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Orlicz equi-integrability for scaled gradients

Piotr Antoni Kozarzewski[1,2] · **Elvira Zappale**[3]

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Abstract In the realm of $3D$–$2D$ dimensional reduction problems, we prove that, up to an extraction, it is possible to decompose a sequence (u_n), whose scaled gradients $\left(\nabla_\alpha u_n, \frac{1}{\varepsilon_n}\nabla_3 u_n\right)$ are bounded in $L^\Phi(\omega \times (-1,1), \mathbb{R}^{3\times 3})$ for a suitable Orlicz function Φ, as $u_n = v_n + z_n$, such that v_n describes the oscillations, $\left(\Phi\left(\left|\nabla_\alpha v_n, \frac{1}{\varepsilon_n}\nabla_3 v_n\right|\right)\right)$, is equi-integrable and the remainder z_n, accounting for concentration effects, converges to zero in measure. In particular, we extend to the Orlicz–Sobolev setting the results contained in Bocea and Fonseca, (ESAIM: COCV 7:443–470, 2002) and Braides and Zeppieri (Calc Var 29:231–238, 2007).

Keywords Equi-integrability · Scaled gradients · Orlicz–Sobolev spaces

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✉ Piotr Antoni Kozarzewski
piotr.kozarzewski@mimuw.edu.pl

Elvira Zappale
ezappale@unisa.it

[1] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

[2] Faculty of Cybernetics, Military University of Technology, ul. gen. Sylwestra Kaliskiego 2, 00-908 Warsaw, Poland

[3] Department of industrial Engineering, Universitá di Salerno, via Giovanni Paolo II, 132, 84084 Fisciano, SA, Italy

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1 Introduction

In the study of thin structures, i.e. when one or more dimensions are much smaller than the others, say of order $\varepsilon << 1$, rigorous analysis via dimensional reduction proves to be a useful tool to deduce properties of thin domains starting from thicker models. In this analysis one deals with sequences of functions defined on cylindrical sets with some thin (ε sized) dimension. In the $3D$ setting, thin films are modelled as $\omega \times (-\varepsilon, \varepsilon)$ with $\omega \subset \mathbb{R}^2$ a bounded open set. In order to perform an asymptotic analysis as $\varepsilon \to 0$, with the aim of deducing a theory settled in ω, functions are usually rescaled to an ε-independent reference configuration, so that a new sequence (u_ε) is constructed, satisfying, in the standard Sobolev setting, some 'degenerate' bounds of the form

$$\int_{\omega\times(-1,1)} \left(|\nabla_\alpha u_\varepsilon|^p + \frac{1}{\varepsilon^p} |\nabla_3 u_\varepsilon|^p \right) dx \leq C < +\infty, \tag{1}$$

if the sequence of unscaled gradients (∇w_ε) satisfied some corresponding L^p bound on the unscaled domain $\omega \times (-\varepsilon, \varepsilon)$.

Above and in the sequel ∇_α represents the gradient with respect to the unscaled coordinates (denoted by x_α) and ∇_3 represents the gradient with respect to the thin coordinate direction denoted by x_3. In particular, $\Omega := \omega \times (-1, 1) = \{(x_\alpha, x_3) : (x_\alpha, \varepsilon x_3) \in \omega \times (-\varepsilon, \varepsilon)\}$ and $u_\varepsilon(x_\alpha, x_3) = w_\varepsilon(x_\alpha, \varepsilon x_3)$.

Bocea and Fonseca in [3] (see also Braides and Zeppieri in [4] for any dimension) proved an equi-integrability Lemma for scaled gradients satisfying a bound as (1). Indeed they generalized the Fonseca et al.'s result (see [7, Lemma 1.2], in turn refining the results in [1]) which allows to substitute a sequence (u_n), whose gradients (∇u_n) are bounded in L^p, by a sequence (v_n) with $(|\nabla v_n|^p)$ equi-integrable, such that the two sequences are equal except on a set of vanishing measure. The purpose of such a result is due to the fact that when applying the direct methods of the Calculus of Variations, or some Γ-convergence argument, it is very convenient to replace a given sequence with one having better regularity and integrability properties.

In this note we extend [3, Theorem 1.1, Corollary 1.2] to the Orlicz–Sobolev setting (see Sect. 2 for details and properties about Orlicz spaces L^Φ and Orlicz–Sobolev ones $W^{1,\Phi}$). Our main motivation is to provide new tools, namely the Lipschitz type approximation for scaled gradients, to the asymptotic analysis of thin structures whose stored energy can be modelled in terms of Orlicz–Sobolev functions. Indeed a larger class of materials can be considered, replacing standard coercivity and growth condtions (i.e. of the type $|\cdot|^p$) for the energy density, by convex functions [satisfying suitable properties, as (5) and (6)]. We refer to the recent works [18, 19] aimed to describe thin structures and their bending phenomena, and to the forthcoming paper [16], where optimal design questions are addressed in the same spirit of [5, 6]. We believe that our result can have further applications like those to fluid mechanics and multiscale problems (we refer to [21], where homogenization of integral functionals was treated, in a very similar setting to ours).

Via Young measures techniques, we prove

Theorem 1 *Let $\omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and $\Omega := \omega \times (-1,1)$. Let $\Phi : [0,+\infty) \to [0,+\infty)$ be an Orlicz function satisfying (5) and (6). Let $(u_n) \subset W^{1,\Phi}(\Omega;\mathbb{R}^3)$. Assume that (ε_n)* is a *sequence of numbers converging to 0, such that*

$$\sup_n \int_\Omega (\Phi(|\nabla_\alpha u_n, \frac{1}{\varepsilon_n}\nabla_3 u_n|))dx = C < +\infty. \tag{2}$$

Then there exists a (non-relabelled) subsequence (u_n) and a sequence $(v_n) \subset W^{1,\Phi}(\Omega;\mathbb{R}^3)$ such that

(i) *sequence $(\Phi(|\nabla_\alpha v_n, \frac{1}{\varepsilon_n}\nabla_3 v_n|))$ is equi-integrable,*
(ii) *$v_n \rightharpoonup u_0$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$, where u_0 is the weak limit of (u_n) in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$,*
(iii) *$|\{x \in \Omega : u_n \neq v_n \text{or} \nabla u_n \neq \nabla v_n\}| \to 0$, as $n \to +\infty$,*
(iv) *$v_{n|\partial\omega\times(-1,1)} = u_0$.*

We stress that the above result holds for any sequence of scaled gradients appearing in any *Nd–Kd* dimensional reduction problem, besides the proof is presented for $N = 3$ and $K = 2$.

Having in mind the equilibrium problems related to membranes, where the total energy of the thin film under a deformation $w_\varepsilon : \omega \times (-\varepsilon,\varepsilon) \to \mathbb{R}^3$ is given by

$$E_\varepsilon(w_\varepsilon) := \int_{\omega\times(-\varepsilon,\varepsilon)} W(\nabla w_\varepsilon(y))dy - \int_{\omega\times(-\varepsilon,\varepsilon)} f^\varepsilon(y)\cdot w_\varepsilon(y)dy,$$

with $f^\varepsilon \in L^\Psi(\omega \times (-\varepsilon,\varepsilon),\mathbb{R}^3)$ an appropriate dead loading body force density (we refer to [18] for the asymptotic analysis of the above energy), it is important to prove the existence of an 'attaining' sequence for the limit density, which is Φ-equi-integrable. Indeed the following result holds.

Theorem 2 *Let Ω and Φ be as in Theorem 1. Let $u_0 \in W^{1,\Phi}(\omega,\mathbb{R}^3)$ be an affine mapping with gradient $\xi_0 \in \mathbb{R}^{3\times 2}$ and let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying*

$$\beta\Phi(|\xi|) - c \le W(\xi) \le \beta'\Phi(|\xi|) + C \quad \text{forevery} \xi \in \mathbb{R}^{3\times 3}, \tag{3}$$

for suitable constant $0 < \beta \le \beta'$, $c, C > 0$.

Given any sequence (ε_n) of positive real numbers converging to zero, there exist a subsequence (not relabelled) of (ε_n), and a sequence of functions $(u_n) \subset W^{1,\Phi}(\Omega,\mathbb{R}^3)$ such that

(i) $\lim_{n\to+\infty} \frac{1}{|\Omega|}\int_\Omega W\left(\nabla_\alpha u_n, \frac{1}{\varepsilon_n}\nabla_3 u_n\right)dx = Q\overline{W}(\xi_0)$, *where* $\overline{W}(\xi_0) = \min_{z\in\mathbb{R}^3} W(\xi_0|z)$ *and $Q\overline{W}$ denotes the quasiconvex envelope of $\overline{W}$, namely*

$$Q\overline{W}(\xi_0) = \inf_{\varphi \in W_0^{1,\infty}(Q_b,\mathbb{R}^3)} \left\{ |Q_b|^{-1} \int_{Q_b} \overline{W}(\xi_0 + \nabla_\alpha \varphi(x_\alpha)) dx_\alpha \right\} \tag{4}$$

for any cube $Q_b \subseteq \omega$,

(ii) $\lim_{n\to+\infty} \|u_n - u_0\|_{L^\Phi(\Omega;\mathbb{R}^3)} = 0$,

(iii) $u_{n|\partial\omega\times(-1,1)} = u_0$.

(iv) $\Phi\left(\left|\nabla_\alpha u_n, \frac{1}{\varepsilon_n}\nabla_3 u_n\right|\right)$ *is equi-integrable.*

It is worth to observe that such a result can be seen as a counterpart of the characterization of the Young measures generated by scaled gradients in the Orlicz–Sobolev setting. Indeed formula (i) is entirely analogous to [14, formula before (1.16)].

The proof of Theorem 1 develops first by proving a Decomposition Lemma for standard gradients (see Theorem 4) which relies on properties of maximal functions, and exploits the Fundamental Theorem of Young measures (see Theorem 3). Then the proof of Theorem 1 follows as a consequence making use of the fine homogenization technique introduced in [4]. These are the subject of Sect. 3, while all the preliminary results, together with properties of Hardy maximal operator are contained in Sect. 2.

2 Notation and preliminaries

We will use the following notation:

- $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^N$, $N \geq 2$, and it will be clear from the context;
- the symbol dx will also be used to denote integration with respect to the Lebesgue measure $\mathcal{L}^N$, $N \geq 3$;
- the symbol dx_α will be used to denote integration with respect to the Lebesgue measure $\mathcal{L}^2$;
- the symbol $\nabla_\alpha u$ denotes the derivatives with respect to $x_\alpha := (x_1, x_2)$ of a given field u;
- C represents a generic positive constant that may change from line to line;
- a matrix $\xi \in \mathbb{R}^{3\times3}$, will be often written as (ξ_α, ξ_3) where ξ_α stands for the first two columns and ξ_3 represents the third;
- the Euclidean norm of a vector or of a matrix will be described as $|\cdot|$ and it will be clear from the context;
- a sequence (f_n) is said to be Φ-equi-integrable if the sequence $(\Phi(|f_n|))$ is equi-integrable.

We say that $\Phi : [0, +\infty) \to [0, +\infty)$ is an Orlicz function whenever it is continuous, strictly increasing, convex, vanishes only at 0 and $\lim_{t\to0^+} \Phi(t)/t = 0; \lim_{t\to+\infty} \Phi(t)/t = +\infty$. This statement is equivalent to demanding that $\Phi(t) =$

$\int_0^t \phi(s)ds$ for some right-continuous, non-decreasing ϕ s.t. $\phi(t) = 0 \iff t = 0$ and $\lim_{t\to+\infty} \phi(t) = +\infty$.

We say that Φ satisfies Δ_2 (denoted by $\Phi \in \Delta_2$) condition whenever

$$\text{there exist } C > 0 \text{ and } t \geq t_0 \text{ such that } \Phi(2t) < C\Phi(t) \text{ for all } t \geq t_0. \tag{5}$$

Orlicz functions Φ possess the complementary Orlicz function $\Psi(s) := \Phi^\star(s)$, where the latter denotes the standard Fenchel's conjugate of Φ, i.e.

$$\Psi(s) := \sup_{t\geq 0}\{st - \Phi(t)\}, \quad s \geq 0,$$

and, it results that $\Psi(s) = \int_0^s \phi^{-1}(\tau)d\tau$, where ϕ^{-1} stands for right inverse function of ϕ.

Clearly $\Psi^\star = (\Phi^\star)^\star = \Phi$.

If $\Psi \in \Delta_2$ then (see [17, Theorem 4.2])

$$\text{there exist } C > 0 \text{ and } t_0 \geq 0 \text{ such that } \Phi(t) \leq 1/(2C)\ \Phi(Ct) \text{ for any } t > t_0. \tag{6}$$

Given two Orlicz functions Φ and Φ', Φ dominated Φ' near infinity ($\Phi' \prec \Phi$ or $\Phi \succ \Phi'$ in symbols) if there exists $C > 1$ and $t_0 > 0$ such that $\Phi'(t) \leq \Phi(Ct)$ for all $t > t_0$.

For an arbitrary set of positive Lebesgue measure $E \subset \mathbb{R}^N$ we define the Orlicz class $L_\Phi(E)$ of functions u on E as functions satisfying inequality

$$\int_E \Phi(|u|)dx < +\infty$$

In general the class $L_\Phi(E)$ is not a linear space, and the Orlicz space $L^\Phi(E)$ is defined as the linear hull of $L_\Phi(E)$. It is easy to check that (see [17, Theorem 8.2]) Orlicz class $L_\Phi(E)$ coincides with its Orlicz space $L^\Phi(E)$ if and only if $\Phi \in \Delta_2$.

Orlicz spaces are equipped with the Luxemburg norm, namely

$$||u||_{L^\Phi(E)} = \inf_{k>0} \int_E \Phi(|u|/k) \leq 1 \tag{7}$$

and are complete (see [17, Theorems 9.2 and 9.5]).

The following properties hold.

Lemma 1 *Let Φ be an Orlicz function satisfying the Δ_2 condition* (*i.e.* (5)) *and let E be a bounded open set in $\mathbb{R}^N$. Then*

(i) *$C_c^\infty(E)$ is dense in $L^\Phi(E)$* [10, Theorem 1];

(ii) *$L^\Phi(E)$ is separable* [17, point 4 at page 85] *and it is reflexive when Φ satisfies* (6) [17, Theorem 14.2];

(iii) *the dual of $L^\Phi(E)$ is identified with $L^\Psi(E)$,* ($\Psi = \Phi^\star$) *and the dual norm on $L^\Psi(E)$ is equivalent to $\|\cdot\|_{L^\Psi}$* [17, Theorem 14.2];

(iv) *given* $u \in L^\Phi(E)$ *and* $v \in L^\Psi(E)$, *then* $uv \in L^1(E)$ *and the following generalized Hölder inequality holds* [17, Theorem 9.3 and formula (9.24)]

$$\left| \int_E uv dx \right| \leq 4 \|u\|_{L^\phi} \|v\|_{L^\Psi};$$

(v) *for every* $v \in L^\Phi(E)$ *the linear functional* I_v *on* $L^\Psi(E)$ *defined as*

$$I_v(u) := \int_E u(x)v(x)dx$$

belongs to the dual of $L^\Psi(E)$ *with* $\|v\|_{L^\Phi} \leq \|L_v\|_{[L^\Psi(E)]'} \leq 2\|v\|_{L^\Phi}$ [17, Theorem 9.5, formula 9.24];

(vi) *given* Φ *and* $\tilde{\Phi}$, *the continuous embedding* $L^\Phi(E) \hookrightarrow L^{\tilde{\Phi}}(E)$ *holds iff* $\Phi \succ \tilde{\Phi}$ [17, Theorem 8.1];

(vii) *in view of* (vi) $L^\Phi(E) \hookrightarrow L^1(E) \hookrightarrow L^1_{\text{loc}}(E) \hookrightarrow \mathcal{D}'(E)$;

(viii) *the product of d identical copies of* $L^\Phi(E)$, $(L^\Phi(E))^d := L^\Phi(E) \times \ldots \times L^\Phi(E)$ *endowed with the norm* $\|v\|_{(L^\Phi(E))^d} := \sum_{i=1}^d \|v_i\|_{L^\Phi(E)}$ *is an Orlicz space* (*i.e. the norm is equivalent to the* $L^\Phi(\sqcup_1^d E)$ *norm, where* $\sqcup$ *stays for sum of disjoint copies of the set*).

Sobolev–Orlicz spaces $W^{1,\Phi}(E)$ are defined as follows

$$W^{1,\Phi}(E) := \{u \in \mathcal{D}'(E) : u \in L^\Phi(E), \nabla u \in (L^\Phi(E))^N\}$$

endowed with the norm

$$||u||_{W^{1,\Phi}(E)} := ||u||_{L^\Phi(E)} + ||\nabla u||_{(L^\Phi(E))^N},$$

thus they are Banach spaces.

The Sobolev–Orlicz space $W^{1,\Phi}(E; \mathbb{R}^d)$, $d \in \mathbb{N}$ is defined as the Banach space of $\mathbb{R}^d$ valued functions $u \in L^\Phi(E; \mathbb{R}^d)$ with distributional derivative $\nabla u \in L^\Phi(E; \mathbb{R}^{N\times d})$, equipped with the norm

$$\|u\|_{W^{1,\Phi}(E;\mathbb{R}^d)} := \|u\|_{L^\Phi(E;\mathbb{R}^d)} + \|\nabla u\|_{L^\Phi(E;\mathbb{R}^{N\times d})},$$

where the meaning of the norm $\| \cdot \|_{L^\Phi(E;\mathbb{R}^\cdot)}$ is easily understood from (viii) in Lemma 1. On the other hand, all the other properties in Lemma 1 extend with obvious meaning to the vectorial setting.

If E has Lipschitz boundary, then the embedding

$$W^{1,\Phi}(E; \mathbb{R}^d) \hookrightarrow L^\Phi(E; \mathbb{R}^d) \tag{8}$$

is compact (see [2] and [9, Theorems 2.2 and Proposition 2.1]).

For Sobolev–Orlicz space $W^{1,\Phi}(E)$, where E has a Lipschitz boundary and $\Phi \in \Delta_2$, there exists a linear continuous trace operator $\text{Tr} : W^{1,\Phi}(E) \to L^\Phi(\partial E)$ [11, Theorem 3.13].

Let $\mathcal{M}$ be a (centred) Hardy maximal operator, i.e. for any $f \in L^1_{loc}(E) \cap L^\Phi(E)$ let

$$\mathcal{M}f(x) := \sup_r |B(x,r)|^{-1} \int_{B(x,r)\cap E} |f(x)|dx.$$

The following result will be exploited in the sequel.

Proposition 1 (Weak estimate on Hardy's operator) *Let Φ be an Orlicz function satisfying (5) and (6). For any $f \in L^\Phi(E)$ there exists a constant $C = C\big(E,\Phi\big)$ such that*

$$|\{\mathcal{M}f > t\}| \leq \frac{C}{\Phi(t)} \int_E \Phi(|f|)dx, \tag{9}$$

for every $t > 0$.

Proof We start with standard Chebyshev inequality

$$|\{\mathcal{M}f > t\}| = \int_{\{\mathcal{M}f\,>\,t\}dx \leq \int_{\{\mathcal{M}f\,>\,t\}}} \underline{\Phi(\mathcal{M}f)\Phi(t)dx,}$$

where we use the fact that Orlicz function Φ is increasing and $\Phi(\mathcal{M}f)$ is integrable. This latter property, in turn, relying on the integrability of $\Phi(|f|)$, (5) and result the continuity of Hardy's operator in [8]. Assuming that Φ satisfies (5), (6), [12, Theorem 1] (with applied weight $w \equiv \chi_{\{\mathcal{M}f\,>\,t\}}$ note that condition (2) is obviously satisfied) shows that there exists a constant $C > 0$ such that

$$\int_{\{\mathcal{M}f\,>\,t\}\Phi(\mathcal{M}f)\Phi(t)dx} \leq \frac{C}{\Phi(t)} \int_{\{\mathcal{M}f\,>\,t\}\Phi(|f|)dx,}$$

for every $t > 0$. □

It is worth to observe that the result holds with the same proof in the vectorial case.

We quote the Fundamental Theorem on Young measures, which will be invoked in the proof of our main results, for more details we refer to [20] (and regarding Young measures generated by gradients to [13, 15]).

Theorem 3 *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let (z_n) be a sequence of measurable functions, $z_n : E \to \mathbb{R}^m$. Then there exists a subsequence (z_{n_k}) and a weak * measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ such that the following hold:*

(i) $\nu_x \geq 0, \|\nu_x\|_{\mathcal{M}(R^m) = \int_{\mathbb{R}^m} d\nu_x \leq 1}$ *for a.e. $x \in E$;*

(ii) *one has* (i′) $\|\nu_x\|_{\mathcal{M}=1}$ *for a.e. $x \in E$ if and only if*

$$\lim_{R\to+\infty} \sup_k |\{|z_{n_k}| \geq R\}| = 0$$

(iii) *if* $K \subset \mathbb{R}^m$ *is a compact subset and* $\operatorname{dist}(z_{n_k}, K) \to 0$ *in measure, then* $\operatorname{supp}\nu_x \subset K$ *for a.e.* $x \in E$*;*

(iv) *if* (i′) *holds, then in* (iii) *one may replace ''f' with ''if and only if';*

(v) *if* $f : E \times \mathbb{R}^m \to \mathbb{R}$ *is a normal integrand, bounded from below, then*

$$\liminf_{n\to+\infty} \int_E f(x, z_{n_k}(x))dx \geq \int_E \int_{\mathbb{R}^m} f(x,y)d\nu_x(y)dx$$

(vi) *if* (i′) *holds and if* $f : E \times \mathbb{R}^m \to \mathbb{R}$ *is Carathéodory and bounded from below, then*

$$\lim_{n\to+\infty} \int_E f(x, z_{n_k}(x))dx = \int_E \int_{\mathbb{R}^m} f(x,y)d\nu_x(y)dx$$

if and only if $(f(x, z_{n_k}(x)))$ *is equi-integrable. In this case*

$$f(x, z_{n_k}(x)) \rightharpoonup \int_{\mathbb{R}^d} f(x,y)d\nu_x(y) \text{in} L^1(E).$$

The map $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ is called the Young measure generated by (z_{n_k}).

3 Proofs of Theorems 1 and 2

This section is devoted to the proof of our main result.

We start by proving a Lemma which generalizes [20, Lemma 8.13] to the Orlicz setting.

Lemma 2 *Let* Φ *be an Orlicz function satisfying* (5) *and* (6). *Let* $E \subset \mathbb{R}^N$ *be a Lebesgue measurable set of finite measure and let* (u_n) *be a uniformly bounded sequence in* $L^\Phi(E;\mathbb{R}^m)$. *For any* $r > 0$ *define the standard truncature operators* $\tau_r : \mathbb{R} \to \mathbb{R}$ *as*

$$\tau_r(t) := \begin{cases} t & \text{whenever} |t| \leq r, \\ r\dfrac{t}{|t|} & \text{otherwise.} \end{cases} \tag{10}$$

Then there exist a (non-relabelled) subsequence (u_n) *and an increasing sequence of positive numbers* $r_n \to +\infty$ *such that* $\tau_{r_n} \circ u_n$ *are* Φ*-equi-integrable and the measure* $|\{x \in E : \tau_{r_n} \circ u_n \neq u_n\}| \to 0$.

Proof By (i) in Theorem 3, we may assume that (u_n) generates the Young measure ν_x and (iii) therein guarantees that

$$\int_E \int_{\mathbb{R}^m} \Phi(|z|)d\nu_x(z)dx < +\infty.$$

So we have

$$\lim_{r\to+\infty}\lim_{n\to\infty}\int_E \Phi(|\tau_r\circ u_n|)dx = \lim_{r\to+\infty}\int_E\int_{\mathbb{R}^m}\Phi\big(|\tau_r(z)|\big)d\nu_x(z)dx = \int_E\int_{\mathbb{R}^m}\Phi(|z|)d\nu_x(z)dx.$$

where the first equality relies on (vi) of Theorem 3, and the second one on Lebesgue Monotone Convergence theorem. Take r_n such that

$$\lim_{n\to+\infty}\int_E \Phi(|\tau_{r_n}\circ u_n|)dx = \int_E\int_{\mathbb{R}^m}\Phi(|z|)d\nu_x(z)dx.$$

As $r_n \to +\infty$ and (u_n) is bounded, one has

$$|\{x\in E : \tau_{r_n}\circ u_n \neq u_n\}| \to 0.$$

Thus, we can conclude that $(\tau_{r_n}\circ u_n)$ generates the same Young measure as (u_n) (see [20, Corollary 8.7]).

Finally (vi) in Theorem 3 ensures Φ-equi-integrability. □

Now we prove a Decomposition Lemma for gradients and then we extend this result to scaled ones.

Theorem 4 *Let $E\subset\mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let Φ be an Orlicz function satisfying (5) and (6), and let $(u_n)\subset W^{1,\Phi}(E;\mathbb{R}^d)$ be a sequence of functions converging to u_0 weakly in $W^{1,\Phi}(E;\mathbb{R}^d)$. Then there exists a subsequence (u_{n_k}) and a sequence $(v_k)\subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that (v_k) converges to u_0 weakly in $W^{1,\Phi}(E;\mathbb{R}^d)$, and*

$$|\{x\in E : v_k(x)\neq u_k(x) \text{or} \nabla u_k(x)\neq\nabla v_k(x)\}| \to 0 \text{as} k\to+\infty$$

and $(\Phi(|\nabla v_k|))$ is equi-integrable.

Proof Since

$$\sup_n \|u_n\|_{W^{1,\Phi}(E;\mathbb{R}^d)} \le C$$

and by (5),

$$\sup_n\left\{\int_E(\Phi(|u_n|)+\Phi(|\nabla u_n|))dx\right\}\le C,$$

it follows that from the continuity of the maximal operator [8, Theorem 2.1], and the passage to an equivalent norm, that

$$\sup_n\left\{\int_{\mathbb{R}^N}\Phi(\mathcal{M}(|u_n|+|\nabla u_n|)\chi_E)dx\right\}\le C,$$

where $\mathcal{M}((|u_n|+|\nabla u_n|)\chi_E)$ is the maximal function of $(|u_n|+|\nabla u_n|)\chi_E$. By Lemma 2, there exists an increasing sequence $t_n\to+\infty$ such that $(\Phi(|\tau_{t_n}\circ(\mathcal{M}((|u_n|+|\nabla u_n|)\chi_E))|))$ is equi-integrable, where τ_{t_n} is as in (10).

Define

$$A_n := \{x \in E : |\mathcal{M}\big((|u_n| + |\nabla u_n|)\chi_E\big)| > t_n\}. \tag{11}$$

By [20, Theorem 4.32], there exists $(v_n) \subset W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^m)$ such that

$$\|v_n\|_{W^{1,\infty}} \leq Ct_n,$$

where C depends on E and N, and such that $v_n = u_n$ $\mathcal{L}^N$ a.e. on $E \setminus A_n$ and by (9)

$$|A_n| \leq \frac{C}{\Phi(t)} \int_{\mathbb{R}^N} \Phi(|u_n| + |\nabla u_n|)dx.$$

In order to show that $(\Phi(|\nabla v_n|))$ is equi-integrable we observe that for $\mathcal{L}^N$ a.e. x in $E \setminus A_n$

$$|\nabla v_n| = |\nabla u_n| \leq \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E) = |\tau_{t_n} \circ \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E)|$$

while if $x \in A_n$ then

$$|\nabla v_n| \leq Ct_n \leq C|\tau_{t_n} \circ \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E)|.$$

It remains to prove the weak convergence of (v_n) to u_0 in $W^{1,\Phi}(E; \mathbb{R}^d)$. To this end, first we observe that (11) and (9) ensure

$$\begin{aligned}\int_E \Phi(|v_n| + |\nabla v_n|)dx &= \int_{E\setminus A_n} \Phi(|u_n| + |\nabla u_n|)dx + \int_{A_n} \Phi(|v_n| + |\nabla v_n|)dx \\ &\leq \int_{E\setminus A_n} \Phi(|u_n| + |\nabla u_n|)dx + C\Phi(t_n)|A_n| \\ &\leq C\int_E \Phi(|u_n| + |\nabla u_n|)dx.\end{aligned}$$

Next the reflexivity of $W^{1,\Phi}(E; \mathbb{R}^d)$ under (5), (6) (see Lemma 1) and the Banach–Alaoglu–Bourbaki theorem ensure that $v_n \rightharpoonup v_0$ in $W^{1,\Phi}(E; \mathbb{R}^d)$. Thus, since $|\{x \in E : v_n \neq u_n \text{or} \nabla u_n \neq v_n\}| \to 0$ as $n \to +\infty$ we can conclude, via the compact imbedding (see (8)) that $v_0 = u_0$ $\mathcal{L}^N$-a.e. in E. □

Proof of Theorem 1 The proof of the claims (i) and (iii) follows line by line as in [4, Theorem 3.1]. Namely, we define $\hat{u}_n := u_n(x_1, x_2, \frac{x_3}{\varepsilon_j} - 1)$ (so it is a shifted and scaled version of u_n, and it is defined on $\omega \times (0, 2\varepsilon_n)$) and observe that

$$\sup_j \varepsilon_j^{-1} \int_{\omega\times(0,2\varepsilon_n)} \Phi(|\nabla \hat{u}_n|)dx = C, \quad \text{where} C \text{isexactlylikein}(1.2).$$

We now extend $\hat{u}_n$ by reflection to $\omega \times (-2\varepsilon_n, 2\varepsilon_n)$ and then produce its periodic extension to $\omega \times (-1, 1)$.

For such constructed sequence one can obtain the uniform bound of the norm in $W^{1,\Phi}(\omega \times (-1, 1))$ as in [4, formula (3.6)]. Thus we apply Theorem 4 and obtain a sequence $(\hat{v}_n)$ with $(\nabla \hat{v}_n)$ Φ-equi-integrable. The use of de la Vallée Poussin Criterion (see [20, Theorem 2.29]) and an ingenious computation (see [4, formula

(3.7)]) gives us the sequence $(\bar{v}_n)$ satisfying claim (i) and (iii).

Up to an extraction of a subsequence one may immediately deduce claim (ii).

To get (iv) we argue as in [3, Corollary 1.2]. We define sets

$$\omega_j := \{x \in \omega : \mathrm{dist}(x, \partial\omega) < 1/j\} \tag{12}$$

and cut-off functions $\theta_j \in C_0^\infty(\omega, [0,1])$, equal to 1 on $\omega \setminus \omega_j$, vanishing in a neighbourhood of $\partial\omega$, and such that $|\nabla\theta_j| < Cj$ for some constant C. We set then $v_{n,j} := u_0 + \theta_j \bar{v}_n$. Via compact imbedding (see (8)) and diagonal argument we may find a sequence $n(j)$ such that $n(j) \to +\infty$ as $j \to +\infty$ and

$$||v_{n(j),j} - u_0||_{L^\Phi(\Omega;\mathbb{R}^3)} \to 0 \quad \text{and} \quad ||v_{n(j),j}||_{L^\Phi(\Omega;\mathbb{R}^3)} < \frac{1}{j^2}.$$

To obtain (iv), it suffices to define $v_j := v_{n(j),j}$. It remains to deduce (i)–(iii) for this latter sequence. To prove (iii) we just observe that

$$\begin{aligned} &|\{x \in \Omega : u_j \neq v_j \mathrm{or} \nabla u_j \neq \nabla v_j\}| \\ &\quad \leq |\{x \in \Omega : u_j \neq \bar{v}_j \mathrm{or} \nabla u_j \neq \nabla \bar{v}_j\}| + |\{x \in \Omega : \bar{v}_j \neq v_j \mathrm{or} \nabla u_j \neq \nabla \bar{v}_j\}|, \end{aligned}$$

and the claim follows from the control of the latter two sets. For (i), it suffices to exploit the definition of u_j and the Φ-equi-integrability of $\bar{v}_j$, (see also [3, formula (4.8)]). Up to the extraction of the subsequence we may know deduce (ii). □

Proof of Theorem 2 It can be deduced from [3, Corollary 1.2]. We sketch the main points for the readers' convenience. First let us observe that from density of smooth functions and properties of quasiconvex envelope and definition of $\overline{W}$ it can be easily proven that

$$\inf_{\varepsilon, u|_{\partial\omega\times(-1,1)} \equiv u_0} \frac{1}{|\Omega|} \int_\Omega W(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u) dx = Q\overline{W}(\xi_0). \tag{13}$$

Now let us assume that ω is a square $(-c/2, c/2)^2$. Let (w_n, L_n) be the infimizing sequence of the left-hand side in (13). We may thus assume that, up to a reflection and then a periodic extension, functions $(w_n - u_0)$ are already defined on $\mathbb{R}^2 \times (-1,1)$. We define $w_{n,j}(x) := \varepsilon_j L_n (w_n - u_0)\left((\varepsilon_j L_n)^{-1} x_\alpha, x_3\right)$ and observe that $w_{n,j} \rightharpoonup 0$ and

$$\lim_{n\to\infty} \lim_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega W\left(\nabla_\alpha u_0 + \nabla_\alpha w_{n,j}, \frac{1}{\varepsilon_j} \nabla_3 w_{n,j}\right) = Q\overline{W}(\xi_0).$$

By a diagonal procedure and (8) we may choose $j(n)$ such that (denoting $w_{n,j(n)}$ as $\tilde{w}_n$ and $\varepsilon_{j(n)}$ as $\tilde{\varepsilon}_n$), $\lim \tilde{w}_n = 0$ in $L^\Phi(\Omega)$, and

$$\lim_{n\to\infty} \frac{1}{|\Omega|} \int_\Omega W(\nabla_\alpha \tilde{w}_n, \frac{1}{\tilde{\varepsilon}_n} \nabla_3 \tilde{w}_n) dx = Q\overline{W}(\xi_0).$$

The latter equality together with (2) gives us bound on the norm of $\tilde{w}_n$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$. Up to an extraction of the subsequence (not relabelled) we may still assume that $\tilde{w}_n \rightharpoonup 0$.

Applying Theorem 1 we obtain a sequence (v_n) satisfying (ii)–(iv). (i) follows from triangle inequality, Φ-equi-integrability of (v_n), point (iii) and the fact that $|\omega_j| \to 0$ (see (12)).

To generalize the result to ω with Lipschitz boundary we refer to the second step of the proof of [3, Corollary 1.2]. □

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