^{N-1}$ and if the cone Σ_ω has a point of convexity in the sense of Definition 2.6, then the problem (1.1) has an axially antisymmetric least energy solution, which is nonradial and changes sign; see Theorem 2.8. As far as we know, this is the first existence result of a nodal solution to (1.1).

Next, we investigate the existence of positive nonradial solutions. In this case we do not require the cone to have any particular symmetry. We establish the existence of a positive nonradial solution to (1.1) under some conditions involving the local convexity of Σ_ω at a boundary point and the measure of the bounded cone Λ_ω; see Corollary 3.5 and Theorem 3.6. We refer to Section 3 for the precise statements and further remarks.

2. A nonradial sign-changing solution

If Ω is a domain in $\mathbb{R}^N$ we consider the Sobolev space

$$D^{1,2}(\Omega) := \{u \in L^{2^*}(\Omega) : \nabla u \in L^2(\Omega, \mathbb{R}^N)\}$$

with the norm
$$\|u\|_\Omega^2 := \int_\Omega |\nabla u|^2.$$
We denote by $J_\Omega : D^{1,2}(\Omega) \to \mathbb{R}$ the functional given by
$$J_\Omega(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*},$$
and its Nehari manifold by
$$\mathcal{N}(\Omega) := \left\{ u \in D^{1,2}(\Omega) : u \neq 0 \ \text{ and } \ \int_\Omega |\nabla u|^2 = \int_\Omega |u|^{2^*} \right\}.$$
For $u \in D^{1,2}(\Omega) \smallsetminus \{0\}$ let $t_u \in (0,\infty)$ be such that $t_u u \in \mathcal{N}(\Omega)$. Then,
$$J_\Omega(t_u u) = \frac{1}{N}[Q_\Omega(u)]^{\frac{N}{2}}, \qquad \text{where } \ Q_\Omega(u) := \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{2^*}\right)^{2/2^*}}. \tag{2.1}$$
Hence,
$$c_\Omega := \inf_{u\in\mathcal{N}(\Omega)} J_\Omega(u) = \inf_{u\in D^{1,2}(\Omega)\smallsetminus\{0\}} \frac{1}{N}[Q_\Omega(u)]^{\frac{N}{2}}. \tag{2.2}$$
We set $c_\infty := c_{\mathbb{R}^N}$. It is well known that this infimum is attained at the function
$$U(x) = a_N\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}}, \qquad a_N := (N(N-2))^{\frac{N-2}{4}}, \tag{2.3}$$
which is called the standard bubble, and at every rescaling and translation of it, and that
$$c_\infty = J_{\mathbb{R}^N}(U) = \frac{1}{N}S^{\frac{N}{2}},$$
where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ and let ω be a smooth domain in $\mathbb{S}^{N-1}$ with nonempty boundary, i.e., ω is connected and open in $\mathbb{S}^{N-1}$ and its boundary $\partial\omega$ is a smooth $(N-2)$-dimensional submanifold of $\mathbb{S}^{N-1}$. The nontrivial solutions to the Neumann problem (1.1) in the unbounded cone
$$\Sigma_\omega := \{tx : x \in \omega \text{ and } t > 0\}$$
are the critical points of J_{Σ_ω} on $\mathcal{N}(\Sigma_\omega)$.

To produce a nonradial sign-changing solution for (1.1) we introduce some symmetries. We write a point in $\mathbb{R}^N$ as $x = (x', x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$, and consider the reflection $\varrho(x', x_N) := (-x', x_N)$. Then, a subset X of $\mathbb{R}^N$ will be called **ϱ-invariant** if $\varrho x \in X$ for every $x \in X$, and a function $u : X \to \mathbb{R}$ will be called **ϱ-equivariant** if
$$u(\varrho x) = -u(x) \quad \forall x \in X.$$
Note that every nontrivial ϱ-equivariant function is nonradial and changes sign.

Throughout this section we will assume that ω is ϱ-invariant. Note that $(0, \pm 1) \notin \partial\omega$ because $\partial\omega$ is smooth. Hence, $\varrho x \neq x$ for every $x \in \partial\Sigma_\omega \smallsetminus \{0\}$. Our aim is to show that (1.1) has a ϱ-equivariant solution. We set

$$D^{1,2}_\varrho(\Sigma_\omega) := \{u \in D^{1,2}(\Sigma_\omega) : u \text{ is } \varrho\text{-equivariant}\},$$
$$\mathcal{N}^\varrho(\Sigma_\omega) := \{u \in \mathcal{N}(\Sigma_\omega) : u \text{ is } \varrho\text{-equivariant}\}$$

and

$$c^\varrho_{\Sigma_\omega} := \inf_{u\in\mathcal{N}^\varrho(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u\in D^{1,2}_\varrho(\Sigma_\omega)\smallsetminus\{0\}} \frac{1}{N}[Q_{\Sigma_\omega}(u)]^{\frac{N}{2}}. \tag{2.4}$$

Define

$$\Lambda_\omega := \{tx : x \in \omega \text{ and } 0 < t < 1\}$$

and set $\Gamma_1 := \partial\Lambda_\omega \smallsetminus \overline{\omega}$. In Λ_ω we consider the mixed boundary value problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Lambda_\omega, \\ u = 0 & \text{on } \omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{2.5}$$

We point out that (2.5) does not have a nontrivial solution. Indeed, by the well known Pohozhaev identity, a solution to (2.5) must satisfy

$$\begin{aligned} 0 &= \int_{\Gamma_1\cup\omega}\left((\nabla u\cdot s)\frac{\partial u}{\partial\nu} - \frac{|\nabla u|^2}{2} s\cdot\nu + F(u)s\cdot\nu\right)\mathrm{d}s \\ &= \int_\omega\left((\nabla u\cdot s)\frac{\partial u}{\partial\nu} - \frac{|\nabla u|^2}{2} s\cdot\nu\right)\mathrm{d}s = \frac{1}{2}\int_\omega\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}s \end{aligned} \tag{2.6}$$

because $s\cdot\nu = 0$ for every $s \in \Gamma_1$ and $s\cdot\nu = 1$ for every $s \in \omega$. Therefore $\frac{\partial u}{\partial\nu}$ vanishes on ω. But then the trivial extension of u to the infinite cone Σ_ω solves (1.1), contradicting the unique continuation principle.

Let $V(\Lambda_\omega)$ be the space of functions in $D^{1,2}(\Lambda_\omega)$ whose trace vanishes on ω. Note that $V(\Lambda_\omega) \subset D^{1,2}(\Sigma_\omega)$ via trivial extension. Let $J_{\Lambda_\omega} : V(\Lambda_\omega) \to \mathbb{R}$ be the restriction of J_{Σ_ω} to $V(\Lambda_\omega)$ and set

$$\mathcal{N}^\varrho(\Lambda_\omega) := \mathcal{N}^\varrho(\Sigma_\omega) \cap V(\Lambda_\omega) \qquad \text{and} \qquad c^\varrho_{\Lambda_\omega} := \inf_{u\in\mathcal{N}^\varrho(\Lambda_\omega)} J_{\Lambda_\omega}(u).$$

To produce a sign-changing solution for the problem (1.1) we will study the concentration behavior of ϱ-equivariant minimizing sequences for (2.5). We start with the following lemmas.

Lemma 2.1. $0 < c^\varrho_{\Lambda_\omega} = c^\varrho_{\Sigma_\omega} \leq c_\infty$.

Proof. It is shown in [8, Theorem 2.1] that $c^\varrho_{\Lambda_\omega} > 0$.

Since $\mathcal{N}^\varrho(\Lambda_\omega) \subset \mathcal{N}^\varrho(\Sigma_\omega)$, we have that $c^\varrho_{\Lambda_\omega} \geq c^\varrho_{\Sigma_\omega}$. To prove the opposite inequality, let $\varphi_k \in \mathcal{N}^\varrho(\Sigma_\omega) \cap C^\infty(\overline{\Sigma}_\omega)$ be such that φ_k has compact support and $J(\varphi_k) \to c^\varrho_{\Sigma_\omega}$ as $k \to \infty$. Then, we may choose $\varepsilon_k > 0$ such that the support of $\widetilde{\varphi}_k(x) := \varepsilon_k^{-(N-2)/2}\varphi_k(\varepsilon_k^{-1}x)$ is contained in $\overline{\Lambda}_\omega \smallsetminus \overline{\omega}$. Thus, $\widetilde{\varphi}_k \in \mathcal{N}^\varrho(\Lambda_\omega)$ and, hence,

$$c^\varrho_{\Lambda_\omega} \leq J(\widetilde{\varphi}_k) = J(\varphi_k) \quad \text{for all } k.$$

Letting $k \to \infty$ we conclude that $c^{\varrho}_{\Lambda_\omega} \leq c^{\varrho}_{\Sigma_\omega}$.

To prove that $c^{\varrho}_{\Sigma_\omega} \leq c_\infty$ we fix a point $\xi \in \partial\Sigma_\omega \smallsetminus \{0\}$ and a sequence of positive numbers $\varepsilon_k \to 0$, and we set $\Sigma_k := \varepsilon_k^{-1}(\Sigma_\omega - \xi)$. Since $\partial\Sigma_\omega \smallsetminus \{0\}$ is smooth, the limit of the sequence of sets (Σ_k) is the half-space

$$\mathbb{H}_\nu := \{z \in \mathbb{R}^N : z \cdot \nu < 0\}, \tag{2.7}$$

where ν is the exterior unit normal to Σ_ω at ξ. Let $u_k(x) := \varepsilon_k^{(2-N)/2} U(\frac{x-\xi}{\varepsilon_k})$, where U is the standard bubble (2.3). Then,

$$\lim_{k\to\infty} \int_{\Sigma_\omega} |\nabla u_k|^2 = \lim_{k\to\infty} \int_{\Sigma_k} |\nabla U|^2 = \int_{\mathbb{H}_\nu} |\nabla U|^2 = \frac{1}{2N} S^{\frac{N}{2}}, \tag{2.8}$$

$$\lim_{k\to\infty} \int_{\Sigma_\omega} |u_k|^{2^*} = \lim_{k\to\infty} \int_{\Sigma_k} |U|^{2^*} = \int_{\mathbb{H}_\nu} |U|^{2^*} = \frac{1}{2N} S^{\frac{N}{2}}. \tag{2.9}$$

The function

$$\widehat{u_k}(x) = u_k(x) - u_k(\varrho x) = \varepsilon_k^{\frac{2-N}{2}} U\left(\frac{x-\xi}{\varepsilon_k}\right) - \varepsilon_k^{\frac{2-N}{2}} U\left(\frac{x-\varrho\xi}{\varepsilon_k}\right)$$

is ϱ-equivariant, and from (2.4), (2.8) and (2.9) we obtain

$$c^{\varrho}_{\Sigma_\omega} \leq \lim_{k\to\infty} \frac{1}{N} [Q_{\Sigma_\omega}(\widehat{u_k})]^{\frac{N}{2}} = \frac{1}{N} S^{\frac{N}{2}} = c_\infty.$$

This conclude the proof. □

Lemma 2.2. *Given a domain Ω in $\mathbb{R}^N$ and $\varepsilon > 0$, we set $\Omega_\varepsilon := \{\varepsilon^{-1}x : x \in \Omega\}$. If $\partial\Omega$ is Lipschitz continuous, then there exist linear* ***extension operators*** *$P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to D^{1,2}(\mathbb{R}^N)$ and a positive constant C, independent of ε, such that*

(i) *$(P_\varepsilon u)(x) = u(x)$ for every $x \in \Omega_\varepsilon$.*
(ii) *$\int_{\mathbb{R}^N} |\nabla(P_\varepsilon u)|^2 \leq C \int_{\Omega_\varepsilon} |\nabla u|^2$.*
(iii) *$\int_{\mathbb{R}^N} |P_\varepsilon u|^{2^*} \leq C \int_{\Omega_\varepsilon} |u|^{2^*}$.*
(iv) *If Ω is ϱ-invariant, then $P_\varepsilon u$ is ϱ-equivariant if u is ϱ-equivariant.*

Proof. The existence of an extension operator $P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to D^{1,2}(\mathbb{R}^N)$ satisfying (i) − (iii) is well known, and the fact that the constant C does not depend on ε was proved in [6, Lemma 2.1]. To obtain (iv) we replace $P_\varepsilon u$ by the function $x \mapsto \frac{1}{2}[(P_\varepsilon u)(x) - (P_\varepsilon u)(\varrho x)]$. □

The following proposition describes the behavior of minimizing sequences for J_{Λ_ω} on $\mathcal{N}^{\varrho}(\Lambda_\omega)$.

Proposition 2.3. *Let $u_k \in \mathcal{N}^{\varrho}(\Lambda_\omega)$ be such that*

$$J_{\Lambda_\omega}(u_k) \to c^{\varrho}_{\Lambda_\omega} \qquad \text{and} \qquad J'_{\Lambda_\omega}(u_k) \to 0 \ \text{ in } \ (V(\Lambda_\omega))'.$$

Then, after passing to a subsequence, one of the following statements holds true:

(i) *There exist a sequence of positive numbers (ε_k), a sequence of points (ξ_k) in Γ_1 and a function $w \in D^{1,2}(\mathbb{R}^N)$ such that $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \bar{\omega} \cup \{0\}) \to \infty$, $w|_{\mathbb{H}}$ solves the Neumann problem*

$$-\Delta w = |w|^{2^*-2}w, \qquad w \in D^{1,2}(\mathbb{H}), \tag{2.10}$$

in some half-space $\mathbb{H}$, $J_{\mathbb{H}}(w) = \frac{1}{2}c_\infty$,

$$\lim_{k\to\infty}\left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot - \xi_k}{\varepsilon_k}\right) + \varepsilon_k^{\frac{2-N}{2}} (w\circ\varrho)\left(\frac{\cdot - \varrho\xi_k}{\varepsilon_k}\right)\right\|_{\Sigma_\omega} = 0,$$

and $c^{\varrho}_{\Sigma_\omega} = c^{\varrho}_{\Lambda_\omega} = c_\infty$.

(*ii*) *There exist a sequence of positive numbers* (ε_k) *with* $\varepsilon_k \to 0$, *and a* ϱ*-equivariant solution* $w \in D^{1,2}(\Sigma_\omega)$ *to the problem* (1.1) *such that*

$$\lim_{k\to\infty}\left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot}{\varepsilon_k}\right)\right\|_{\Sigma_\omega} = 0,$$

and $J_{\Sigma_\omega}(w) = c^{\varrho}_{\Sigma_\omega} = c^{\varrho}_{\Lambda_\omega} \leq c_\infty$.

Proof. Since

$$\frac{1}{N}\|u_k\|^2_{\Lambda_\omega} = J_{\Lambda_\omega}(u_k) - \frac{1}{2^*}J'_{\Lambda_\omega}(u_k)u_k \leq C + o(1)\|u_k\|_{\Lambda_\omega}, \tag{2.11}$$

the sequence (u_k) is bounded and, after passing to a subsequence, $u_k \rightharpoonup u$ weakly in $V(\Lambda_\omega)$. Then, $J'_{\Lambda_\omega}(u) = 0$. Since the problem (2.5) does not have a nontrivial solution, we conclude that $u = 0$.

Fix $\delta \in (0, \frac{N}{2}c^{\varrho}_{\Lambda_\omega})$. As

$$\int_{\Lambda_\omega} |u_k|^{2^*} = N\left(J_{\Lambda_\omega}(u_k) - \frac{1}{2}J'_{\Lambda_\omega}(u_k)u_k\right) \to Nc^{\varrho}_{\Lambda_\omega},$$

there are bounded sequences (ε_k) in $(0,\infty)$ and (x_k) in $\mathbb{R}^N$ such that, after passing to a subsequence,

$$\delta = \sup_{x\in\mathbb{R}^N}\int_{\Lambda_\omega\cap B_{\varepsilon_k}(x)} |u_k|^{2^*} = \int_{\Lambda_\omega\cap B_{\varepsilon_k}(x_k)} |u_k|^{2^*},$$

where $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$. Note that, as $\delta > 0$, we have that $\mathrm{dist}(x_k, \Lambda_\omega) < \varepsilon_k$. We claim that, after passing to a subsequence, there exist $\xi_k \in \bar{\Lambda}_\omega$ and $C_0 > 0$ such that

$$\varepsilon_k^{-1}|x_k - \xi_k| < C_0 \qquad \forall k \in \mathbb{N}, \tag{2.12}$$

and one of the following statements holds true:

(*a*) $\xi_k = 0$ for all $k \in \mathbb{N}$.
(*b*) $\xi_k \in \partial\omega = \overline{\omega} \cap \overline{\Gamma}_1$ for all $k \in \mathbb{N}$.
(*c*) $\xi_k \in \Gamma_1$ for all $k \in \mathbb{N}$ and $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \bar{\omega} \cup \{0\}) \to \infty$.
(*d*) $\xi_k \in \omega$ for all $k \in \mathbb{N}$ and $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \Gamma_1) \to \infty$.
(*e*) $\xi_k \in \Lambda_\omega$ for all $k \in \mathbb{N}$, $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \partial\Lambda_\omega) \to \infty$ and, either $\varepsilon_k^{-1}|\xi_k - \varrho\xi_k| \to \infty$, or $\xi_k = \varrho\xi_k$ for all $k \in \mathbb{N}$.

This can be seen as follows: If the sequence $(\varepsilon_k^{-1}|x_k|)$ is bounded, we set $\xi_k := 0$. Then, (2.12) and (*a*) hold true. If $(\varepsilon_k^{-1}\mathrm{dist}(x_k, \partial\omega))$ is bounded, we take $\xi_k \in \partial\omega$ such that $|x_k - \xi_k| = \mathrm{dist}(x_k, \partial\omega)$. Then, (2.12) and (*b*) hold true. If both $(\varepsilon_k^{-1}|x_k|)$ and $(\varepsilon_k^{-1}\mathrm{dist}(x_k, \partial\omega))$ are unbounded and $(\varepsilon_k^{-1}\mathrm{dist}(x_k, \Gamma_1))$ is bounded, we take $\xi_k \in \Gamma_1$ with $|x_k - \xi_k| = \mathrm{dist}(x_k, \Gamma_1)$. Then, (2.12) and (*c*) hold true. If $(\varepsilon_k^{-1}\mathrm{dist}(x_k, \Gamma_1))$

is unbounded and $(\varepsilon_k^{-1}\mathrm{dist}(x_k,\omega))$ is bounded, we take $\xi_k \in \omega$ with $|x_k - \xi_k| = \mathrm{dist}(x_k,\omega)$. Then, (2.12) and (d) hold true. Finally, if $(\varepsilon_k^{-1}\mathrm{dist}(x_k,\partial\Lambda_\omega))$ is unbounded, we set $\xi_k := \frac{x_k+\varrho x_k}{2}$ if $(\varepsilon_k^{-1}|x_k-\varrho x_k|)$ is bounded and $\xi_k := x_k$ if $(\varepsilon_k^{-1}|x_k-\varrho x_k|)$ is unbounded. Then, (2.12) and (e) hold true.

Set $C_1 := C_0 + 1$. Inequality (2.12) yields

$$\delta = \int_{\Lambda_\omega\cap B_{\varepsilon_k}(x_k)} |u_k|^{2^*} \leq \int_{\Lambda_\omega\cap B_{C_1\varepsilon_k}(\xi_k)} |u_k|^{2^*}. \tag{2.13}$$

We consider u_k as a function in $D^{1,2}(\Sigma_\omega)$ via trivial extension, and we define $\widehat{u}_k \in D^{1,2}(\Sigma_\omega)$ as $\widehat{u}_k(z) := \varepsilon_k^{(N-2)/2}u_k(\varepsilon_k z)$. Since $\widehat{u}_k$ is ϱ-equivariant, so is its extension $P_{\varepsilon_k}\widehat{u}_k \in D^{1,2}(\mathbb{R}^N)$ given by Lemma 2.2. Let

$$w_k(z) := (P_{\varepsilon_k}\widehat{u}_k)(z + \varepsilon_k^{-1}\xi_k) \in D^{1,2}(\mathbb{R}^N).$$

Then,

$$w_k(z) = \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k z + \xi_k) \qquad \text{if } \ z \in \Lambda_k := \varepsilon_k^{-1}(\Lambda_\omega - \xi_k), \tag{2.14}$$

$$w_k\left(z - \varepsilon_k^{-1}\xi_k\right) = -w_k\left(\varrho z - \varepsilon_k^{-1}\xi_k\right) \qquad \text{for every } z \in \mathbb{R}^N, \tag{2.15}$$

$$\delta = \sup_{z\in\mathbb{R}^N} \int_{\Lambda_k\cap B_1(z)} |w_k|^{2^*} \leq \int_{\Lambda_k\cap B_{C_1}(0)} |w_k|^{2^*}, \tag{2.16}$$

and (w_k) is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence, a subsequence satisfies that $w_k \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$, $w_k \to w$ a.e. in $\mathbb{R}^N$ and $w_k \to w$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Choosing δ sufficiently small and using (2.16), a standard argument shows that $w \neq 0$; see, e.g., [10, Section 8.3]. Moreover, we have that $\xi_k \to \xi$ and $\varepsilon_k \to 0$, because $u_k \rightharpoonup 0$ weakly in $V(\Lambda_\omega)$ and $w \neq 0$.

Let $\mathbb{E}$ be the limit of the domains Λ_k. Since (w_k) is bounded in $D^{1,2}(\mathbb{R}^N)$, using Hölder's inequality we obtain

$$\left|\int_{\mathbb{E}\smallsetminus\Lambda_k} \nabla w_k\cdot\nabla\varphi\right| \leq C\left(\int_{\mathbb{E}\smallsetminus\Lambda_k} |\nabla\varphi|^2\right)^{\frac{1}{2}} = o(1),$$

$$\left|\int_{\mathbb{E}\smallsetminus\Lambda_k} |w_k|^{2^*-2}w_k\varphi\right| \leq C\left(\int_{\mathbb{E}\smallsetminus\Lambda_k} |\varphi|^{2^*}\right)^{\frac{1}{2^*}} = o(1),$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, and similarly for the integrals over $\Lambda_k \smallsetminus \mathbb{E}$. Therefore, as $w_k \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{E})$, rescaling and using (2.14) we conclude that

$$\begin{aligned}
\int_{\mathbb{E}} \nabla w\cdot\nabla\varphi - \int_{\mathbb{E}} |w|^{2^*-2}w\varphi &= \int_{\mathbb{E}} \nabla w_k\cdot\nabla\varphi - \int_{\mathbb{E}} |w_k|^{2^*-2}w_k\varphi + o(1) \\
&= \int_{\Lambda_k} \nabla w_k\cdot\nabla\varphi - \int_{\Lambda_k} |w_k|^{2^*-2}w_k\varphi + o(1) \\
&= \int_{\Lambda_\omega} \nabla u_k\cdot\nabla\varphi_k - \int_{\Lambda_\omega} |u_k|^{2^*-2}u_k\varphi_k + o(1),
\end{aligned} \tag{2.17}$$

where $\varphi_k(x) := \varepsilon_k^{(2-N)/2}\varphi(\frac{x-\xi_k}{\varepsilon_k})$. Next, we analyze all possibilities, according to the location of ξ_k.

(*a*) If $\xi_k = 0$ for all $k \in \mathbb{N}$, then $\mathbb{E} = \Sigma_\omega$ and w_k is ϱ-equivariant. Hence, w is ϱ-equivariant. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then, $\varphi_k|_{\Lambda_\omega} \in V(\Lambda_\omega)$ for large enough k, and from (2.17) we obtain

$$J'_{\Sigma_\omega}(w)[\varphi|_{\Sigma_\omega}] = \int_{\Sigma_\omega} \nabla w \cdot \nabla \varphi - \int_{\Sigma_\omega} |w|^{2^*-2} w\varphi = J'_{\Lambda_\omega}(u_k)[\varphi_k|_{\Lambda_\omega}] = o(1).$$

This shows that $w|_{\Sigma_\omega}$ solves (1.1). Therefore,

$$c^\varrho_{\Sigma_\omega} \leq \frac{1}{N}\|w\|^2_{\Sigma_\omega} \leq \liminf_{k\to\infty} \frac{1}{N}\|w_k\|^2_{\Sigma_\omega} = \lim_{k\to\infty} \frac{1}{N}\|u_k\|^2_{\Lambda_\omega} = c^\varrho_{\Lambda_\omega}.$$

Together with Lemma 2.1, this implies that $J_{\Sigma_\omega}(w) = c^\varrho_{\Sigma_\omega} = c^\varrho_{\Lambda_\omega} \leq c_\infty$ and

$$o(1) = \|w_k - w\|_{\Sigma_\omega} = \left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot}{\varepsilon_k}\right)\right\|_{\Sigma_\omega}.$$

So, in this case, we obtain statement (*ii*).

(*b*) If $\xi_k \in \partial\omega$ for all $k \in \mathbb{N}$, then $\mathbb{E} = \mathbb{H}_\xi \cap \mathbb{H}_\nu$, where $\xi = \lim_{k\to\infty} \xi_k$, ν is the exterior unit normal to Σ_ω at ξ, and $\mathbb{H}_\xi$ and $\mathbb{H}_\nu$ are half-spaces defined as in (2.7). If $\varphi \in C_c^\infty(\mathbb{H}_\xi)$, then $\varphi_k|_{\Lambda_\omega} \in V(\Lambda_\omega)$ for large enough k, and using (2.17) we conclude that $w|_{\mathbb{E}}$ solves the mixed boundary value problem

$$-\Delta w = |w|^{2^*-2}w, \qquad w = 0 \text{ on } \partial\mathbb{E} \cap \partial\mathbb{H}_\xi, \qquad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\mathbb{E} \cap \partial\mathbb{H}_\nu.$$

Since ξ and ν are orthogonal, extending $w|_{\mathbb{E}}$ by reflection on $\partial\mathbb{E} \cap \partial\mathbb{H}_\nu$, yields a nontrivial solution to the Dirichlet problem

$$-\Delta w = |w|^{2^*-2}w, \qquad w \in D_0^{1,2}(\mathbb{H}_\xi). \tag{2.18}$$

It is well known that this problem does not have a nontrivial solution, so (*b*) cannot occur.

(*c*) If $\xi_k \in \Gamma_1$ for all $k \in \mathbb{N}$ and $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \bar\omega \cup \{0\}) \to \infty$, then $\mathbb{E} = \mathbb{H}_\nu$, where ν is the exterior unit normal to Σ_ω at $\xi = \lim_{k\to\infty} \xi_k$. Using (2.17) we conclude that $w|_{\mathbb{H}_\nu}$ solves the Neumann problem (2.10) in $\mathbb{H}_\nu$. Since $\varepsilon_k^{-1}|\xi_k| \to \infty$, we have that $\varepsilon_k^{-1}|\xi_k - \varrho\xi_k| \to \infty$. Therefore,

$$w_k - (w \circ \varrho)(\,\cdot\, + \varepsilon_k^{-1}(\xi_k - \varrho\xi_k)) \rightharpoonup w \qquad \text{weakly in } D^{1,2}(\mathbb{R}^N).$$

Note also that $w_k \circ \varrho \rightharpoonup w \circ \varrho$ weakly in $D^{1,2}(\mathbb{R}^N)$. Using these facts and performing suitable rescalings and translations we obtain

$$\begin{aligned}
&\left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot - \xi_k}{\varepsilon_k}\right) + \varepsilon_k^{\frac{2-N}{2}} (w\circ\varrho)\left(\frac{\cdot - \varrho\xi_k}{\varepsilon_k}\right)\right\|^2_{\Sigma_\omega} \\
&= \left\| \widehat{u}_k - w(\,\cdot\, - \varepsilon_k^{-1}\xi_k) + (w\circ\varrho)(\,\cdot\, - \varepsilon_k^{-1}\varrho\xi_k)\right\|^2_{\Sigma_\omega} \\
&= \left\| w_k - w + (w\circ\varrho)\left(\,\cdot\, + \varepsilon_k^{-1}(\xi_k - \varrho\xi_k)\right)\right\|^2_{\Sigma_\omega - \varepsilon_k^{-1}\xi_k} \\
&= \left\| w_k + (w\circ\varrho)\left(\,\cdot\, + \varepsilon_k^{-1}(\xi_k - \varrho\xi_k)\right)\right\|^2_{\Sigma_\omega - \varepsilon_k^{-1}\xi_k} - \|w\|^2_{\mathbb{H}_\nu} + o(1) \\
&= \|-w_k\circ\varrho + w\circ\varrho\|^2_{\Sigma_\omega - \varepsilon_k^{-1}\varrho\xi_k} - \|w\|^2_{\mathbb{H}_\nu} + o(1)
\end{aligned}$$

$$= \left\|\widehat{u_k}\right\|^2_{\Sigma_\omega} - 2\|w\|^2_{\mathbb{H}_\nu} + o(1)$$
$$= \|u_k\|^2_{\Lambda_\omega} - 2\|w\|^2_{\mathbb{H}_\nu} + o(1) = Nc^{\varrho}_{\Lambda_\omega} - 2\|w\|^2_{\mathbb{H}_\nu} + o(1).$$

Since $J_{\mathbb{H}_\nu}(w) = \frac{1}{N}\|w\|^2_{\mathbb{H}_\nu} \geq \frac{1}{2}c_\infty$, applying Lemma 2.1 we conclude that $J_{\mathbb{H}_\nu}(w) = \frac{1}{2}c_\infty$, $c^{\varrho}_{\Sigma_\omega} = c^{\varrho}_{\Lambda_\omega} = c_\infty$, and

$$\lim_{k\to\infty}\left\|u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot - \xi_k}{\varepsilon_k}\right) + \varepsilon_k^{\frac{2-N}{2}} (w \circ \varrho)\left(\frac{\cdot - \varrho\xi_k}{\varepsilon_k}\right)\right\|^2_{\Sigma_\omega} = 0.$$

So, in this case we obtain statement (i).

(d) If $\xi_k \in \omega$ for all $k \in \mathbb{N}$ and $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \Gamma_1) \to \infty$, then $\mathbb{E} = \mathbb{H}_\xi$ and using (2.17) we conclude that $w|_{\mathbb{H}_\xi}$ solves the Dirichlet problem (2.18). So this case does not occur.

(e) If $\xi_k \in \Lambda_\omega$ for all $k \in \mathbb{N}$ and $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \partial\Lambda_\omega) \to \infty$, then $\mathbb{E} = \mathbb{R}^N$ and w solves the problem (1.2). If $\rho\xi_k = \xi_k$ for every k, then w_k is ϱ-equivariant, and so is w. Since w is a sign-changing solution to (1.2) we have that

$$2c_\infty < \frac{1}{N}\|w\|^2_{\mathbb{R}^N} \leq \lim_{k\to\infty}\frac{1}{N}\|w_k\|^2_{\mathbb{R}^N} = \lim_{k\to\infty}\frac{1}{N}\|u_k\|^2_{\Lambda_\omega} = c^{\varrho}_{\Lambda_\omega},$$

contradicting Lemma 2.1. On the other hand, if $\varepsilon_k^{-1}|\varrho\xi_k - \xi_k| \to \infty$, then, arguing as in case (c), we conclude that

$$2c_\infty \leq \frac{2}{N}\|w\|^2_{\mathbb{R}^N} \leq \lim_{k\to\infty}\frac{1}{N}\|w_k\|^2_{\mathbb{R}^N} = \lim_{k\to\infty}\frac{1}{N}\|u_k\|^2_{\Lambda_\omega} = c^{\varrho}_{\Lambda_\omega},$$

contradicting Lemma 2.1 again. So (e) cannot occur.

We are left with (a) and (c). This concludes the proof. □

Proposition 2.3 immediately yields the following result.

Corollary 2.4. *If $c^{\varrho}_{\Sigma_\omega} < c_\infty$, then the problem* (1.1) *has a ϱ-equivariant least energy solution in $D^{1,2}(\Sigma_\omega)$.*

Equality is not enough, as the following example shows. Set

$$\mathbb{S}^{N-1}_+ := \{(x_1, \ldots, x_N) \in \mathbb{S}^{N-1} : x_N > 0\}.$$

Example 2.5. *If $\omega = \mathbb{S}^{N-1}_+$, then problem* (1.1) *does not have a ϱ-equivariant least energy solution in $D^{1,2}(\Sigma_\omega)$.*

Proof. Σ_ω is the upper half-space $\mathbb{R}^N_+ := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. If u were a ϱ-equivariant least energy solution to (1.1) in $\mathbb{R}^N_+$ then, extending u by reflection on $\partial(\mathbb{R}^N_+)$, would yield a sign-changing solution $\widetilde{u}$ to the problem (1.2) in $\mathbb{R}^N$ with $J_{\mathbb{R}^N}(\widetilde{u}) \leq 2c_\infty$. But the energy of any sign-changing solution to (1.2) is $> 2c_\infty$; see [9]. □

The following local geometric condition guarantees the existence of a minimizer. It was introduced by Adimurthi and Mancini in [1].

Definition 2.6. *A point $\xi \in \partial\omega$ is a* ***point of convexity of*** *Σ_ω* ***(of radius*** *$r > 0$) if $B_r(\xi) \cap \Sigma_\omega \subset \mathbb{H}_\nu$ and the mean curvature of $\partial\Sigma_\omega$ at ξ with respect to the exterior unit normal ν at ξ is positive.*

As in [1] we make the convention that the curvature of a geodesic in $\partial\Sigma_\omega$ is positive at ξ if it curves away from the exterior unit normal ν. The half-space $\mathbb{H}_\nu$ is defined as in (2.7). Examples of cones having a point of convexity are given as follows.

Proposition 2.7. *If $\overline{\omega} \subset \mathbb{S}^{N-1}_+$, then Σ_ω has a point of convexity.*

Proof. Let β be the smallest geodesic ball in $\mathbb{S}^{N-1}$, centered at the north pole $(0,\dots,0,1)$, which contains ω. Then, $\partial\omega \cap \partial\beta \neq \emptyset$ and $\overline{\beta} \subset \mathbb{S}^{N-1}_+$. Hence, every point on $\partial\beta$ is a point of convexity of Σ_β. As $\omega \subset \beta$, we have that any point $\xi \in \partial\omega \cap \partial\beta$ is a point of convexity of Σ_ω. □

Theorem 2.8. *If Σ_ω has a point of convexity, then $c^\varrho_{\Sigma_\omega} < c_\infty$. Consequently, the problem* (1.1) *has a ϱ-equivariant least energy solution in $D^{1,2}(\Sigma_\omega)$. This solution is nonradial and changes sign.*

Proof. Let $\xi \in \partial\omega$ be a point of convexity of Σ_ω of radius $r > 0$. It is shown in [1, Lemma 2.2] that, after fixing r small enough and a radial cut-off function $\psi \in C^\infty_c(\mathbb{R}^N)$ with $\psi(x) = 1$ if $|x| \leq \frac{r}{4}$ and $\psi(x) = 0$ if $|x| \geq \frac{r}{2}$, the function $u_{\varepsilon,\xi}(x) := \psi(x-\xi)\varepsilon^{(2-N)/2}U(\varepsilon^{-1}(x-\xi))$, with U as in (2.3), satisfies

$$Q_{\Sigma_\omega}(u_{\varepsilon,\xi}) = \begin{cases} \frac{S}{2^{2/N}} - d_N H_\omega(\xi) S\, \varepsilon \ln(\varepsilon^{-2}) + O(\varepsilon) & \text{if } N = 3, \\ \frac{S}{2^{2/N}} - d_N H_\omega(\xi) S\, \varepsilon + O(\varepsilon^2 \ln(\varepsilon^{-2})) & \text{if } N \geq 4, \end{cases} \tag{2.19}$$

where d_N is a positive constant depending only on N and $H_\omega(\xi)$ is the mean curvature of $\partial\Sigma_\omega$ at ξ. Hence, for ε small enough,

$$J_{\Sigma_\omega}(t_{\varepsilon,\xi}u_{\varepsilon,\xi}) = \frac{1}{N}[Q_{\Sigma_\omega}(u_{\varepsilon,\xi})]^{\frac{N}{2}} < \frac{1}{2N}S^{\frac{N}{2}} = \frac{1}{2}c_\infty,$$

where $t_{\varepsilon,\xi} > 0$ is such that $t_{\varepsilon,\xi}u_{\varepsilon,\xi} \in \mathcal{N}(\Sigma_\omega)$; see (2.1). Choosing r so that $B_r(\xi) \cap B_r(\varrho\xi) = \emptyset$ we conclude that $t_{\varepsilon,\xi}(u_{\varepsilon,\xi} - u_{\varepsilon,\xi} \circ \varrho) \in \mathcal{N}^\varrho(\Sigma_\omega)$ and

$$c^\varrho_{\Sigma_\omega} \leq J_{\Sigma_\omega}(t_{\varepsilon,\xi}(u_{\varepsilon,\xi} - u_{\varepsilon,\xi} \circ \varrho)) < c_\infty.$$

The existence of a ϱ-equivariant least energy solution to (1.1) follows from Corollary 2.4. □

3. A positive nonradial solution

In this section ω is not assumed to have any symmetries.

We are interested in positive solutions to the problem (1.1). Note that this problem has always a positive *radial* solution given by the restriction to Σ_ω of the standard bubble U defined in (2.3). The question we wish to address in this section is whether problem (1.1) has a positive *nonradial* solution.

Recall the notation introduced in Section 2 and set

$$c_{\Sigma_\omega} := \inf_{u \in \mathcal{N}(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u \in D^{1,2}(\Sigma_\omega)\smallsetminus\{0\}} \frac{1}{N}[Q_{\Sigma_\omega}(u)]^{\frac{N}{2}},$$

$$\mathcal{N}(\Lambda_\omega) := \mathcal{N}(\Sigma_\omega) \cap V(\Lambda_\omega) \qquad \text{and} \qquad c_{\Lambda_\omega} := \inf_{u \in \mathcal{N}(\Lambda_\omega)} J_{\Lambda_\omega}(u).$$

It is shown in [8, Theorem 2.1] that $c_{\Lambda_\omega} > 0$. As in Lemma 2.1 one shows that $c_{\Sigma_\omega} = c_{\Lambda_\omega} \leq \frac{1}{2}c_\infty$. We start by describing the behavior of minimizing sequences for J_{Λ_ω} on $\mathcal{N}(\Lambda_\omega)$.

Proposition 3.1. *Let $u_k \in \mathcal{N}(\Lambda_\omega)$ be such that*

$$J_{\Lambda_\omega}(u_k) \to c_{\Lambda_\omega} \qquad \text{and} \qquad J'_{\Lambda_\omega}(u_k) \to 0 \;\; \text{in} \;\; (V(\Lambda_\omega))'.$$

Then, after passing to a subsequence, one of the following statements holds true:

(*i*) *There exist a sequence of positive numbers (ε_k), a sequence of points (ξ_k) in Γ_1 and a function $w \in D^{1,2}(\mathbb{R}^N)$ such that $\varepsilon_k^{-1}\mathrm{dist}(\xi_k, \bar{\omega} \cup \{0\}) \to \infty$, $w|_{\mathbb{H}}$ solves the Neumann problem*

$$-\Delta w = |w|^{2^*-2}w, \qquad w \in D^{1,2}(\mathbb{H}),$$

in some half-space $\mathbb{H}$, $J_{\mathbb{H}}(w) = \frac{1}{2}c_\infty$,

$$\lim_{k\to\infty} \left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot - \xi_k}{\varepsilon_k}\right) \right\|_{\Sigma_\omega} = 0,$$

and $c_{\Sigma_\omega} = c_{\Lambda_\omega} = \frac{1}{2}c_\infty$.

(*ii*) *There exist a sequence of positive numbers (ε_k) with $\varepsilon_k \to 0$ and a solution $w \in D^{1,2}(\Sigma_\omega)$ to the problem* (1.1) *such that*

$$\lim_{k\to\infty} \left\| u_k - \varepsilon_k^{\frac{2-N}{2}} w\left(\frac{\cdot}{\varepsilon_k}\right) \right\|_{\Sigma_\omega} = 0,$$

and $J_{\Sigma_\omega}(w) = c_{\Sigma_\omega} = c_{\Lambda_\omega} \le \frac{1}{2}c_\infty$.

Proof. The proof is similar, but simpler than that of Proposition 2.3. □

The following statement is an immediate consequence of this proposition.

Corollary 3.2. *If $c_{\Sigma_\omega} < \frac{1}{2}c_\infty$, then the problem* (1.1) *has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$.*

Theorem 3.3. *If Σ_ω has a point of convexity, then $c_{\Sigma_\omega} < \frac{1}{2}c_\infty$. Consequently, the problem* (1.1) *has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$.*

Proof. The proof is similar to that of Theorem 2.8. □

Let $D^{1,2}_{\mathrm{rad}}(\Sigma_\omega)$ be the subspace of radial functions in $D^{1,2}(\Sigma_\omega)$, and define $\mathcal{N}^{\mathrm{rad}}(\Sigma_\omega) := \mathcal{N}(\Sigma_\omega) \cap D^{1,2}_{\mathrm{rad}}(\Sigma_\omega)$ and

$$c^{\mathrm{rad}}_{\Sigma_\omega} := \inf_{u\in\mathcal{N}^{\mathrm{rad}}(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u\in D^{1,2}_{\mathrm{rad}}(\Sigma_\omega)\smallsetminus\{0\}} \frac{1}{N}[Q_{\Sigma_\omega}(u)]^{\frac{N}{2}}.$$

It was shown in [8, Theorem 2.4] that, if Σ_ω is convex, then $c^{\mathrm{rad}}_{\Sigma_\omega} = c_{\Sigma_\omega}$ and the only positive minimizers are the restrictions of the rescalings

$$U_\varepsilon(x) = a_N\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}, \qquad \varepsilon > 0, \tag{3.1}$$

of the standard bubble to Σ_ω. In fact, the proof of [8, Theorem 2.4] shows that these are the only positive solutions of (1.1) in a convex cone. Moreover, the following statement holds true.

Proposition 3.4. *For any cone Σ_ω, the restrictions to Σ_ω of the functions U_ε defined in* (3.1) *are minimizers of J_{Σ_ω} on $\mathcal{N}^{\text{rad}}(\Sigma_\omega)$. These are the only nontrivial radial solutions to* (1.1)*, up to sign. Moreover,*

$$c^{\text{rad}}_{\Sigma_\omega} = b_N|\Lambda_\omega|, \qquad \textit{where } b_N = \frac{c_\infty}{|B_1(0)|}$$

and $|X|$ is the Lebesgue measure of X. In particular, $c^{\text{rad}}_{\Sigma_\omega}$ increases with $|\Lambda_\omega|$.

Proof. A radial function u solves (1.1) in Σ_ω if and only if the function $\bar{u}$ given by $\bar{u}(r) := u(x)$ with $r = \|x\|$ solves

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^{N-1}\bar{u}'(r)) = r^{N-1}|\bar{u}(r)|^{N-2}\bar{u}(r) \text{ in } (0,\infty), \quad \bar{u}(0) = u(0), \quad \bar{u}'(0) = 0.$$

This last problem does not depend on ω. It is well known that, up to sign, the functions U_ε are the only nontrivial radial solutions to the problem (1.2) in $\mathbb{R}^N = \Sigma_{\mathbb{S}^{N-1}}$. Hence, their restrictions to Σ_ω are the only nontrivial radial solutions to (1.1).

As in Lemma 2.1 one shows that $c^{\text{rad}}_{\Sigma_\omega} = c^{\text{rad}}_{\Lambda_\omega} := \inf_{u\in\mathcal{N}^{\text{rad}}(\Lambda_\omega)} J_{\Lambda_\omega}(u)$. For $u \in V_{\text{rad}}(\Lambda_\omega) := D^{1,2}_{\text{rad}}(\Lambda_\omega) \cap V(\Lambda_\omega), u \neq 0$, we have that

$$Q_{\Lambda_\omega}(u) = \frac{\int_{\Lambda_\omega} |\nabla u|^2}{\left(\int_{\Lambda_\omega} |u|^{2^*}\right)^{2/2^*}} = \frac{N|\Lambda_\omega| \int_0^1 |\bar{u}'(r)|^2 r^{N-1}\mathrm{d}r}{\left(N|\Lambda_\omega| \int_0^1 |\bar{u}(r)|^{2^*} r^{N-1}\mathrm{d}r\right)^{2/2^*}}.$$

Therefore,

$$\begin{aligned} c^{\text{rad}}_{\Lambda_\omega} &= \inf_{u\in V_{\text{rad}}(\Lambda_\omega)\setminus\{0\}} \frac{1}{N}[Q_{\Lambda_\omega}(u)]^{\frac{N}{2}} \\ &= \inf_{u\in V_{\text{rad}}(\Lambda_\omega)\setminus\{0\}} \frac{\int_0^1 |\bar{u}'(r)|^2 r^{N-1}\mathrm{d}r}{\left(\int_0^1 |\bar{u}(r)|^{2^*} r^{N-1}\mathrm{d}r\right)^{2/2^*}} |\Lambda_\omega| =: b_N|\Lambda_\omega|. \end{aligned}$$

The same formula holds true when we replace ω by $\mathbb{S}^{N-1}$. In this case, the left-hand side is c_∞. Hence, $b_N = \frac{c_\infty}{|B_1(0)|}$, as claimed. □

Corollary 3.5. *If Σ_ω has a point of convexity and $|\Lambda_\omega| \geq \frac{1}{2}|B_1(0)|$, then*

(*i*) *the problem* (1.1) *has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$,*
(*ii*) *every least energy solution of* (1.1) *is nonradial.*

Proof. From Theorem 3.3 and Proposition 3.4 we get that c_{Σ_ω} is attained and

$$c_{\Sigma_\omega} < \frac{1}{2}c_\infty = c^{\text{rad}}_{\mathbb{R}^N_+} = \frac{b_N}{2}|B_1(0)| \leq b_N|\Lambda_\omega| = c^{\text{rad}}_{\Sigma_\omega},$$

where $\mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$. So every least energy solution is nonradial. □

Note that the hypothesis that $|\Lambda_\omega| \geq \frac{1}{2}|B_1(0)|$ implies that Σ_ω is not convex.

A closer look at the estimate (2.19) allows to refine Corollary 3.5 and to produce examples of cones Σ_ω with $|\Lambda_\omega| < \frac{1}{2}|B_1(0)|$ for which the problem (1.1) has a positive nonradial solution.

To this end, we fix a smooth domain ω_0 in $\mathbb{S}^{N-1}$ for which Σ_{ω_0} has a point of convexity $\xi \in \partial\omega_0$ of radius $r > 0$, and we define

$$\ell(\omega_0, \xi, r) := \{\omega : \omega \text{ is a smooth domain in } \mathbb{S}^{N-1},\ B_r(\xi) \cap \Sigma_{\omega_0} \subset B_r(\xi) \cap \Sigma_\omega \\ \text{and } \operatorname{dist}(B_r(\xi) \cap \Sigma_{\omega_0},\ B_r(\xi) \cap (\Sigma_\omega \setminus \Sigma_{\omega_0})) > 0\}.$$

Then, we have the following result.

Theorem 3.6. *There exists $\alpha_\xi \in (0, \frac{1}{2}|B_1(0)|)$, depending only on $B_r(\xi) \cap \Sigma_{\omega_0}$, such that, for every $\omega \in \ell(\omega_0, \xi, r)$ with $|\Lambda_\omega| > \alpha_\xi$, the following statements hold true:*

(*i*) *the problem* (1.1) *has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$,*
(*ii*) *every least energy solution of* (1.1) *is nonradial,*
(*iii*) *Σ_ω is not convex.*

Proof. Recall that the functions $u_{\varepsilon,\xi}$, introduced in the proof of Theorem 2.8, vanish outside the ball $B_{r/2}(0)$. Moreover, the value $Q_{\Sigma_{\omega_0}}(u_{\varepsilon,\xi})$ and the estimate (2.19) depend only on the value of $u_{\varepsilon,\xi}$ in $B_r(\xi) \cap \Sigma_{\omega_0}$. We fix $\varepsilon_0 > 0$ small enough so that

$$Q_\xi := Q_{\Sigma_{\omega_0}}(u_{\varepsilon_0,\xi}) < \frac{S}{2^{2/N}},$$

and we set $\alpha_\xi := \frac{1}{Nb_N} Q_\xi^{N/2}$ with b_N as in Proposition 3.4. Then,

$$\alpha_\xi < \frac{1}{2Nb_N} S^{\frac{N}{2}} = \frac{1}{2}|B_1(0)|.$$

Given $\omega \in \ell(\omega_0, \xi, r)$, we fix a function $\widehat{u}_{\varepsilon_0,\xi} \in C_c^\infty(B_r(0))$ such that $\widehat{u}_{\varepsilon_0,\xi}(x) = u_{\varepsilon_0,\xi}(x)$ if $x \in B_r(\xi) \cap \Sigma_{\omega_0}$ and $\widehat{u}_{\varepsilon_0,\xi}(x) = 0$ if $x \in B_r(\xi) \cap (\Sigma_\omega \setminus \Sigma_{\omega_0})$. So, if $|\Lambda_\omega| > \alpha_\xi$, we have that

$$c_{\Sigma_\omega} \le \frac{1}{N}[Q_{\Sigma_\omega}(\widehat{u}_{\varepsilon_0,\xi})]^{\frac{N}{2}} = \frac{1}{N} Q_\xi^{\frac{N}{2}} = b_N \alpha_\xi < b_N |\Lambda_\omega| = c_{\Sigma_\omega}^{\mathrm{rad}}.$$

Note that ξ is a point of convexity of ω. Hence, by Theorem 3.3 and the previous inequality, c_{Σ_ω} is attained at a nonradial solution of (1.1). Finally, recall that, if Σ_ω were convex, then $c_{\Sigma_\omega} = c_{\Sigma_\omega}^{\mathrm{rad}}$; see [8, Theorem 2.4]. This completes the proof. □

Corollary 3.7. *There exists a smooth domain $\omega \subset \mathbb{S}_+^{N-1}$ such that the problem* (1.1) *has a positive nonradial solution in Σ_ω.*

Proof. Let ω_0 be the geodesic ball in $\mathbb{S}^{N-1}$ of radius $\pi/4$ centered at the north pole and let ξ be any point on $\partial\omega_0$. Fix $r > 0$ such that $B_r(\xi) \cap \mathbb{S}^{N-1} \subset \mathbb{S}_+^{N-1}$. Clearly, ξ is a point of convexity of Σ_{ω_0} of radius r, so we may fix $\alpha_\xi > 0$ as in Theorem 3.6. As $\alpha_\xi < \frac{1}{2}|B_1(0)|$, there exists $\omega \in \ell(\omega_0, \xi, r)$ with $\omega \subset \mathbb{S}_+^{N-1}$ and $|\Lambda_\omega| > \alpha_\xi$. Now, Theorem 3.6 yields a positive nonradial solution to problem (1.1) in Σ_ω. □

Remark 3.8. *Let $\omega \neq \mathbb{S}_+^{N-1}$ be such that Σ_ω is convex. Then, every point $\xi \in \partial\omega$ is a point of convexity of radius r for any $r > 0$. Fix $r = 1$, and fix $\varepsilon > 0$ such that*

$$Q_\xi := Q_{\Sigma_\omega}(u_{\varepsilon,\xi}) < \frac{S}{2^{2/N}} \qquad \forall \xi \in \partial\omega.$$

Now, define $\alpha_\xi := \frac{1}{Nb_N} Q_\xi^{N/2}$*, as in* Theorem 3.6. *Since* Σ_ω *is convex, we must have that*

$$|\Lambda_\omega| \leq \alpha_\xi = \frac{|B_1(0)|}{S^{N/2}} Q_\xi^{N/2}, \qquad \forall \xi \in \partial\omega,$$

where the equality follows from the definition of b_N*; see* Proposition 3.4. *Hence, for any convex cone* Σ_ω*, we obtain the upper bound*

$$|\Lambda_\omega| \leq \frac{|B_1(0)|}{S^{N/2}} \min_{\xi \in \partial\omega} Q_\xi$$

for the measure of Λ_ω*, which is given in terms of the Sobolev constant and the local energy of the standard bubbles.*

Acknowledgments

M. Clapp was partially supported by UNAM-DGAPA-PAPIIT grant IN100718 (Mexico) and CONACYT grant A1-S-10457 (Mexico). F. Pacella was partially supported by PRIN 2015 (Italy) and INDAM-GNAMPA (Italy).

Conflict of interest

The authors declare no conflict of interest.

References

1. Adimurthi A, Mancini G (1991) The Neumann problem for elliptic equations with critical nonlinearity. *Nonlinear Anal*, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 9–25.
2. Clapp M (2016) Entire nodal solutions to the pure critical exponent problem arising from concentration. *J Differ Equations* 261: 3042–3060.
3. del Pino M, Musso M, Pacard F, et al. (2011) Large energy entire solutions for the Yamabe equation. *J Differ Equations* 251: 2568–2597.
4. Ding WY (1986) On a conformally invariant elliptic equation on $\mathbb{R}^n$. *Commun Math Phys* 107: 331–335.
5. Fernández JC, Petean J (2020) Low energy nodal solutions to the Yamabe equation. *J Differ Equations* 268: 6576–6597.
6. Grossi M, Pacella F (1990) Positive solutions of nonlinear elliptic equations with critical Sobolev exponent and mixed boundary conditions. *P Roy Soc Edinb A* 116: 23–43.
7. Lions PL, Pacella F (1990) Isoperimetric inequalities for convex cones. *P Am Math Soc* 109: 477–485.
8. Lions PL, Pacella F, Tricarico M (1988) Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions. *Indiana U Math J* 37: 301–324.
9. Weth T (2006) Energy bounds for entire nodal solutions of autonomous superlinear equations. *Calc Var Partial Dif* 27: 421–437.

10. Willem M (1996) *Minimax Theorems*, Boston: Birkhäuser Boston.