

SECTIONAL SYMMETRY OF SOLUTIONS OF ELLIPTIC SYSTEMS IN CYLINDRICAL DOMAINS

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Abstract. In this paper we prove a kind of rotational symmetry for solutions of semilinear elliptic systems in some bounded cylindrical domains. The symmetry theorems obtained hold for low-Morse index solutions whenever the nonlinearities satisfy some convexity assumptions. These results extend and improve those obtained in [6, 9, 16, 18].

1. INTRODUCTION

We consider the Dirichlet problem for a semilinear elliptic system of the type

$$(1.1) \quad \begin{cases} -\Delta U = F(x, U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 2$ and $F = (f_1, \dots, f_m)$ is a function belonging to $C^1(\bar{\Omega} \times \mathbb{R}^m; \mathbb{R}^m)$, $m \geq 1$. Here $U = (u_1, \dots, u_m)$ is a vector valued function.

When $m = 1$, i.e. the equation in (1.1) is a scalar semilinear elliptic equation, the famous symmetry result by B. Gidas, W.M. Ni and L. Nirenberg [14], based on the moving planes method, asserts that if Ω is a ball then every positive solution of (1.1) is radial if the nonlinear term $f = f(|x|, u)$ is monotone decreasing with respect to $r = |x|$. The result of [14] was then extended to systems in [22], [11], [12].

It is well known that the radial symmetry of a solution does not hold, in general, when Ω is an annulus or if sign changing solutions are considered and even if f does not have the right monotonicity with respect to $|x|$ (see for example [17]). Nevertheless when the hypotheses of the theorem of Gidas, Ni and Nirenberg fail another kind of symmetry can be recovered, namely the foliated Schwarz symmetry for solutions of (1.1) in a ball or in annulus having low Morse index and assuming that the nonlinear term has some convexity properties in the U -variable. We refer to Section 2 for the definition of Morse index.

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A symmetry result of this type was first proved in [16] in the case $m = 1$ for solutions having Morse index one and assuming that the nonlinearity $f = f(|x|, s)$ is convex in the second variable. Later it was extended in [18] to solutions having Morse index not larger than the dimension N and assuming that the derivative $\frac{\partial f}{\partial s}$ is a convex function in the s -variable. Finally in [6] and [9] the foliated Schwarz symmetry was proved for low Morse index solutions of cooperative elliptic systems, i.e. when $m \geq 2$. Let us point out that the extension of the results in [16] and [18] to systems is nontrivial. Indeed the results of [6] could not be proved for any convex nonlinearity $F = F(|x|, U)$ but some additional hypotheses were required.

In this paper we extend the above results by considering more general symmetric domains and not just balls or annulus. As a consequence we will get less symmetry of the solutions, depending also on a tighter bound on their Morse index. Moreover we are able to improve the results in [6] by allowing any convex nonlinearity in (1.1).

To state precisely our results we need some preliminary definitions. The first one concerns the domains we consider.

Let $N \geq 2$ and $2 \leq k \leq N$. If $k < N$, let us denote by $x = (x', x'')$ a point in \mathbb{R}^N , with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{N-k}$, and for a bounded domain Ω let us denote by Ω'' the set

$$\Omega'' = \{x'' \in \mathbb{R}^{N-k} : \exists x' \in \mathbb{R}^k : (x', x'') \in \Omega\}$$

We will consider domains of the following type.

DEFINITION 1.1. Assume that $N \geq 2$, $2 \leq k \leq N$. We say that a bounded domain Ω in \mathbb{R}^N , is k -rotationally symmetric if either $k = N$ and Ω is a ball or an annulus, or $2 \leq k < N$ and the sets

$$\Omega^h = \Omega \cap \{x = (x', x'') \in \mathbb{R}^N : x'' = h\}$$

are either k -dimensional balls or k -dimensional annulus with the center on $(0, h)$ for every $h \in \Omega''$.

In other words we require that the set Ω^h , which represents a section of Ω at the level $x'' = h$ is either a ball or an annulus in dimension k .

The symmetry we will get for solutions of (1.1) when Ω is k -rotationally symmetric is a variant of the foliated Schwarz symmetry considered in several previous paper (see [1], [6], [9], [10], [15], [16], [18], [20], [23] and the references therein).

We will call it k -sectional foliated Schwarz symmetry.

DEFINITION 1.2. Let Ω be a bounded k -rotationally symmetric domain in \mathbb{R}^N , $2 \leq k \leq N$, and let $U : \Omega \rightarrow \mathbb{R}^m$ a continuous function. We say that U is k -sectionally foliated Schwarz symmetric if there exists a vector $p' = (p_1, \dots, p_k, 0, \dots, 0) \in \mathbb{R}^N$, $|p'| = 1$, such that $U(x) = U(x', x'')$ depends only on x'' , $r = |x'|$ and $\vartheta = \arccos(\frac{x'}{|x'|} \cdot p')$ and U is nonincreasing in ϑ .

When $k = N$ the previous definition coincides with that of foliated Schwarz symmetry. Definition 1.2 just means that the functions $x' \mapsto U(x', h)$ defined in Ω^h are either radial for any $h \in \Omega''$, or nonradial but foliated Schwarz symmetric for any $h \in \Omega''$, with the same axis of symmetry. In the case $k = N - 1$ the sectional foliated Schwarz symmetry was defined in [7] to study some elliptic problems with nonlinear mixed boundary conditions.

In order to prove symmetry of solutions we also need some symmetry on the nonlinearity. Therefore from now on we assume that Ω is a smooth bounded k -rotationally symmetric domain in \mathbb{R}^N and we rewrite the system (1.1) as

$$(1.2) \quad \begin{cases} -\Delta U = F(|x'|, x'', U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. we require that F depends radially on x' . As in (1.1) $F = F(r, x'', S) = (f_1(r, x'', S), \dots, f_m(r, x'', S))$ satisfies

$$(1.3) \quad F \in C^1([0, +\infty) \times \Omega'' \times \mathbb{R}^m; \mathbb{R}^m)$$

The symmetry results we get are the following (the definition of Morse index and fully coupled systems will be recalled in Section 2).

THEOREM 1.1. *Let Ω be a k -rotationally symmetric domain in \mathbb{R}^N , $2 \leq k \leq N$, and let $U \in C^2(\overline{\Omega}; \mathbb{R}^m)$ be a solution of (1.2) with F satisfying (1.3). Assume that*

- i) *the system (1.2) is fully coupled along U in Ω*
- ii) *for any $i = 1, \dots, m$ the scalar function $f_i(|x'|, x'', S)$ is convex in the variable $S = (s_1, \dots, s_m) \in \mathbb{R}^m$.*

If $m(U) \leq k$, where $m(U)$ is the Morse index of U , then U is k -sectional foliated Schwarz symmetric, and if the functions $x' \mapsto U(x', h)$, $h \in \Omega''$, are not radial then they are strictly decreasing in the angular variable.

This theorem not only extends the results in [6] to k -rotationally symmetric domains but also improves the result of [6] for the case $k = N$ since it only requires that the components f_i of the nonlinearity are convex without further assumptions.

The next theorem concerns the case when the nonlinearity has convex first derivatives.

THEOREM 1.2. *Let Ω be a k -rotationally symmetric in \mathbb{R}^N , $2 \leq k \leq N$, and let $U \in C^2(\overline{\Omega}; \mathbb{R}^m)$ be a solution of (1.2). Assume that:*

- i) *the system (1.2) is fully coupled along U in Ω*
- ii) *for any $i, j = 1, \dots, m$ the function $\frac{\partial f_i}{\partial s_j}(|x'|, x'', S)$ is convex in $S = (s_1, \dots, s_m)$.*

If $m(U) \leq k - 1$ then a solution U is k -sectionally foliated Schwarz symmetric and if the functions $x' \mapsto U(x', h)$, $h \in \Omega''$, are not radial then they are strictly decreasing in the angular variable.

The previous theorem extends to k -rotationally symmetric domains the result in [9] and we provide a different proof which also simplifies the one given in [9]. Note that in Theorem 1.2 the bound on the Morse index $m(U) \leq k - 1$ is stricter than in Theorem 1.1. When $m = 1$, i.e. in the scalar case, it is possible to improve it to $m(U) \leq k$ (adapting the proof in [18] for $k = N$). However in the vectorial case serious difficulties arise when $m(U) = k$, which prevent to use the same approach, though we believe that the symmetry result should be true also in this case.

It is interesting to see in the previous theorems how the Morse index of a solution is related to the "dimension" of the sectional symmetry of the domain.

Remark 1.1. In the particular case of stable solutions (see Section 2 for the definition) we get the radial symmetry on each section Ω^h without requiring any convexity on F . This can be proved easily as in the proof of Theorem 1.5 of [15].

We will deduce from the proof of Theorem 1.1 and Theorem 1.2 that for nonradial Morse index one solutions the following condition holds.

COROLLARY 1.1. *Under the assumptions of Theorem 1.1 or Theorem 1.2 if a solution U has Morse index one and is not radial then*

$$(1.4) \quad \sum_{j=1}^m \frac{\partial f_i}{\partial s_j}(r, x'', U(r, \vartheta)) \frac{\partial u_j}{\partial \vartheta}(r, x'', \vartheta) = \sum_{j=1}^m \frac{\partial f_j}{\partial s_i}(r, x'', U(r, \vartheta)) \frac{\partial u_j}{\partial \vartheta}(r, x'', \vartheta)$$

for any $i = 1, \dots, m$, with (r, ϑ) as in Definition 1.2.

In particular if $m = 2$ then (1.4) implies that

$$(1.5) \quad \frac{\partial f_1}{\partial s_2}(|x'|, x'', U(x)) = \frac{\partial f_2}{\partial s_1}(|x'|, x'', U(x)), \quad \forall x \in \Omega$$

Remark 1.2. Under the assumptions of Theorem 1.2, the conditions (1.4) and (1.5) hold more generally for solutions having Morse index $m(U) \leq k - 1$ if for some $i_0, j_0 \in \{1, \dots, m\}$ the function $\frac{\partial f_{i_0}}{\partial s_{j_0}}(|x|, S)$ satisfies a strict convexity assumption as in Theorem 1.3 in [9].

The paper is organized as follows.

In Section 2 we recall suitable versions of weak and strong maximum principles as well as comparison principles for systems. Moreover we state some results from the spectral theory for an eigenvalue problem related to a symmetrized version of the system (1.1). Finally we define the Morse index. In Section 3 we give some sufficient conditions for k -sectional foliated Schwarz symmetry and prove Theorem 1.1, Theorem 1.2 and Corollary 1.1.

2. PRELIMINARIES

2.1. Spectral theory for linear elliptic systems.

Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, and D a $m \times m$ matrix with bounded entries:

$$(2.1) \quad D = (d_{ij})_{i,j=1}^m, \quad d_{ij} \in L^\infty(\Omega)$$

We consider the linear elliptic system

$$(2.2) \quad \begin{cases} -\Delta U + D(x)U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 + d_{11}u_1 + \cdots + d_{1m}u_m = f_1 & \text{in } \Omega \\ \dots\dots\dots & \dots \\ -\Delta u_m + d_{m1}u_1 + \cdots + d_{mm}u_m = f_m & \text{in } \Omega \\ u_1 = \cdots = u_m = 0 & \text{on } \partial\Omega \end{cases}$$

where $F = (f_1, \dots, f_m) \in (L^2(\Omega))^m$, $U = (U_1, \dots, U_m)$.

DEFINITION 2.1. The matrix D or the associated system (2.2) is said to be

- *cooperative* or *weakly coupled* in Ω if

$$(2.3) \quad d_{ij} \leq 0 \quad \text{a.e. in } \Omega, \quad \text{whenever } i \neq j$$

- *fully coupled* in Ω if it is weakly coupled in Ω and the following condition holds:

$$(2.4) \quad \begin{aligned} & \forall I, J \subset \{1, \dots, m\}, I, J \neq \emptyset, I \cap J = \emptyset, I \cup J = \{1, \dots, m\} \\ & \exists i_0 \in I, j_0 \in J : \text{meas}(\{x \in \Omega : d_{i_0 j_0} < 0\}) > 0 \end{aligned}$$

Before going on we fix some notations and definitions.

- Inequalities involving vectors should be understood to hold componentwise, e.g. if $\Psi = (\psi_1, \dots, \psi_m)$, Ψ nonnegative means that $\psi_j \geq 0$ for any index $j = 1, \dots, m$.
- If $m \geq 2$ and $1 \leq p \leq \infty$ we will consider the Banach spaces

$$\mathbf{L}^p(\Omega) = (L^p(\Omega))^m, \quad \mathbf{W}^{1,p}(\Omega) = (W^{1,p}(\Omega))^m$$

If $p = 2$ in particular we have the Hilbert spaces

$$\mathbf{L}^2(\Omega) = (L^2(\Omega))^m, \quad \mathbf{H}^1(\Omega) = (H^1(\Omega))^m$$

and the space $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^m$, i.e. the closure in $\mathbf{H}^1(\Omega)$ of the subspace $C_c^1(\Omega; \mathbb{R}^m)$. If $f = (f_1, \dots, f_m)$, $g = (g_1, \dots, g_m)$,

the related scalar products are

$$(2.5) \quad \begin{aligned} (f, g)_{\mathbf{L}^2(\Omega)} &= \sum_{i=1}^m (f_i, g_i)_{L^2(\Omega)} = \sum_{i=1}^m \int_{\Omega} f_i g_i dx \\ (f, g)_{\mathbf{H}_0^1(\Omega)} &= \sum_{i=1}^m (f_i, g_i)_{H_0^1(\Omega)} = \sum_{i=1}^m \int_{\Omega} \nabla f_i \cdot \nabla g_i dx \end{aligned}$$

- If $U = (u_1, \dots, u_m)$, $\Psi = (\psi_1, \dots, \psi_m) \in \mathbf{H}_0^1(\Omega)$, and the matrix D satisfies (2.1) we set

$$(2.6) \quad \nabla U \cdot \nabla \Psi = \sum_{i=1}^m \nabla u_i \cdot \nabla \psi_i$$

$$(2.7) \quad D(x)(U, \Psi) = \sum_{i,j=1}^m d_{ij}(x) u_i \psi_j$$

$$(2.8) \quad \begin{aligned} B(U, \Psi) &= B^D(U, \Psi) = \int_{\Omega} [\nabla U \cdot \nabla \Psi + D(U, \Psi)] dx \\ &= \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \psi_i + \sum_{i,j=1}^m d_{ij} u_i \psi_j \right] dx \end{aligned}$$

i.e. $D(x)(U, \Psi)$ is the action of the bilinear form associated to the matrix D on the pair (U, Ψ) , and B is the bilinear form in $\mathbf{H}_0^1(\Omega)$ associated to the operator $-\Delta + D$.

- If $U = (u_1, \dots, u_m) \in \mathbf{H}^1(\Omega)$ we say that U weakly satisfies

$$U \leq 0 \text{ on } \partial\Omega \quad (U \geq 0 \text{ on } \partial\Omega)$$

if $U^+ \in \mathbf{H}_0^1(\Omega)$ ($U^- \in \mathbf{H}_0^1(\Omega)$), i.e. if $u_i^+ \in H_0^1(\Omega)$ ($u_i^- \in H_0^1(\Omega)$) for any $i = 1, \dots, m$.

- If $U = (u_1, \dots, u_m) \in \mathbf{H}^1(\Omega)$ and D satisfies (2.1) we say that U weakly satisfies the inequality

$$(2.9) \quad -\Delta U + D(x)U \geq 0 \text{ in } \Omega$$

if

$$(2.10) \quad \int_{\Omega} \nabla U \cdot \nabla \Psi + D(x)(U, \Psi) = \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \psi_i + \sum_{i,j=1}^m d_{ij}(x) u_i \psi_j \right] dx \geq 0$$

for any nonnegative $\Psi = (\psi_1, \dots, \psi_m) \in \mathbf{H}_0^1(\Omega)$ (which is equivalent to require that $\int_{\Omega} \left(\nabla u_i \cdot \nabla \psi + \sum_{j=1}^m d_{ij} u_j \psi \right) dx \geq 0$ for any $\psi \in H_0^1(\Omega)$ with $\psi \geq 0$ and any $i = 1, \dots, m$).

It is well known that either condition (2.3) or conditions (2.3) and (2.4) together are needed in the proofs of maximum principles for systems (see [11], [13], [21] and the references therein). In particular if both are fulfilled the strong maximum principle holds as it is stated in the next theorem (see [8], [11], [13], [21] for the proof).

THEOREM 2.1. *(Strong Maximum Principle and Hopf's Lemma). Suppose that (2.1) and (2.3) hold and $U = (u_1, \dots, u_m) \in C^1(\Omega; \mathbb{R}^m)$ is a weak solution of the inequalities*

$$-\Delta U + D(x)U \geq 0 \text{ in } \Omega \quad \text{and} \quad U \geq 0 \text{ in } \Omega$$

Then:

- (1) for any $k \in \{1, \dots, m\}$ either $u_k \equiv 0$ or $u_k > 0$ in Ω ; in the latter case if $u_k \in C^1(\overline{\Omega}; \mathbb{R}^m)$, Ω satisfies the interior sphere condition at $P \in \partial\Omega$ and $u_k(P) = 0$ then $\frac{\partial u_k}{\partial \nu}(P) < 0$, where ν is the unit exterior normal vector at P .
- (2) if in addition (2.4) holds, then the same alternative holds for all $k = 1, \dots, m$, i.e. either $U \equiv 0$ in Ω or $U > 0$ in Ω . In the latter case if $U \in C^1(\overline{\Omega}; \mathbb{R}^m)$, Ω satisfies the interior sphere condition at $P \in \partial\Omega$ and $U(P) = 0$ then $\frac{\partial U}{\partial \nu}(P) < 0$, where ν is the unit exterior normal vector at P .

Together with the bilinear form (2.8) we consider the quadratic form

$$(2.11) \quad Q(\Psi) = B^D(\Psi, \Psi) = \int_{\Omega} (|\nabla \Psi|^2 + D(x)(\Psi, \Psi)) dx = \int_{\Omega} \left(\sum_{i=1}^m |\nabla \psi_i|^2 + \sum_{i,j=1}^m d_{ij}(x) \psi_i \psi_j \right) dx$$

for $\Psi = (\psi_1, \dots, \psi_m) \in \mathbf{H}_0^1(\Omega)$.

Sometimes we will also write $Q(\Psi; \Omega)$ instead of $Q(\Psi)$ specifying the domain.

It is easy to see that this quadratic form coincides with the quadratic form B^C associated to the symmetric linear operator $-\Delta + C$ where $C = \frac{1}{2}(D + D^t)$ and D^t is the transpose of D , i.e.

$$(2.12) \quad C = (c_{ij}), \quad c_{ij} = \frac{1}{2}(d_{ij} + d_{ji})$$

Let us observe that if the matrix D is cooperative, respectively fully coupled, so is the associate matrix D .

Thus, let us review some results for a symmetric linear operator $-\Delta + C$, with C such that

$$(2.13) \quad c_{ij} \in L^\infty(\Omega) \quad , \quad c_{ij} = c_{ji} \quad \text{a.e. in } \Omega$$

Let us consider the bilinear form

(2.14)

$$B(U, \Phi) = \int_{\Omega} [\nabla U \cdot \nabla \Phi + C(U, \Phi)] = \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \phi_i + \sum_{i,j=1}^m c_{ij} u_i \phi_j \right]$$

Using the theory of compact selfadjoint operators we get that there exists a sequence $\{\lambda_j\} = \{\lambda_j(-\Delta + C)\}$ of eigenvalues, with $-\infty < \lambda_1 \leq \lambda_2 \leq \dots$, $\lim_{j \rightarrow +\infty} \lambda_j = +\infty$, and a corresponding sequence of eigenfunctions $\{W^j\}$ which weakly solve the systems

$$(2.15) \quad \begin{cases} -\Delta W^j + CW^j = \lambda_j W^j & \text{in } \Omega \\ W^j = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. if $W^j = (w_1, \dots, w_m)$

$$\begin{cases} -\Delta w_1 + c_{11}w_1 + \dots + c_{1m}w_m & = \lambda_j w_1 \\ \dots & \dots \\ -\Delta w_m + c_{m1}w_1 + \dots + c_{mm}w_m & = \lambda_j w_m \end{cases}$$

Moreover by (scalar) elliptic regularity theory applied iteratively to each equation, the eigenfunctions W^j belong at least to $C^1(\Omega; \mathbb{R}^m)$ and the eigenvalues can be given a variational formulation. We refer to [6], [8] for the construction of the sequences λ_j and $\{W^j\}$ as well as for the proof of the following theorem, which gives some variational properties of eigenvalues and eigenfunctions.

THEOREM 2.2. *Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$. Suppose that $C = (c_{ij})_{i,j=1}^m$ satisfies (2.13), and let $\{\lambda_j\}$, $\{W^j\}$ be the sequences of eigenvalues and eigenfunctions satisfying (2.15).*

Define the Rayleigh quotient

$$(2.16) \quad R(V) = \frac{Q(V)}{(V, V)_{L^2(\Omega)}} \quad \text{for } V \in \mathbf{H}_0^1(\Omega), \quad V \neq 0$$

with $Q(V) = B(V, V)$ and B as in (2.14). Then the following properties hold, where \mathbf{H}_k denotes a k -dimensional subspace of $\mathbf{H}_0^1(\Omega)$ and the orthogonality conditions $V \perp W^k$ or $V \perp \mathbf{H}_k$ stand for the orthogonality in $L^2(\Omega)$.

i)

$$\lambda_1 = \min_{\substack{V \in \mathbf{H}_0^1(\Omega) \\ V \neq 0}} R(V) = \min_{\substack{V \in \mathbf{H}_0^1(\Omega) \\ (V, V)_{L^2} = 1}} Q(V)$$

ii) *if $k \geq 2$ then*

$$\lambda_k = \min_{\substack{V \in \mathbf{H}_0^1(\Omega) \\ V \neq 0 \\ V \perp W^1, \dots, V \perp W^{k-1}}} R(V) = \min_{\substack{V \in \mathbf{H}_0^1(\Omega) \\ (V, V)_{L^2(\Omega)} = 1 \\ V \perp W^1, \dots, V \perp W^{k-1}}} Q(V)$$

$$= \min_{\mathbf{H}_k} \max_{\substack{V \in \mathbf{H}_k \\ V \neq 0}} R(V) = \max_{\mathbf{H}_{k-1}} \min_{\substack{V \perp \mathbf{H}_{k-1} \\ V \neq 0}} R(V)$$

- iii) if $W \in \mathbf{H}_0^1(\Omega)$, $W \neq 0$, and $R(W) = \lambda_1$, then W is an eigenfunction corresponding to λ_1 .
- iv) if the system is fully coupled in Ω , then the first eigenfunction does not change sign in Ω and the first eigenvalue is simple, i.e. up to scalar multiplication there is only one eigenfunction corresponding to the first eigenvalue.
- v) if $X = (C_c^1(\Omega))^m$ and we denote by X_k a k -dimensional subspace of X then

$$\lambda_k = \inf_{X_k} \max_{\substack{v \in X_k \\ v \neq 0}} R(v)$$

- vi) let us consider an open subset $\Omega' \subset \Omega$ and set $\lambda_1(\Omega') = \lambda_1(-\Delta + C; \Omega')$. Then

$$\lim_{\text{meas}(\Omega') \rightarrow 0} \lambda_1(\Omega') = +\infty$$

2.2. Weak Maximum Principle for Cooperative systems.

Let us turn back to the (possibly) nonsymmetric cooperative system (2.2) with D satisfying (2.1) and (2.3). We consider the associated symmetric matrix C given by (2.12) and we denote by $\lambda_j^{(s)} = \lambda_j^{(s)}(-\Delta + D; \Omega)$ the eigenvalues of the corresponding symmetric linear operator $-\Delta + C$ and by $W_j^{(s)}$ the corresponding eigenfunctions.

The eigenvalues $\lambda_j^{(s)}$ will be called **symmetric eigenvalues** of the (possibly nonsymmetric) operator $-\Delta + D$.

The bilinear form corresponding to the symmetric operator will be denoted by $B^s(U, \Phi)$, i.e. B^s is as (2.14).

As already remarked, the quadratic form (2.11) corresponding to the linear operator $-\Delta + D$ coincides with that associated to the symmetric linear operator $-\Delta + C$.

DEFINITION 2.2. We say that the maximum principle holds for the operator $-\Delta + D$ in an open set $\Omega' \subseteq \Omega$ if for any $U \in \mathbf{H}^1(\Omega')$ such that $U \leq 0$ on $\partial\Omega'$ (i.e. $U^+ \in \mathbf{H}_0^1(\Omega')$) and $-\Delta U + D(x)U \leq 0$ in Ω' (i.e. $\int \nabla U \cdot \nabla \Phi + D(x)(U, \Phi) \leq 0$ for any nonnegative $\Phi \in \mathbf{H}_0^1(\Omega')$) it holds that $U \leq 0$ a.e. in Ω .

Let us denote by $\lambda_j^{(s)}(\Omega')$, $j \in \mathbb{N}^+$, the sequence of the symmetric eigenvalues of the linear operator $-\Delta + D$ (i.e. the eigenvalues of $-\Delta + C$) in an open set $\Omega' \subseteq \Omega$.

THEOREM 2.3. [Sufficient condition for weak maximum principle] Under the hypothesis (2.1) and (2.3), if $\lambda_1^{(s)}(\Omega') > 0$ then the maximum principle holds for the operator $-\Delta + D$ in $\Omega' \subseteq \Omega$.

Proof. By the variational characterization of the first eigenvalue given in Theorem 2.2 we have

$$\lambda_1^{(s)} = \min_{\substack{V \in \mathbf{H}^1(\Omega') \\ V \neq 0}} R(V) > 0$$

so that $Q(V) = B^s(V, V) > 0$ for any $V \neq 0$ in $\mathbf{H}_0^1(\Omega')$.

Assume that $U \leq 0$ on $\partial\Omega'$ and $-\Delta U + D(x)U \leq 0$ in Ω' . Then, testing the equation with $U^+ = (u_1^+, \dots, u_m^+)$, writing in the i -th equation $u_j = u_j^+ - u_j^-$ for $i \neq j$, and recalling that $-c_{ij}u_i^+u_j^- \geq 0$ if $i \neq j$, we obtain that $B^s(U^+, U^+) \leq 0$, which implies $U^+ \equiv 0$ in Ω' . \square

As an almost immediate consequence we get a quick proof of the following "classical" and "small measure" forms of the weak maximum principle (see [5], [13], [19], [21]).

THEOREM 2.4. *Assume that (2.1) and (2.3) hold.*

- i) *If D is a.e. nonnegative definite in Ω' then the maximum principle holds for $-\Delta + D$ in Ω' .*
- ii) *There exists $\delta > 0$, depending on D , such that for any subdomain $\Omega' \subseteq \Omega$ the maximum principle holds for $-\Delta + D$ in $\Omega' \subseteq \Omega$ provided $|\Omega'| \leq \delta$.*

Proof. i) If the matrix D is nonnegative definite then

$$Q(\Psi) = B^s(\Psi, \Psi) \geq \int_{\Omega} |\nabla \Psi|^2 > 0 \text{ for any } \Psi \in \mathbf{H}_0^1(\Omega') \setminus \{0\}.$$

Hence $\lambda_1^{(s)}(\Omega') > 0$, and by Theorem 2.3 we get i).

ii) It is a consequence of Theorem 2.3 and Theorem 2.2, vi). \square

Remark 2.1. Obviously the converse of Theorem 2.3 holds if $D = C$ i.e. if D is symmetric: if the maximum principle holds for $-\Delta + C$ in Ω' then $\lambda_1^{(s)}(\Omega') > 0$. Indeed if $\lambda_1^{(s)}(\Omega') \leq 0$, since the system is cooperative (and symmetric), there exists a corresponding nontrivial nonnegative first eigenfunction $\Phi_1 \geq 0$, $\Phi \not\equiv 0$, and the maximum principle does not hold, since $-\Delta\Phi_1 + C\Phi_1 = \lambda_1\Phi_1 \leq 0$ in Ω' , $\Phi_1 = 0$ on $\partial\Omega'$, while $\Phi_1 \geq 0$ and $\Phi_1 \neq 0$. However the converse of Theorem 2.3 is not true for general nonsymmetric systems, since there is an equivalence between the validity of the maximum principle for the operator $-\Delta + D$ and the positivity of another eigenvalue, the *principal eigenvalue* $\tilde{\lambda}_1$, (we recall below the definition given in [3]), and the inequality $\tilde{\lambda}_1(\Omega') \geq \lambda_1^{(s)}(\Omega')$, which can be strict, holds.

DEFINITION 2.3. The **principal eigenvalue** of the operator $-\Delta + D$ in an open set $\Omega' \subseteq \Omega$ is defined as

$$(2.17) \quad \begin{aligned} \tilde{\lambda}_1(\Omega') = \sup \{ \lambda \in \mathbb{R} : \exists \Psi \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m) \text{ s.t.} \\ \Psi > 0, -\Delta\Psi + D(x)\Psi - \lambda\Psi \geq 0 \text{ in } \Omega' \} \end{aligned}$$

Let us recall some of the properties of the principal eigenvalue. We refer to [5] for the proofs of items i) – iii), as well as for references on the subject, and to [6], [8] for the proof of iv).

THEOREM 2.5. *Assume that the matrix D is fully coupled in an open set $\Omega' \subseteq \Omega$, i.e. (2.4) holds. Then:*

- i) *there exists a positive eigenfunction $\Psi_1 \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m)$ which satisfies*

$$(2.18) \quad \begin{cases} -\Delta \Psi_1 + D(x)\Psi_1 = \tilde{\lambda}_1(\Omega')\Psi_1 & \text{in } \Omega' \\ \Psi_1 > 0 & \text{in } \Omega' \\ \Psi_1 = 0 & \text{on } \partial\Omega' \end{cases}$$

Moreover the principal eigenvalue is simple, i.e. any function that satisfy (2.18) must be a multiple of Ψ_1

- ii) *the maximum principle holds for the operator $-\Delta + D$ in Ω' if and only if $\tilde{\lambda}_1(\Omega') > 0$*
- iii) *if there exists $\Psi \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m)$ such that $\Psi > 0$ and $-\Delta\Psi + D(x)\Psi \geq 0$ in Ω' , then either $\tilde{\lambda}_1(\Omega') > 0$ or $\tilde{\lambda}_1(\Omega') = 0$ and $\Psi = c\Psi_1$ for some $c > 0$*
- iv) *$\tilde{\lambda}_1(\Omega') \geq \lambda_1^{(s)}(\Omega')$, with equality if and only if Ψ_1 is also the first eigenfunction of the symmetric operator $-\Delta + C$ in Ω' , $C = \frac{1}{2}(D + D^t)$. If this is the case the equality $C(x)\Psi_1 = D(x)\Psi_1$ holds and, if $m = 2$, this implies that $d_{12} = d_{21}$.*

2.3. Comparison principles for semilinear elliptic systems.

Let us consider a semilinear elliptic system of the type

$$(2.19) \quad \begin{cases} -\Delta U = F(x, U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 = f_1(x, u_1, \dots, u_m) & \text{in } \Omega \\ \dots\dots\dots & \dots \\ -\Delta u_m = f_m(x, u_1, \dots, u_m) & \text{in } \Omega \\ u_1 = \dots = u_m = 0 & \text{on } \partial\Omega \end{cases}$$

for the unknown vector valued function $U = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$, where Ω is a bounded domain in \mathbb{R}^N and $F = (f_1, \dots, f_m) : \bar{\Omega} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a C^1 function.

A weak solution of (2.19) is a function $U \in \mathbf{H}_0^1(\Omega)$ such that the function $x \mapsto F(x, U(x))$ belongs to $\mathbf{L}^q(\Omega)$, with $q > 1$ if $N = 2$, $q = \frac{2N}{N+2}$ if $N \geq 3$ (note that $\frac{2N}{N+2}$ is the conjugate exponent of the critical Sobolev exponent $2^* = \frac{2N}{N-2}$) and

$$(2.20) \quad \int_{\Omega} \nabla U \cdot \nabla \Phi \, dx = \int_{\Omega} F(x, U) \cdot \Phi \, dx \quad \forall \Phi \in \mathbf{H}_0^1(\Omega)$$

If $U, V \in \mathbf{H}^1(\Omega)$ we write $U \leq V$ on $\partial\Omega$, if the difference $U - V$ weakly satisfies the inequality $U - V \leq 0$ on $\partial\Omega$, i.e. if $(U - V)^+ \in \mathbf{H}_0^1(\Omega)$.

Moreover we say that U satisfies in a weak sense the inequality

$$(2.21) \quad -\Delta U \geq (\leq) F(x, U) \quad \text{in } \Omega$$

if for any $i = 1, \dots, m$ the component u_i of U weakly satisfies

$$-\Delta u_i \geq (\leq) f_i(x, U) \quad \text{in } \Omega, \quad \text{i.e.}$$

$$(2.22) \quad \int_{\Omega} \nabla u_i \cdot \nabla \varphi \, dx \geq (\leq) \int_{\Omega} f_i(x, U) \varphi \, dx$$

for any $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ in Ω . This is equivalent to require that

$$(2.23) \quad \int_{\Omega} \nabla U \cdot \nabla \Phi \, dx \geq (\leq) \int_{\Omega} F(x, U) \cdot \Phi \, dx$$

for any $\Phi \in \mathbf{H}_0^1(\Omega)$ with $\Phi \geq 0$ in Ω .

DEFINITION 2.4. We say that the system (2.19) is

- **cooperative or weakly coupled** in an open set $\Omega' \subseteq \Omega$ if

$$(2.24) \quad \frac{\partial f_i}{\partial s_j}(x, s_1, \dots, s_m) \geq 0 \quad \text{for every } (x, s_1, \dots, s_m) \in \Omega' \times \mathbb{R}^m$$

and every $i, j = 1, \dots, m$ with $i \neq j$.

- **fully coupled** in an open set $\Omega' \subseteq \Omega$ along $U \in \mathbf{H}_0^1(\Omega) \cap C^0(\Omega; \mathbb{R}^m)$ if it is cooperative in Ω' and in addition $\forall I, J \subset \{1, \dots, m\}$ such that $I \neq \emptyset, J \neq \emptyset, I \cap J = \emptyset, I \cup J = \{1, \dots, m\}$ there exist $i_0 \in I, j_0 \in J$ such that

$$(2.25) \quad \text{meas}(\{x \in \Omega' : \frac{\partial f_{i_0}}{\partial s_{j_0}}(x, U(x)) > 0\}) > 0$$

As a consequence of Theorem 2.4 and Theorem 2.1 the following comparison principles hold (see [8] for the proof).

THEOREM 2.6 (Weak comparison principle in small domains for systems). *Let Ω be a domain in \mathbb{R}^N , $F : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ a C^1 function and assume that (2.24) holds. Let $A > 0$ and $U, V \in \mathbf{H}^1(\Omega) \cap L^\infty(\Omega)$ such that*

$$\|U\|_{L^\infty(\Omega)} \leq A, \quad \|V\|_{L^\infty(\Omega)} \leq A$$

Then there exists $\delta > 0$, depending on F and A such that the following holds:

if $\Omega' \subseteq \Omega$ is a bounded subdomain of Ω , $\text{meas}_N([u > v] \cap \Omega') < \delta$ and

$$(2.26) \quad \begin{cases} -\Delta U \leq F(x, U), & -\Delta V \geq F(x, V) & \text{in } \Omega' \\ U \leq V & & \text{on } \partial\Omega' \end{cases}$$

then $U \leq V$ in Ω' .

THEOREM 2.7 (Strong Comparison Principle for systems). *Let Ω be a (bounded or unbounded) domain in \mathbb{R}^N , and let $U, V \in C^1(\Omega)$ weakly satisfy*

$$(2.27) \quad \begin{cases} -\Delta U \leq F(x, U) & ; & -\Delta v \geq F(x, V) & \text{in } \Omega \\ U \leq V & & & \text{in } \Omega \end{cases}$$

where $F(x, U) : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a C^1 function and (2.24) holds.

- (1) For every $i \in \{1, \dots, m\}$ the following holds: either $u_i \equiv v_i$ in Ω or $u_i < v_i$ in Ω and, in the latter case, if $u_i, v_i \in C^1(\overline{\Omega})$, $u_i(x_0) = v_i(x_0)$ at a point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied then $\frac{\partial u_i}{\partial s}(x_0) < \frac{\partial v_i}{\partial s}(x_0)$ for any inward directional derivative.
- (2) If moreover $U \in C^1(\Omega; \mathbb{R}^m)$ is a solution of (2.19) and the system is fully coupled along U in Ω (i.e. also (2.25) with $\Omega' = \Omega$ holds) then either $U \equiv V$ in Ω or $U < V$ in Ω (i.e. the same alternative holds for any component u_i). In the latter case assume that $U, V \in C^1(\overline{\Omega})$ and let $x_0 \in \partial\Omega$ a point where $U(x_0) = V(x_0)$ and the interior sphere condition is satisfied. Then $\frac{\partial U}{\partial s}(x_0) < \frac{\partial V}{\partial s}(x_0)$ for any inward directional derivative.

2.4. Morse index of a solution.

DEFINITION 2.5.

- i) Let $U \in \mathbf{H}_0^1(\Omega) \cap \mathbf{L}^\infty(\Omega)$ be a weak solution of (1.1). We say that U is linearized stable (or has zero Morse index) if the quadratic form

$$(2.28) \quad Q_U(\Psi; \Omega) = \int_{\Omega} [|\nabla \Psi|^2 - J_F(x, U(x))(\Psi, \Psi)] dx = \int_{\Omega} \left[\sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial s_j}(x, U(x)) \psi_i \psi_j \right] dx \geq 0$$

for any $\Psi = (\psi_1, \dots, \psi_m) \in C_c^1(\Omega; \mathbb{R}^m)$ where $J_F(x, U(x))$ is the jacobian matrix of $F(x, S)$ with respect to the variables $S = (s_1, \dots, s_m)$ computed at $S = U(x)$.

- ii) U has (linearized) Morse index equal to the integer $m = m(U) \geq 1$ if m is the maximal dimension of a subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.
- iii) U has infinite (linearized) Morse index if for any integer k there exists a k -dimensional subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.

The crucial, simple remark that allowed to extend some of the symmetry results known for equations to the case of systems in [6] and [9], is that the quadratic form associated to the linearized operator at a

solution U , i.e. to the linear operator

$$(2.29) \quad L_U(V) = -\Delta V - J_F(x, U)V$$

which in general is not selfadjoint, coincides with the quadratic form corresponding to the selfadjoint operator

$$(2.30) \quad L_U^s(V) = -\Delta V - \frac{1}{2} (J_F(x, U) + J_F^t(x, U)) V$$

where J_F^t is the transpose of the matrix J_F .

Therefore the *symmetric eigenvalues* of L , i.e. the eigenvalues of L_U^s , as defined in Section 2.2 can be exploited to study the symmetry of the solution U , using the information on its Morse index.

As in section 2.2 we denote by $\lambda_k^s = \lambda_k(-\Delta - \frac{1}{2}(J_F(x, U) + J_F^t(x, U)); \Omega)$ and W^k , $k \in \mathbb{N}^+$, the *symmetric eigenvalues* and eigenfunctions of $L_U = -\Delta V - J_F(x, U)$ in an open set Ω . Then we have

PROPOSITION 2.1. *Let Ω be a bounded domain in \mathbb{R}^N . Then the Morse index of a solution U to (2.19) equals the number of negative symmetric eigenvalues of the linearized operator L_U .*

Proof. Let us denote by $\mu(U)$ the number of negative symmetric eigenvalues of L_U . If the quadratic form Q_U defined in (2.28) is negative definite on a k -dimensional subspace of $C_c^1(\Omega \cup \Gamma)$, then, by *iii*) of Theorem 2.2 we have that the k -th eigenvalue $\lambda_k(-\Delta + C; \Omega)$ is negative. Hence $\mu(u) \geq m(u)$. On the other hand if there are k negative symmetric eigenvalues, by *v*) of Theorem 2.2 there is a k -dimensional subspace of $C_c^1(\Omega \cup \Gamma)$ where the quadratic form Q_u is negative definite, hence $m(u) \geq \mu(u)$. \square

3. PROOF OF THE SYMMETRY RESULTS

3.1. On the k -sectional foliated Schwarz symmetry.

From now on we will consider the case of system (1.2) in a bounded k -rotationally symmetric domain. Let us fix some notations.

For a unit vector $e \in S^{N-1}$ we consider the hyperplane

$$H(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}$$

orthogonal to the direction e and the open half domain

$$\Omega(e) = \{x \in \Omega : x \cdot e > 0\}$$

We then set

$$\sigma_e(x) = x - 2(x \cdot e)e, \quad x \in \Omega,$$

i.e. $\sigma_e : \Omega \rightarrow \Omega$ is the *reflection with respect to the hyperplane $H(e)$* . Finally if $U : \Omega \rightarrow \mathbb{R}^m$ is a continuous function we define the *reflected function* $U^{\sigma(e)} : \Omega \rightarrow \mathbb{R}^m$ defined by

$$U^{\sigma(e)}(x) = U(\sigma_e(x))$$

We will use in the sequel the rotating plane method, a variant of the moving plane method in the version of Berestycki and Nirenberg (see [2]), exploited e.g. in [18] and subsequent papers on foliated Schwarz symmetry of solutions of equations.

THEOREM 3.1 (Rotating Planes method for systems). *Let Ω be a bounded k -rotationally symmetric domain in \mathbb{R}^N , $F \in C^1([0, \infty) \times \mathbb{R}^{N-k} \times \mathbb{R}^m; \mathbb{R}^m)$ and $U \in \mathbf{H}_0^1(\Omega) \cap C^0(\bar{\Omega}; \mathbb{R}^m)$ a weak solution of (1.2). Assume that the system (1.2) is fully coupled along U in Ω and there exists a direction $e_{\vartheta_0} = (\cos(\vartheta_0), \sin(\vartheta_0), 0, \dots, 0)$ such that*

$$U < U^{\sigma(e_{\vartheta_0})} \quad \text{in } \Omega(e_{\vartheta_0})$$

Then there exists a direction $e_{\vartheta_1} = (\cos(\vartheta_1), \sin(\vartheta_1), 0, \dots, 0)$, with $\vartheta_1 > \vartheta_0$, such that

$$U \equiv U^{\sigma(e_{\vartheta_1})} \quad \text{in } \Omega(e_{\vartheta_1})$$

and

$$U < U^{\sigma(e_{\vartheta})} \quad \text{in } \Omega(e_{\vartheta}) \quad \forall \vartheta \in (\vartheta_0, \vartheta_1)$$

Proof. Let us observe that the functions $U^{\sigma(e_{\vartheta})}$ satisfy the same equation as U , namely $-\Delta U^{\sigma(e_{\vartheta})} = F(|x'|, x'', U^{\sigma(e_{\vartheta})})$ in Ω , and both $\|U\|_{L^\infty(\Omega(e_{\vartheta}))}$ and $\|U^{\sigma(e_{\vartheta})}\|_{L^\infty(\Omega(e_{\vartheta}))}$ are bounded by $\|U\|_{L^\infty(\Omega)} =: A$.

Let us fix $\delta = \delta(A)$ as in Theorem 2.6 and observe that δ is independent of ϑ , and the functions $U, U^{\sigma(e_{\vartheta})}$ satisfy

$$(3.1) \quad \begin{cases} -\Delta U = F(|x'|, x'', U) & ; & -\Delta U^{\sigma(e_{\vartheta})} = F(|x'|, x'', U^{\sigma(e_{\vartheta})}) & \text{in } \Omega(e_{\vartheta}) \\ U = U^{\sigma(e_{\vartheta})} & & & \text{on } \partial\Omega(e_{\vartheta}) \end{cases}$$

Let us set $\Theta = \{\vartheta \geq \vartheta_0 : U < U^{\sigma(e_{\vartheta'})}$ in $\Omega(e_{\vartheta'}) \forall \vartheta' \in (\vartheta_0, \vartheta)\}$ and let us show that the set Θ is nonempty and contains an interval $[\vartheta_0, \vartheta_0 + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Indeed we can take a compact set $K \subset \Omega(e_{\vartheta_0})$ such that $|\Omega(e_{\vartheta_0}) \setminus K| \leq \frac{\delta}{2}$ and $m = \min_K(U^{\sigma(e_{\vartheta_0})} - U) > 0$. By continuity if ϑ is close to ϑ_0 we have that $K \subset \Omega(e_{\vartheta})$, $(U^{\sigma(e_{\vartheta})} - U) \geq \frac{m}{2} > 0$ in K , $|\Omega(e_{\vartheta}) \setminus K| \leq \delta$ and $(U^{\sigma(e_{\vartheta})} - U) \geq 0$ on $\partial(\Omega(e_{\vartheta}) \setminus K)$. Then by the weak comparison principle in small domains (Theorem 2.6) we get that $U \leq U^{\sigma(e_{\vartheta})}$ in $\Omega' = \Omega(e_{\vartheta}) \setminus K$ and hence in $\Omega(e_{\vartheta})$. Moreover $U < U^{\sigma(e_{\vartheta})}$ in $\Omega(e_{\vartheta})$ by the strong comparison principle (Theorem 2.7). So the set Θ is nonempty, and is bounded from above by $\vartheta_0 + \pi$, since, considering the opposite direction, the inequality between U and the reflected function gets reversed. Let us set $\vartheta_1 = \sup \Theta$.

We claim that $U \equiv U^{\sigma(e_{\vartheta_1})}$ in $\Omega(e_{\vartheta_1})$. Indeed, if this is not the case, we get $U < U^{\sigma(e_{\vartheta_1})}$ in $\Omega(e_{\vartheta_1})$ by the strong comparison principle (Theorem 2.7), since by continuity $U \leq U^{\sigma(e_{\vartheta_1})}$ in $\Omega(e_{\vartheta_1})$. Then, using again the weak comparison principle in small domains and the previous technique we get $U < U^{\sigma(e_{\vartheta})}$ in $\Omega(e_{\vartheta})$ for $\vartheta > \vartheta_1$ and close to ϑ_1 , contradicting the definition of ϑ_1 . \square

Let us define, with a little abuse of notations,

$$(3.2) \quad S^{k-1} = \{e \in S^{N-1} : e \cdot e_j = 0, j = k+1, \dots, N\}$$

A sufficient condition for the k -sectional foliated Schwarz symmetry is the following.

PROPOSITION 3.1. *Let Ω be a k -rotationally symmetric domain in \mathbb{R}^N , $2 \leq k \leq N$, and $U \in \mathbf{H}_0^1(\Omega) \cap C^0(\overline{\Omega})$ a weak solution of (1.2) where $F = F(r, x'', S) \in C^1([0, \infty) \times \Omega'' \times \mathbb{R}^m; \mathbb{R}^m)$. Assume that the system is fully coupled along U in Ω and that $\forall e \in S^{k-1}$*

$$(3.3) \quad \text{either } U \geq U^{\sigma(e)} \text{ or } U \leq U^{\sigma(e)} \text{ in } \Omega(e)$$

Then U is k -sectionally foliated Schwarz symmetric.

The proof is similar to the one given, for the case $k = N$, in [6], with some obvious change.

Let us consider a pair of orthogonal directions $\eta_1, \eta_2 \in S^{k-1}$, the polar coordinates (ρ, ϑ) in the plane spanned by them and the corresponding cylindrical coordinates $(\rho, \vartheta, \tilde{y})$, with $\tilde{y} \in \mathbb{R}^{N-2}$. Then we define for $U \in C^2(\overline{\Omega}; \mathbb{R}^m)$ the angular derivative

$$(3.4) \quad U_\vartheta = U_{\vartheta(\eta_1, \eta_2)}$$

(trivially extended if $\rho = 0$) which solves the linearized system

$$(3.5) \quad \begin{cases} -\Delta U_\vartheta - J_F(|x|, U)U_\vartheta = 0 & \text{in } \Omega \\ U_\vartheta = 0 & \text{on } \partial\Omega \end{cases}$$

and, if $e \in \text{span}(\eta_1, \eta_2)$ and $U \equiv U^{\sigma(e)}$ in $\Omega(e)$, also the system

$$(3.6) \quad \begin{cases} -\Delta U_\vartheta - J_F(|x|, U)U_\vartheta = 0 & \text{in } \Omega(e) \\ U_\vartheta = 0 & \text{on } \partial\Omega(e) \end{cases}$$

Using the properties of the principal eigenvalue and of the corresponding eigenfunction we deduce, as in the case $k = N$, (see [6], [8], [9]), the following sufficient conditions for the k -sectionally foliated Schwarz symmetry.

THEOREM 3.2 (Sufficient conditions for sectional FSS-Systems). *Let Ω be a k -rotationally symmetric domain in \mathbb{R}^N , $2 \leq k \leq N$, and $U \in C^2(\overline{\Omega}; \mathbb{R}^m)$ a solution of (1.2), where $F \in C^1([0, R] \times \mathbb{R}^m; \mathbb{R}^m)$. Then U is k -sectionally foliated Schwarz symmetric provided one of the following conditions holds:*

- i) *there exists a direction $e \in S^{k-1}$ such that $U \equiv U^{\sigma(e)}$ in $\Omega(e)$ and the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ of the linearized operator $L_U = -\Delta - J_F(x, U)$ in $\Omega(e)$ is nonnegative.*
- ii) *there exists a direction $e \in S^{k-1}$ such that either $U < U^{\sigma(e)}$ or $U > U^{\sigma(e)}$ in $\Omega(e)$*

Remark 3.1. Let us observe that in Theorem 3.2 it is the nonnegativity of the *principal* eigenvalue the crucial hypothesis, while the information we get in the sequel will concern the *symmetric* eigenvalues of the linearized system. Therefore in the proofs that follow there will be an interplay and a comparison between the principal eigenvalue and the first symmetric eigenvalue in the cap $\Omega(e)$.

If U is a solution of (1.2), $e \in S^{k-1}$ and the system is fully coupled along U in Ω , then the difference $W = W^e = U - U^{\sigma(e)} = (w_1, \dots, w_m)$ satisfies a linear system in Ω , which is fully coupled in Ω and $\Omega(e)$:

LEMMA 3.1. *The following assertions hold.*

- i) *Assume that $U \in C^1(\bar{\Omega}; \mathbb{R}^m)$ is a solution of (1.2) and that the system is fully coupled along U in Ω . Let us define for any direction $e \in S^{k-1}$ the matrix $B^e(x) = (b_{ij}^e(x))_{i,j=1}^m$, where*

$$(3.7) \quad b_{ij}^e(x) = - \int_0^1 \frac{\partial f_i}{\partial s_j} (|x|, tU(x) + (1-t)U^{\sigma(e)}(x)) dt$$

Then for any $e \in S^{k-1}$ the function $W^e = U - U^{\sigma(e)}$ satisfies in $\Omega(e)$ the linear system

$$(3.8) \quad \begin{cases} -\Delta W^e + B^e(x)W^e & = 0 & \text{in } \Omega(e) \\ W^e & = 0 & \text{on } \partial\Omega(e) \end{cases}$$

which is fully coupled in $\Omega(e)$.

- ii) *If $\Psi = (\psi_1, \dots, \psi_m) \in \mathbf{H}_0^1(\Omega(e))$ let $Q^e(\Psi; \Omega(e))$ denote the quadratic form associated to the system (3.8) in $\Omega(e)$, i.e.*

$$(3.9) \quad \begin{aligned} Q^e(\Psi; \Omega(e)) &= \int_{\Omega(e)} (|\nabla \Psi|^2 + B^e(\Psi, \Psi)) dx \\ &= \int_{\Omega(e)} \left(\sum_{i=1}^m |\nabla \psi_i|^2 + \sum_{i,j=1}^m b_{ij}^e \psi_i \psi_j \right) dx \end{aligned}$$

Then

$$(3.10) \quad Q^e(W^e; \Omega(e)) = \int_{\Omega(e)} [|\nabla(W^e)|^2 + B^e(W^e, W^e)] dx = 0$$

while for the positive and negative parts of W^e the following holds:

$$(3.11) \quad Q^e((W^e)^\pm; \Omega(e)) = \int_{\Omega(e)} [|\nabla(W^e)^\pm|^2 + B^e((W^e)^\pm, (W^e)^\pm)] dx \leq 0$$

Proof. From the equation $-\Delta U = F(|x|, U(x))$ we deduce that the reflected function $U^{\sigma(e)}$ satisfies the equation $-\Delta U^{\sigma(e)} = F(|x|, U^{\sigma(e)}(x))$ and hence the difference $W^e = U - U^{\sigma(e)} = (w_1, \dots, w_m)$ satisfies

$$-\Delta W^e = F(|x|, U) - F(|x|, U^{\sigma(e)})$$

Let us set $V = U^{\sigma(e)}$. For any $i = 1, \dots, m$ we have that

$$f_i(|x|, U(x)) - f_i(|x|, V(x)) = \sum_{j=1}^m \int_0^1 \frac{\partial f_i}{\partial s_j} (|x|, tU(x) + (1-t)V(x)) (u_j(x) - v_j(x)) dt$$

As a consequence W^e satisfies (3.8). Moreover if $i \neq j$ then $b_{ij}^e(x) \leq 0$ by (2.24), so that the linear system (3.8) is weakly coupled.

If $U \in C^1(\bar{\Omega}; \mathbb{R}^m)$ is a solution of (1.2) and the system is fully coupled along U then the linear system associated to the matrix B^e is fully coupled in Ω . Indeed if $i_0 \neq j_0$ and $\frac{\partial f_{i_0}}{\partial s_{j_0}}(x, U(x)) > 0$ then, since $\frac{\partial f_i(y)}{\partial s_j} \geq 0$ for every $y \in \Omega$, we get that

$$b_{ij}(x) = - \int_0^1 \frac{\partial f_i}{\partial s_j} [|x|, tU(x) + (1-t)V(x)] dt < 0.$$

Since B^e is symmetric with respect to the reflection σ_e , (3.8) is fully coupled in $\Omega(e)$ as well and i) is proved.

To get (3.10) it is enough to multiply the i -th equation of the system for w_i and integrate. Instead, multiplying the i -th equation of (3.8) for w_i^+ , we get

$$0 = \int_{\Omega(e)} (|\nabla w_i^+|^2 + \sum_{j=1}^m b_{ij}^e w_j w_i^+) dx \geq \int_{\Omega(e)} (|\nabla w_i^+|^2 + \sum_{j=1}^m b_{ij}^e w_j^+ w_i^+) dx$$

since $w_i w_i^+ = |w_i^+|^2$, while $w_j w_i^+ \leq w_j^+ w_i^+$ and $b_{ij} \leq 0$ if $i \neq j$.

Summing on i we get

$$0 \geq \int_{\Omega(e)} \sum_{i=1}^m |\nabla w_i^+|^2 + \sum_{i,j=1}^m b_{ij}^e w_j^+ w_i^+ dx$$

i.e. (3.11) in the case of the positive part.

For the negative part we proceed analogously multiplying the i -th equation of (3.8) for w_i^- and integrating. We get

$$\begin{aligned} 0 &= - \int_{\Omega(e)} |\nabla w_i^-|^2 + \sum_{j=1}^m b_{ij}^e w_j w_i^- dx \leq - \int_{\Omega(e)} |\nabla w_i^-|^2 + \sum_{j=1}^m b_{ij}^e (-w_j^-) w_i^- dx \\ &= - \int_{\Omega(e)} |\nabla w_i^-|^2 - \sum_{j=1}^m b_{ij}^e (w_j^-) w_i^- dx \end{aligned}$$

since $w_i w_i^- = -|w_i^-|^2$, while $w_j w_i^- \geq -(w_j^-) w_i^-$ and $b_{ij} \leq 0$ if $i \neq j$.

Summing on i we obtain

$$0 \geq \int_{\Omega(e)} \sum_{i=1}^m |\nabla w_i^-|^2 + \sum_{i,j=1}^m b_{ij}^e w_j^- w_i^- dx$$

i.e. (3.11) in the case of the negative part. \square

Remark 3.2. Note that the inequalities in (3.11) could be strict. Indeed the products $w_i^+ w_j^-$ could be not identically zero if $i \neq j$, and therefore $Q(W^e)$ does not coincide in general with $Q((W^e)^+) + Q((W^e)^-)$, as it happens in the scalar case.

3.2. Nonlinearities having convex components.

We will prove Theorem 1.1 by several auxiliary results.

LEMMA 3.2. *Assume that U is a solution of (1.2) and that the hypotheses $i)$ – $ii)$ of Theorem 1.1 hold. Then for any direction $e \in S^{k-1}$*

$$Q_U((W^e)^+; \Omega(e)) \leq 0$$

where Q_U is the quadratic form defined in (2.28) and W^e is as in Lemma 3.1.

Proof. For any $i = 1, \dots, m$ we have

$$-\Delta w_i = f_i(|x|, U) - f_i(|x|, U^{\sigma(e)}) \quad \text{in } \Omega(e)$$

Testing the equation with w_i^+ we obtain

$$(3.12) \quad \int_{\Omega(e)} |\nabla(w_i^+)|^2 dx = \int_{\Omega(e)} (f_i(|x|, U) - f_i(|x|, U^{\sigma(e)})) w_i^+ dx$$

Observe that $f_i(|x|, S)$ is convex in S , so that

$$\begin{aligned} (f_i(|x|, U(x)) - f_i(|x|, U^{\sigma(e)}(x))) w_i^+ &\leq (\nabla f_i(|x|, U(x)) \cdot (U(x) - U^{\sigma(e)}(x))) w_i^+ \\ &= (\nabla f_i(|x|, U(x)) \cdot W^e) w_i^+ = \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U(x)) w_j w_i^+ \end{aligned}$$

where ∇ stands for the gradient of f_i with respect to the variables $S = (s_1, \dots, s_m)$. Moreover

$$\frac{\partial f_i}{\partial s_i} w_i w_i^+ = \frac{\partial f_i}{\partial s_i} |w_i^+|^2, \quad \text{while} \quad \frac{\partial f_i}{\partial s_j} w_j w_i^+ \leq \frac{\partial f_i}{\partial s_j} w_j^+ w_i^+ \quad \text{if } i \neq j$$

because $\frac{\partial f_i}{\partial s_j} \geq 0$ by the weak coupling assumption.

By (3.12), taking into account the previous inequalities, we get

$$\int_{\Omega(e)} |\nabla(w_i^+)|^2 dx \leq \int_{\Omega(e)} \sum_{j=1}^m \frac{\partial f_i}{\partial s_j}(|x|, U(x)) w_j^+ w_i^+ dx$$

Thus, summing on $i = 1, \dots, m$, we obtain

$$(3.13) \quad \int_{\Omega(e)} \left(\sum_{i=1}^m |\nabla(w_i^+)|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial s_j}(|x|, U(x)) w_i^+ w_j^+ \right) dx \leq 0$$

i.e. $Q_U((W^e)^+; \Omega(e)) \leq 0$. □

If $e \in S^{k-1}$ is a direction orthogonal to $\mathbf{e}_{k+1}, \dots, \mathbf{e}_N$ and $C = \frac{1}{2}(J_F(x) + J_F(x)^t)$, let us denote the eigenvalues and the eigenfunctions of the operator $-\Delta - C$ in the cap $\Omega(e)$ by

$$(3.14) \quad \lambda_k^e = \lambda_k(-\Delta - C; \Omega(e)) \quad ; \quad \Phi_k^e = \Phi_k(-\Delta - C; \Omega(e))$$

LEMMA 3.3. *Suppose that U is a solution of (1.2) with Morse index $m(U) \leq k$ and assume that the hypothesis i) of Theorem 1.1 holds. Then there exists a direction $e \in S^{k-1}$ such that $\lambda_1^e \geq 0$, hence also the corresponding principal eigenvalue $\tilde{\lambda}_1(L_U, \Omega(e))$ is nonnegative, by Theorem 2.5, so that*

$$Q_U(\Psi; \Omega(e)) \geq 0$$

for any $\Psi \in C_c^1(\Omega(e); \mathbb{R}^m)$.

Proof. The assertion is immediate if the Morse index of the solution satisfies $m(u) \leq 1$. Indeed in this case for any direction e at least one among λ_1^e and λ_1^{-e} must be nonnegative. Indeed if this would not be the case then the quadratic form $Q_U(\Psi) = \int_{\Omega} (|\nabla \Psi|^2 - C(x)(\Psi, \Psi)) dx$ would be negative definite on the 2-dimensional space spanned by the trivial extensions of the eigenfunctions Φ_1^e and Φ_1^{-e} and hence $m(u) \geq 2$.

So let us assume that $2 \leq j = m(u) \leq k$.

Denote by Φ_k the $L^2(\Omega)$ normalized eigenfunctions of the operator $L_U = -\Delta - C$ in Ω , with Φ_1 positive in Ω , and for any direction $e \in S^{k-1}$ let us consider the function

$$\Psi^e(x) = \begin{cases} \left(\frac{(\Phi_1^{-e}, \Phi_1)_{L^2(\Omega)}}{(\Phi_1^e, \Phi_1)_{L^2(\Omega)}} \right)^{\frac{1}{2}} \Phi_1^e(x) & \text{if } x \in \Omega(e) \\ - \left(\frac{(\Phi_1^e, \Phi_1)_{L^2(\Omega)}}{(\Phi_1^{-e}, \Phi_1)_{L^2(\Omega)}} \right)^{\frac{1}{2}} \Phi_1^{-e}(x) & \text{if } x \in \Omega(-e) \end{cases}$$

where Φ_1^e is the first positive L^2 -normalized eigenfunction in $\Omega(e)$, as in (3.14).

The mapping $e \mapsto \Psi_e$ is odd and continuous from S^{k-1} to $H_0^1(\Omega)$ and, by construction,

$$(3.15) \quad (\Psi^e, \Phi_1)_{L^2(\Omega)} = 0$$

The function $h : S^{k-1} \rightarrow \mathbb{R}^{j-1}$ defined by

$$(3.16) \quad h(e) = ((\Psi^e, \Phi_2)_{L^2(\Omega)}, \dots, (\Psi^e, \Phi_j)_{L^2(\Omega)})$$

is also odd and continuous. Since $(j-1) < k$, by the Borsuk-Ulam Theorem it must have a zero. This means that there exists a direction $e \in S^{k-1}$ such that Ψ^e is orthogonal to all the eigenfunctions Φ_1, \dots, Φ_j . Since $m(u) = j$, by Theorem 2.2 ii) we deduce that $Q_U(\Psi^e; \Omega) \geq 0$, which in turn implies that either $Q_U(\Phi_1^e; \Omega(e)) \geq 0$ or $Q_U(\Phi_1^{-e}; \Omega(-e)) \geq 0$, i.e. either λ_1^e or λ_1^{-e} is nonnegative, so the assertion is proved. \square

Proof of Theorem 1.1. By Lemma 3.3 there exists a direction $e \in S^{k-1}$ such that the first symmetric eigenvalue $\lambda_1^s(L_U, \Omega(e))$ of the linearized operator is nonnegative, so that the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ is nonnegative as well. Moreover by Lemma 3.2 we have that $Q_U((W^e)^+) \leq$

0, so that either $(W^e)^+ \equiv 0$, or $\lambda_1^s(L_U, \Omega(e)) = 0$ and $(W^e)^+$ is the positive first symmetric eigenfunction in $\Omega(e)$. In any case either $U \leq U^{\sigma(e)}$ or $U \geq U^{\sigma(e)}$ in $\Omega(e)$ holds.

Thus, by the strong maximum principle, either $U \equiv U^{\sigma(e)}$ in $\Omega(e)$, and the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ is nonnegative, or $U < U^{\sigma(e)}$ in $\Omega(e)$ or $U > U^{\sigma(e)}$ in $\Omega(e)$. Hence, by Theorem 3.2 U is foliated Schwarz symmetric. \square

Remark 3.3. In the previous proof when $(U - U^{\sigma(e)})^+ \equiv 0$ we also have by construction that $\lambda_1^s(L_u, \Omega(e)) \geq 0$ and therefore the principal eigenvalue satisfies $\tilde{\lambda}_1(\Omega(e)) = \tilde{\lambda}_1(\Omega(-e)) \geq 0$.

In the case when $U < U^{\sigma(e)}$ in $\Omega(e)$ or $U > U^{\sigma(e)}$ in $\Omega(e)$, by rotating the planes we find a different direction e' such that $U \equiv U^{\sigma(e')}$ in $\Omega(e')$ and it could happen that $\lambda_1^s(\Omega(e')) < 0$. However let us observe explicitly that the sign of the principal eigenvalue is preserved in the rotation, i.e. $\tilde{\lambda}_1(\Omega(e')) = \tilde{\lambda}_1(\Omega(-e')) \geq 0$, and actually $\tilde{\lambda}_1(\Omega(e')) = \tilde{\lambda}_1(\Omega(-e')) = 0$.

Indeed since $U < U^{\sigma(g)}$ for any direction g between e and e' , we have that 0 is the principal eigenvalue of the system satisfied by $U - U^{\sigma(g)}$, namely (3.8), with coefficients

$$b_{ij}^g(x) = - \int_0^1 \frac{\partial f_i}{\partial s_j} [|x|, tU(x) + (1-t)U^{\sigma(g)}(x)] dt$$

As $g \rightarrow e'$, where e' is the symmetry position, the coefficients b_{ij} approach the coefficients of the linearized system, namely $c_{ij} = - \frac{\partial f_i}{\partial s_j}$, so by continuity $\tilde{\lambda}_1(\Omega(e')) = \tilde{\lambda}_1(\Omega(-e')) = 0$.

3.3. Nonlinearities with convex derivatives.

The proof of Theorem 1.2 follows the scheme of the proof of Theorem 1.1, and it is based upon the following results.

LEMMA 3.4. *Assume that U is a solution of (1.2) and the hypotheses of Theorem 1.2 hold. Let $B^e(x) = (b_{ij}^e(x))_{i,j=1}^m$ be the matrix associated to the fully coupled system (3.8) defined by (3.7), i.e.*

$$b_{ij}^e(x) = - \int_0^1 \frac{\partial f_i}{\partial s_j} [|x|, tU(x) + (1-t)U^{\sigma(e)}(x)] dt$$

and let us define the matrix $B^{e,s}(x) = (b_{ij}^{e,s}(x))_{i,j=1}^m$, where

$$(3.17) \quad b_{ij}^{e,s}(x) = -\frac{1}{2} \left(\frac{\partial f_i}{\partial s_j}(|x|, U(x)) + \frac{\partial f_i}{\partial s_j}(|x|, U^{\sigma(e)}(x)) \right)$$

Then the linear system with matrix $B^{e,s}$ is fully coupled in Ω and $\Omega(e)$ for any $e \in S^{N-1}$. Moreover for any $i, j = 1, \dots, m$ and $x \in \Omega$ it holds

$$(3.18) \quad b_{ij}^e(x) \geq b_{ij}^{e,s}(x)$$

Finally for the quadratic forms Q^e and $Q^{e,s}$ associated to the matrices B^e and $B^{e,s}$ we have that

$$(3.19) \quad \begin{aligned} 0 &\geq Q^e((W^e)^\pm; \Omega(e)) = \int_{\Omega(e)} [|\nabla(W^e)^\pm|^2 + B^e((W^e)^\pm, (W^e)^\pm)] dx \\ &\geq \int_{\Omega(e)} [|\nabla(W^e)^\pm|^2 + B^{e,s}((W^e)^\pm, (W^e)^\pm)] dx = Q^{e,s}((W^e)^\pm; \Omega(e)) \end{aligned}$$

for $W^e = U - U^{\sigma(e)}$,

Proof. By hypothesis ii) of Theorem 1.2 we get

$$(3.20) \quad \begin{aligned} -b_{ij}^e(x) &= \int_0^1 \frac{\partial f_i}{\partial s_j} [|x|, tU(x) + (1-t)U^{\sigma(e)}(x)] dt \\ &\leq \int_0^1 \left(t \frac{\partial f_i}{\partial s_j} [|x|, U(x)] + (1-t) \frac{\partial f_i}{\partial s_j} [|x|, U^{\sigma(e)}(x)] \right) dt \\ &= \frac{1}{2} \left(\frac{\partial f_i}{\partial s_j} (|x|, U(x)) + \frac{\partial f_i}{\partial s_j} (|x|, U^{\sigma(e)}(x)) \right) = -b_{ij}^{e,s}(x) \end{aligned}$$

This implies (3.18) and hence the full coupling of the system with matrix $B^{e,s}$, since, by Lemma 3.1, the system with matrix B^e is fully coupled. From (3.11) and (3.18), since $w_k^\pm \geq 0$, we get

$$\begin{aligned} 0 &\geq \int_{\Omega(e)} \left(\sum_{i=1}^m |\nabla w_i^+|^2 + \sum_{i,j=1}^m b_{ij}^e w_j^+ w_i^+ \right) dx \geq \\ &\int_{\Omega(e)} \left(\sum_{i=1}^m |\nabla w_i^+|^2 + \sum_{i,j=1}^m b_{ij}^{e,s} w_j^+ w_i^+ \right) dx \end{aligned}$$

i.e. (3.19) in the case of the positive part of W^e . Analogously we get the corresponding inequality for the negative part of W^e . \square

LEMMA 3.5. *Suppose that U is a solution of (1.2) with Morse index $m(U) \leq k-1$ and assume that the hypothesis i) of Theorem 1.2 holds. Let $Q^{e,s}$ be the quadratic form associated to the operator $L^{e,s}(V) = -\Delta V + B^{e,s}V$, $B^{e,s}$ being the matrix defined in (3.17) :*

$$(3.21) \quad \begin{aligned} Q^{e,s}(\Psi; \Omega') &= \int_{\Omega'} [|\nabla \Psi|^2 + B^{e,s}(\Psi, \Psi)] dx = \\ &\int_{\Omega} \left[\sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{1}{2} \left(\frac{\partial f_i}{\partial s_j} (|x|, U(x)) + \frac{\partial f_i}{\partial s_j} (|x|, U^{\sigma(e)}(x)) \right) \psi_i \psi_j \right] dx \end{aligned}$$

Then there exists a direction $e \in S^{k-1}$ such that

$$Q^{e,s}(\Psi; \Omega(e)) \geq 0 \quad \forall \Psi \in C_c^1(\Omega(e); \mathbb{R}^m)$$

Equivalently the first symmetric eigenvalue $\lambda_1^s(L^{e,s}, \Omega(e))$ of the operator $L^{e,s}(V) = -\Delta V + B^{e,s}V$ in $\Omega(e)$ is nonnegative (and hence also the principal eigenvalue $\tilde{\lambda}_1(L^{e,s}, \Omega(e))$ is nonnegative).

Proof. Let us assume that $1 \leq j = m(U) \leq k-1$ and let Φ_1, \dots, Φ_j be mutually orthogonal eigenfunctions corresponding to the negative symmetric eigenvalues $\lambda_1^s(L_U, \Omega), \dots, \lambda_j^s(L_U, \Omega)$ of the linearized operator $L_U(V) = -\Delta V - J_F(x, U)V$ in Ω .

For any $e \in S^{k-1}$ let $\phi^{e,s}$ be the first positive L^2 -normalized eigenfunction of the symmetric system associated to the linear operator $L^{e,s}$ in $\Omega(e)$. We observe that $\phi^{e,s}$ is uniquely determined since the corresponding system is fully coupled in $\Omega(e)$. Let $\Phi^{e,s}$ be the odd extension of $\phi^{e,s}$ to Ω , and let us observe that $\Phi^{-e,s} = -\Phi^{e,s}$, because $B^{e,s}$ is symmetric with respect to the reflection σ_e .

The mapping $e \mapsto \Phi^{e,s}$ is a continuous odd function from S^{k-1} to $H_0^1(\Omega \cup \Gamma)$, therefore the mapping $h : S^{k-1} \rightarrow \mathbb{R}^j$ defined by

$$h(e) = ((\Phi^{e,s}, \Phi_1)_{L^2(\Omega)}, \dots, (\Phi^{e,s}, \Phi_j)_{L^2(\Omega)})$$

is an odd continuous mapping, and since $j \leq k-1$, by the Borsuk-Ulam Theorem it must have a zero. This means that there exists a direction $e \in S^{k-1}$ such that $\Phi^{e,s}$ is orthogonal to all the eigenfunctions Φ_1, \dots, Φ_j . This implies that $Q_U(\Phi^{e,s}; \Omega) \geq 0$, because $m(U) = j$, and since $\Phi^{e,s}$ is an odd function, we obtain that $0 \leq Q_U(\Phi^{e,s}; \Omega) = Q^{e,s}(\Phi^{e,s}, \Omega) = 2Q^{e,s}(\phi^{e,s}, \Omega(e)) = 2\lambda_1^s(L^{e,s}, \Omega(e))$ \square

Proof of Theorem 1.2. By Lemma 3.5 there exists a direction e such that the first symmetric eigenvalue $\lambda_1^s(L^{e,s}, \Omega(e))$ of the operator $L^{e,s}(V) = -\Delta V + B^{e,s}V$ in $\Omega(e)$ is nonnegative, and hence also the principal eigenvalue $\tilde{\lambda}_1(L^{e,s}, \Omega(e))$ is nonnegative.

Since $Q^{e,s}((W^e)^\pm; \Omega(e)) \leq 0$ by Lemma 3.4, we have two alternatives. The first one is that $(W^e)^+$ and $(W^e)^-$ both vanish, in which case $W^e \equiv 0$ in $\Omega(e)$, and this implies in turn that $L^{e,s} = L_U$. Then U is symmetric and the principal eigenvalue $\tilde{\lambda}_1(L_U, \Omega(e))$ is nonnegative, so that the hypothesis i) of Theorem 3.2 holds and we get that U is foliated Schwarz symmetric. The second alternative is that one among $(W^e)^+$ and $(W^e)^-$ does not vanish and $\lambda_1^s(L^{e,s}, \Omega(e)) = 0$. Then either $(W^e)^+$ or $(W^e)^-$ is a first symmetric eigenfunction of the operator $L^{e,s}(V)$ in $\Omega(e)$. If $(W^e)^+$ is a first symmetric eigenfunction of the operator $L^{e,s}(V) = -\Delta V + B^{e,s}V$ in $\Omega(e)$ then it is positive in $\Omega(e)$, i.e. $U > U^{\sigma_e}$ in $\Omega(e)$. In the case when $(W^e)^-$ is the first symmetric eigenfunction we get the reversed inequality. Then, by the sufficient condition ii) given by Theorem 3.2, u is foliated Schwarz symmetric. \square

Proof of Corollary 1.1. By the proof of Theorem 1.1 and Remark 3.3 we can find $e \in S^{k-1}$ such that U is symmetric with respect to the hyperplane $H(e)$ and the principal eigenvalue $\tilde{\lambda}_1(\Omega(e)) = \tilde{\lambda}_1(\Omega(-e)) \geq 0$. As in the proof of Lemma 3.3 it is easy to see that if U is a Morse index one solution then for any direction e either $\lambda_1^s(L_U, \Omega(e))$ or $\lambda_1^s(L_U, \Omega(-e))$ must be nonnegative. On the other hand, by symmetry, $\lambda_1^s(\Omega(e)) = \lambda_1^s(\Omega(-e))$, so that $\tilde{\lambda}_1(\Omega(e)) = \tilde{\lambda}_1(\Omega(-e)) \geq \lambda_1^s(\Omega(e)) = \lambda_1^s(\Omega(-e)) \geq 0$.

Then, if $\tilde{\lambda}_1(\Omega(e)) > 0$, the angular derivative U_θ must vanish (since it satisfies (3.6) and the maximum principle holds in $\Omega(e)$). Hence U is radial.

So if $U_\theta \not\equiv 0$ necessarily $\tilde{\lambda}_1(\Omega(e)) = \lambda_1^s(\Omega(e)) = 0$ and by iv) of Proposition 2.5 the derivative U_θ is a negative first eigenfunction of the symmetrized system in $\Omega(e)$, as well as a solution of (3.6). Thus we get that

$$J_F(|x|, U)U_\theta = \frac{1}{2} (J_F(|x|, U) + J_F^t(|x|, U)) U_\theta$$

i.e. (1.4) and if $m = 2$ we get (1.5), since U_θ is strictly negative.

The proof in the case when the hypotheses of Theorem 1.2 hold is the same. \square

REFERENCES

- [1] T. Bartsch, T. Weth, M. Willem *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math 96, 2005, 1-18
- [2] H. Berestycki, L. Nirenberg, *On the method of moving Planes and the Sliding Method*, Bol. Soc. Bras. Mat. 22, 1991, pp. 1-22
- [3] H. Berestycki, L. Nirenberg, S.R.S Varadhan, *The principal eigenvalue and maximum principle for second order elliptic operators in general domains*, Comm.Pure Appl. Math. 47 (1), 1994, pp. 47-92
- [4] J. Busca, B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space*, Journal Diff. Eq. 163, 2000, pp. 41-56
- [5] J. Busca, B. Sirakov, *Harnack type estimates for nonlinear elliptic systems and applications*, Ann. Inst. H. Poincaré Anal. Non Lin. 21 (5), 2004, pp. 543-590
- [6] L. Damascelli, F. Pacella, *Symmetry results for cooperative elliptic systems via linearization*, SIAM J. Math. Anal. 45 (2013), no. 3, 1003-1026
- [7] L. Damascelli, F. Pacella, *Morse index and symmetry for elliptic problems with nonlinear mixed boundary conditions*, Proc. Royal S. Edinburgh Sect. A, 2019, <https://doi.org/10.1017/prm.2018.29>
- [8] L. Damascelli, F. Pacella, *Morse Index of Solutions of Semilinear Elliptic Equations and Applications*, De Gruyter Studium, 2019, in press
- [9] L. Damascelli, F. Gladiali, F. Pacella, *A symmetry result for semilinear cooperative elliptic systems*, in Recent trends in nonlinear partial differential equations. II. Stationary problems, Amer. Math. Soc., Providence, RI, Contemporary Mathematics 595 (2013), 187-204
- [10] L. Damascelli, F. Gladiali, F. Pacella, *Symmetry results for cooperative elliptic systems in unbounded domains*, Indiana Univ. Math. J. 63 No. 3 (2014), 615-649

- [11] D.G. de Figueiredo, *Monotonicity and symmetry of solutions of elliptic systems in general domains*, NoDEA 1, 1994, pp. 119-123
- [12] D.G. de Figueiredo, *Semilinear Elliptic Systems: existence, multiplicity, symmetry of solutions*, Handbook of Differential Equations: stationary partial differential equations. Vol. V Eq. 2008, 1-48
- [13] D.G. de Figueiredo, E. Mitidieri, *Maximum principles for linear elliptic systems*, Rend. Inst. Mat. Univ. Trieste, 1992, pp. 36-66
- [14] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68, 1979, pp. 209-243
- [15] F. Gladiali, F. Pacella, T. Weth, *Symmetry and Nonexistence of low Morse index solutions in unbounded domains*, J. Math. Pures Appl. (9) 93 (5), 2010, pp. 536-558.
- [16] F. Pacella, *Symmetry of Solutions to Semilinear Elliptic Equations with Convex Nonlinearities*, J. Funct. Anal. 192 (1), 2002, pp. 271-282
- [17] F. Pacella, M. Ramaswamy, *Symmetry of solutions of elliptic equations via maximum principle*, Handbook of Differential Equations: stationary partial differential equations. Vol. VI 2008, 269-312
- [18] F. Pacella, T. Weth, *Symmetry Results for Solutions of Semilinear Elliptic Equations via Morse index*, Proc. AMS 135 (6), 2007, pp 1753-1762
- [19] M.H. Protter, H.F. Weinberger, *Maximum Principle in Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.
- [20] D. Smets, M. Willem *Partial symmetry and asymptotic behaviour for some elliptic variational problem*, Calc. Var. Part. Diff. Eq. 18, 2003, 57-75
- [21] B. Sirakov, *Some estimates and maximum principles for weakly coupled systems of elliptic PDE*, Nonlinear Analysis 70 (8) , 2009, pp. 3039-3046
- [22] Troy, W.C. *Symmetry properties in Systems of Semilinear Elliptic Equations*, J. Differential Eq. 42, 1981, pp. 400-413
- [23] T. Weth, *Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods*, Jahresber. Dtsch. Math - Ver 112 , 2010, pp. 119-158

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