



Radon Transforms in Hyperbolic Spaces and Their Discrete Counterparts

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Abstract

In the hyperbolic disc (or more generally in real hyperbolic spaces) we consider the horospherical Radon transform R and the geodesic Radon transform X . Composition with their respective dual operators yields two convolution operators on the disc (with respect to the hyperbolic measure). We describe their convolution kernels in comparison with those of the corresponding operators on a homogeneous tree T , separately studied as acting on functions on the vertices or on the edges. This leads to a new theory of spherical functions and Radon inversion on the edges of a tree.

Keywords Real hyperbolic spaces · Homogeneous trees · Radon transform · X-ray transform · Convolution operators · Horospheres · Spherical functions on trees · Plancherel measure

Mathematics Subject Classification Primary 44A05; Secondary 51M10 · 05C05 · 53C65

To the memory of our good friend Carlos.

Communicated by Irene Sabadini.

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1 Introduction

The aim of this article is to compare various types of Radon transforms on hyperbolic spaces and on homogeneous trees, a subject on which the authors collaborated with Carlos Berenstein, coauthoring with him a preliminary survey. Our goal is to present these ideas in a renewed version with the addition of several original results on Radon transforms on homogeneous trees: in particular, a theory of spherical functions, the Plancherel measure and the Radon inversion on the edges.

The Radon transform was introduced in [33] as the operator on continuous compactly supported functions in the plane defined by integrating over lines. This operator and its inversion formula, studied later in [28], have become of wide interest in applied mathematics because of Computerized Axial Tomography (C.A.T.) and its need of fast hardware implementations of the inversion formula. Many different variants of the Radon transform have been investigated since then, in a long list of articles in pure and in applied mathematics. Despite its long history, the Radon transform is still in the focus of current research, both in continuous and discrete settings: see, for instance, [2,10,16,34] and their references lists.

For instance, it is of interest to consider Radon transforms on the hyperbolic disc. One motivation for these extensions, introduced and deeply investigated in a series of papers by S. Helgason (see [25] for references), is due to the fact that there are two natural analogs of lines in hyperbolic geometry, and consequently two extensions of the Radon transform. The first is to geodesics in the hyperbolic metric: that is, arcs of Euclidean circles in the Poincaré disc which cross the boundary orthogonally. The second extension is to horospheres, that are circles in the disc tangent to the boundary. Following Helgason, we shall call *Radon transform* the latter, and *X-ray transform* the former, and denote them by R and X , respectively.

The typical approach to the inversion of the Radon transform in the flat plane is based upon some type of radial average. In order to reconstruct the value of f at a point x , one considers the average of Rf over all lines through x . This amounts to applying to Rf the dual Radon transform R^* . Unfortunately R^*Rf does not coincide with f ; however, the operator R^*R commutes with translations, therefore it is a convolution operator on functions on the plane (it also commutes with rotations, hence the convolution kernel is radial). Thus, in order to invert R , it is enough to invert this convolution operator on an appropriate function space, a task which is made easier by spectral theory (i.e., Fourier analysis).

The same approach works in the hyperbolic disc. The set of horospheres through a point x can be identified with the boundary Ω of the disc, and the set of geodesics through x with the set of unordered pairs $(\Omega \times \Omega \setminus \text{diagonal})/\{\pm 1\}$. Hence these spaces inherit natural Borel structures and Borel measures which are covariant under the group of automorphisms of the disc. Here the dual transforms R^* and X^* are defined in terms of these measures (and of the hyperbolic measure on the disc). The operators R^*R and X^*X are convolution operators on the disc (with respect to the hyperbolic measure). By inverting these convolvers we obtain inversion formulas for R and X . A similar approach works for the geodesic transform and the horospherical transform in n -dimensional hyperbolic space.

Explicit inversion formulas can be found in [5,23] for X , and in [25] for R . We shall explore the intriguing similarity, observed in [1,6,7,14,15], between Radon transforms on the disc and analogous operators defined on homogeneous trees. This led to regard homogeneous trees as discrete counterparts for the hyperbolic disc and semisimple rank-one symmetric spaces in the study of representation theory, harmonic functions, Hardy spaces, and integral geometry. In some cases, this resemblance has been extended to general trees; for a detailed bibliography, see [11,20,29] and references therein. In particular, the horospherical Radon transform on trees has been studied in [7,12,15,20] and the X-ray transform in [1,6]. Homogeneous trees do not provide a discrete model for n -dimensional hyperbolic space except when $n = 2$ (that is, for the disc); discrete models for higher dimensional hyperbolic spaces should be obtained from higher-rank Bruhat-Tits buildings, see [3,4].

From the point of view of Radon transforms, the analogy between the hyperbolic disc and homogeneous trees is best seen by comparing the continuous and the discrete convolution kernels of R^*R and X^*X . We start this comparison in Sect. 2 by deriving the explicit expression of the radial convolution kernels J_R of R^*R and J_X of X^*X on the hyperbolic disc (the general expression in n -dimensional hyperbolic space will also be stated). The corresponding kernels on the vertices of a homogeneous tree T are obtained in Sect. 3. We shall see, both for the disc and for T , that the value $J_X(d)$ of the kernel J_X at pairs of points at distance d is inversely proportional to the ‘length’ of the circle of radius d , whereas $J_R(d)$ is inversely proportional to the length of the circle of radius $d/2$.

The computation of the convolution kernel J_R on the set V of vertices of T leads to a direct computation of the inverse convolution operator (Corollary 3.4), hence to the inversion of R . On the other hand, no such direct computation seems feasible on the set E of edges of T . Because of this, we prove many preliminary results in analysis for functions on E .

Analysis on functions on V is well-known, and makes extensive use of spherical functions, the spectrum of the Laplace operator and its Plancherel measure, and the spherical Fourier transform. Here we need to study similar problems and techniques for functions on E . Spherical functions and the spherical Fourier transform on E have been indirectly computed in [27], where E is regarded as a graph, that is not a tree but the dual graph of T ; hence those results may be transported to functions on E . On the other hand, the Plancherel measure and the Radon transform on E have not yet appeared in the literature. The geodesic and the horospherical Radon transforms on V are dealt with in Sects. 3.2,3.3, respectively. The geodesic Radon transform on E is studied in Sect. 4, where a simple inversion formula is found. The Poisson kernel for edges of T , the edge-spherical functions and the spectrum of the Laplacian on E are studied in the first part of Sect. 5. The Plancherel measure on E is computed in Sect. 5.4, and the horospherical Radon transform on E is introduced in Sect. 5.5. Finally, the Radon back-projection on E is studied in Sect. 6.1 and shown to give rise to a convolution operator acting on a Schwartz class on E to a space of distributions, whose inversion is obtained in Sect. 6.2.

Unlike the hyperbolic disk and rank-one symmetric spaces, trees appear in several different setups. Indeed, trees may be non-homogeneous, hence discrete analogs of Riemannian spaces that are not group-invariant. There is an intermediate case:

semi-homogeneous trees, that are trees with two different alternating degrees of homogeneity. The isometry groups of these trees are not transitive but have index 2. The homogeneous tree whose degree is a prime number p has a transitive group of isometries isomorphic to the p -adic semisimple group $PGL(2, \mathbf{Q}_p)$ and is the typical lowest-rank Bruhat-Tits building. A combinatorial generalization (Tits buildings) includes all homogeneous trees. Semi-homogeneous trees give all the other examples of combinatorial Tits buildings of the same rank. Therefore these trees can be appropriately included in the environment of discrete structures where Radon transforms bear strong analogies to continuous symmetric spaces. This paper describes also the kernels of R^*R and X^*X on vertices of semi-homogeneous trees, in comparison to the continuous environment. A more extensive account of integral geometry on trees will appear in [16].

2 Radon Transforms in the Hyperbolic Disc

2.1 The X-Ray Transform

The results of this subsection are adapted from [5,25]. As a model for the real hyperbolic space \mathbf{H}^2 we shall take the open unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$ of \mathbf{C} endowed with the Poincaré metric

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2},$$

which is conformal to the Euclidean metric $|dz|^2$ and has constant curvature -1 . The relation between the hyperbolic and the Euclidean distance $d(z, 0)$ of z from the origin is

$$|z| = \tanh \frac{d(z, 0)}{2}, \quad \text{i.e.,} \quad d(z, 0) = \log \frac{1 + |z|}{1 - |z|}.$$

The elements of the group $\text{Aut}^+ U$ of orientation-preserving isometries of U are the Möbius transformations

$$U \ni z \mapsto \tau_{\theta,w}(z) = e^{i\theta} \frac{z + w}{1 + \bar{w}z}$$

parametrized by $w \in U$ and $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. Observing that $\tau_{0,w} = \tau_{0,-w}^{-1}$ maps 0 to w , one easily computes the hyperbolic distance between arbitrary z, w as

$$d(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}.$$

The *geodesics* of U are arcs of (Euclidean) circles which intersect the unit circle ∂U of \mathbf{C} perpendicularly. Taking geodesic polar coordinates centered at 0 on U , we write $z = e^{i\theta} \tanh(r/2)$, where $r = d(0, z)$ and $\theta \in \mathbf{R}$. The Poincaré metric is then

expressed by

$$ds^2 = dr^2 + \sinh^2 r d\theta^2. \tag{1}$$

Therefore a geodesic circle of radius r (the set of points at hyperbolic distance r from a fixed center) has hyperbolic length

$$A(r) = 2\pi \sinh r.$$

$\text{Aut}^+ U$ acts transitively on the set of geodesics Γ in U , hence Γ has a non-zero invariant measure. The *X-ray transform* X is defined on functions on U having sufficiently fast decay at ∂U as

$$Xf(\gamma) = \int_{\gamma} f(z) ds(z) \quad \text{for every } \gamma \in \Gamma.$$

For simplicity we shall assume f to be in $\mathcal{S}(U)$, the space of smooth functions rapidly decreasing with all their derivatives. To define the dual X^* of X , let γ_{θ} be the geodesic through 0 and $e^{i\theta}$, and set

$$X^*\phi(0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\gamma_{\theta}) d\theta$$

for all, e.g., essentially bounded functions on Γ . For arbitrary $z \in U$ set $X^*\phi(z) = X^*(\phi \circ \tau_{0,z})(0)$.

We now compute X^*X and show that it is the convolution operator (with respect to the convolution product induced by $\text{Aut}^+ U$) with the radial kernel $J_X(r) = 2/A(r) = 1/\pi \sinh r$. The convolution product on a metric space induced by a transitive group of automorphisms is explained at the beginning of the ‘‘Appendix’’, where for simplicity we shall restrict attention to the setup of homogeneous trees. Here we briefly explain the setup of the disk U . Denote by $d\mu$ the Haar measure on the unimodular group $G = \text{Aut}^+ U$ and by dv the quotient measure on G/G_0 , that is, the volume element on U , namely

$$dv = \sinh r d\theta dr \tag{2}$$

Let G_0 be the isotropy subgroup at 0 and $g \mapsto \tilde{g} = gG_0$ the canonical projection onto G/G_0 . Via this projection, a function f on $U = G/G_0$ can be bijectively lifted to a right- G_0 -invariant function on G ; the function f is radial around 0 if and only if the lifted function is bi- G_0 -invariant. If f, k are functions on U with k radial, then the convolution $k * f$ becomes

$$k * f(\tilde{g}) = \int_{G/G_0} k(\tilde{h}^{-1}\tilde{g}) f(\tilde{h}) dv(\tilde{h})$$

(see more details in the ‘‘Appendix’’). If $z = e^{i\theta} \tanh(r/2)$, $z' = e^{i\theta'} \tanh(r'/2) \in U$, and $g_z \in G$ verifies $g_z(0) = z$, then

$$k * f(z) = k * f(g_z(0)) = \int_{G/G_0} k(\widetilde{h^{-1}g_z(0)}) f(\widetilde{h}(0)) dv(\widetilde{h}).$$

Since G is unimodular,

$$k * f(z) = \int_{G/G_0} k(\widetilde{h}(z)) f(\widetilde{h^{-1}(0)}) dv(\widetilde{h}) = \int_U k(h_{z'}(z)) f(h_{z'}^{-1}(0)) dv(z').$$

In particular, since $h_{z'}^{-1} = h_{-\theta',r'}$ and k is radial,

$$\begin{aligned} k * f(0) &= \int_U k(z') f(h_{z'}^{-1}(0)) dv(z') \\ &= \int_0^{2\pi} \int_0^\infty k\left(\tanh \frac{r'}{2}\right) f\left(e^{-i\theta'} \tanh \frac{r'}{2}\right) \sinh r' d\theta' dr'. \end{aligned} \tag{3}$$

Theorem 2.1 *We have*

$$X^*Xf = J_X * f, \quad \text{where } J_X(r) = \frac{1}{\pi \sinh r}.$$

Proof Both X^*X and the operator of convolution with J_X commute with the action of $\text{Aut}^+ U$ on $\mathcal{S}(U)$, therefore it is enough to prove that they take the same value at 0 when applied to $f \in \mathcal{S}(U)$. Indeed, by (1) ds coincides with dr on a geodesic through the origin. Then, by (3),

$$\begin{aligned} X^*Xf(0) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{\gamma_\theta} f(z) ds(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty f\left(e^{i\theta} \tanh \frac{r}{2}\right) dr d\theta \\ &= \int_U f\left(e^{i\theta} \tanh \frac{r}{2}\right) \frac{1}{\pi \sinh r} dv = J_X * f(0). \end{aligned} \quad \square$$

J_X is the symbol of the operator $L^{-1/2}$, where L is the Laplace–Beltrami operator. This computation was used with Carlos Berenstein [5, Theorem 4.3] to obtain an inversion formula for X that factors through X^*X . The inverse is obtained by the left action on X^*X of a convolution operator followed by the Laplace–Beltrami operator L . The convolution operator is given by the radial function $(1 - \coth r)/4\pi$ on U . In higher dimensional hyperbolic spaces the behavior of X^*X is similar. In fact in \mathbf{H}^n we have $X^*Xf = J_X * f$, where $J_X(r) = \pi^{-n/2} \sinh^{1-n} r$, except for $n = 3$ (where $J_X(r) = \cosh r \sinh^{-3} r/2\pi^3$ and the Laplace–Beltrami operator is replaced by $L + 1$). See also [25, Chapter II, Table II-1] with an outline of other inversion formulas that do not factor through X^*X .

2.2 The Horospherical Radon Transform

In \mathbf{R}^2 with the Euclidean metric, a straight line can be regarded either as a distance-minimizing curve, i.e., a geodesic, or else as a ‘circle of infinite radius’, that is, the limit of a circle constrained to contain a fixed point and whose center moves to infinity along a half-geodesic originating at that point; i.e., as a *horosphere*. (In a symmetric Riemannian space M , horospheres—also called *horocycles*—are the orbits of maximal nilpotent subgroups of the connected component of the identity in the group of isometries of M .) Therefore, as pointed out in the introduction, there are two counterparts of straight lines in the setting of the hyperbolic disc U (a realization of \mathbf{H}^2): geodesics, discussed in the previous subsection, and horospheres, i.e., Euclidean circles tangent to ∂U internally. A horosphere is orthogonal to all geodesics which touch its point of tangency with ∂U . Denote with \mathcal{H} the set of all horospheres of U . $\text{Aut}^+ U$ acts transitively on \mathcal{H} , hence \mathcal{H} is equipped with an invariant measure.

As done for the geodesic case, the (*horospherical*) Radon transform R is given on (say) rapidly decreasing functions on U as

$$Rf(h) = \int_h f(z) ds(z) \quad \text{for every } h \in \mathcal{H}.$$

The *dual Radon transform* R^* is defined as follows. If h_θ is the horosphere through 0 and $e^{i\theta}$, set

$$R^*\phi(0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(h_\theta) d\theta$$

for all, e.g., essentially bounded functions on \mathcal{H} ; then, for general $z \in U$, set $R^*\phi(z) = R^*(\phi \circ \tau_{0,z})(0)$.

Theorem *On \mathbf{H}^2 , R^*R is the convolution operator with the radial kernel $1/A(r/2)$, that is,*

$$R^*Rf = J_R * f, \quad \text{where } J_R(r) = \frac{1}{2\pi \sinh(r/2)}.$$

Proof As in the proof of Theorem 2.1, without loss of generality we can restrict our attention to 0. The length element ds on a horosphere h_θ through 0 equals $dr/\cos \psi$ at $z \neq 0$, where $d(0, z) = r$, and ψ is the angle between h_θ and the geodesic $\gamma_{\arg z}$ through 0 and z (see Fig. 1).

It is easy to show that

$$\cos \psi = \sqrt{1 - \tanh^2(r/2)} = \frac{2 \sinh(r/2)}{\sinh r},$$

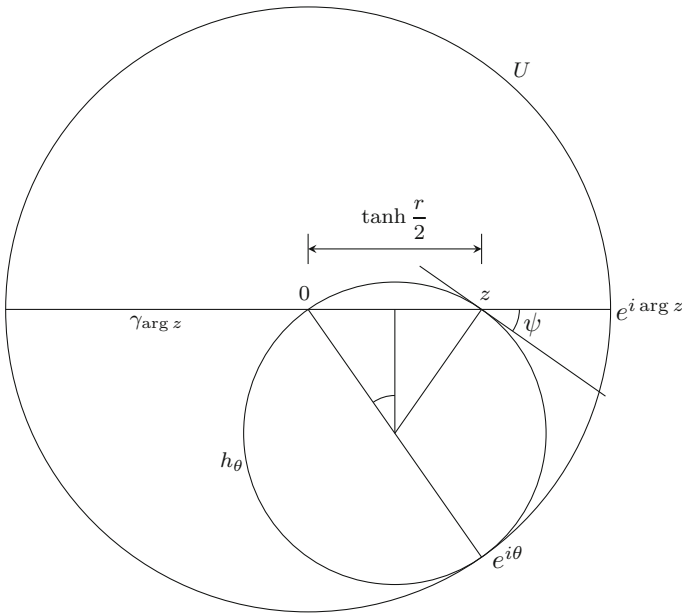


Fig. 1 Computation of the length element of a horosphere in the hyperbolic disc.

so that, taking into account that each horosphere through 0 has two ‘branches’ stemming from 0, we see from (1),(2) that

$$\begin{aligned}
 R^* R f(0) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{h_\theta} f(z) ds(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty f(e^{i\theta} \tanh(r/2)) \frac{dr}{\cos \psi} d\theta \\
 &= \int_U f(e^{i\theta} \tanh(r/2)) \frac{1}{2\pi \sinh(r/2)} dv = J_R * f(0). \quad \square
 \end{aligned}$$

Horospheres (or horocycles) in \mathbf{H}^n are (Euclidean) spheres tangent to the boundary internally. In a fashion very similar to the above, one proves that $R^* R$ is the convolution operator with the radial kernel

$$J_R(r) = \frac{\Gamma(n/2)}{2\sqrt{\pi}\Gamma((n-1)/2) \sinh(r/2)}.$$

As in the case of the X-ray transform, this leads to an inversion formula for R that factors through $R^* R$, and holds for every n ; see a comparative outline in [25, Chapter III, Theorem 3.1], and full details in [24, Chapter I, §4, in particular Theorem 4.5]. The inverse operator is always given by the left action on $R^* R$ of $Q(L)$ for a suitable polynomial Q . For \mathbf{H}^2 , up to suitable normalization, the polynomial is the identity. In

\mathbf{H}^n for larger n , again up to normalization, one has

$$Q(L) = \begin{cases} \prod_{j=1}^{n/2-1} \left(L + \frac{(n-2j)j}{n+2} \right) & \text{if } n \text{ is even,} \\ \prod_{j=1}^{(n-1)/2} \left(L + \frac{(n-2j)(2j-1)}{2n} \right) & \text{if } n \text{ is odd.} \end{cases}$$

Again as in the case of the X-ray transform, there are also direct inversion formulas that do not factor through R^*R , generalizations of the original inversion formulas on Euclidean spaces [33].

3 Radon Transforms on the Vertices of a Homogeneous Tree

3.1 Vertices of Homogeneous Trees

Let T be a homogeneous tree in which each vertex belongs to $q + 1$ edges, with $q \geq 2$. Since T has no loops, it is the Cayley graph of the free product \mathcal{G} of $q + 1$ copies of the two-element group \mathbf{Z}_2 . In particular, \mathcal{G} acts simply transitively on T and induces a convolution product on functions on the set V of vertices and on the set E of edges of T . \mathcal{G} is not the only subgroup of $\text{Aut } T$ acting simply transitively. The properties of the convolution, however, do not depend on the choice of this subgroup, but only on the geometry of the tree. We give more details in the ‘‘Appendix’’ on how to define the convolution on certain algebras of functions on T independently of the choice of \mathcal{G} .

The set V is equipped with the discrete topology. Two distinct vertices are *adjacent*, or *neighbors*, if they belong to the same edge: we shall write $v \sim v'$. In general, the *chain* of edges from v to v' is the minimal finite sequence $v = v_0, v_1, \dots, v_n = v'$ in V such that $v_{j-1} \sim v_j$ for every $j = 1, \dots, n$. The *distance* $d(v, v')$ is the non-negative integer n . We choose once and for all a reference vertex v_0 and write $|v| = d(v, v_0)$, the *length* of v . For $v, v' \in V$ we denote by $S(v, v')$ the set of all vertices v'' such that the path from v to v'' contains v' . We write $S(v) = S(v_0, v)$, and call *sectors* the sets of this type.

A *ray* is a sequence $[w_0, w_1, w_2, \dots]$ of distinct vertices such that $w_i \sim w_{i+1}$ for every i . A *geodesic* in V is the union of two rays that share only the initial vertex. This coincides with the notion of geodesic induced by the distance d . Denote by Γ the set of geodesics of V , and by Ω the set of rays starting at v_0 . It does not matter which starting vertex is chosen, since we can regard rays as equivalent if they coincide except for a finite number of vertices. In each equivalence class there is a unique representative starting at v_0 .

Both Γ and Ω are equipped with a Hausdorff topology as follows. Let $\Omega_{w,v}$ be the set of rays which start at w and pass through v . A sub-base for the topology of Ω consists of all *arcs* $\Omega_v = \Omega_{v_0,v}$ for v ranging in V . In particular, $\Omega_{v_0} = \Omega$. With this topology, Ω is a compact space, usually called the *boundary* of T . On the other hand, we have $\Gamma = (\Omega \times \Omega \setminus \text{diagonal}) / \{\pm 1\}$, because each geodesic is obtained by

an unordered pair of distinct rays sharing a chain starting at v_0 (only the last vertex of the chain belongs to the geodesic). Therefore the topology of Ω induces a (non-compact) topology on Γ . By the same token, one lifts Borel measures from Ω to Γ . In particular, for every $v \in V$ there exists a unique probability measure ν_v on Ω which is invariant under the isotropy subgroup $(\text{Aut } T)_v$ of $\text{Aut } T$ at v ; denote by ρ_v the induced probability measure on Γ .

Denote by $\Gamma_{w,v}$ the set of geodesics which contain $w, v \in V$, and by $\Gamma_w = \Gamma_{w,w}$ the set of geodesics through w . If j is a nonnegative integer, the number N_j of vertices at distance j from a given vertex is given by

$$N_j = \begin{cases} 1 & \text{if } j = 0, \\ (q + 1)q^{j-1} & \text{if } j \geq 1. \end{cases}$$

Then the measure ν_w is described by

$$\nu_w(\Omega_{w,v}) = \frac{1}{N_{d(w,v)}} \quad \text{for every } v \in V.$$

Proposition 3.1

$$\rho_w(\Gamma_{w,v}) = \begin{cases} \frac{q}{q + 1} & \text{if } v = w, \\ \frac{2}{(q + 1)^2 q^{d(w,v)-2}} & \text{if } v \neq w. \end{cases}$$

Proof If $v = w$, decompose

$$\Gamma_w = \left(\bigcup_{\substack{u,t \sim w \\ u \neq t}} \Omega_{w,u} \times \Omega_{w,t} \right) / \{\pm 1\}.$$

Since there are $(q + 1)q$ different choices of u, t as distinct neighbors of w , and $\nu_w(\Omega_{w,u}) = \nu_w(\Omega_{w,t}) = 1/(q + 1)$, we get $\rho_w(\Gamma_w) = q/(q + 1)$.

For $v \neq w$, let $n = d(v, w)$ and let s be the neighbor of w in V which lies on the path from w to v . Then $\rho_w(\Gamma_{w,v}) = 2\nu_w(\Omega_{w,v})(1 - \nu_w(\Omega_{w,s}))$ (the factor 2 comes from counting geodesics twice according to whether their beginning endpoint is chosen on the side of w or of v), $\nu_w(\Omega_{w,v}) = 1/N_n$, and $1 - \nu_w(\Omega_{w,s}) = q/(q + 1)$. □

3.2 The X-Ray Transform on Vertices

The results of this subsection are taken from [6, §3]; see also [1]. The *X-ray transform* is the operator $X: \ell^1(V) \rightarrow \ell^\infty(\Gamma)$ given by

$$Xf(\gamma) = \sum_{v \in \gamma} f(v) \quad \text{for each } f \in \ell^1(V) \text{ and } \gamma \in \Gamma.$$

The dual transform is the operator $X^* : \ell^\infty(\Gamma) \rightarrow \ell^\infty(V)$ defined as

$$X^* \phi(v) = \frac{1}{\rho_v(\Gamma_v)} \int_{\Gamma_v} \phi(\gamma) d\rho_v(\gamma) \quad \text{for all } \phi \in \ell^\infty(\Gamma).$$

Let χ_j be the sum operator at distance j , defined by

$$\chi_j f(v) = \sum_{w: d(v,w)=j} f(w) \quad \text{for } v \in V,$$

i.e., $\chi_j f(v)$ is the sum of f on the circle of radius j centered at v (in particular, χ_0 is the identity operator); χ_j is a radial convolution operator (with respect to the convolution product induced by $\text{Aut } T$). By abuse of notation, we shall denote its convolution kernel again by χ_j : it is given by

$$\chi_j(v, w) = \begin{cases} 1 & \text{if } d(v, w) = j, \\ 0 & \text{if } d(v, w) \neq j. \end{cases}$$

Theorem 3.2 X^*X acts on functions on V as the convolution operator with the radial kernel $J_X = \chi_0 + 2 \sum_{j=1}^\infty (1/N_j) \chi_j$. That is,

$$X^*X = \chi_0 + 2 \sum_{j=1}^\infty \frac{1}{(q+1)q^{j-1}} \chi_j \quad \text{on } \ell^1(V).$$

Proof Set $\phi = Xf$. If f vanishes outside $\{v\}$, then $X^*\phi(v) = f(v) = \chi_0 f(v)$, while $\chi_j f(v) = 0$ for $j \geq 1$. On the other hand, if f vanishes outside $\{u\}$, with $u \neq v$, and $n = d(u, v)$, then

$$X^*\phi(v) = \frac{\rho_v(\Gamma_{v,u})}{\rho_v(\Gamma_v)} f(u) = \frac{2}{N_n} \chi_n f(v)$$

by Proposition 3.1, while $\chi_j f(v) = 0$ for $j \neq n$. The case of arbitrary f follows by linearity and the density of finitely supported functions in $\ell^1(V)$. \square

The series in the above statement is absolutely convergent in the convolution operator norm on $\ell^2(V)$. This follows from Haagerup’s convolution theorem [22, Lemma 1.4]; also [20, Proposition VIII.1.5], [8, Theorem 5.1], [6, proof of Proposition 3.2]. An explicit inversion formula for the X-ray transform can be derived from this fact, see [1,6]. Inversion of X^*X is achieved by showing that the Laplace operator $\Delta_V = \mu_1 - \delta_0$, with $\mu_1 = \chi_1/(q+1)$, is a multiple of a left inverse of X^*X . The left inverse operator turns out to be

$$\Phi_X = \frac{2(q+1)^3}{q(q-1)^2} \left(\chi_0 + \frac{2q}{q+1} \sum_{n=1}^\infty (-1)^n q^{-n} \chi_n \right).$$

If T is not homogeneous, let $q_v + 1$ be the number of neighbors of the vertex v . No group acts on T in general, so the words ‘convolution’ and ‘radiality’ lose their

meaning. Also, one must choose a transient transition operator P on V . It is natural to choose the nearest-neighbor isotropic transition operator at each vertex. Then X^* can be defined as in the homogeneous case with respect to an appropriate choice of the family of boundary probability measures $\{\rho_v : v \in V\}$; a natural choice was made in [6, Proposition 2.3] as the measure induced on Γ by $\nu_v \times \nu_v$ on $\Omega \times \Omega$, where ν_v is the hitting distribution of the Markov process generated by P starting at v . Then X^*X can be written as the operator of summation $X^*Xf = \sum_{v \in V} J_X(\cdot, v)f(v)$. If ρ_v is chosen as above, the kernel is

$$J_X(u, v) = \begin{cases} 1 & \text{for } u = v, \\ \frac{2}{(q_u + 1) \prod_{j=1}^{n-1} q_{v_j}} & \text{for } n > 0, \end{cases}$$

where $[u=v_0, v_1, \dots, v_n=v]$ is the path from u to v . The proof is similar to the homogeneous case and is left to the reader. We just mention that on a semi-homogeneous tree, if we write q_+ for q_u and denote by q_- the opposite homogeneity degree, the kernel $J_X(u, v)$, becomes

$$J_X(u, v) = \begin{cases} 1 & \text{for } u = v, \\ \frac{2}{(q_+ + 1)\mathbf{q}^{d(u,v)-1}} & \text{for } d(u, v) \text{ odd,} \\ \frac{2}{(q_+ + 1)q_- \mathbf{q}^{d(u,v)-2}} & \text{for } d(u, v) \text{ non-zero even,} \end{cases}$$

where $\mathbf{q} = \sqrt{q_+q_-}$ is the average growth of the semi-homogeneous tree.

3.3 The Horospherical Radon Transform on Vertices

The results of this subsection are taken from [7,9,15]. We start by introducing the notion of horosphere of vertices in a tree. As we only need to deal with homogeneous trees, it is natural, although somewhat less elegant, to give a definition which apparently depends on the choice of the reference vertex v_0 . For a reference-free presentation see [12].

Let $v \in V$ and $\omega \in \Omega$, regarded as a ray that starts at v_0 . The *horospherical index* $h_\omega(v)$ is defined as $d(v_0, u) - d(v, u)$, where u is the branching vertex between the ray ω and the finite path from v_0 to v . In other words, u is the vertex of ω which is closest to v , and $d(v, u)$ equals $d(v, \omega)$, the distance between v and the ray ω . Therefore $h_\omega(v) = d(v, v_0) - 2d(v, \omega)$. For every $k \in \mathbf{Z}$, the *horosphere* $h(k, \omega)$ is the set $\{v \in V : h_\omega(v) = k\}$. Every two vertices in the same horosphere are at even distance. The horospheres through v_0 have index 0.

We equip the space \mathcal{H} of horospheres with the product topology of $\mathbf{Z} \times \Omega$. This parametrization depends on the choice of v_0 , whereas the induced topology on \mathcal{H} does not, as can be easily seen. Denote by $\mathcal{H}_{v,v'}$ the set of horospheres that contain v and v' , an open and closed subset of \mathcal{H} that is empty if $d(v, v')$ is odd (as remarked earlier). For simplicity, set $\mathcal{H}_v = \mathcal{H}_{v,v}$. We equip \mathcal{H} with the $(\text{Aut } T)$ -invariant measure

$\sigma = m \times \nu_{v_0}$, where m is the measure on \mathbf{Z} given by $m(\{n\}) = q^n$, see [16]. While $\sigma(\mathcal{H}) = \infty$, nevertheless σ is a probability measure on $\mathcal{H}_{v_0} = \{0\} \times \Omega$ (the set of horospheres through v_0) as well as on \mathcal{H}_v for every v .

The (horospherical) Radon transform $R: \ell^1(V) \rightarrow L^\infty(\mathcal{H})$, in analogy with the geodesic case, is given by

$$Rf(h) = \sum_{v \in h} f(v) \quad \text{for every } f \in \ell^1(V) \text{ and } h \in \mathcal{H}.$$

The dual Radon transform $R^*: L^\infty(\mathcal{H}) \rightarrow \ell^\infty(V)$ is defined as follows. Set

$$R^*\phi(v) = \int_{\mathcal{H}_v} \phi(h) d\sigma(h) \quad \text{for every } \phi \in L^\infty(\mathcal{H}).$$

The operator R^*R therefore maps $\ell^1(V)$ to $\ell^\infty(V)$, and commutes with the action of $\text{Aut } T$ on $\ell^1(V)$.

Theorem 3.3 *The horospherical Radon transform on V satisfies*

$$R^*R = \chi_{v_0} + (1 - 1/q) \sum_{j=1}^{\infty} \frac{1}{N_j} \chi_{2j} \quad \text{on } \ell^1(V).$$

Proof It is enough to compute R^*R on every Dirac delta δ_v for $v \in V$ and evaluate at v_0 . We have $R^*R\delta_v(v_0) = \sigma(\mathcal{H}_{v_0,v})$. This vanishes if v_0 and v are an odd distance apart. If the distance is nonzero and even, let $[v_0, v_1, \dots, v_{2n}=v]$ be the finite path from v_0 to v , and the end ω corresponding to each horosphere h in \mathcal{H}_v must belong to $\Omega_{v_n} \setminus \Omega_{v_{n+1}}$; indeed, v_n must be the closest vertex to v of the ray ω , in order for the index $h_\omega(v)$ to vanish. Since $\Omega_{v_{n+1}} \subset \Omega_{v_n}$,

$$\sigma(\mathcal{H}_{v_0,v_{2n}}) = \nu_{v_0}(\Omega_{v_n}) - \nu_{v_0}(\Omega_{v_{n+1}}) = \frac{1}{N_n} - \frac{1}{N_{n+1}} = \frac{q - 1}{(q + 1)q^n}.$$

Since $\sigma(\mathcal{H}_{v_0}) = 1$, as remarked before, R^*R acts as the convolution operator with the radial kernel $J_R = \chi_{v_0} + (1 - 1/q) \sum_{j=1}^{\infty} (1/N_j) \chi_{2j}$. □

The inverse operator can be computed by solving a second-order difference equation for the values of the radial kernel J_R , with the result hereunder, appeared in [15, §5] following an outline of [7] that makes use of the resolvent of the normalized nearest-neighbor transition operator on T and the Carleman formula.

Corollary 3.4 *The inverse of R^*R on V is the convolution operator with kernel*

$$\Phi_R(w, v) = \begin{cases} \frac{3q + 1}{2(q + 1)} & \text{if } w = v, \\ 0 & \text{if } d(v, w) \text{ is odd,} \\ -\frac{q - 1}{2(q + 1)q^{d(v,w)-1}} & \text{if } d(v, w) \text{ is nonzero even.} \end{cases}$$

We shall give a similar argument in Sect. 5.5, in the set-up of functions on edges instead of vertices, not studied before.

The kernel representation of Theorem 3.3 has an extension to semi-homogeneous trees with homogeneity degrees q_+ at w and q_- at vertices of the other parity. The following expression for the kernel can be derived from [15, Proposition 4.3]:

$$J_R(w, v) = \begin{cases} 1 & \text{for } w = v, \\ 0 & \text{for } d(w, v) \text{ odd,} \\ \frac{q_- - 1}{(q_+ + 1)q_-^{d(w,v)/2-1}} & \text{for } d(w, v) \equiv 2 \pmod{4}, \\ \frac{q_+ - 1}{(q_+ + 1)q_+^{d(w,v)/2}} & \text{for } 0 < d(w, v) \equiv 0 \pmod{4}. \end{cases}$$

4 Functions on Tree Edges and the X-Ray Transform

4.1 The Graph of Tree Edges and Convolution Operators

When a tree is regarded as a simplicial complex there are two types of simplices: vertices and edges. So far we have given a survey of Radon transforms on functions defined on vertices. We shall henceforth look at the other set-up, where all the results are new.

Two edges e, e' are adjacent if they share exactly one vertex. So, every edge has $2q$ neighbors. The concepts of chain, distance, ray, geodesic, boundary and automorphisms are defined for edges as done for vertices in Sect. 3.1. Indeed, these concepts are intrinsic to T , not to V and E . We shall fix a reference edge e_0 and set $|e| = d(e, e_0)$.

Let us study the graph associated to E under the natural notion of adjacency. Every $e \in E$ becomes a vertex of this graph, that by abuse of notation we still denote by E . Two edges e, e' are adjacent in T if and only if they are adjacent regarded as vertices of this graph. In other words, E is the dual graph of T . The group $\mathcal{G} = \mathbf{Z}_{q+1} * \mathbf{Z}_{q+1}$ acts as a simply transitive group of isometries of E (see the ‘‘Appendix’’). The distance on E lifts to a distance on \mathcal{G} , called *block distance*. The graph E has the same group of isometries of T , hence \mathcal{G} is a subgroup of $\text{Aut } T = \text{Aut } E = \text{Aut } V$, simply transitive on edges but with two orbits on V and non-trivial isotropy subgroups. The convolution product on $\ell^2(\mathcal{G})$ corresponds to a convolution product on $\ell^2(E)$, with the same operator norm.

An estimate for the norm of convolution operators on \mathcal{G} was given in [26] in analogy to Haagerup’s well-known convolution estimate [22], hence it also holds for convolution operators on $\ell^2(E)$. The statement is more general than we need here. The part relevant to the rest of this article is the following:

Lemma (Haagerup’s inequality [22,26]) *If $f : \mathcal{G} \rightarrow \mathbf{C}$ is supported on the circle of words of block distance $n > 0$ from the identity, then*

$$\|f * g\|_2 \leq (q - 1)(n + 1) \|f\|_2 \|g\|_2 \text{ for every } g \in \ell^2(\mathcal{G}). \tag{4}$$

Here the constant is larger than the one originally claimed in [26, Theorem 1.(ii)], although asymptotically equivalent; that argument actually leads to the constant written here.

Spherical functions on a polygonal graph were studied in [27] whence the properties of spherical functions on the edges of a homogeneous tree can be derived. We give some of the statements in the rest of this Section.

4.2 The Algebra of Radial Functions on Edges

By abuse of notation, a convolution operator with a radial convolution kernel f will be denoted again by f , by setting $f(e, e') = f(d(e, e'))$. If $C_n = \{e \in E : |e| = n\}$, then

$$|C_n| = 2q^n \quad \text{for } n > 0. \tag{5}$$

Remark 4.1 Every edge except e_0 has q forward neighbors (farther from e_0), one backward neighbor (closer to e_0) and $q - 1$ neighbors at the same distance from e_0 as e . (Instead, on V the neighbors have constant parity: this yields a slightly different relation for vertex convolution, see [20] and its references).

Lemma 4.2 *Let χ_n be the characteristic function of the set of edges at distance n from e_0 . Then $\{\chi_n\}$ satisfies the convolution product rule*

$$\chi_1 * \chi_n = \begin{cases} \chi_1 & \text{if } n = 0; \\ 2q\chi_0 + (q - 1)\chi_1 + \chi_2 & \text{if } n = 1; \\ q\chi_{n-1} + (q - 1)\chi_n + \chi_{n+1} & \text{if } n > 1. \end{cases}$$

Proof The case $n = 0$ is trivial. Since $\chi_1 = \sum_{e' \sim e_0} \delta_{e'}$,

$$\chi_1 * \chi_1 = \sum_{e'' \sim e'} \sum_{e' \sim e_0} \delta_{e''}.$$

Each edge e'' in this double sum has length 0, 1, or 2:

- $e'' = e_0$ appears in the sum as many times as there are neighbors of e_0 (namely $2q$ times);
- each $e'' \sim e_0$ appears as many times as there are edges $e' \sim e_0$ on the same side of e'' with respect to e_0 , except e'' itself (namely $q - 1$ times);
- each e'' of length 2 appears only once.

Therefore the statement is proved for $n = 1$.

For $n > 1$

$$\chi_1 * \chi_n = \sum_{e'' \sim e'} \sum_{|e'|=n} \delta_{e''}.$$

The right-hand side vanishes unless $|e''| = n - 1, n$ or $n + 1$. If $|e''| = n - 1$, then $e'' \sim e'$ gives a non-zero contribution to the sum if and only if $|e'| = n$ and e'' belongs

to the path from e_0 to e' ; for each e'' there are exactly q such edges e' . If $|e''| = n$, then e'' appears in the sum as many times as there are neighbors e' of e'' with $|e'| = n$, namely the $q - 1$ edges $e' \neq e''$ that share with e'' the vertex closer to e_0 . If $|e''| = n + 1$, every such edge appears as many times as there are edges e' with $|e'| = n$ in the path from e_0 to e'' , namely once. \square

Corollary 4.3 *Normalize χ_n by $\eta_n = \chi_n/|C_n|$. Then*

$$\eta_1 * \eta_n = \frac{1}{2q}\eta_{n-1} + \frac{q-1}{2q}\eta_n + \frac{1}{2}\eta_{n+1} \quad \text{for every } n \geq 1.$$

In particular the convolution algebra $\mathcal{R}_\#$ of radial finitely supported functions on E (with identity) is generated by η_1 , hence it is commutative.

The radialization operator \mathcal{E} around e_0 acts on a function g on E by

$$\mathcal{E}g(e) = \sum_{e': |e'|=|e|} \frac{g(e')}{|C|_e|}.$$

If f, g are functions on E , set $\langle f, g \rangle = \sum_e f(e)g(e)$ whenever the series is absolutely convergent; $(f, g)_2 = \langle f, \bar{g} \rangle$ is the ℓ^2 -inner product. An obvious, yet useful property satisfied by the radialization operator is the following: if f, g are functions on E with f radial such that fg is summable, then

$$\langle f, g \rangle = \langle f, \mathcal{E}g \rangle. \tag{6}$$

4.3 The X-Ray Transform on Edges

Geodesics and rays of edges are defined in the same way as for vertices, and naturally identified with geodesics, respectively rays, of vertices, so they give rise to a boundary canonically identifiable with the boundary for V . Each pair of edges $e \neq e'$ induces an arc $\Omega_{e,e'} \subset \Omega$ that consists of the equivalence classes of all rays starting at e and containing e' ; it is clear that the arcs induced by E are the same as those induced by V . We shall write $\Omega_e = \Omega_{e_0,e}$.

We denote by ν_e the probability measure on Ω that is invariant under the isotropy subgroup $(\text{Aut } T)_e$ of $\text{Aut } T$ at e . Then we have, in the same way as in Proposition 3.1, a measure ρ_e on the space of geodesics of edges containing e , equivariant under $(\text{Aut } T)_e$. For $e, e' \in E$ let $\Gamma_{e,e'}$ be the set of geodesics of edges that contain both e and e' .

Proposition 4.4

$$\rho_e(\Gamma_{e,e'}) = \frac{1}{2}q^{-d(e,e')}.$$

Proof Each edge $e = [v, w]$ splits Ω into the disjoint union of the subsets $\Omega_{v,w}$ and $\Omega_{w,v}$, that have the same ν_e -measure because they are interchanged by any element of $(\text{Aut } T)_e$ that interchanges the endpoints of e .

Fix a boundary point ω and consider the subset $\Omega_{\omega,e}$ of Ω given by all ω' such that the geodesic with endpoints ω, ω' contains e . Then $\Omega_{\omega,e}$ is one of the two subsets of the splitting of Ω induced by e , hence it has ν_e -measure $1/2$.

For each geodesic that contains distinct e, e' consider its boundary point ω closest to e . The set of these boundary points is $\Omega_{e',e}$, of ν_e -measure $1/2$. The other boundary point of these geodesics varies in $\Omega_{e,e'}$, a set of ν_e -measure $q^{-d(e',e)}/2$ by (5). Therefore the set of geodesics with these ordered endpoints has measure $q^{-d(e',e)}/4$. Counting also the geodesics with reversed endpoints we obtain the expression in the statement. □

The *X-ray transform* on E is the operator $X: \ell^1(E) \rightarrow \ell^\infty(\Gamma)$ given by

$$Xf(\gamma) = \sum_{e \in \gamma} f(e) \quad \text{for each } f \in \ell^1(E) \text{ and } \gamma \in \Gamma.$$

Its dual is the operator $X^*: \ell^\infty(\Gamma) \rightarrow \ell^\infty(E)$ defined as

$$X^*\phi(e) = \frac{1}{\rho_e(\Gamma_e)} \int_{\Gamma_e} \phi(\gamma) d\rho_e(\gamma).$$

It is clear that, for every $e, e' \in E$, the convolution operator with kernel χ_j satisfies

$$\chi_j \delta_e(e') = \begin{cases} 1 & \text{if } d(e, e') = j, \\ 0 & \text{if } d(e, e') \neq j. \end{cases}$$

This fact, via the same argument as for Theorem 3.2, using Proposition 4.4 instead of 3.1 yields:

Theorem *For the X-ray transform X on E , the expression of the operator X^*X in terms of the radial kernels χ_j is:*

$$X^*X = \sum_{j=0}^{\infty} q^{-j} \chi_j \quad \text{on } \ell^1(E).$$

The series converges absolutely in the operator norm on $\ell^2(E)$, by Haagerup's inequality (4): therefore it converges to a bounded operator on ℓ^2 .

Corollary *Let $\Delta_E = \eta_1 - \eta_0$ be the Laplace operator on E . Then $(q/(1 - q))\Delta_E$ is a left inverse of X^*X , hence $(q/(1 - q))\Delta_E X^*$ is a left inverse of the edge X-ray transform X .*

Proof Since $\chi_0 = \eta_0 = \mathbf{I}$, the identity, and by (5) $\chi_j = 2q^j \eta_j$ for $j > 0$, we have

$$X^*X = \eta_0 + 2 \sum_{j=1}^{\infty} \eta_j.$$

By Corollary 4.3,

$$(\eta_0 - \eta_1)X^*X = \eta_0 - \eta_1 + 2 \sum_{j \geq 1} \eta_j - \frac{1}{q} \sum_{j \geq 1} (\eta_{j-1} + (q-1)\eta_j + q\eta_{j+1}) = \frac{q-1}{q} \eta_0.$$

□

5 Spherical Functions on Edges and the Spectrum of the Laplacian

5.1 The Tree Boundary and the Poisson Kernel

The Poisson kernel on V , when T is equipped with the nearest-neighbor isotropic stochastic transition operator μ_1 , has been extensively studied in [11,13,20,21] and their bibliographic lists. Here we introduce and study it on edges. The harmonic functions on E are the eigenfunctions with eigenvalue 1 of η_1 . Since T has exponential growth, it is trivial to construct non-constant harmonic functions, including positive ones; equivalently, the random walk induced by η_1 is transient. Ω is the Poisson boundary of η_1 regarded as the generator of a random walk. It is well known that the probability measure ν_e introduced in Sect. 4.3 is the reproducing measure at e of harmonic functions, in the sense that, for every harmonic function h on E that extends continuously to $E \cup \Omega$, one has $h(e) = \int_{\Omega} h(\omega) d\nu_e(\omega)$. If $\nu = \nu_{e_0}$, then $\nu(\Omega_{e_1}) = \nu(\Omega_{e_2})$ if $|e_1| = |e_2|$. Then, by (5),

$$\nu(\Omega_e) = \frac{1}{2}q^{-|e|} \text{ for } |e| > 0. \tag{7}$$

Since all Σ_n are finite σ -algebras, ν_e is transformed by each $\lambda \in \text{Aut } T$ to the measure $\nu_{\lambda e}$ which is absolutely continuous with respect to ν_e . It is easy to compute the Radon-Nikodym derivative $d\nu_{e_1}/d\nu_{e_2}(\omega)$. It is enough to compute $d\nu_e/d\nu(\omega)$. We call *Poisson kernel* the limit

$$K(e, \omega) = \lim_{e' \rightarrow \omega} \frac{\nu_e(\Omega_{e'})}{\nu(\Omega_{e'})}.$$

The rays starting at the edges e_0, e , respectively, and belonging to the same equivalence class $\omega \in \Omega$ merge after a finite number of steps: call e_ω the first common edge. Define the *horospherical index* as

$$h(e, \omega) = d(e_\omega, e_0) - d(e_\omega, e).$$

The *horosphere* $\mathbf{h}(\omega, n)$ tangent at ω with index $n \in \mathbf{Z}$ with respect to e_0 is the set of all edges e where $h(e, \omega) = n$, namely those that are reached from e_0 moving a certain number m_+ of steps towards ω and then m_- steps away from ω , with $m_+ - m_- = n$. We have $h(\lambda e, \lambda \omega) = h(e, \omega)$ if $\lambda \in (\text{Aut } T)_{e_0}$.

The space \mathcal{H} of horospheres is identified with $\Omega \times \mathbf{Z}$ via the global coordinates defined by e_0 . In this identification, \mathcal{H} is equipped with the product topology induced by $\Omega \times \mathbf{Z}$, that is easily seen to be independent of the choice of e_0 .

Proposition For every e, ω ,

$$K(e, \omega) = q^{h(e, \omega)}.$$

5.2 Spherical Functions on Edges

The properties of spherical functions on E are proved as in the general theory of spherical functions for a Gel'fand pair, see for instance [20, Ch. III]). We shall skip most of the proofs. Similar arguments for spherical functions on polygonal graphs are in [27].

A function ϕ on E is *spherical* if it is radial, $\phi * u = c_\phi u$ for every u in the ℓ^1 -completion $\ell^1_\#$ of $\mathcal{R}_\#$ with some constant c_ϕ independent of u , and $\phi(e_0) = 1$.

Proposition 5.1 A radial function ϕ is spherical if and only if $h \mapsto \langle \phi, h \rangle$ is a multiplicative functional on the convolution algebra $\mathcal{R}_\#$. Thus, the set of bounded spherical functions is the Gel'fand spectrum of the Banach algebra $\ell^1_\#$.

Every spherical function is uniquely determined by its value $\lambda(\phi) = \phi_{(1)}$ attained at distance 1 from e_0 .

Corollary Let h be a radial function. If $\eta_1 * h = ch$ with $c \neq 0$, then h is a multiple of the spherical function ϕ such that $\lambda(\phi) = c$. If $\eta_1 * h = 0$ then $h = 0$.

Proof If $e \sim e_0$, then by radially $h(e) = \langle h, \eta_1 \rangle = h * \eta_1(e_0) = ch(e_0)$. Hence, if $h(e_0) = 0$, then $h \equiv 0$ by Lemma 4.2, otherwise $h/h(e_0)$ is a normalized radial eigenfunction of η_1 , hence of $\mathcal{R}_\#$, and therefore it is a spherical function. \square

Proposition 5.2 For $z \in \mathbf{C}$, the functions

$$\phi_z(e) = \int_\Omega K^z(e, \omega) dv(\omega)$$

are spherical functions and

$$\eta_1 * \phi_z = \gamma(z)\phi_z, \tag{8}$$

$$\text{with } \gamma(z) = \frac{q^z + q - 1 + q^{1-z}}{2q} = \frac{\cosh((z - 1/2) \log q)}{\sqrt{q}} + \frac{q - 1}{2q}. \tag{9}$$

Conversely, every spherical function arises in this way. Finally, we have the majorization principle

$$|\phi_z| \leq \phi_{\Re z}.$$

Corollary The map γ is periodic of period $2\pi i / \log q$ and we have $\gamma(1/2 + it) = \gamma(1/2 - it)$. Therefore

$$\gamma(1/2 + i\mathbf{R}) = \{ \gamma(1/2 + it) : 0 \leq t \leq \pi / \log q \}.$$

The properties of spherical functions on E follow immediately from multiplicativity on $\mathcal{R}_\#$ and the recurrence relation of Lemma 4.2 and are similar to those proved for V in [20].

Proposition *The following are equivalent for a function ϕ on E :*

- $\phi = \phi_z$;
- ϕ is radial and, if $\phi_{(n)} = \phi(e)$ whenever $|e| = n$, then $\phi_0 = 1$, $\phi_1 = \gamma(z)$ and

$$\phi_{(1)}\phi_{(n)} = \frac{1}{2q}\phi_{(n-1)} + \frac{q-1}{2q}\phi_{(n)} + \frac{1}{2}\phi_{(n+1)} \quad \text{for every } n > 0. \quad (10)$$

- $\phi(e_0) = 1$ and $\eta_1 * \phi = \gamma(z)\phi$.

As a consequence we obtain a nice expansion of edge-spherical functions as linear combinations of exponentials.

Corollary 5.3 *Set*

$$c(z) = \frac{q-1 + q^{1-z} - q^z}{2(q^{1-z} - q^z)},$$

for which $\overline{c(z)} = c(\bar{z})$, therefore

$$\overline{c(1/2 + it)} = c(1/2 - it) \quad \text{for every } t \in \mathbf{R}. \quad (11)$$

If $\Pi = \{z: q^{2z-1} = 1\} = \{1/2 + ik\pi/\log q: k \in \mathbf{Z}\}$ and $y_z(e) = q^{-z|e|}$ for $e \in E$:

$$\phi_z(e) = \begin{cases} c(z)y_z(e) + c(1-z)y_{1-z}(e) \sim q^{-|e|\min\{\Re z, 1-\Re z\}} & \text{for } z \notin \Pi, \\ \left(1 + \frac{q-1}{2\sqrt{q}}|e|\right)y_z(e) \\ = (-1)^k|e|\left(1 + \frac{q-1}{2\sqrt{q}}|e|\right)q^{-|e|/2} \sim |e|q^{-|e|/2} & \text{for } z \in \Pi. \end{cases}$$

Furthermore

$$\phi_z \in \begin{cases} \ell^\infty & \text{if and only if } 0 \leq \Re z \leq 1, \\ \ell^p \text{ for } p > 2 & \text{if and only if } 1/p \leq \Re z \leq 1 - 1/p, \\ \bigcap_{p>2} \ell^p & \text{if and only if } \Re z = 1/2. \end{cases}$$

Finally, the ℓ^1 spectrum of η_1 , that is the Gel'fand spectrum of $\ell_\#^1$, is the ellipse

$$\{\gamma(z): 0 \leq \Re z \leq 1\} = \left\{ \left(\frac{2q\Re\gamma(z) - (q-1)}{q+1} \right)^2 + \left(\frac{2q\Im\gamma(z)}{q-1} \right)^2 \leq 1 \right\}.$$

Definition 5.4 It follows from Lemma 4.2 that there are polynomials Q_n of degree n such that $\eta_n = Q_n(\eta_1)$. On the other hand, the functionals $g \mapsto \langle \phi_z, g \rangle$ are multiplicative on the radial algebra (Proposition 5.1), hence

$$\phi_z(n) = \langle \phi_z, \eta_n \rangle = \langle \phi_z, Q_n(\eta_1) \rangle = Q_n(\langle \phi_z, \eta_1 \rangle) = Q_n(\gamma(z)). \tag{12}$$

The functions Q_n are called *spherical polynomials*. By abuse of terminology, sometimes also the functions $\tilde{Q}_n(z) = Q_n(\gamma(z))$ are called spherical polynomials (although they are not polynomials in the variable z).

Corollary 5.5 For every $z \in \mathbf{C}$, $|\tilde{Q}_n(z)| \leq |\tilde{Q}_n(\Re z)|$.

Proof It follows from the majorization principle of Proposition 5.2: $|\tilde{Q}_n(z)| = |\phi_z(n)| \leq |\phi_{\Re z}(n)| = |\tilde{Q}_n(\Re z)|$. □

5.3 Spectrum of the Laplace Operator on Edges

The following results link spherical functions with the ℓ^2 -spectral theory of the Laplace operator on edges η_1 .

Theorem 5.6 The spectrum of η_1 on $\ell^2(E)$ is the real segment

$$\begin{aligned} S &= \{ \gamma(z) : \Re z = 1/2 \} = [\gamma(1/2 + i\pi / \log q), \gamma(1/2)] \\ &= \left[\frac{q-1}{2q} - \frac{1}{\sqrt{q}}, \frac{q-1}{2q} + \frac{1}{\sqrt{q}} \right], \end{aligned}$$

its spectral radius $\rho(\eta_1)$ is $(q-1)/2q + 1/\sqrt{q}$, and its resolvent $\xi_{\gamma(z)}$ at the eigenvalue $\gamma(z)$ is the convolution operator on ℓ^2 given by the function

$$r_z(e) = \frac{2q}{q^{1-z} - q^z - (q-1)} q^{-z|e|} = \frac{1}{c(1-z)(q^{-z} - q^{z-1})} q^{-z|e|} \tag{13}$$

with $\Re z > 1/2$. In other words, r_z satisfies the resolvent equation

$$\eta_1 r_z - \gamma(z) r_z = \delta_{e_0}. \tag{14}$$

For $\Re z < 1/2$, $r_z(e)$ grows with $|e|$, hence it does not act as a convolution operator on ℓ^2 .

Proof By identifying E with the free product $\mathcal{G} = \mathbf{Z}_{q+1} * \mathbf{Z}_{q+1}$ introduced in Sect. 4.1 we give a meaning to expressions such as e^{-1} where e is an edge. If a spherical function ϕ_z is positive definite, then $\gamma(z) \in S$. Indeed, for every $e \in E$, the edges e and e^{-1} clearly have the same length, by construction of \mathcal{G} . So, since ϕ_z is radial, $\phi_z(e^{-1}) = \phi_z(e)$, but if it is positive definite, $\phi_z(e^{-1}) = \overline{\phi_z(e)}$. Hence such ϕ_z is real valued. Being positive definite, it is also bounded by $\phi(e_0)$; but ϕ_z is bounded if and only if $0 \leq \Re(z) \leq 1$, by Corollary 5.3, and it is real only if $\gamma(z) \in \mathbf{R}$, by the recurrence relation (10) (because $\gamma(z) = \phi_z(e)$ for $|e| = 1$).

Now we show the opposite implication: if $\gamma(z) \in \mathbf{R}$ and $0 \leq \Re z \leq 1$ then ϕ_z is positive definite, that is, it induces a functional $f \mapsto \langle f, \phi_z \rangle$ that takes positive values on the positive elements f of the involutive algebra ℓ^1 . These positive elements form the positive cone generated by functions of the type $f = h^* * h$, where $h^*(e) = \overline{h(e^{-1})}$, and are positive definite functions. The radialization maps positive definite functions (that is, positive elements of the involutive algebra ℓ^1), to radial positive definite functions (that is, positive elements of the involutive algebra $\ell^1_{\#}$). Thus $\mathcal{E}f$ is a linear combination with positive coefficients of functions of the type $g^* * g$ for some radial g . Without loss of generality, we can restrict attention to a positive element f such that $\mathcal{E}f = g^* * g$. Then

$$\langle f, \phi_z \rangle = \langle \mathcal{E}f, \phi_z \rangle = \langle g^* * g, \phi_z \rangle = \overline{\langle g, \phi_z \rangle} \langle g, \phi_z \rangle \geq 0$$

by (6), Proposition 5.1 and the fact that ϕ_z is real valued for $\Re z = 1/2$. So we have shown that ϕ_z is positive definite if $z = z_t = 1/2 + it$ with $t \in \mathbf{R}$ or if $z = \sigma$ or $\sigma + i\pi/\log q$ with $1/2 \leq \sigma \leq 1$. We need to determine which of these positive definite functions are multiplicative linear functionals continuous on the C^* -algebra C^*_λ of convolution operators on ℓ^2 . For the relevant background, the reader is referred to [18].

By Corollary 5.3 we have $\phi_{1/2+it} \in \ell^{2+\epsilon}$, and, by the asymptotic rate of decay of spherical functions, the product $\phi_\sigma \phi_{1/2+it}$ is positive definite and belongs to ℓ^2 , hence to the Fourier algebra $A(\mathcal{G}) = \ell^2(\mathcal{G}) * \ell^2(\mathcal{G})$. In particular, $\phi_\sigma \phi_{1/2+it}$ has norm 1 in the Banach algebra $B(\mathcal{G})$ of positive definite functions. On the other hand, ϕ_σ converges pointwise to 1 as $\sigma \rightarrow 1$ because ϕ_1 is the constant function 1. Therefore $\phi_{1/2+it}$ is the pointwise limit of functions $\phi_\sigma \phi_{1/2+it}$ of norm 1 in $B(\mathcal{G})$, hence it defines a multiplicative linear functional continuous on C^*_λ . This shows that $\gamma(1/2 + it/\log q)$ belongs to the spectrum of η_1 in C^*_λ . The extreme points of the spectrum of η_1 are $\rho = \gamma(1/2) = 1/\sqrt{q} + (q-1)/2q$, $\rho' = \gamma(1/2 + it/\log q) = -1/\sqrt{q} + (q-1)/2q$ and $\{\gamma(z) : \Re z = 1/2\} = [\rho', \rho]$.

Suppose that $\gamma(z) \notin [-\rho, \rho]$. Then $\Re z \neq 1/2$: since $\gamma(z) = \gamma(1-z)$ we may assume $\Re z > 1/2$. As before, write $y_z(e) = q^{-z|e|}$. We know from (8) that $\eta_1 y_z(e) = \gamma(z) y_z(e)$ for every $e \neq e_0$. Instead, $\eta_1 y_z(e_0)$ is the value of y_z on edges of length 1, that is q^{-z} ; moreover, $y_z(e_0) = 1$. Hence

$$(\eta_1 - \gamma(z)\mathbf{I})y_z(e_0) = q^{-z} - \gamma(z) = \frac{q^{1-z} - q^z - (q-1)}{2q}.$$

Therefore

$$(\eta_1 - \gamma(z)\mathbf{I})y_z = \frac{q^{1-z} - q^z - (q-1)}{2q} \delta_{e_0}.$$

Thus, for $\Re z > 1/2$, the function $r_z(e) = 2q y_z^{-z|e|} / (q^{1-z} - q^z - (q-1))$ is the resolvent of η_1 provided that it is a bounded convolution operator on ℓ^2 . This follows from Haagerup's inequality for edges (4), $\|r_z\|_{C^*_\lambda} \leq (q-1) \sum_{n \geq 0} (n+1) \|\chi_n r_z\|_2$. So it is enough to prove that the series converges. If $|e| = n$, then $r_z(e) \sim q^{-nz}$, and

so, by (5),

$$\sum_{n \geq 0} (n + 1) \|\chi_n r_z\|_2 \sim \sum_{n \geq 0} (n + 1) |q^{-nz}| \|\chi_n\|_2 = \sum_{n \geq 0} (n + 1) q^{(1/2 - \Re z)n} < \infty. \tag{15}$$

The last identity in (13) follows from the definition of $c(z)$. □

It was shown in [20] that the spectrum of μ_1 as a convolution operator on $\ell^2(V)$ is the real segment

$$S_V = \{\gamma(z) : \Re z = 1/2\} = [\gamma(1/2 + i\pi / \log q), \gamma(1/2)] = \left[-\frac{2\sqrt{q}}{q + 1}, \frac{2\sqrt{q}}{q + 1} \right].$$

Therefore the spectral radius of μ_1 on $\ell^2(V)$ is $\rho(\mu_1) = 2\sqrt{q}/(q + 1)$, and its resolvent at the eigenvalue $\gamma(z)$ is the convolution operator by the function

$$s_z(v) = \frac{q + 1}{q^{-z} - q^z} q^{-z|v|} = \frac{1}{c(1 - z)(q^{-z} - q^{z-1})} q^{-z|v|}$$

with $\Re z > 1/2$, i.e., $\mu_1 s_z - \gamma(z) s_z = \delta_{v_0}$, and the c -function is

$$c(z) = \frac{q^{1-z} - q^{z-1}}{(q + 1)(q^{-z} - q^{z-1})}.$$

In particular, the spectrum of μ_1 on $\ell^2(V)$ is symmetric with respect to the origin. This confirms the well known property [32] of symmetry of the spectrum of the adjacency operator on a bipartite graph (i.e., a graph whose set of vertices is the union of two disjoint sets and each edge is represented by a pair of vertices in the two different sets). On the other hand, the spectrum of η_1 on ℓ^2 is not symmetric, as seen in Theorem 5.6, and the edges can be regarded as “vertices” in the graph described in the “Appendix”, that is not bipartite except in the trivial case of the tree given by the integers. Indeed, for larger numbers of neighbors there are at least three edges in E connecting at a vertex in V , hence each of them touches the other two and so the “vertices” of the graph form triangles.

Remark 5.7 The alternating function $\epsilon(v) = (-1)^{|v|}$, i.e., the parity of vertices, is an eigenfunction of μ_1 with eigenvalue -1 , and multiplication by $\epsilon(v)$ maps eigenfunctions of μ_1 to other eigenfunctions of μ_1 , since $\epsilon(u) = -\epsilon(v)$ for all neighbors u of each vertex v . Indeed, if h is an eigenfunction with eigenvalue γ_V , then ϵh has eigenvalue $-\gamma_V$, and the spectrum of μ_1 is invariant under reflection around the origin, as already observed. We have $-\gamma_V(z) = \gamma_V(z + i\pi / \log q)$, and so the resolvent at the eigenvalue $-\gamma_V(z)$ is $\epsilon s_z = s_{z+i\pi / \log q}$. Moreover, ϵ does not change if we move the reference vertex v_0 to another vertex at even distance, and becomes $-\epsilon$ if we move the reference vertex by an odd distance.

The same is not true for edges. Unless $e = e_0$, there are edges $e' \sim e$ such that $|e'| = |e|$. If we set $\epsilon(e) = (-1)^{|e|}$, ϵ depends on the reference edge in a non-trivial

way, and in the environment of edges $\gamma(z + i\pi/\log q) \neq -\gamma(z)$. We show that $\epsilon r_z = r_{\tilde{z}}$ for some $\tilde{z} \in \mathbf{C}$. Indeed, we claim that ϵr_z is a multiple of the resolvent at an eigenvalue $\gamma(\tilde{z}) = \sigma(\gamma(z))$, where σ is the central reflection of \mathbf{C} around the midpoint $(q - 1)/2q$ of the ℓ^2 spectrum S , i.e., $\sigma(w) = (q - 1)/q - w$. That is,

$$\gamma(\tilde{z}) = \frac{q - 1}{q} - \gamma(z). \tag{16}$$

So $\tilde{z} \in S$ if (and only if) $z \in S$. Indeed, by Remark 4.1, for every $e \neq e_0$,

$$\eta_1(\epsilon r_z)(e) = \frac{-q^{1-z} - q^z + (q - 1)}{2q} \epsilon r_z(e) = \left(\frac{q - 1}{q} - \gamma(z) \right) \epsilon r_z(e),$$

which proves the claim: ϵr_z satisfies the resolvent equation (14) at the eigenvalue $(q - 1)/q - \gamma(z)$. Moreover, since $|\epsilon r_z| = |r_z|$, by (15) ϵr_z satisfies Haagerup’s estimate (4), hence it is the ℓ^2 -resolvent of η_1 at this eigenvalue.

By the expression 9 of γ , (16) reduces to $q^{\tilde{z}} + q^{1-\tilde{z}} = -(q^z + q^{1-z})$. Equivalently, by writing $z, \tilde{z} \in S$ as $z = 1/2 + it, \tilde{z} = 1/2 + is$ with $t, s \in [0, \pi/\log q]$, one has $\cos(t \log q) = -\cos(s \log q)$. The solutions are $t \log q = \pi \pm s \log q$, that is, restricting to $[0, \pi/\log q]$,

$$\tilde{z} = z + ik\pi/\log q \quad \text{or} \quad \tilde{z} = \bar{z} + ik\pi/\log q \quad \text{with } k = 0, 1. \tag{17}$$

5.4 Spherical Fourier Transform on Edges and the Plancherel Formula

The spherical Fourier transform $\widehat{h}(z)$ of a function $h \in \ell^1(E)$ at $z \in \mathbf{C}$ is defined as

$$\widehat{h}(z) = \langle h, \phi_z \rangle = \sum_{e \in E} h(e) \phi_z(e).$$

Corollary *We have*

$$\widehat{\eta_1}(z) = \gamma(z). \tag{18}$$

For every $h \in \ell^1_{\#}$ and $\phi_z \in \ell^\infty$ one has

$$\phi_z * h = h * \phi_z = \widehat{h}(z) \phi_z$$

If ϕ_z is bounded and real (that is, if $\gamma(z) \in \mathbf{R}$, or, equivalently, if $\Re z = 1/2$ or if $\Im z = in\pi/\log q, n \in \mathbf{Z}$ and $0 \leq \Re z \leq 1$), then, for every $h \in \ell^1_{\#}$, the ℓ^2 inner product $\langle \phi_z * h, h \rangle_2 = \langle \phi_z * h, \bar{h} \rangle$ is equal to $|\widehat{h}(z)|^2$.

Proof Identity (18) follows from Proposition 5.2. By (8), $\phi_z * \eta_1 = \eta_1 * \phi_z = \gamma(z) \phi_z$, hence, by (12), all functions h in $\mathcal{R}_{\#}$ satisfy the second identity, and the same holds for the ℓ^1 -completion. In particular, if h is radial, $\langle \phi_z, h \rangle = \phi_z * h(e_0) = \widehat{h}(z)$. Then, if ϕ_z is real, $\langle \phi_z, \bar{h} \rangle = \overline{\widehat{h}(z)}$ and $\langle \phi_z * h, \bar{h} \rangle = \widehat{h}(z) \langle \phi_z, \bar{h} \rangle = \widehat{h}(z) \overline{\widehat{h}(z)}$. \square

A celebrated Harish-Chandra theorem expresses the Plancherel measure for the spherical Fourier transform on semi-simple Lie groups in terms of the c -function appearing in the asymptotic expansion of spherical functions as sum of exponentials. An analog for free groups acting on vertices of T has been proved in [20, Chapter 3, §4] via a direct calculation. Although a direct proof of the analogous statement for edges is possible, we opt for a more elegant approach based on the Carleman formula [17, p. 192]: for any $h \in \ell^2_{\#}$ and any continuous compactly supported u on \mathbf{R} , the spectral resolution $E(d\lambda)$ of the operator η_1 on ℓ^2 is expressed as

$$\int_{-\infty}^{\infty} u(\lambda)(E(d\lambda)h, h)_2 = -\frac{1}{2i} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} u(\lambda)((\xi_{\lambda+i\epsilon} - \xi_{\lambda-i\epsilon}) * h, h)_2 d\lambda, \tag{19}$$

where the resolvent function ξ_{λ} of η_1 on ℓ^2 at the eigenvalue λ (cf. Theorem 5.6) is holomorphic outside the spectrum S , hence the limit vanishes outside S and the integration can be limited to this interval.

Our argument in the next theorem is inspired by the approach of [19] for polygonal graphs (see also [30]), although our proof is more precise and accurate.

Theorem *For every $h \in \ell^2_{\#}$, the following Plancherel formula holds:*

$$\|h\|_2^2 = \frac{\log q}{4} \int_0^{\pi/\log q} |\widehat{h}(1/2 + it)|^2 |c(1/2 + it)|^{-2} dt,$$

Proof If $z = 1/2 + it$, from (9) we have $\gamma(z) = (q - 1)/2q + \sqrt{q} \cos(t \log q)/q$ and

$$\begin{aligned} \gamma(z + \delta) &= \gamma(z) + \frac{(q^{\delta} - 1)q^z + (q^{-\delta} - 1)q^{1-z}}{2q} \\ &= \gamma(z) + \frac{(\cosh(\delta \log q) - 1) \cos(t \log q) + i \sinh(\delta \log q) \sin(t \log q)}{\sqrt{q}}. \end{aligned}$$

Setting $\epsilon = i(\gamma(z) - \gamma(z + \delta))$, from the relation

$$\gamma(z) + i\epsilon = \gamma(z + \delta) \tag{20}$$

we obtain

$$\begin{aligned} \epsilon &= \frac{\sinh(\delta \log q) \sin(t \log q)}{\sqrt{q}} - i \frac{(\cosh(\delta \log q) - 1) \cos(t \log q)}{\sqrt{q}} \\ &= \frac{\sin(t \log q)}{\sqrt{q}} \delta \log q + O(\delta^2), \end{aligned} \tag{21}$$

and $\delta(\epsilon) = \epsilon \sqrt{q}/(\log q) \sin(t \log q) + O(\epsilon^2)$. Therefore δ is asymptotically proportional to ϵ for $\epsilon \rightarrow 0$, with a non-negative constant of proportionality. Hence, as $\mathbf{R} \ni \epsilon \rightarrow 0^+$, $\delta(\epsilon)$ tends to 0 and is asymptotically tangent to the real axis, unless the coefficient of proportionality vanishes. The coefficient vanishes only if

$\sin(t \log q) = 0$, that is at the extreme points of the spectrum S , and this does not affect the spectral measure in the interior of the spectral interval (it might only produce atoms at the endpoints). Note that $\gamma(1/2 + w) = \gamma(1/2 - w)$. By (21),

$$\gamma(1/2 + it) - i\epsilon = \gamma(1/2 + it - \delta + O(\delta^2)) = \gamma(1/2 - it + \delta + O(\delta^2)).$$

Therefore we can rewrite the right-hand side of Carleman’s formula (19) as

$$\begin{aligned} &-\frac{1}{2i} \lim_{\epsilon \rightarrow 0^+} \int_S u(\gamma)((\xi_{\gamma+i\epsilon} - \xi_{\gamma-i\epsilon}) * h, h)_2 d\gamma \\ &= -\frac{1}{2i} \lim_{\delta \rightarrow 0^+} \int_0^{\pi/\log q} u(1/2 + it)((\xi_{\gamma(1/2+it+\delta)} - \xi_{\gamma(1/2+it-\delta+O(\delta^2))}) * h, h)_2 \\ &\quad \cdot \gamma'(1/2 + it) dt. \end{aligned}$$

It follows from the expression (13) of r_z that, for $z = 1/2 + it$,

$$\xi_{\gamma(z+\delta)} = r_{z+\delta} = \frac{q^{-(1/2+it+\delta)} |e|}{c(1/2 - it - \delta)(q^{-1/2-it-\delta} - q^{-1/2+it+\delta})}.$$

By Corollary 5.3, the denominator in the last identity has a non-zero limit as $\delta \rightarrow 0$, hence $\lim_{\delta \rightarrow 0} r_{1/2+it\pm\delta} = \lim_{\epsilon \rightarrow 0} \xi_{\gamma(1/2+it)\pm i\epsilon}$ exists and is non-zero.

Because of the rate of decay of $\xi_{\gamma(z)}$, in the equality $\xi_{\gamma(z+\delta)} = r_{z+\delta}$, if $\delta \in \mathbf{R}$ we must limit attention to $\delta > 0$ because the ℓ^2 -resolvent $r_{z+\delta}$ must belong to ℓ^2 (and in general we should limit attention to $\Re\delta > 0$). This means that, when approaching $\gamma(1/2 + it) \in S$ with

$$\gamma(1/2 + it) \pm i\epsilon = \gamma(1/2 + it \pm \delta + O(\delta^2)) = \gamma(1/2 \pm it + \delta + O(\delta^2)),$$

we are approaching the point $z = 1/2 \pm it$ in the z -plane always from the right-hand half-space. In terms of the variable $\gamma(z)$, although $\delta > 0$ implies that $\epsilon \rightarrow 0^+$, the subscript $\gamma(1/2 + it + \delta)$ in the integrand, for $0 < t < \pi/\log q$, approaches $\gamma(1/2 + it)$ from above, and the subscript $\gamma(1/2 + it - \delta + O(\delta^2))$ approaches the same eigenvalue, regarded as $\gamma(1/2 + i(-t))$, from below as requested.

Note also that, by (9),

$$d\gamma(z) = \frac{(q^z - q^{1-z}) \log q}{2q} dz = \frac{\log q}{2} (q^{z-1} - q^{-z}) dz.$$

Therefore, for $0 < t < \pi/\log q$,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} (\xi_{\gamma(1/2+it)+i\epsilon} - \xi_{\gamma(1/2+it)-i\epsilon}) d\gamma \\ &= \frac{\log q}{2} \left(\frac{q^{-(1/2+it)}|e|}{c(1/2-it)(q^{-1/2-it} - q^{-1/2+it})} - \frac{q^{-(1/2-it)}|e|}{c(1/2+it)(q^{-1/2+it} - q^{-1/2-it})} \right) \\ & \quad \cdot i(q^{-1/2+it} - q^{-1/2-it}) dt \\ &= -i \frac{\log q}{2} \left(\frac{q^{-(1/2+it)}|e|}{c(1/2-it)} + \frac{q^{-(1/2-it)}|e|}{c(1/2+it)} \right) dt = -i(\log q) \frac{\phi_{1/2+it}(e)}{2|c(1/2+it)|^2} dt; \end{aligned}$$

the last identity follows from the expansion of the spherical function in Corollary 5.3 and from identity (11). Then, again by the same Corollary,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} ((\xi_{\gamma(1/2+it)+i\epsilon} - \xi_{\gamma(1/2+it)-i\epsilon}) * h, h)_2 d\gamma &= -i(\log q) \frac{(\phi_{1/2+it} * h, h)_2}{2|c(1/2+it)|^2} dt \\ &= -i(\log q) \frac{|\widehat{h}(1/2+it)|^2}{2|c(1/2+it)|^2} dt. \end{aligned}$$

Since γ' vanishes at the endpoints of the spectrum, and $\lim_{\epsilon \rightarrow 0} \xi_{\gamma(1/2+it)\pm i\epsilon}$ is finite for every $t \in \mathbf{R}$, the integrand in Carleman’s formula is bounded, and the dominated convergence theorem allows us to take the limit for $\delta \rightarrow 0$ inside the integral. This yields the Plancherel formula of the statement, because, as u is a continuous function on \mathbf{R} that is constantly 1 on S , by definition of spectral measure,

$$\int_{-\infty}^{\infty} u(\gamma)(E(d\gamma)h, h)_2 = \int_S (E(d\gamma)h, h)_2 = \|h\|_2^2.$$

In particular, the Plancherel measure has no poles at the endpoints of the interval $[1/2, 1/2 + i\pi/\log q]$. Let us give a direct proof of this fact. By (20), (21), $\gamma(1/2 + \delta) - \gamma(1/2) = i\epsilon = (\cosh(\delta \log q) - 1)/\sqrt{q}$ so, if $\delta > 0$, then $\delta \mapsto \gamma(1/2 + \delta)$ approaches $\gamma(1/2)$ from the right-hand, that is from the positive real semi-axis. Instead, again for $\delta > 0$, $\gamma(1/2 + i\pi/\log q + \delta) - \gamma(1/2 + i\pi/\log q) = -(\cosh(\delta \log q) - 1)/\sqrt{q}$, and the other endpoint $\gamma(1/2 + i\pi/\log q)$ is approached from the negative real semi-axis. Since we need $\epsilon \in \mathbf{R}$, we cannot require any longer that $\delta > 0$, although, as mentioned above, we need at least $\Re\delta > 0$. We have shown that the expressions $\gamma(1/2 + \delta) - \gamma(1/2)$ and $\gamma(1/2 + i\pi/\log q + \delta) - \gamma(1/2 + i\pi/\log q)$ are asymptotically proportional to δ^2 . Therefore the displacements $\gamma(1/2)\pm i\epsilon$ with $\epsilon > 0$ are obtained by the two curves in the z -plane $\epsilon \mapsto 1/2 + \delta(\pm i\epsilon) \approx 1/2 + C\sqrt{\pm i\epsilon}$, with $C = (\log q)^2/2\sqrt{q}$ (here $\delta(\epsilon)$ is as after (21), and the determination of the complex square root is $\sqrt{\pm i\epsilon} = e^{\pm i\pi/4}\sqrt{\epsilon}$ so that $\Re\delta = C\Re\sqrt{\pm i\epsilon} > 0$). These curves are asymptotically tangent to the half-lines at slope ± 1 respectively given by $s \mapsto 1/2 + (1 \pm i)s$ with $s > 0$. Similarly, $\gamma(1/2 + i\pi/\log q)$ is approached by the curves $\epsilon \mapsto 1/2 + i\pi/\log q + \delta(\pm i\epsilon) = 1/2 + i\pi/\log q - C\sqrt{\pm i\epsilon}$, asymptotically tangent to the half-lines with slopes ± 1 given by $s \mapsto 1/2 + i\pi/\log q + (1 \pm i)s$ with $s < 0$. Here the determination of the square root is the same as before. Now

it is easy to see from the expression of r_z that, along these curves, the difference $(r_{1/2+(1+i)\delta} - r_{1/2+(1-i)\delta})(e)$ tends to zero as $\delta \rightarrow 0$. The same fact happens at $z = 1/2 + i\pi/\log q$. Hence there are no poles even at the extreme points, and so no atoms for the Plancherel measure. \square

Corollary 5.8 *For every $h \in \ell^1$ and for every $e \in E$,*

$$h(e) = \frac{\log q}{4} \int_0^{\pi/\log q} \widehat{h}(1/2 + it) \phi_{1/2+it}(e) |c(1/2 + it)|^{-2} dt.$$

Proof We first deduce this inversion formula at $e = e_0$ from the Plancherel formula. By (6), $\widehat{h} = \widehat{\mathcal{E}h}$, hence in order to compute $h(e_0)$ we can limit attention to the radialization of h . It is enough to prove the formula for positive elements $h = g^* * g \in \ell^1_{\#}$ as in the proof of Theorem 5.6. For these elements, by radialization, the inversion formula at e_0 follows immediately from the Plancherel formula since $\|g\|_2^2 = h(e_0)$ and $|\widehat{g^2}| = \widehat{g^* * g} = \widehat{h}$. For $e \neq e_0$, as in Proposition 5.1 let λ_e be the element of the subgroup $\mathcal{G} \subset \text{Aut } T$ that maps e_0 to e . Then the statement follows by applying the inversion formula at the reference edge to $h \circ \lambda_e$ and making use of the multiplicative rule of Proposition 5.1. \square

5.5 The Schwartz Class on Edges and the Paley–Wiener Theorem

For any function f on E and any $r \in \mathbf{R}$, consider the seminorms

$$|f|_r = \sup_{e \in E} (1 + |e|)^r |f(e)| q^{|e|/2}.$$

The Schwartz class \mathcal{S} consists of all complex functions f such that $|f|_r < \infty$ for every r . Since $|f|_r \leq |f|_s$ for $r \leq s$, \mathcal{S} is equipped with a topology induced by a countable family of seminorms, hence it is a Fréchet space. The factor $q^{|e|/2}$ in the definition of the Schwartz class is an ℓ^2 norm compensation for the growth of circles, see (5).

Corollary *The dual space of \mathcal{S} is the space of distributions \mathcal{S}' consisting of all functions g on E such that $|g|_r < \infty$ for some r (the supremum of these exponents r is the order of g). Whenever $p < 2 < s$ we have $\ell^p \subset \mathcal{S} \subset \ell^2 \subset \mathcal{S}' \subset \ell^s$.*

Proof If $f \in \mathcal{S}$, then

$$|f(e)| \leq |f|_s (1 + |e|)^{-s} q^{-|e|/2} \quad \text{for every } s. \tag{22}$$

On the other hand, $g \in \mathcal{S}'$ satisfies the same inequality for some r . Hence

$$|\langle f, g \rangle| \leq |f|_s |g|_r \sum_{e \in E} q^{-|e|} (1 + |e|)^{-(r+s)} \quad \text{for every } s$$

and the right-hand side is finite for $r + s > 1$. Therefore we have $|\langle f, g \rangle| \leq |f|_s |g|_r \sum_{e \in E} q^{-|e|/2} (1 + |e|)^{-(r+s)}$, so $g \mapsto \langle f, g \rangle$ is a continuous functional on \mathcal{S} . The claim on inclusions follows immediately from (22). \square

The following is an analog of a result outlined in [7, Theorem 3.2]. We give a complete and detailed proof.

Proposition 5.9 *Denote by $\mathcal{S}_\#$, $\mathcal{S}'_\#$ the subspaces of \mathcal{S} , respectively \mathcal{S}' , consisting of radial functions (with respect to the fixed reference element). Then*

$$\begin{aligned} \mathcal{S}_\# * \mathcal{S} &\subset \mathcal{S}, \\ \mathcal{S}_\# * \mathcal{S}' &\subset \mathcal{S}', \\ \mathcal{S}'_\# * \mathcal{S} &\subset \mathcal{S}', \end{aligned}$$

and the left convolution operators defined by $\mathcal{S}_\#$ and $\mathcal{S}'_\#$ are continuous.

Proof By Corollary 5.3, we know that $q^{-|e|/2} = \phi_{1/2}(e)/(1 + (q - 1)|e|/2q) \approx |e|q^{-|e|/2}$. To keep the notation simpler, in this proof we write $\phi = \phi_{1/2}$, hence $q^{-d(e,e')/2} = \delta_e * \phi(e')/(1 + (q - 1)d(e, e')/2q)$. Then, by the same Corollary, for every function h on E all seminorms $|h|_s$ are finite if and only if so are the seminorms

$$|h|'_s = \sup_{e \in E} (1 + |e|)^s \frac{|h(e)|}{\phi(e)}.$$

Then, by (30), (22), in order to prove the first inclusion it is enough to show that, for all $f \in \mathcal{S}$, $g \in \mathcal{S}_\#$, $|g * f|'_t < \infty$ for all $t \in \mathbf{R}$. Since the seminorm $|\cdot|'_t$ is monotonically increasing with t , it is enough to restrict attention to $t > 0$. Since g is radial, we have $g * f(e) = \sum_{e' \in E} g(d(e, e'))f(e')$ and

$$|g * f(e)| \leq |f|'_r |g|'_s \sum_{e' \in E} (1 + d(e, e'))^{-s} (1 + |e'|)^{-r} \phi(e') \phi(d(e, e')).$$

We have $|g|'_s < \infty$ for every s because $g \in \mathcal{S}$. As $|e| = d(e, e_0)$, it follows from the triangle inequality that $1 + |e| \leq (1 + d(e, e'))(1 + |e'|)$. Then, if we choose $s = t > 0$,

$$(1 + d(e, e'))^{-s} \leq (1 + |e'|)^s (1 + |e|)^{-s},$$

and the previous inequality gives

$$\begin{aligned} |g * f(e)| &\leq |f|'_r |g|'_s (1 + |e|)^{-s} \sum_{e' \in E} (1 + |e'|)^{s-r} \phi(e') \phi(d(e, e')) \\ &= |f|'_r |g|'_s (1 + |e|)^{-s} \sum_{n \geq 0} (1 + n)^{s-r} \phi(n) \sum_{|e'|=n} \phi(d(e, e')). \end{aligned}$$

By the multiplicative property of Proposition 5.1,

$$\sum_{|e'|=n} \phi(d(e, e')) = \sum_{|e'|=n} (\delta_e * \phi)(e') = |C(e_0, n)|\phi(n)\phi(e). \tag{23}$$

By (5)

$$\begin{aligned} |g * f(e)| &\leq C|f|'_r|g|'_s(1 + |e|)^{-s}\phi(e) \sum_{n \geq 0} (1 + n)^{s-r}\phi(n)^2q^n \\ &< C'|f|'_r|g|'_s(1 + |e|)^{-s}\phi(e) \sum_{n \geq 0} (1 + n)^{s-r}n^2 \end{aligned}$$

again by the asymptotics of spherical functions in Corollary 5.3. For $s < r - 3$ the series converges. Hence, for $0 < s = t < r - 3$, $|g * f(e)| < C|f|'_r|g|'_s(1 + |e|)^{-s}\phi(e)$. We have just proved that, if $|f|'_r, |g|'_s < \infty$ and $s > 0$, then $|g * f|'_s < \infty$ for $r > s + 3$. The first inclusion follows because r is arbitrary.

In the second inclusion, $|f|'_r, |g|'_s < \infty$ for some r and all s . Choose $s > 0$ and use the triangle inequality in the form $1 + |e'| \leq (1 + d(e, e'))(1 + |e|)$, hence $(1 + d(e, e'))^{-s} \leq (1 + |e|)^s(1 + |e'|)^{-s}$. We make use again of (23) and get $|g * f(e)| \leq C''|f|'_r|g|'_s(1 + |e|)^s\phi(e) \sum_{n \geq 0} n^2(1 + n)^{-(s+r)}$. The series converges for $s > 3 - r$. Since s is arbitrary, this inequality is satisfied for suitable $s > 0$. This proves the second inclusion.

For the third inclusion, $|f|'_r, |g|'_s < \infty$ for all r and some s . Choose $r > 0$, and use the triangle inequality in the form $1 + d(e, e') \leq (1 + |e|)(1 + |e'|)$. Since $r > 0$, $(1 + |e'|)^{-r} \leq (1 + d(e, e'))^{-r}(1 + |e|)^r$. The same steps lead to $|g * f(e)| \leq C|f|'_r|g|'_s(1 + |e|)^r\phi(e) \sum_{n \geq 0} n^2(1 + n)^{-(s+r)}$, and the series converges for $r > 3 - s$. Since s is fixed and r is arbitrary, this inequality is satisfied for suitable $r > 0$. \square

By Theorem 5.6, the spectrum of η_1 as a convolution operator on ℓ^2 is $S = \gamma(1/2 + i\mathbf{R}) = [a_-, a_+]$ with $a_{\pm} = (q - 1)/2q \pm 1/\sqrt{q}$; moreover, $\gamma(1/2) = a_+$. We need to estimate the spherical polynomials \tilde{Q}_n on S .

Lemma For every $k \in \mathbf{N}$,

$$|D^k \tilde{Q}_n(x)| \leq q^{-(n-k)/2} \left(1 + n \frac{q-1}{2\sqrt{q}}\right) \prod_{j=n-k+1}^n j^2 \text{ for } x \in S.$$

Proof From Definition 5.4 and Corollary 5.5, one has that $|\tilde{Q}_n(\gamma(1/2 + it))| = |\phi_{1/2+it}(n)| \leq \phi_{1/2}(n)$, that is, $|\tilde{Q}_n(x)| \leq \phi_{1/2}(n) = M_n$ for all $x \in S$, with $M_n = (1 + n(q - 1)/2\sqrt{q})q^{-n/2}$. It follows from [31, p. 40] that, if a polynomial p of degree n satisfies the inequality $|p| \leq M$ on $[a_-, a_+]$, then its derivative satisfies $|Dp| \leq 2Mn^2/(a_+ - a_-)$ in $[a_-, a_+]$. Here $a_+ - a_- = 2/\sqrt{q}$. Hence $|D\tilde{Q}_n| \leq M_n n^2 \sqrt{q} = n^2(1 + n(q - 1)/2\sqrt{q})q^{-(n-1)/2}$ on S . The polynomial $D\tilde{Q}_n$ has degree $n - 1$, so the same argument shows that $D^2\tilde{Q}_n$ is bounded by

$qn^2(n-1)^2M_n = q^{-(n-2)/2}n^2(n-1)^2(1+n(q-1)/2\sqrt{q})$. We conclude by iteration. \square

Theorem 5.10 *The spherical Fourier transform is an isomorphism of $\mathcal{S}_\#$ onto $C^\infty(S)$, and of $\mathcal{S}'_\#$ onto the space of distributions with support in S .*

Proof It is enough to show the first part of the statement, as the second follows by duality. By (5), the spherical Fourier transform of $h \in \mathcal{S}_\#$ is $\widehat{h}(z) = 2 \sum_{n \geq 0} q^n h(n) \underline{Q}_n(z)$. If $x = \gamma(z)$, it is enough to prove the convergence of the series $\sum_{n \geq 0} q^n h(n) \widetilde{Q}_n(x)$ in $C^\infty(S)$, that is, that for each $k \geq 0$ the series $D^k \widehat{h}(x) = \sum_{n \geq 0} q^n h(n) D^k \widetilde{Q}_n(x)$ converges uniformly. The assumption $h \in \mathcal{S}_\#$ implies that $|h(n)| \leq c_r(1+n)^{-r} q^{-n/2}$ for every r . If $r = 2k + 3$, by the previous lemma the summands decay as n^{-2} and uniform convergence follows.

To prove the converse, take any $g \in C^\infty(S)$. By the inversion formula of Corollary 5.8 we must show that the radial function

$$h(n) = \frac{\log q}{4} \int_0^{\pi/\log q} \phi_{1/2+it}(n) g(\gamma(1/2 + it)) |c(1/2 + it)|^{-2} dt$$

belongs to $\mathcal{S}_\#$. By the expansion in Corollary 5.3,

$$\begin{aligned} h(n) &= \frac{\log q}{4} q^{-n/2} \int_0^{\pi/\log q} g(\gamma(1/2 + it)) \left(\frac{q^{int}}{c(1/2 + it)} + \frac{q^{-int}}{c(1/2 - it)} \right) dt \\ &= \frac{\log q}{2} q^{-n/2} \int_0^{\pi/\log q} q^{int} \frac{g(\gamma(1/2 + it))}{c(1/2 + it)} dt, \end{aligned}$$

where the last identity follows from the fact that the first integral is the sum of two terms, and these terms are interchanged by the reflection $t \mapsto -t$, that leaves the integral invariant because of the periodicity and symmetry properties of the functions γ and c . Therefore the two terms are equal.

Since $c(1/2 + it) \neq 0$ for $0 < t < \pi/\log q$ and c has simple poles at $t = 0$ and $t = \pi/\log q$, the sequence $h(n)q^{n/2}$ is the n -th Fourier coefficient of a function in $C^\infty[0, \pi/\log q]$, and so it is rapidly decreasing. This means that $h \in \mathcal{S}_\#$. \square

6 The Horospherical Radon Transform on Edges

6.1 The Radon Transform on Edges and Its Dual

The Radon transform R maps any function f on E to a function Rf on the space \mathcal{H} of horospheres by

$$Rf(\omega, n) = \sum_{e \in \mathbf{h}(\omega, n)} f(e).$$

We look at the volumes of the intersections of circles and horospheres: for $n \in \mathbf{Z}$ and $m \in \mathbf{N}$, let $k(n, m) = |\mathbf{h}(\omega, n) \cap C_m|$ (because of radial symmetry, $k(n, m)$ does

not depend on the tangency point ω). The numbers $k(n, m)$ can be easily computed as in [16]; here we need only their asymptotics. We state the result.

Lemma 6.1

$$k(n, m) = \begin{cases} 1 & \text{if } n = m, \\ (q - 1)q^{(m-n-1)/2} & \text{if } |n| < m \text{ and } n + m \text{ is odd,} \\ q^m & \text{if } n = -m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $k(n, m)$ vanishes if $m < |n|$ and is $O(q^{(m-n)/2})$ if n is fixed and $m \rightarrow \infty$.

If $|e| = n$, the set of tangency points of horospheres in \mathcal{H} through e of index n is the boundary arc Ω_e , with measure proportional to q^{-n} . As we did for \mathcal{S} , we can normalize with the square root of this factor and define the *Schwartz class* $\mathcal{S}(\mathcal{H})$ as the space of all continuous functions F on \mathcal{H} such that, for all $r \in \mathbf{R}$,

$$|F|_r = \sup_{\substack{\omega \in \Omega \\ n \in \mathbf{Z}}} (1 + |n|)^r q^{n/2} |F(\omega, n)| < \infty. \tag{24}$$

The left-hand side is a seminorm on the space $\mathcal{S}(\mathcal{H})$. Equipped with these seminorms, $\mathcal{S}(\mathcal{H})$ is a Fréchet space.

Theorem 6.2 *The Radon transform is a continuous linear operator from \mathcal{S} to $\mathcal{S}(\mathcal{H})$.*

Proof By (22), if $f \in \mathcal{S}$, $r \in \mathbf{R}$ and $\mathbf{h} = \mathbf{h}(\omega, n) \in \mathcal{H}$,

$$|Rf(\mathbf{h})| \leq |f|_r \sum_{e \in \mathbf{h}} (1 + |e|)^{-r} q^{-|e|/2} = |f|_r R w_r(\mathbf{h}),$$

where $w_r(e) = (1 + |e|)^{-r} q^{-|e|/2}$. Therefore it is enough to show that $|R w_r|_{r-1}$ is finite. One has

$$R w_r(\omega, n) = \sum_{e \in \mathbf{h}(\omega, n)} (1 + |e|)^{-r} q^{-|e|/2} = \sum_{m=|n|}^{\infty} k(n, m) (1 + m)^{-r} q^{-m/2}.$$

The series on the right-hand side does not depend on ω , and by Lemma 6.1 its terms are bounded by $C(1 + m)^{-r} q^{-n/2}$ for some $C > 0$. Hence, for each $r > 1$, one has $\sum_{m \geq 0} (1 + |n| + m)^{-r} < C(1 + |n|)^{1-r}$. Therefore $|R w_r(\omega, n)| < C q^{-n/2} (1 + |n|)^{1-r}$ and so $|R w_r|_{r-1}$ is finite for $r > 1$. □

With notation as before (7), the space \mathcal{H} is equipped with the $(\text{Aut } T)$ -invariant measure $\sigma = m \times \nu_{e_0}$, where m is the measure on \mathbf{Z} given by $m(\{n\}) = q^n$ (see [16]).

For any $e \in E$, let \mathcal{H}_e be the set of horospheres containing e . If F is a bounded measurable function on \mathcal{H} , its dual Radon transform is defined as

$$R^*F(e) = \int_{\mathcal{H}_e} F(\mathbf{h}) d\sigma(\mathbf{h}).$$

Set $B(e, n) = \{\omega : e \in \mathbf{h}(\omega, n)\}$. Then the last expression becomes

$$R^*F(e) = \sum_{n=-\infty}^{\infty} \int_{B(e,n)} F(\omega, n) dv_e(\omega) = \sum_{n=-\infty}^{\infty} q^n \int_{B(e,n)} F(\omega, n) dv(\omega), \tag{25}$$

because, by (7), the factor q^n is the value of $dv_e/dv(S_e)$.

Theorem 6.3 *The dual Radon transform is a continuous linear operator from $\mathcal{S}(\mathcal{H})$ to \mathcal{S}' .*

Proof Let $F \in \mathcal{S}(\mathcal{H})$. For every $r \in \mathbf{R}$, by (24), (25),

$$|R^*F(e)| \leq \sum_{n=-\infty}^{\infty} q^n \int_{B(e,n)} |F(\omega, n)| dv(\omega) \leq |F|_r \sum_{n=-\infty}^{\infty} q^{n/2} (1 + |n|)^{-r} v(B(e, n)).$$

For each e of fixed length, $v(B(e, n))$ is independent of e , by rotational invariance. Therefore $v(B(e, n))$ is proportional to the intersection cardinality $k(n, |e|)$, i.e., $v(B(e, n)) = k(n, |e|)/|C_{|e}|$. Since $|C_{|e}| \approx q^{|e|}$, by Lemma 6.1 we have $v(B(e, n)) \approx q^{-(n+|e|)/2}$ if $|n| \leq |e|$, and 0 otherwise. Therefore

$$|R^*F(e)| \leq C|F|_r q^{-|e|/2} \sum_{n=-|e|}^{|e|} (1 + |n|)^{-r}.$$

If $r \geq 0$, the terms of the sum with positive indices are non-increasing, hence $\sum_{n=1}^k (1 + |n|)^{-r} \approx \int_0^k (1 + x)^{-r} dx = 1 - (1 + k)^{1-r}$. The right-hand side is bounded for each $r \geq 1$. Hence, for all $r \geq 1$ and some $C > 0$, $|R^*F(e)| \leq C|F|_r q^{-|e|/2}$. Thus $|R^*F|_0 < C|F|_r$ if $r > 1$. □

6.2 Radon Transform Inversion on Edges

By their covariant definition, the Radon transform and its dual transform commute with the action of $\text{Aut } T$. Therefore the same is true for R^*R , bounded from \mathcal{S} to \mathcal{S}' by Theorems 6.2 and 6.3:

Corollary 6.4 *R^*R is a bounded convolution operator from \mathcal{S} to \mathcal{S}' .*

Its convolution kernel $\Psi = R^*R\delta_{e_0}$ is in \mathcal{S}' . Being invariant under $(\text{Aut } T)_{e_0}$, it is a radial function. Our goal is to find a continuous left convolution inverse, that is, a function Φ , necessarily radial, such that $\Phi * \Psi = \delta_{e_0}$ and that the convolution operator with kernel Φ maps the image $R^*R\mathcal{S}$ to \mathcal{S} and is continuous in the topology of $\mathcal{S}'(E)$.

For $|e| = n$, it is easy although tedious to verify, as in [16], that the value ψ_n of $\Psi(e) = R^*R\delta_{e_0}(e) = R^*\chi_{\mathcal{H}_{e_0}} = \sigma(\mathcal{H}_e \cap \mathcal{H}_{e_0})$ is

$$\psi_n = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \text{ non-zero even,} \\ \frac{q-1}{2q^{(n+1)/2}} & \text{for } n \text{ odd.} \end{cases} \tag{26}$$

We know that $\Psi \in \mathcal{S}'_{\#}$, and, by Corollary 5.3, all the spherical functions ϕ_z belong to $\mathcal{S}'_{\#}$ but not to $\mathcal{S}_{\#}$. Therefore $\widehat{\Psi}(z) = \langle \Psi, \phi_z \rangle$ does not converge in the ordinary sense: we shall compute it by functional analysis and the resolvents.

Lemma For $e \in E$, let $y_z(e) = q^{-z|e|}$, $\widetilde{y}_z(e) = (-1)^{|e|}y_z(e) = y_{\bar{z}}(e)$ (notation as in Remark 5.7) and $M_z = (y_z - \widetilde{y}_z)/2$. Then, if r_z is as in Theorem 5.6, one has

$$M_z(e) = \frac{q^{1-z} - q^z - (q-1)}{2q}r_z(e) = q^{-z|e|}$$

if $|e|$ is odd, and zero otherwise. In particular;

$$\Psi = \delta_{e_0} + \frac{q-1}{2\sqrt{q}}M_{1/2}. \tag{27}$$

Moreover,

$$\widehat{M}_z(w) = \frac{(q^{1-z} - q^z)(\gamma(w) - (q-1)/2q) - (q-1)(\gamma(z) - (q-1)/2q)}{2q(\gamma(w) + \gamma(z) - (q-1)/q)(\gamma(w) - \gamma(z))}. \tag{28}$$

Proof $M_z(e) = q^{-(2k+1)z}$ if $|e| = 2k + 1$ and $M_z(e) = 0$ if $|e| = 2k$. Then, by (26), $M_{1/2}(e) = 2\Psi(e)\sqrt{q}/(q-1)$ for $e \neq e_0$ and $M_{1/2}(e_0) = 0$. This proves (27).

Since $\widehat{\eta}_1(w) = \sum_{e \in E} \eta_1(e)\phi_w(e) = \gamma(w)$, it follows from (18) and the resolvent equation (14) that $\widehat{r}_z(w) = 1/(\gamma(w) - \gamma(z))$. Hence, by (13),

$$\widehat{y}_z(w) = \frac{q^{1-z} - q^z - (q-1)}{2q(\gamma(w) - \gamma(z))}.$$

On the other hand, by (16),(17),

$$\widehat{\widetilde{y}}_z(w) = \widehat{y}_{z+i\pi/\log q}(w) = \frac{q^z - q^{1-z} - (q-1)}{2q} \frac{1}{\gamma(w) + (\gamma(z) - (q-1)/q)}.$$

Now $\widehat{M}_z = (\widehat{y}_z - \widehat{\bar{y}}_z)/2$ yields (28). □

By Theorem 5.6, $\gamma(1/2) = \rho(\eta_1) = (q - 1)/2q + 1/\sqrt{q}$. Hence the expression $\gamma(1/2) - (q - 1)/2q$ in the numerator of (28) of $\widehat{M}_{1/2}$ equals $1/\sqrt{q}$, and, by (27),

$$\begin{aligned} \widehat{\Psi}(w) &= 1 - \frac{(q - 1)^2}{4q^2(\gamma(w) - (q - 1)/2q + 1/\sqrt{q})(\gamma(w) - (q - 1)/2q - 1/\sqrt{q})} \\ &= 1 - \frac{(q - 1)^2}{4q^2(\gamma(w) - (q - 1)/2q)^2 - 4q}. \end{aligned} \tag{29}$$

By Corollary 6.4 and the Paley-Wiener theorem 5.10, the multiplication operator by $\widehat{\Psi}$ is bounded on $S'(S)$. Since $\widehat{\Psi}$ is smooth in the interior of S and has simple poles at the endpoints, this multiplication maps $C^\infty(S)$ to the space C of smooth functions with simple poles at the endpoints (therefore to the distribution space $S'(S)$). Hence C is the spherical Fourier transform of the image of R^*R acting on $S(E)$.

If there exists an inverse convolution operator Φ , then its symbol must be $\widehat{\Phi} = 1/\widehat{\Psi}$, and multiplication by this function should give rise to a bounded operator from $S'(S)$ to $C^\infty(S)$, that of course is impossible. However, we claim that multiplication by $1/\widehat{\Psi}$ is a continuous operator on $S'(S)$. Indeed, we first show that $1/\widehat{\Psi} \in C^\infty(S)$. Since γ is holomorphic, by (29) it is enough to show that $\widehat{\Psi}$ never vanishes in S . This amounts to

$$4q^2 \left(\gamma(w) - \frac{q - 1}{2q} \right)^2 - 4q \neq (q - 1)^2$$

for $w = 1/2 + it$, where we can restrict attention to $0 \leq t \leq \pi/\log q$. As $\gamma(1/2 + it) = \cos(t \log q)/\sqrt{q} + (q - 1)/2q$ by (9), the non-vanishing condition becomes

$$4q \cos^2(t \log q) \neq 4q + (q - 1)^2.$$

This is true for every t when $q \neq 1$, that is, on non-trivial trees.

A similar argument shows that $1/\widehat{\Psi} \neq 0$ in the interior of S and has a zero of order one at the endpoints $\pm\rho$. So the multiplication by $1/\widehat{\Psi}$ maps $C^\infty(S)$ to the proper subspace of $C^\infty(S)$ of functions vanishing at the endpoints with zeros of order one, and maps the functions that are smooth in the interior of S and have simple poles at the endpoints, that is the image C of $(R^*R)\widehat{}$, surjectively onto $C^\infty(S)$. So we have proved the following statement, thereby achieving our goal:

Theorem $1/\widehat{\Psi}$ is the symbol of a left inverse of $R^*R: S(E) \rightarrow S'(E)$, a continuous operator in the topology of $S'(E)$.

An easy computation shows that

$$\widehat{\Psi}(1/2 + it) = 1 + \frac{(q - 1)^2}{4q \sin^2(t \log q)} = \left(\frac{4q \sin^2(t \log q)}{(q - 1)^2 + 4 \sin^2(t \log q)} \right)^{-1}$$

as well as

$$c(1/2 + it) = \frac{1}{2} \left(1 + i \frac{q - 1}{2\sqrt{q} \sin(t \log q)} \right),$$

$$|c(1/2 + it)|^{-2} = \frac{16q \sin^2(t \log q)}{4q \sin^2(t \log q) + (q - 1)^2}$$

and, for $t \neq ki / \log q$,

$$\phi_{1/2+it}(e) = q^{-|e|/2} \left(\cos(t|e| \log q) + \frac{q - 1}{2\sqrt{q} \sin(t \log q)} \sin(t|e| \log q) \right).$$

Therefore, by the inversion formula (5.8) applied to $1/\widehat{\Psi}$,

$$\Phi(e) = q^{-|e|/2} (\log q) \int_0^{\pi/\log q} \frac{4q \sin^2(t \log q)}{4 \sin^2(t \log q) + (q - 1)^2} \frac{16q \sin^2(t \log q)}{4q \sin^2(t \log q) + (q - 1)^2} \cdot \left(\cos(t|e| \log q) + \frac{q - 1}{2\sqrt{q} \sin(t \log q)} \sin(t|e| \log q) \right) dt.$$

The denominators in the first two factors of the integrand never vanish and are bounded away from 0, so these factors are positive and bounded (moreover, each tends to zero quadratically at the endpoints of the interval). The third factor is bounded in absolute value by $1 + (q - 1)|e|/2\sqrt{q}$, attained when t tends to 0 or to $\pi / \log q$, and therefore $|\Phi(e)| < Cq^{-|e|/2}$ for some constant C . So the seminorm of order 0 of ϕ is finite, hence ϕ belongs to $S'_\#$ and to ℓ^p for $p > 2$. If this were the exact rate of decay, then Hölder’s inequality would not imply that the series that defines $\Phi * \Psi$ is absolutely convergent, because, by (26), $\Psi \in \ell^r$ only for $r > 2$.

The actual decay of Φ is faster, thanks to the third factor in the integral. Indeed, its first summand, $\cos(t|e| \log q)$, multiplied by the product of the first two factors, yields its cosine Fourier coefficient of order $|e|$ of this product, that is a C^∞ periodic function; hence this term of the integral vanishes faster than polynomially as $|e| \rightarrow \infty$. The second summand of the last factor yields a sine Fourier coefficient that behaves in the same way, because the sine function in its denominator is canceled by the numerators of the other factors. Therefore $\Phi(e) = \zeta(|e|)q^{-|e|/2}$ with ζ vanishing at infinity faster than polynomially, hence $\Phi \in S'_\#$ and the series that defines $\Phi * \Psi$ is convergent because $\Psi \in S'_\#$. By Proposition 5.9, not only does Φ define a bounded operator $S'(E) \rightarrow S'(E)$ (mapping in particular Ψ to $\Phi * \Psi \in S'_\#$), but it gives also a bounded convolution operator on $S(E)$. Note, however, that $\Phi \notin \ell^2$ and does not define a bounded convolution operator on ℓ^2 to ℓ^∞ , and the convolution with ℓ^2 functions, and even with distributions, as for instance Ψ , is convergent although it may be not absolutely convergent.

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Appendix: Automorphisms of Trees and Convolution Products

The isotropy subgroup K_v of $G = \text{Aut } T$ at $v \in V$ is compact, and G/K_v is discrete. Since T is homogeneous, G acts transitively upon V , and G/K_v is in bijection with V . Moreover, K_v acts transitively upon the circle C_n of vertices at any distance n from v , or equivalently, the action of G is doubly transitive.

By this bijection, summable functions on the discrete space V lift to summable functions on G (with respect to its Haar measure). Hence, the convolution product on G gives rise to a convolution product on L^1 , and in particular on finitely supported functions therein; we shall often restrict attention to this space. The liftings from G/K_{v_0} to G identify right- K_{v_0} -invariant functions on G with functions on V . Write $K = K_{v_0}$; then the bi- K -invariant functions on G are *radial* functions on V , in the sense that they depend only on the distance from v_0 . We denote the corresponding L^1 spaces by $L^1_{\#}$. We shall show that (G, K) is a Gel’fand pair, that is, $L^1_{\#}$ is an abelian convolution algebra. All this also works word by word for vertices.

Let μ denote the Haar measure on G normalized on K . The usual definition of convolution for functions on G is $u_1 * u_2(\tau) = \int_G u_1(\lambda^{-1}\tau)u_2(\lambda) d\mu(\lambda)$, since G is unimodular. Let $\lambda \mapsto \tilde{\lambda}$ be the canonical projection of G to G/K , $\tilde{\mu}$ the quotient measure of μ (that is, the counting measure) on G/K . If u_2 is right- K -invariant and u_1 bi- K -invariant, we have

$$\begin{aligned} u_1 * u_2(\tilde{\tau}) &= \int_{G/K} \int_K u_1(\kappa^{-1}\lambda^{-1}\tau)u_2(\lambda\kappa) d\mu(\kappa) d\tilde{\mu}(\tilde{\lambda}) \\ &= \int_{G/K} u_1(\lambda^{-1}\tau)u_2(\lambda) d\tilde{\mu}(\tilde{\lambda}) = \int_{G/K} u_1(\widetilde{\lambda^{-1}\tau})u_2(\tilde{\lambda}) d\tilde{\mu}(\tilde{\lambda}). \end{aligned}$$

Convolution can be handled in a simpler way if we produce a suitable cross-section in G/K , and precisely a subgroup \mathcal{G} of G such that the quotient map $\mathcal{G} \rightarrow G/K$ is one-to-one and onto. We would like to choose such subgroup \mathcal{G} based on the geometry of the tree. The way to do so has been known for a long time. Label every edge that touches a reference vertex $v_0 \in V$ with a letter a_0, \dots, a_q . Then label with the same set of letters all edges stemming from a neighbor v_1 of v_0 : to the edge $[v_0, v_1]$ we assign the same letter that was assigned at the first step, and so on. Label each edge-path $\{v_0, v_1, v_2\}$ of length 2 by the word $a_{j_1}a_{j_2}$, where a_{j_1} is the letter associated to $[v_0, v_1]$ and a_{j_2} is associated to $[v_1, v_2]$. Iterate this procedure to label all finite paths of edges starting at v_0 (that is, all vertices) with words in the letters a_0, \dots, a_q : the labeling is well defined because the tree has no loops.

The semigroup of all words in the letters a_0, \dots, a_q with the juxtaposition product is made into a group by setting $a_j^{-1} = a_j$; the empty word, its identity element, is associated to v_0 . Limiting attention to reduced words, that is, canceling every occurrence of two consecutive identical letters, we see that V is in bijective correspondence with this group, isomorphic to the direct product $\mathcal{G} = \ast_{j=0}^q \mathbf{Z}_2$ because T has no loops. Each reduced word $w = a_{j_1} \cdots a_{j_n}$ can be regarded as a finite chain in V starting at v_0 : each element $w_j \neq v_0$ of this chain is obtained multiplying its predecessor w_{j-1} by a_{k_j} on the right. The left regular action of \mathcal{G} onto itself becomes an action on this set of words, isometric in the metric of V (the natural distance in T), that is, preserving adjacency. Thus $\mathcal{G} \subset \text{Aut } T$ and $\mathcal{G} \cap K_{v_0}$ is the identity element, that is the empty word. \mathcal{G} is transitive, and clearly T is the Cayley graph of \mathcal{G} with generators a_0, \dots, a_q . If the homogeneity degree q is odd, i.e., there is an even number $q + 1 = 2r$ of neighbors, then, by a similar argument, another simply transitive subgroup of $\text{Aut } T$ is isomorphic to the free group $\mathbf{F}_r = \ast_{j=1}^r \mathbf{Z}$ [13,20].

We can build a similar labeling for edges, although there is a difference. We can regard E as the set of vertices of some graph E , whose edges correspond to pairs of adjacent edges in E . Then E has loops, and this leads to a simply transitive group of isometries that is not a free product of $q + 1$ copies of \mathbf{Z}_2 or a free group, but the free product $\mathbf{Z}_{q+1} \ast \mathbf{Z}_{q+1}$, where \mathbf{Z}_{q+1} is the cyclic group with $q + 1$ elements.

Indeed, the Cayley graph of this group (with respect to the generators of its factors) has E as its set of vertices. More precisely, E can be regarded as a graph via the dual graph construction, that identifies each edge with a vertex of the associated graph. Two vertices of this graph are contiguous if the corresponding edges in E join at a vertex $v \in V$. Then the graph E is the symmetric graph of complete polygons of $q + 1$ sides, defined in [26]. The vertices of each polygon in this graph correspond to the $q + 1$ edges in E that share a given endpoint; each vertex in the graph belongs to exactly two polygons, namely those corresponding to the two endpoints in V of the edge in E associated to that vertex of the graph. For each pair of vertices in the graph, the shortest path connecting them lies in only one chain of consecutively adjacent polygons. This “tree of polygons” is the Cayley graph of $\mathbf{Z}_{q+1} \ast \mathbf{Z}_{q+1}$ if we choose all the $2q$ non-zero elements of each factor \mathbf{Z}_{q+1} as generators. Clearly, this graph is hyperbolic in the sense of Gromov. The block distance between two vertices of the graph is the number of “polygons” visited by the shortest path joining them plus 1: two vertices belonging to the same polygon have distance 1, two vertices in adjoining polygons have distance 2 and so on.

Nevertheless, every group \mathcal{G} of automorphisms acting simply transitively on either V or E induces a convolution product on functions that depends on the choice of \mathcal{G} . On the other hand, the full automorphism group G does not act simply transitively. To show in which sense G defines a convolution product we shall consider its action on E ; the statements for V are exactly the same. Take f bi- K -invariant, g right- K -invariant functions on G . Both can be regarded as functions on E , with f radial around e_0 . Their convolution on E becomes

$$f \ast g(e) = \sum_{e' \in E} f(d(e, e'))g(e'). \tag{30}$$

If f and g are both bi- K -invariant, this is

$$f * g(e) = f * g(d(e, e_0)) = \sum_{y \in E} f(d(e, e'))g(d(e', e_0)).$$

The convolution of bi- K -invariant functions is bi- K -invariant, hence $L_{\#}^1$ is a convolution algebra, the closure in the L^1 norm of the algebra $\mathcal{R}_{\#}$ of radial finitely supported functions. Radial functions on E are bi- K -invariant functions on G and are constant on the circles in E with center e_0 . For every functions f, g on E with f radial, $\tau(f * g) = f * \tau g$, where τg is defined by $\tau g(e) = g(\tau^{-1}e)$. Indeed,

$$\begin{aligned} \tau(f * g)(e) &= \sum_{e'} g(e')f(d(\tau^{-1}e, e')) = \sum_{e''} g(\tau^{-1}e'')f(d(\tau^{-1}e, \tau^{-1}e'')) \\ &= \sum_{e''} g(\tau^{-1}e'')f(d(e, e'')) = f * \tau g(e). \end{aligned}$$

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