SYMMETRY AND SPECTRAL PROPERTIES FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR EQUATIONS

ISABEAU BIRINDELLI, FABIANA LEONI & FILOMENA PACELLA

Abstract. We study properties of viscosity solutions in bounded domains of fully nonlinear uniformly elliptic equations of the form $F(x, D^2u) + f(x, u) = 0$, where f is convex in the second variable. The main results consist in showing connections between symmetry or other qualitative properties of the solutions and the sign of some principal eigenvalue of the operator $\mathcal{L}_u = \mathcal{M}^+ + \frac{\partial f}{\partial u}(x, u)$, which plays the role of the linearized operator at u, with $\mathcal{M}^+$ standing for the Pucci's sup-operator. We apply our results to obtain bounds on the eigenvalues of the uniformly elliptic operator F and to deduce properties of its possible nodal eigenfunctions.

1. Introduction

This paper studies qualitative properties of solutions of fully nonlinear equations related to spectral properties of what, improperly, will be called the linearized operator. A question we would like to answer is which symmetry features of the domain and the operator are inherited by the viscosity solutions of the homogeneous Dirichlet problem

$$\begin{cases} -F(x, D^2u) = f(x, u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain and F is a fully nonlinear uniformly elliptic operator.

Starting with Alexandrov [2] and after the fundamental works of Serrin [30] and Gidas, Ni, Nirenberg [19] most results on symmetry of solutions rely on the moving plane method. It is impossible to even start mentioning all the results obtained via that method, be they for semilinear, quasilinear or fully nonlinear equations. Let us just quote here the results obtained for positive solutions of fully nonlinear equations in [14], [8] and [31].

For the purpose of this introduction, let us emphasise its limit of application. Indeed, as it is well known by the experts, the moving plane method cannot be applied if the domain is not convex in the symmetry direction, say e.g. if Ω is an annulus, or if the nonlinear term $f(x, u)$ does not have the right monotonicity in the x–variable (see e.g. [25] for several counterexamples). The moving plane method does not apply also to sign changing solutions. Of course, even when Ω is a ball and F is the Laplacian, one cannot expect sign

2010 *Mathematics Subject Classification.* 35J60.

Key words and phrases. Fully nonlinear elliptic equations, Pucci's extremal operators, maximum principle, principal eigenvalues, symmetry of solutions, nodal eigenfunctions.

changing solutions to be radially symmetric, as it is clearly exhibited by the fact that there are non radial eigenfunctions. In these cases, some other notion of symmetry is required.

In a more philosophical understanding, the moving plane method is the tool that allows to extend the symmetry of the principal eigenfunctions, which are the only constant sign eigenfunctions, to all positive solutions of nonlinear equations. It is quite natural to wonder if this analogy can be continued, i.e. under which conditions can one expect solutions of semilinear equations to share the same symmetry of other eigenfunctions, in particular of the second eigenfunctions.

Indeed, in balls or annuli, linear operators of the type $\Delta + c(x)$ do have nodal eigenfunctions, in particular the second eigenfunctions, which are symmetric though they are not radial. For problems in non convex domains, one can imagine, and sometimes observe numerically, that even some positive solutions, like least-energy solutions, inherit only part of the symmetry of the domain, for instance, axial symmetry. In all these cases, if the domain is rotationally symmetric, the solutions are proved to be foliated Schwarz symmetric, according to the following

Definition 1.1. *Let B be a ball or an annulus in $\mathbb{R}^n$, $n \geq 2$. A function $u : \overline{B} \to \mathbb{R}$ is foliated Schwarz symmetric if there exists a unit vector $p \in S^{n-1}$ such that $u(x)$ only depends on $|x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$, and u is non increasing with respect to $\theta \in (0, \pi)$.*

In other words, a foliated Schwarz symmetric function is axially symmetric with respect to the axis $\mathbb{R}p$ and non increasing with respect to the polar angle θ. Note that a radially symmetric function is in particular foliated Schwarz symmetric with respect to any direction p, and for a not radial foliated Schwarz symmetric function the symmetry direction p is unique.

In the last decades, some work has been devoted to understanding under which conditions solutions of semilinear elliptic equations are foliated Schwarz symmetric. This line of research, which strongly relies on the maximum principle, was started in the paper [24] and then developed in [26], [20], [25] and [34]. In the semilinear elliptic case, by using symmetrization techniques, some results about foliated Schwarz symmetry of minimizers of associated functionals were obtained in [32], [4] and [10].

Let us recall some results occurring when the diffusion operator is the Laplacian, i.e. for solutions of

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } B\,, \\ u = 0 & \text{on } \partial B\,. \end{cases} \tag{1.2}$$

Under some convexity hypotheses on f, it was proved in [24] and [26] that a sufficient condition for the foliated Schwarz symmetry of a solution u of (1.2) is that the first eigenvalue $\lambda_1(\mathcal{L}_u, B(e))$ of the linearized operator $\mathcal{L}_u = \Delta + \frac{\partial f}{\partial u}(|x|, u)$ at the solution u, in the half domain $B(e) = \{x \in B \,:\, x \cdot e > 0\}$ is nonnegative, for a direction $e \in S^{n-1}$.

In this line of thought, the first question is: what plays the role of the linearized operator for the fully nonlinear problem (1.1)?

In the whole paper we will suppose that F is uniformly elliptic (in the sense of condition (2.1)) and Lipschitz continuous in x (see condition (2.2)). Let us recall that uniform ellipticity is equivalent to

$$\mathcal{M}^-_{\alpha,\beta}(M-N) \le F(x,M) - F(x,N) \le \mathcal{M}^+_{\alpha,\beta}(M-N) \quad \forall\, x \in \Omega\,,\ M, N \in \mathcal{S}_n\,, \tag{1.3}$$

where $\mathcal{M}^-_{\alpha,\beta}$ and $\mathcal{M}^+_{\alpha,\beta}$ are the Pucci's extremal operators with ellipticity constants $0 < \alpha \le \beta$ (for a precise definition, see Section 2) and $\mathcal{S}_n$ is the set of $n \times n$ symmetric matrices.

Let us emphasize that condition (1.3) implies Lipschitz continuity of the operator with respect to the Hessian but in general it does not imply that the operator be differentiable. In particular the Pucci's operators are only Lipschitz continuous functions of the Hessian. Hence we will use (1.3) to deduce (see Lemma 3.1) that any "derivative" v of u will satisfy

$$-\mathcal{M}^+_{\alpha,\beta}(D^2 v) \le \frac{\partial f}{\partial u}(x,u)\, v \quad \text{ in } \Omega$$

but, in general, v is not a solution as in the semilinear case. This suggests to define as "linearized" operator at the solution u the fully nonlinear operator

$$\mathcal{L}_u(v) \, := \mathcal{M}^+_{\alpha,\beta}(D^2 v) + \frac{\partial f}{\partial u}(x,u)\, v\,.$$

and to use the properties of this operator to deduce qualitative properties for solutions of (1.1). In analogy with the linear case, see [5, 6, 11, 21], in a domain D one may define, through the maximum principle, the principal eigenvalues $\lambda_1^+ = \lambda_1^+(\mathcal{L}_u, D)$ and $\lambda_1^- = \lambda_1^-(\mathcal{L}_u, D)$. Associated with these values, there are principal eigenfunctions $\phi_1^\pm \in C(\overline{D}) \cap C^2(D)$, defined up to positive constant multiples, which satisfy respectively

$$\begin{cases} -\mathcal{L}_u[\phi_1^+] = \lambda_1^+\, \phi_1^+ & \text{in } D \\ \phi_1^+ > 0 \text{ in } D\,,\ \phi_1^+ = 0 \text{ on } \partial D \end{cases} \quad \text{and} \quad \begin{cases} -\mathcal{L}_u[\phi_1^-] = \lambda_1^-\, \phi_1^- & \text{in } D \\ \phi_1^- < 0 \text{ in } D\,,\ \phi_1^- = 0 \text{ on } \partial D. \end{cases} \tag{1.4}$$

However, besides the principal eigenvalues and their corresponding eigenfunctions, almost nothing is known about other eigenvalues, not even existence. A completeness result of a spectral basis holds only for radial eigenfunctions, see [11, 16, 17, 21, 22]. Indeed one of the main purpose of our paper is to derive information on nodal non radial eigenfunctions in radially symmetric domains.

Nonetheless by analyzing the sign of the principal eigenvalue of $\mathcal{L}_u$ in half domains we obtain some qualitative properties of solutions for any symmetric operator F satisfying (1.3).

In order to describe our results, let us introduce a few notations that will always be valid in the sequel. B will always denote a bounded radial domain, that is a ball or an annulus centered at the origin. For any unit vector $e \in S^{n-1}$, we further denote by $H(e) = \{x \in \mathbb{R}^n \,:\, x \cdot e = 0\}$ the hyperplane orthogonal to e and by $B(e) = \{x \in B \,:\, x \cdot e > 0\}$ the open half domain on the side of $H(e)$ which contains e. Moreover, we indicate with $\sigma_e : \mathbb{R}^n \to \mathbb{R}^n$ the reflection with respect to $H(e)$, that is the map $\sigma_e(x) = x - 2(x \cdot e)e$. Accordingly, for any domain Ω, we will set $\Omega(e) = \{x \in \Omega \,:\, x \cdot e > 0\}$.

A first simple, but useful, result that we get is a sufficient condition for the symmetry of a viscosity solution u of (1.1) in a domain Ω symmetric with respect to a certain hyperplane $H(e)$. More precisely, we will show that if $f(x,s)$ is convex in the s–variable, then the positivity of the principal eigenvalues $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))$ in both domains $\Omega(\pm e)$ implies that $u(x)=u(\sigma_e(x))$ for all $x\in\Omega$, see Proposition 3.3.

Next, our main result, concerning the foliated Schwarz symmetry of viscosity solutions of (1.1), is

Theorem 1.2. *Suppose that F is invariant with respect to any reflection σ_e and by rotations. Let u be a viscosity solution of problem* (1.1)*, with $\Omega = B$ and $f(x,\cdot)=f(|x|,\cdot)$ convex in $\mathbb{R}$. If there exists $e\in S^{n-1}$ such that*

$$\lambda_1^+(\mathcal{L}_u,B(e))\geq 0,$$

then u is foliated Schwarz symmetric.

So, under the convexity assumption on f, the knowledge of the sign of the principal eigenvalue $\lambda_1^+(\mathcal{L}_u,B(e))$ in only one of the subdomains $B(e)$ is sufficient for the foliated Schwarz symmetry of a solution u of (1.1), for any fully nonlinear uniformly elliptic operator F with ellipticity constants $0<\alpha\leq\beta$.

Next we prove an interesting connection between the sign of the principal eigenvalue of $\mathcal{L}_u$ in half domains and the nodal set $\mathcal{N}(u)$, i.e. the closure of the zero set of u. Namely, we prove that if u is a sign changing viscosity solution of (1.1) with f independent of x and with u and F symmetric with respect to an hyperplane $H(e)$, then the non negativity of the eigenvalue $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))$ implies that the nodal set $\mathcal{N}(u)$ intersects the boundary of Ω, see Proposition 3.5. As a consequence of the above result, we obtain that for any radial sign changing solution u one has $\lambda_1^+(\mathcal{L}_u,B(e))<0$ for any direction e, see Corollary 3.6.

In $\mathbb{R}^2$, the above result can be extended to a larger class of domains, i.e. domains which are symmetric with respect to two orthogonal directions and convex in those directions. Interestingly, besides the ball, the only two dimensional domains for which the eigenvalues of $\mathcal{M}^+_{\alpha,\beta}$ are known explicitly have these symmetry, see [9].

The qualitative properties obtained allow to prove some results about nodal eigenvalues (i.e. eigenvalues that are not the principal ones which are the only ones having eigenfunctions that do not change sign). Indeed, following [3], for any uniformly elliptic and positively one homogeneous operator F one can define

$$\Lambda_2(F)=\inf\{\lambda>\max\{\lambda_1^-(F),\lambda_1^+(F)\}\,:\,\lambda \text{ is an eigenvalue of } F\ \}. \tag{1.5}$$

It was proved in [3] that $\Lambda_2(F)>\max\{\lambda_1^-(F),\lambda_1^+(F)\}$ and that for any

$$\max\{\lambda_1^-(F),\lambda_1^+(F)\}<\mu<\Lambda_2(F))$$

and for any continuous f, there exists a solution of the Dirichlet problem

$$\begin{cases} F(x,D^2u)+\mu u=f(x) & \text{in } B\\ u=0 & \text{on } \partial B.\end{cases}$$

Hence the importance of any estimate on $\Lambda_2(F) = \Lambda_2(F, B)$. The next result relates the principal eigenvalue of $\mathcal{M}^+_{\alpha,\beta}$ in any half domain $B(e)$ i.e. $\lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e))$ with $\Lambda_2(F, B)$ and $\lambda_2^r(F, B)$, which denotes the smallest radial nodal eigenvalue of F in B.

Theorem 1.3. *Let F be as in Theorem 1.2 and positively one homogeneous, then the following inequalities hold*

$$\lambda_2^r(F, B) > \lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e)) \quad \text{and} \quad \Lambda_2(F, B) \geq \lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e))\,.$$

In any bounded domain Ω, one can define

$$\mu_2^+(\mathcal{L}_u, \Omega) = \inf_{D\subset\Omega} \max\left\{\lambda_1^+(\mathcal{L}_u, D), \lambda_1^+(\mathcal{L}_u, \Omega \setminus \overline{D})\right\} \tag{1.6}$$

where the infimum is taken on all open subsets D contained in Ω, such that both D and $\Omega \setminus \overline{D}$ are connected. When $\mathcal{L}_u = \Delta + f'(|x|, u)$, $\mu_2^+ = \Lambda_2$ i.e. it is just the second eigenvalue of $\mathcal{L}_u$. It turns out that, in the currently considered fully nonlinear case, μ_2^+ is not an eigenvalue for $\mathcal{L}_u$ in Ω, as shown in Proposition 5.1.

Nevertheless, for $\Omega = B$, we immediately obtain

$$\mu_2^+(\mathcal{L}_u, B) \geq 0 \Rightarrow \forall e \in S^{n-1} \;\text{ either } \lambda_1^+(\mathcal{L}_u, B(e)) \geq 0 \;\text{ or } \lambda_1^+(\mathcal{L}_u, B(-e)) \geq 0$$

so that, by applying Theorem 1.2, the following corollary holds.

Corollary 1.4. *Under the assumptions of Theorem 1.2, if u is a viscosity solution of* (1.1) *and $\mu_2^+(\mathcal{L}_u, B) \geq 0$, then u is foliated Schwarz symmetric.*

Two final remarks are in order.

First, let us finally point out that our symmetry results apply to viscosity solutions, and not only to classical solutions, of (1.1). This is essential in view of the fact that, in general, axially symmetric viscosity solutions of fully nonlinear equations may not be of class C^2, as proved by Nadirashvili and Vlăduţ [23].

Second, let us emphasize that our results are in the spirit of elliptic regularity theory for fully nonlinear equations (see e.g. [12]), where results for the extremal Pucci's operators are extended to all operators in the same ellipticity class.

The paper is organized in the following way. The hypotheses and some preliminaries are recalled in the next section. In the third section we prove some symmetry results. Foliated Schwarz symmetry is then studied in the fourth section. Finally, in the last section, we give some applications, in particular to the study of spectral properties.

2. Preliminaries on fully nonlinear elliptic equations

We assume that $F : \Omega \times \mathcal{S}_n \to \mathbb{R}$ is a continuous function, with $\mathcal{S}_n$ denoting the set of symmetric $n \times n$ matrices equipped with the usual partial ordering

$$M \geq N \Longleftrightarrow M - N \geq 0 \Longleftrightarrow (M - N)\xi \cdot \xi \geq 0 \quad \forall \xi \in \mathbb{R}^n\,.$$

We will always assume that F is uniformly elliptic, that is

$$\alpha\,\mathrm{tr}(P) \leq F(x, M + P) - F(x, M) \leq \beta\,\mathrm{tr}(P)\,, \quad \forall\, x \in \Omega\,,\; M, P \in \mathcal{S}_n,\; P \geq 0\,, \tag{2.1}$$

for positive constants $0 < \alpha \leq \beta$. Let us recall that condition (2.1) is equivalent to

$$\mathcal{M}^-_{\alpha,\beta}(M-N) \leq F(x,M) - F(x,N) \leq \mathcal{M}^+_{\alpha,\beta}(M-N) \quad \forall\, x \in \Omega\,,\ M,N \in \mathcal{S}_n\,,$$

where $\mathcal{M}^-_{\alpha,\beta}$ and $\mathcal{M}^+_{\alpha,\beta}$ are the Pucci's extremal operators defined respectively as

$$\begin{aligned}\mathcal{M}^-_{\alpha,\beta}(M) &= \inf_{A\in\mathcal{A}_{\alpha,\beta}} \operatorname{tr}(AM) = \alpha \sum_{\mu_i>0} \mu_i + \beta \sum_{\mu_i<0} \mu_i \\ \mathcal{M}^+_{\alpha,\beta}(M) &= \sup_{A\in\mathcal{A}_{\alpha,\beta}} \operatorname{tr}(AM) = \beta \sum_{\mu_i>0} \mu_i + \alpha \sum_{\mu_i<0} \mu_i\end{aligned}$$

where $\mathcal{A}_{\alpha,\beta} = \{A \in \mathcal{S}_n \,:\, \alpha\, I_n \leq A \leq \beta\, I_n\}$, I_n being the identity matrix in $\mathcal{S}_n$, and $\mu_1,\ldots,\mu_n$ being the eigenvalues of the matrix $M \in \mathcal{S}_n$. Thus, Pucci's extremal operators act as barriers for the whole class of uniformly elliptic operators, and for a detailed analysis of the crucial role they play in the regularity theory for elliptic equations we refer to [12]. Clearly, $F(x,M) = \mathcal{M}^+_{\alpha,\beta}(M)$ or $F(x,M) = \mathcal{M}^-_{\alpha,\beta}(M)$ are special cases which can be considered as our model cases; in particular since they are invariant with respect to rotation and reflection. From now on, we intend the ellipticity constants $\beta \geq \alpha$ fixed once and for all, and we will write just $\mathcal{M}^-$ and $\mathcal{M}^+$ for the Pucci's operators with ellipticity constants α and β.

As for the dependence on x of F, we assume Lipschitz continuity, i.e. the existence of $L > 0$ such that, for all $x, y \in \Omega$ and $M \in \mathcal{S}_n$,

$$|F(x,M) - F(y,M)| \leq L\, \|M\| |x-y|\,. \tag{2.2}$$

On the zero order nonlinearity f we assume that it is of class C^1 on $\Omega \times \mathbb{R}$.

By a solution of the Dirichlet problem (1.1), we always mean a viscosity solution $u \in C(\overline{B})$. For the reader's convenience, we recall that a solution in the viscosity sense is both a viscosity subsolution and a viscosity supersolution, as defined below.

Definition 2.1. *A viscosity subsolution (supersolution) of problem* (1.1) *is an upper (lower) semicontinuous function in $\overline{\Omega}$ such that $u \leq (\geq) 0$ on $\partial\Omega$ and for any $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) \leq (\geq) \phi(x)$ for $x \in \Omega$, one has*

$$-F(x_0, D^2\phi(x_0)) \leq (\geq) f(x_0, u(x_0))\,.$$

We refer to [12, 13] the reader not familiar with the viscosity solutions theory for fully nonlinear equations. In the following, all the differential inequalities we are going to consider are always understood in the viscosity sense.

Let us further recall that in the current assumptions, by standard elliptic regularity theory (see [12, 33]), any viscosity solution u of problem (1.1) is of class $C^1(\overline{\Omega})$, provided that Ω is of class C^1. As far as existence of solutions is concerned, we refer to [18, 28].

In the subsequent symmetry results a crucial role will be played by the principal eigenvalues of linear perturbations of Pucci's operators. In particular, given a Lipschitz domain $D \subset \mathbb{R}^n$ and a function $c \in C(\overline{D})$, let us consider the uniformly elliptic operator

$$\mathcal{L} = \mathcal{M}^+ + c(x)\,.$$

In analogy with the linear elliptic case, see [5], one may define

$$\lambda_1^+(\mathcal{L},D) := \sup\{\lambda\in\mathbb{R} : \exists\,\varphi\in C(\overline{D})\,,\ \varphi>0 \text{ in } D\,,\ -\mathcal{L}[\varphi]\geq\lambda\,\varphi \text{ in } D\}$$

and

$$\lambda_1^-(\mathcal{L},D) := \sup\{\lambda\in\mathbb{R} : \exists\,\varphi\in C(\overline{D})\,,\ \varphi<0 \text{ in } D\,,\ -\mathcal{L}[\varphi]\leq\lambda\,\varphi \text{ in } D\}\,.$$

As it is well known, see [6, 11, 21], associated with these values, called principal eigenvalues, there are principal eigenfunctions $\phi_1^\pm\in C(\overline{D})\cap C^2(D)$, defined up to positive constant multiples, which satisfy respectively

$$\begin{cases} -\mathcal{L}[\phi_1^+] = \lambda_1^+(\mathcal{L},D)\,\phi_1^+ \quad \text{in } D \\ \phi_1^+>0 \text{ in } D\,,\ \phi_1^+=0 \text{ on } \partial D \end{cases} \tag{2.3}$$

$$\begin{cases} -\mathcal{L}[\phi_1^-] = \lambda_1^-(\mathcal{L},D)\,\phi_1^- \quad \text{in } D \\ \phi_1^-<0 \text{ in } D\,,\ \phi_1^-=0 \text{ on } \partial D. \end{cases} \tag{2.4}$$

When no ambiguities arise, the eigenvalues will be denoted by λ_1^+ or λ_1^-, and in certain cases we will only specify either the domain D or the choice of the operator.

A few known properties concerning these eigenvalues are used in the paper, we list them here.

Proposition 2.2. *With the above notations, the following properties hold:*

(i) *If $D_1\subset D_2$ and $D_1\neq D_2$, then $\lambda_1^\pm(D_1)>\lambda_1^\pm(D_2)$.*
(ii) *For a sequence of domains $\{D_k\}$ such that $D_k\subset D_{k+1}$, then*
$$\lim_{k\to+\infty}\lambda_1^\pm(D_k)=\lambda_1^\pm(\cup_k D_k)\,.$$
(iii) *If $\alpha<\beta$ then $\lambda_1^+<\lambda_1^-$.*
(iv) *If $\lambda\neq\lambda_1^\pm$ is an eigenvalue then every corresponding eigenfunction changes sign.*
(v) *$\lambda_1^+(D)>0$ ($\lambda_1^-(D)>0$) if and only if the maximum (minimum) principle holds for $\mathcal{L}$ in D.*
(vi) *$\lambda_1^\pm(D)\to+\infty$ as $\mathrm{meas}(D)\to 0$.*

The proof of the different properties can be found e.g. in [5, 7, 11, 29].

Let us recall that the operator $\mathcal{L}$ satisfies the maximum (minimum) principle in Ω if for every function u upper (lower) semicontinuous in $\overline{\Omega}$ satisfying $-\mathcal{L}[u]\leq 0$ in Ω and $u\leq 0$ on $\partial\Omega$ (resp. $-\mathcal{L}[u]\geq 0$ in Ω and $u\geq 0$ on $\partial\Omega$) one has $u\leq 0$ in $\overline{\Omega}$ ($u\geq 0$ in $\overline{\Omega}$).

Finally we recall that the principal eigenfunctions are the only positive (negative) supersolutions of (2.3) (subsolutions of (2.4)) and that the following proposition, which will be used frequently in the sequel, holds true.

Proposition 2.3. *Assume that there exists u lower semicontinuous and positive such that*

$$-\mathcal{L}[u]\geq 0 \quad \textit{in } \Omega\,.$$

If there exists a function v upper semicontinuous in $\overline{\Omega}$ satisfying $-\mathcal{L}[v] \leq 0$ in Ω, $v \leq 0$ on $\partial\Omega$, and such that $v(\hat{x}) > 0$ for some $\hat{x} \in \Omega$, then, for some $t > 0$,

$$v \equiv t\, u \ \textit{and} \ -\mathcal{L}[u] = 0.$$

For the proof see Theorem 3.3 in [3] inspired by the result in [5] in the linear case. See also [29] and for a generalization to more general boundary condition see [5, 27].

3. First symmetry results

Here and in the sequel we set $f'(x,s) = \frac{\partial f}{\partial s}(x,s)$ and we use the notations fixed in the Introduction. Moreover, for any two linearly independent unit vectors $e, e' \in S^{n-1}$, we denote by $\Pi(e,e')$ the plane spanned by e and e', and by $\theta_{e,e'}$ any polar angle coordinate in $\Pi(e,e')$. If $u : B \to \mathbb{R}$ is a differentiable function, we set $u_{\theta_{e,e'}}$ to indicate the partial derivative of u with respect to $\theta_{e,e'}$, defined as zero at the origin if B is a ball.

The following technical lemma is the starting point of all our symmetry results.

Lemma 3.1. *Assume that F satisfies* (2.1) *and* (2.2) *and let $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ be a viscosity solution of* (1.1)

(i) *Assume that Ω is symmetric with respect to the hyperplane $H(e)$, F is invariant with respect to the reflection σ_e, i.e.*

$$F\left(\sigma_e(x), (I_n - 2e\otimes e)M(I_n - 2e \otimes e)\right) = F(x,M) \quad \forall\, x \in \Omega\,,\ M \in \mathcal{S}_n\,, \tag{3.1}$$

and that f satisfies

$$f(\sigma_e(x), s) = f(x,s)\,, \quad f(x,\cdot) \ \textit{is convex in}\ \mathbb{R}\,, \quad \forall\, x \in \Omega\,,\ s \in \mathbb{R}\,. \tag{3.2}$$

Then, the function $w = u - u \circ \sigma_e$ satisfies

$$-\mathcal{M}^+(D^2 w) \leq f'(x,u)\, w \quad \textit{in}\ \Omega$$

in the viscosity sense. Moreover, if $f(x,\cdot)$ is strictly convex, then either $w \equiv 0$ or w is a strict subsolution.

(ii) *Assume that $\Omega = B$ is a bounded radial domain, F is invariant by rotations, i.e. for every orthogonal matrix O one has*

$$F(O^t x, O^t M O) = F(x,M) \quad \forall\, x \in B\,,\ M \in \mathcal{S}_n\,, \tag{3.3}$$

and that f is radially symmetric in x. Then, for any pair of linearly independent unit vectors $e,\ e' \in S^{n-1}$, the functions $u_{\theta_{e,e'}}$ and $-u_{\theta_{e,e'}}$ both satisfy

$$\begin{aligned} -\mathcal{M}^+(D^2 u_{\theta_{e,e'}}) &\leq f'(|x|,u)\, u_{\theta_{e,e'}} \quad \textit{in}\ B \\ -\mathcal{M}^+(D^2(-u_{\theta_{e,e'}})) &\leq f'(|x|,u)\,(-u_{\theta_{e,e'}}) \quad \textit{in}\ B \end{aligned}$$

in the viscosity sense.

(iii) *Assume that f does not depend on x. Then, for every $1 \leq i \leq n$ both the partial derivative $u_i = \frac{\partial u}{\partial x_i}$ and $-u_i$ satisfy*

$$-\mathcal{M}^+\left(D^2 u_i\right) \leq f'(u)\, u_i \quad \text{in } \Omega$$
$$-\mathcal{M}^+\left(D^2(-u_i)\right) \leq f'(u)\,(-u_i) \quad \text{in } \Omega$$

in the viscosity sense.

Proof. (i) Let $u \in C(\overline{\Omega})$ be a viscosity solution of (1.1). By the invariance of the equation with respect to the reflection σ_e, $u \circ \sigma_e$ is also a viscosity solution of (1.1). Then, the difference $w = u - u \circ \sigma_e$ is a viscosity subsolution of

$$-\mathcal{M}^+(D^2 w) \leq f(x,u) - f(x, u \circ \sigma_e) \quad \text{in } \Omega\,. \tag{3.4}$$

If u and $u \circ \sigma_e$ are classical solutions of (1.1), then (3.4) is an immediate consequence of the uniform ellipticity of F. In the general case, this follows from assumptions (2.1) and (2.2) by means of the standard regularization procedure by sup/inf–convolution, in the spirit of Theorem 5.3 of [12]. For a detailed proof we refer to the proof of Proposition 2.1 in [14]. By (3.4) and the convexity of $f(x,\cdot)$, we immediately get the conclusion.

(ii) Let us fix $e, e', \theta_{e,e'}$ as in the statement. We aim at "differentiating" with respect to $\theta_{e,e'}$ the equation satisfied by u. Let us fix $\theta_0 \in \mathbb{R}$, and let us denote by $\mathcal{R}_0 : \mathbb{R}^n \to \mathbb{R}^n$ the rotation that maps any point x having cylindrical coordinates (r, θ, η) with respect to the plane $\Pi(e,e')$ into the point $\mathcal{R}_0(x)$ with cylindrical coordinates $(r, \theta+\theta_0, \eta)$. Let us further set $u_0(x) = u(\mathcal{R}_0(x))$. Then, by the rotational invariance of F and f, we have that both u and u_0 satisfy

$$-F(x, D^2 u) = f(|x|, u)\,, \quad -F(x, D^2 u_0) = f(|x|, u_0) \quad \text{in } B\,.$$

By uniform ellipticity, arguing as in the proof of (i), we get that the difference $u_0 - u$ satisfies, in the viscosity sense,

$$-\mathcal{M}^+(D^2(u_0 - u)) \leq f(|x|, u_0) - f(|x|, u) \quad \text{in } B\,.$$

Next, by the homogeneity properties of $\mathcal{M}^+$, we also have that for all $\theta_0 > 0$

$$-\mathcal{M}^+\left(D^2\left(\frac{u_0 - u}{\theta_0}\right)\right) \leq \frac{f(|x|, u_0) - f(|x|, u)}{\theta_0} \quad \text{in } B\,,$$

whereas, for all $\theta_0 < 0$,

$$\mathcal{M}^+\left(-D^2\left(\frac{u_0 - u}{\theta_0}\right)\right) \geq \frac{f(|x|, u_0) - f(|x|, u)}{\theta_0} \quad \text{in } B\,.$$

By letting $\theta_0 \to 0^{\pm}$, and using the stability properties of viscosity subsolutions and the fact that $\frac{u_0 - u}{\theta_0} \to u_{\theta_{e,e'}}$ locally uniformly in B, we finally obtain both

$$\mathcal{M}^+(D^2 u_{\theta_{e,e'}}) + f'(|x|, u)\, u_{\theta_{e,e'}} \geq 0 \quad \text{in } B\,,$$

and

$$\mathcal{M}^+(D^2(-u_{\theta_{e,e'}})) + f'(|x|, u)\,(-u_{\theta_{e,e'}}) \geq 0 \quad \text{in } B\,,$$

in the viscosity sense.

(iii) The proof runs as for (ii).

□

Remark 3.2. In statement (i), if $f(x,\cdot)$ is assumed to be concave, then one has

$$-\mathcal{M}^-(D^2w) \geq f'(x,u)\,w \quad \text{in } \Omega\,.$$

We are now ready to prove our first symmetry result for viscosity solutions. If u is a viscosity solution of (1.1), we denote by $\mathcal{L}_u$ the "linearized" fully nonlinear operator

$$\mathcal{L}_u := \mathcal{M}^+ + f'(x,u)\,,$$

and by $\lambda_1^\pm(\mathcal{L}_u,\Omega(e))$, $\lambda_1^\pm(\mathcal{L}_u,\Omega(-e))$ the principal eigenvalues of $\mathcal{L}_u$ in the domains $\Omega(\pm e) = \Omega\cap\{x\cdot(\pm e)>0\}$.

Proposition 3.3. *Assume that Ω is symmetric with respect to the hyperplane $H(e)$, F satisfies* (2.1), (2.2) *and* (3.1) *and that f satisfies* (3.2). *Let u be a viscosity solution of* (1.1) *and assume further that either*

(i) $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))>0$

or

(ii) $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))\geq 0$ *and f is strictly convex.*

Then, u is symmetric with respect to the hyperplane $H(e)$.

Proof. Let us set $w = u - u\circ\sigma_e$ and observe that, by definition, w is antisymmetric with respect to $H(e)$ and satisfies $w=0$ on $\partial\Omega(\pm e)$. By Lemma 3.1, w is a viscosity subsolution of

$$\begin{cases} -\mathcal{M}^+(D^2w) \leq f'(x,u)\,w & \text{in } \Omega(\pm e)\\ w=0 & \text{on } \partial\Omega(\pm e)\end{cases}$$

If $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))>0$, then by the maximum principle both $w\leq 0$ in $\Omega(e)$ and $w\leq 0$ in $\Omega(-e)$, so that, by antisymmetry, $w\equiv 0$ in Ω. If one of the eigenvalues is zero, say $\lambda_1^+(\mathcal{L}_u,\Omega(e))=0$ and $\lambda_1^+(\mathcal{L}_u,\Omega(-e))>0$ and, by contradiction, $w\not\equiv 0$, then $w<0$ in $\Omega(-e)$ by the strong maximum principle. Therefore, $w>0$ in $\Omega(e)$ and by Proposition 2.3 w satisfies

$$-\mathcal{M}^+(D^2w) = f'(x,u)\,w \quad \text{in } \Omega(e)\,,$$

a contradiction to the strict convexity of f by Lemma 3.1 (i). Analogously, if both $\lambda_1^+(\mathcal{L}_u,\Omega(\pm e))=0$, then, either $w\leq 0$ in $\Omega(\pm e)$, and then again $w\equiv 0$ in Ω, or, otherwise, w is a solution either in $\Omega(e)$ or in $\Omega(-e)$, in contrast with the strict convexity of f.

□

Remark 3.4. Since $\lambda^-(\mathcal{M}^- + f'(x,u),D) = \lambda^+(\mathcal{M}^+ + f'(x,u),D)$ for any domain D, by Remark 3.2 the same conclusion of Proposition 3.3 holds if f is concave.

In the remaining part of this section we will exhibit a sufficient condition for the eigenvalue $\lambda_1^+(\mathcal{L}_u, \Omega(e))$ to be negative when u is a sign changing viscosity solution symmetric with respect to the hyperplane $H(e)$.

Let us fix, for simplicity, $e = e_1 = (1, 0, \dots, 0) \in S^{n-1}$ and let Ω be a smooth bounded domain symmetric with respect to $H(e_1)$ and convex in the x_1–direction, i.e. for any two points in Ω having the same x_1–coordinate, the segment joining them is also contained in Ω. We are going to consider a viscosity solution $u \in C^1(\overline{\Omega})$ of the problem

$$\begin{cases} -F(x, D^2u) = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

Note that in (3.5) f does not depend on x. We recall that the nodal set $\mathcal{N}(u)$ of a solution u of (3.5) is defined as

$$\mathcal{N}(u) := \overline{\{x \in \Omega \,:\, u(x) = 0\}}\,.$$

Proposition 3.5. *Let $u \in C^1(\overline{\Omega})$ be a sign changing viscosity solution of* (3.5)*, with F satisfying* (2.1)*,* (2.2) *and* (3.1) *with $e = e_1$ and assume that u is even with respect to x_1. Then*

$$\lambda_1^+(\mathcal{L}_u, \Omega(\pm e_1)) \geq 0 \quad \Longrightarrow \quad \mathcal{N}(u) \cap \partial\Omega \neq \emptyset\,.$$

Proof. We follow the argument used in [1] for semilinear equations, and we prove the equivalent implication

$$\mathcal{N}(u) \cap \partial\Omega = \emptyset \quad \Longrightarrow \quad \lambda_1^+(\mathcal{L}_u, \Omega(\pm e_1)) < 0\,.$$

Let us consider the continuous function $u_1 = \frac{\partial u}{\partial x_1}$ in the subdomain $\Omega(e_1)$. We notice that $u_1 = 0$ on $H(e_1) \cap \overline{\Omega}$ and that u_1 does not change sign on $\partial\Omega \cap \partial\Omega(e_1)$. Indeed, if there were points $Q_1\,,\ Q_2 \in \partial\Omega \cap \partial\Omega(e_1)$ such that $u_1(Q_1) > 0$ and $u_1(Q_2) < 0$, then, since $u = 0$ on $\partial\Omega$, there would exist a sequence of points $\{x_k\} \subset \Omega$ such that $u(x_k) = 0$ for every k and $\text{dist}(x_k, \partial\Omega) \to 0$. This would be a contradiction to the hypothesis $\mathcal{N}(u) \cap \partial\Omega = \emptyset$. Hence, either $u_1 \leq 0$ or $u_1 \geq 0$ on $\partial\Omega \cap \partial\Omega(e_1)$. We can assume without loss of generality to be in the first case, since otherwise we can consider, by the symmetry of u, the opposite set $\Omega(-e_1)$. We further observe that, by Lemma 3.1 (iii), u_1 satisfies in the viscosity sense

$$-\mathcal{L}_u[u_1] \leq 0 \quad \text{in } \Omega\,. \tag{3.6}$$

Furthermore, since u is zero on $\partial\Omega$, changes sign in Ω and is symmetric in the x_1–variable, we deduce that u_1 must change sign in $\Omega(e_1)$. Then, by the previous consideration on the sign of u_1 on $\partial\Omega \cap \partial\Omega(e_1)$, we conclude that there exists an open connected domain $D \subset \Omega(e_1)$ such that $u_1 > 0$ in D and $u_1 = 0$ on ∂D. Thus, if by contradiction $\lambda_1^+(\mathcal{L}_u, \Omega(e_1)) \geq 0$, then $\lambda_1^+(\mathcal{L}_u, D) > 0$ and, by (3.6), the maximum principle would imply the contradiction $u_1 \leq 0$ in D.

□

When $\Omega = B$ is a ball, F satisfies (3.3) and u is a radial sign changing solution of (3.5) with a finite number of nodal regions, then the assumption $\mathcal{N}(u) \cap \partial B = \emptyset$ is obviously satisfied. Hence, we can apply Proposition 3.5 for any direction $e \in S^{n-1}$.

Analogously, if $\Omega = B$ is an annulus, though it is a domain not convex with respect to any direction, a proof similar to that of Proposition 3.5 can be applied (see [1] for more details). Actually, we have the following result.

Corollary 3.6. *If B is a bounded radial domain, F satisfies* (2.1)*,* (2.2) *and* (3.3) *and u is a radial sign changing viscosity solution of* (3.5)*, then*

$$\lambda_1^+(\mathcal{L}_u, B(e)) < 0 \quad \forall\, e \in S^{n-1}\,.$$

Proof. If u has a finite number of nodal regions, then the conclusion follows directly from Proposition 3.5 and the above considerations. If not, there exists a radial subdomain $\mathcal{B} \subset B$ in which u has exactly two nodal regions. Hence, $\lambda_1^+(\mathcal{L}_u, B(e)) < \lambda_1^+(\mathcal{L}_u, \mathcal{B}(e)) < 0$. □

Finally, some extra considerations can be done for the special case of planar domains $\Omega \subset \mathbb{R}^2$ which are symmetric and convex with respect to two orthogonal directions, say $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that this kind of domains need not to be convex, but they can be easily proved to be star–shaped with respect to the origin.

Let us call doubly symmetric a continuous function u which is symmetric with respect to both directions e_i, $i = 1, 2$, i.e. a continuous function u which is even in the variables x_1 and x_2. For such functions we have the following result.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a domain symmetric and convex with respect to e_i, $i = 1, 2$, and let $u \in C(\overline{\Omega})$ be a sign changing, doubly symmetric function with two nodal regions. Then, $\mathcal{N}(u) \cap \partial\Omega = \emptyset$ and $0 \notin \mathcal{N}(u)$, that is the nodal line of u neither touches $\partial\Omega$ nor passes through the origin.*

Proof. Let us define $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ and $\Omega^- = \{x \in \Omega : u(x) < 0\}$. By assumption, both $\Omega^\pm$ are connected open sets, hence connected by arcs, and symmetric with respect to $H(e_i)$, $i = 1, 2$.

Let us consider a point $P_1 \in \Omega^+ \setminus (H(e_1) \cup H(e_2))$ and let $P_2, P_3, P_4 \in \Omega^+$ be the reflected points of P_1 with respect to $H(e_1)$, $H(e_2)$ and to the origin. Then, there exists a simple, closed curve γ^+ joining P_1, P_2, P_3 and P_4 and contained in Ω^+, so that $u > 0$ on γ^+. Obviously we can choose γ^+ not passing through the origin. By the Jordan curve theorem, $\mathbb{R}^2 \setminus \gamma^+$ has two connected components, which we call D_1 and D_2, D_1 being the connected component containing the origin and D_2 the one which contains $\partial\Omega$. Since u has only two nodal regions, it follows that either $\Omega^- \subset D_1 \cap \Omega$ or $\Omega^- \subset D_2 \cap \Omega$. In the former case we immediately deduce that $\mathcal{N}(u) \cap \partial\Omega = \emptyset$. In the latter case, we can repeat the above construction in Ω^-, that is we take in Ω^- four distinct symmetric points Q_i, $i = 1, 2, 3, 4$ as before, and select a simple closed curve $\gamma^- \subset \Omega^-$ passing through them. Again the Jordan curve theorem implies that $\mathbb{R}^2 \setminus \gamma^-$ has two connected components, say A_1 which contains $\partial\Omega$, and A_2 which contains both γ^+ and the origin. Then, Ω^+ must be contained in A_2, so that u is negative in a neighborhood of $\partial\Omega$ and, again, $\mathcal{N}(u) \cap \partial\Omega = \emptyset$. A similar argument shows also that $0 \notin \mathcal{N}(u)$. □

Corollary 3.8. *Assume that $u \in C^1(\overline{\Omega})$ is a viscosity solution of* (3.5)*, with u and Ω as in Lemma 3.7 and F satisfying* (2.1)*,* (2.2) *and* (3.1) *for $e = e_i$, $i = 1, 2$. Then*

$$\lambda_1^+(\mathcal{L}_u, \Omega(\pm e_i)) < 0 \quad \text{for } i = 1, 2\,.$$

Proof. By Lemma 3.7, one has $\mathcal{N}(u) \cap \partial\Omega = \emptyset$ and, by Proposition 3.5, this yields the conclusion.

□

4. Foliated Schwarz symmetry for viscosity solutions

The aim of this section is to establish either full radial symmetry or partial symmetry properties, such as foliated Schwarz symmetry, for viscosity solutions of fully nonlinear elliptic equations in bounded radial domains. Thus, we focus on solutions of the problem

$$\begin{cases} -F(x, D^2u) = f(|x|, u) & \text{in } B\,, \\ u = 0 & \text{on } \partial B\,, \end{cases} \tag{4.1}$$

and the operator F will be always assumed to satisfy (2.1), (2.2) and (3.3).

As a first result, which easily follows from Lemma 3.1 (ii), let us prove the radial symmetry of the usually called "stable" solutions.

Theorem 4.1. *Let u be a viscosity solution of problem* (4.1) *such that $\lambda_1^+(\mathcal{L}_u, B) \geq 0$. Then, u is radially symmetric in B.*

Proof. Let us fix any pair of linearly independent unit vectors $e, e' \in S^{n-1}$, and let us set $\theta = \theta_{e,e'}$. Then, by Lemma 3.1 (ii) and the boundary condition in (4.1), the derivative u_θ satisfies, in the viscosity sense,

$$\begin{cases} -\mathcal{L}_u[u_\theta] \leq 0 & \text{in } B \\ u_\theta = 0 & \text{on } \partial B \end{cases}$$

The assumption $\lambda_1^+(\mathcal{L}_u, B) \geq 0$ implies that either $u_\theta \leq 0$ in B, or, by Proposition (2.3), $u_\theta > 0$ in B. Moreover, in the first case, by the strong maximum principle, either $u_\theta < 0$ in B or $u_\theta \equiv 0$ in B. Therefore, three are the possible cases: $u_\theta < 0$, $u_\theta > 0$ or $u_\theta \equiv 0$ in B. But since u is 2π–periodic with respect to θ, its derivative u_θ has to vanish somewhere in B. Hence, $u_\theta \equiv 0$ in B, and the arbitrariness of e and e' implies that u is radially symmetric.

□

The definition of foliated Schwarz symmetric functions was recalled in the Introduction, see Definition 1.1. Let us now give some characterizations.

Lemma 4.2. *A function $u \in C(\overline{B})$ is foliated Schwarz symmetric if and only if for every $e \in S^{n-1}$ one has either $u(x) \geq u(\sigma_e(x))$ in $B(e)$ or $u(x) \leq u(\sigma_e(x))$ in $B(e)$. More precisely, u is foliated Schwarz symmetric with respect to the direction $p \in S^{n-1}$ if and only if $u(x) \geq u(\sigma_e(x))$ for all $x \in B(e)$ and for every $e \in S^{n-1}$ such that $e \cdot p \geq 0$.*

This property was first stated in [10] and for a detailed proof we refer to [34]. A different proof for solutions of semilinear elliptic equations can be found in [20] (see also [25]).

On the other hand, for differentiable functions, the foliated Schwarz symmetry can be characterized as a sign property of the derivative $u_{\theta_{e,e'}}$, for linearly independent unit vectors $e, e' \in S^{n-1}$.

Proposition 4.3. *A function $u \in C^1(B) \cap C(\overline{B})$ is foliated Schwarz symmetric if and only if there exists a direction $e \in S^{n-1}$ such that u is symmetric with respect to $H(e)$ and for any other direction $e' \in S^{n-1} \setminus \{\pm e\}$ one has either $u_{\theta_{e,e'}} \geq 0$ in $B(e)$ or $u_{\theta_{e,e'}} \leq 0$ in $B(e)$.*

Let us recall that the sufficiency of this condition was already observed in [15] and [26], but let us include the proof for the sake of completeness.

Proof. Let $u \in C^1(B) \cap C(\overline{B})$ be foliated Schwarz symmetric with respect to a direction $p \in S^{n-1}$, and let us fix $e \in S^{n-1}$ such that $e \cdot p = 0$. Then, u is clearly symmetric with respect to $H(e)$. Moreover, let $e' \in S^{n-1}$, with $e' \neq \pm e$. In order to show that either $u_{\theta_{e,e'}} \geq 0$ or $u_{\theta_{e,e'}} \leq 0$ in $B(e)$, we can assume that $e' \cdot e = 0$ and that $\theta_{e,e'} \in [-\pi, \pi]$ is the angle formed by e' and the orthogonal projection of x on the plane $\Pi(e, e')$. We claim that if $e' \cdot p \geq 0$ then $u_{\theta_{e,e'}} \leq 0$ in $B(e)$, whereas if $e' \cdot p \leq 0$ then $u_{\theta_{e,e'}} \geq 0$ in $B(e)$.

Indeed, using cylindrical coordinates with respect to $\Pi(e, e')$, let $x = (r, \theta, \eta)$ and $x' = (r, \theta', \eta)$ be in $B(e)$, for some $r > 0$, $\eta \in \mathbb{R}^{n-2}$ and $0 < \theta \leq \theta' < \pi$. Then, there exists $\nu \in S^{n-1} \cap \Pi(e, e')$ such that $x' = \sigma_\nu(x)$, $\nu \cdot x > 0$ and $\nu \cdot p > 0$ if $p \cdot e' > 0$, whereas $\nu \cdot p < 0$ if $p \cdot e' < 0$. Hence, by Lemma 4.2, one has $u(x) \geq u(x')$ and therefore $u_{\theta_{e,e'}} \leq 0$ in $B(e)$ provided that $p \cdot e' \geq 0$, as well as $u_{\theta_{e,e'}} \geq 0$ in $B(e)$ if $p \cdot e' \leq 0$.

Conversely, assume that there exists $e \in S^{n-1}$such that u is symmetric with respect to $H(e)$, and for every $e' \in S^{n-1} \setminus \{\pm e\}$ the derivative $u_{\theta_{e,e'}}$ does not change sign in $B(e)$. Up to a rotation, we can assume that $e = e_2 = (0, 1, \ldots, 0)$. Again by Lemma 4.2, we have to prove that for any $e' \in S^{n-1}$ either $u(x) \geq u(\sigma_{e'}(x))$ or $u(x) \leq u(\sigma_{e'}(x))$ for all $x \in B(e')$. By assumption if $e' = \pm e_2$ then $u(x) = u(\sigma_{e'}(x))$, so we can assume $e' \neq \pm e_2$. Moreover, again up to a rotation around the e_2-axis, we can suppose that e' lays on the x_1x_2-plane, with $e' = (\cos\theta_0, \sin\theta_0, \ldots, 0)$ for some $\theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, the plane $\Pi(e, e')$ coincides with the x_1x_2-plane and let us denote just with $\theta \in [-\pi, \pi]$ the polar angle coordinate $\theta_{e,e'}$ given by the angle formed by the projection on the x_1x_2-plane with e_1. By assumption, we have that either $u_\theta(x) \geq 0$ or $u_\theta(x) \leq 0$ for all $x \in B$ with $x_2 > 0$. Moreover, using polar coordinates in the x_1x_2-plane, the reflection map $\sigma_{e'}$ may be written as

$$\sigma_{e'}(r\cos\theta, r\sin\theta, \tilde{x}) = (r\cos(2\theta_0 - \theta + \pi), r\sin(2\theta_0 - \theta + \pi), \tilde{x}),$$

with $\tilde{x} = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$.

Now, we claim that $u(x) \leq u(\sigma_{e'}(x))$ in $B(e')$ if $u_\theta \geq 0$ in $B(e_2)$, whereas $u(x) \geq u(\sigma_{e'}(x))$ in $B(e')$ provided that $u_\theta \leq 0$ in $B(e_2)$. Indeed, assume that $u_\theta \geq 0$ in $B(e_2)$, and let us take first $x \in B(e') \cap B(e_2)$. Thus, we have $x = (r\cos\theta, r\sin\theta, \tilde{x})$ for some $\theta \in (0, \pi)$ such that $|\theta - \theta_0| < \frac{\pi}{2}$. This implies that the angle coordinate of the reflected point satisfies

$2\theta_0 - \theta + \pi > \theta > 0$. Two cases are possible: either $\theta \geq 2\theta_0$ or $\theta < 2\theta_0$. In the first case, both x and $\sigma_{e'}(x)$ belong to $\overline{B(e_2)}$ and, by the non decreasing monotonicity with respect to θ, it follows that $u(x) \leq u(\sigma_{e'}(x))$. In the latter case, by periodicity, symmetry and monotonicity, we again obtain

$$\begin{aligned} u(\sigma_{e'}(x)) &= u(r\cos(2\theta_0 - \theta + \pi), r\sin(2\theta_0 - \theta + \pi), \tilde{x}) \\ &= u(r\cos(2\theta_0 - \theta - \pi), r\sin(2\theta_0 - \theta - \pi), \tilde{x}) \\ &= u(r\cos(-2\theta_0 + \theta + \pi), r\sin(-2\theta_0 + \theta + \pi), \tilde{x}) \geq u(x)\,. \end{aligned}$$

Assume now that $x \in B(e') \setminus B(e_2)$, so that $\theta \in (-\pi, 0]$. Observe that we also have $2\theta_0 - \theta - \pi < \theta \leq 0$. Again, we distinguish two cases: either $\theta \leq 2\theta_0$ or $\theta > 2\theta_0$. In the former case, since $u_\theta \leq 0$ in $B(-e_2)$ by symmetry and both x and $\sigma_{e'}(x)$ belong to $\overline{B(-e_2)}$, we immediately obtain

$$u(x) \leq u(r\cos(2\theta_0 - \theta - \pi), r\sin(2\theta_0 - \theta - \pi), \tilde{x}) = u(\sigma_{e'}(x))\,.$$

On the other hand, if $\theta > 2\theta_0$, by symmetry and monotonicity as before, we have as well

$$\begin{aligned} u(\sigma_{e'}(x)) &= u(r\cos(2\theta_0 - \theta + \pi), r\sin(2\theta_0 - \theta + \pi), \tilde{x}) \\ &= u(r\cos(-2\theta_0 + \theta - \pi), r\sin(-2\theta_0 + \theta - \pi), \tilde{x}) \geq u(x)\,. \end{aligned}$$

Hence, the inequality $u(x) \leq u(\sigma_{e'}(x))$ is proved in all cases. The same arguments can be used to show that $u(x) \geq u(\sigma_{e'}(x))$ if $u_\theta \leq 0$ in $B(e_2)$ and this concludes the proof.

□

By Proposition 4.3 and Lemma 3.1 (ii) we can easily deduce a first symmetry result for viscosity solutions of (4.1), which is the fully nonlinear extension of an analogous result for semilinear elliptic equation, see Proposition 2.3 in [26].

Theorem 4.4. *Let u be a viscosity solution of* (4.1) *and assume that there exists a direction $e \in S^{n-1}$ such that u is symmetric with respect to $H(e)$. If $\lambda_1^+(\mathcal{L}_u, B(e)) \geq 0$, then u is foliated Schwarz symmetric and if $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$, then u is radially symmetric.*

Proof. Let us show that, for any $e' \in S^{n-1} \setminus \{\pm e\}$, $u_{\theta_{e,e'}}$ does not change sign in $B(e)$. By Lemma 3.1 (ii), by the boundary condition in (4.1) and by the assumption of symmetry of u with respect to $H(e)$, the function $u_{\theta_{e,e'}}$ satisfies

$$\begin{cases} -\mathcal{L}_u[u_{\theta_{e,e'}}] \leq 0 & \text{in } B(e) \\ u_{\theta_{e,e'}} = 0 & \text{on } \partial B(e) \end{cases}$$

Now, if $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$, then the maximum principle holds true for operator $\mathcal{L}_u$ and we deduce $u_{\theta_{e,e'}} \leq 0$ in $B(e)$. On the other hand, if $\lambda_1^+(\mathcal{L}_u, B(e)) = 0$, then either $u_{\theta_{e,e'}} \leq 0$ in $B(e)$ or, by Proposition 2.3, $u_{\theta_{e,e'}} > 0$ in $B(e)$. In any case, $u_{\theta_{e,e'}}$ does not change sign in $B(e)$. Then, by Proposition 4.3, u is foliated Schwarz symmetric. Moreover, if $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$, again by Lemma 3.1 (ii) and the maximum principle, we obtain also

$-u_{\theta_{e,e'}} \le 0$ in $B(e)$. Hence, $u_{\theta_{e,e'}} \equiv 0$ in $B(e)$, and by the arbitrariness of e' it follows that u is radially symmetric.

□

We are now ready to prove Theorem 1.2, which states that the a priori symmetry assumption on u in Theorem 4.4 can be dropped, provided that $f(|x|,\cdot)$ is convex in $\mathbb{R}$.

Proof of Theorem 1.2. Let e be the direction for which $\lambda_1^+(\mathcal{L}_u, B(e)) \ge 0$ and let us set $w_e(x) = u(x) - u(\sigma_e(x))$. If $w_e \equiv 0$, then u is symmetric with respect to $H(e)$, and we reach the conclusion by Theorem 4.4. Therefore, we assume in the following $w_e \not\equiv 0$. By Lemma 3.1 (i) w_e in particular satisfies, in the viscosity sense,

$$\begin{cases} -\mathcal{L}_u[w_e] \le 0 & \text{in } B(e) \\ w_e = 0 & \text{on } \partial B(e) \end{cases}$$

Now, if $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$, by the maximum principle one has $w_e \le 0$ in $B(e)$ and then, by the strong maximum principle, $w_e < 0$ in $B(e)$. If $\lambda_1^+(\mathcal{L}_u, B(e)) = 0$, then, by Proposition 2.3, either $w_e \le 0$, and then again $w_e < 0$ in $B(e)$, or $w_e > 0$ (and $\mathcal{L}_u[w_e] = 0$ in $B(e)$). Thus, in any case, we have two possibilities: either $w_e < 0$ or $w_e > 0$ in $B(e)$.

Next, in order to prove that u is foliated Schwarz symmetric, we cannot apply directly Proposition 4.3, since we are not able to find a fixed vector $\tilde{e} \in S^{n-1}$ such that u is symmetric with respect to $H(\tilde{e})$ and for any other $e' \in S^{n-1}$, the function $u_{\theta_{\tilde{e},e'}}$ does not change sign in $B(\tilde{e})$. On the other hand, we can repeat the argument of the proof of Proposition 4.3: for any $\nu \in S^{n-1}$, in order to show that either $u(x) \ge u(\sigma_\nu(x))$ or $u(x) \le u(\sigma_\nu(x))$ in $B(\nu)$, it is enough to show that there exists $e' \in S^{n-1} \cap \Pi(e,\nu)$ such that u is symmetric with respect to $H(e')$ and the derivative $u_{\theta_{e,e'}}$ (or $u_{\theta_{e',\nu}}$) does not change sign in $B(e')$. Thus, the proof will be completed if we show that for any plane Π through e, there exists $e' \in S^{n-1} \cap \Pi$ such that u is symmetric with respect to $H(e')$ and the derivative $u_{\theta_{e,e'}}$ does not change sign in $B(e')$.

We first consider the case $w_e < 0$ in $B(e)$. Without loss of generality, we assume that $e = (0,1,\dots,0)$ and that Π is the plane spanned by $(1,0,\dots,0)$ and e. For $\theta \ge 0$, let us consider the direction $e(\theta) = (\sin\theta, \cos\theta, 0, \dots, 0) \in \Pi$, so that $e(0) = e$. We apply the rotating plane method in order to find $\theta' \in (0,\pi)$ such that u is symmetric with respect to $H(e')$ with $e' = e(\theta')$. We set

$$\theta' := \sup\{\tilde{\theta} \in [0,\pi) \,:\, w_{e(\theta)} < 0 \text{ in } \; B(e(\theta)) \,,\; \forall\, \theta \in [0,\tilde{\theta}]\}\,. \tag{4.2}$$

We notice that θ' is well defined since $w_e < 0$ in $B(e)$, and, by continuity, $w_{e(\theta')} \le 0$ in $B(e(\theta'))$. This implies $\theta' < \pi$, since $w_{e(\pi)} = w_{-e} = -w_e \circ \sigma_e > 0$ in $B(-e)$. We claim that $w_{e(\theta')} \equiv 0$, i.e. u is symmetric with respect to $H(e(\theta'))$. For, assume by contradiction that $w_{e(\theta')} \not\equiv 0$, so that, by the strong maximum principle, $w_{e(\theta')} < 0$ in $B(e(\theta'))$. In this case, we can find $\epsilon > 0$ small enough such that the inequality $w_{e(\theta)} < 0$ in $B(e(\theta))$ holds true for all $\theta \in [0, \theta' + \epsilon)$, and this contradicts the definition of θ'. Indeed, for ϵ sufficiently small, we can select a compact set $K \subset \bigcap_{\theta' \le \theta < \theta' + \epsilon} B(e(\theta))$ such that, for all $\theta \in [\theta', \theta' + \epsilon)$ the measure of the set $B(e(\theta)) \setminus K$ is so small that the operator $\mathcal{L}_u$ satisfies the maximum principle in $B(e(\theta)) \setminus K$. Moreover, by assumption, there exists $\eta > 0$ such that $w_{e(\theta')} \leq -\eta$ in K and then, for ϵ small enough, we have $w_{e(\theta)} \leq -\eta/2$ in K for all $\theta \in [\theta', \theta' + \epsilon)$. Thus, by the maximum principle and the strong maximum principle, we have $w_{e(\theta)} < 0$ in $B(e(\theta)) \setminus K$, and then $w_{e(\theta)} < 0$ in $B(e(\theta))$ for all $\theta \in [0, \theta' + \epsilon)$, in contrast with the choice of θ'.

We further observe that, by Hopf's lemma, for all $\theta \in [0, \theta')$, one has

$$\frac{\partial}{\partial e(\theta)} w_{e(\theta)} = 2\, Du \cdot e(\theta) < 0 \quad \text{on } H(e(\theta)) \cap B\,,$$

$e(\theta)$ being the inner unit normal vector to $B(e(\theta))$ on $\partial B(e(\theta)) \cap B$. This implies that, with respect to the cylindrical coordinates $x = (r\cos\theta, -r\sin\theta, x_3, \ldots, x_n)$, one has

$$u_\theta = \begin{cases} -r\, Du \cdot e(\theta) > 0 & \text{in } B(e') \setminus B(e) \\ r\, Du \cdot e(\theta) < 0 & \text{in } \overline{B(e)} \setminus \overline{B(e')} \\ \pm r\ Du \cdot e(\theta') = 0 & \text{in } H(e') \cap B \end{cases}$$

By using also Lemma 3.1 (ii), it then follows that u_θ in particular satisfies

$$\begin{cases} -\mathcal{L}_u[-u_\theta] \leq 0 & \text{in } B(e) \cap B(e') \\ -u_\theta \leq 0 & \text{on } \partial\left(B(e) \cap B(e')\right) \end{cases}$$

Since $\lambda_1^+\left(\mathcal{L}_u, B(e) \cap B(e')\right) > \lambda_1^+(\mathcal{L}_u, B(e)) \geq 0$, we can apply the maximum principle, and then the strong maximum principle, in order to deduce $-u_\theta < 0$ in $B(e) \cap B(e')$. Summing up, we have proved that $u_\theta > 0$ in $B(e')$, and this concludes the proof in the case $w_e < 0$ in $B(e)$.

On the other hand, if $w_e > 0$ in $B(e)$, one has $w_{-e} < 0$ in $B(-e)$ and we can apply again the rotating plane method starting with $e(0) = -e$ and considering the directions $e(\theta) = (\sin\theta, -\cos\theta, \ldots, 0)$ for $\theta \geq 0$. By defining $\theta' \in (0, \pi)$ as in (4.2), we find a unit vector $e' = e(\theta') \in S^{n-1}$ such that u is symmetric with respect to $H(e')$ and $w_{e(\theta)} < 0$ in $B(e(\theta))$ for all $\theta \in [0, \theta')$. By means of Hopf's lemma as above, we also deduce that, again with respect to the cylindrical coordinates $x = (r\cos\theta, -r\sin\theta, x_3, \ldots, x_n)$, one has

$$u_\theta = \begin{cases} -r\, Du \cdot e(\theta) > 0 & \text{in } B(e') \setminus B(-e) \\ r\, Du \cdot e(\theta) < 0 & \text{in } \overline{B(-e)} \setminus \overline{B(e')} \\ \pm r\, Du \cdot e(\theta') = 0 & \text{in } H(e') \cap B \end{cases}$$

Then, the maximum principle applied to u_θ in $B(e) \setminus \overline{B(e')}$ yields $u_\theta < 0$ in $B(-e')$, so that, by symmetry, $u_\theta > 0$ in $B(e')$.

□

Remark 4.5. Let us observe that the only assumption that there exists a direction $e \in S^{n-1}$ such that $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$, i.e. the positivity of the principal eigenvalue in just one subdomain $B(e)$, does not imply the radial symmetry of u. This is somehow in contrast with the assertion of Theorem 4.4 in the case when $\lambda_1^+(\mathcal{L}_u, B(e)) > 0$; however one should note that in Theorem 4.4 the symmetry of u with respect to $H(e)$ was assumed. A counterexample in the case when the symmetry assumption is dropped can be obtained by considering the least–energy (positive) solution of the semilinear problem

$$\begin{cases} -\Delta u = u^p & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases}$$

where A is an annulus in $\mathbb{R}^n$, $n \geq 3$ and $p < \frac{n+2}{n-2}$ is close to the critical exponent $\frac{n+2}{n-2}$. It has been shown in several papers that u is foliated Schwarz but not radially symmetric. On the other hand, it is easy to see that there are directions (indeed, infinitely many!) $e \in S^{n-1}$ such that $\lambda_1^+(\Delta + p\,|u|^{p-1}, B(e)) > 0$ whereas $\lambda_1^+(\Delta + p\,|u|^{p-1}, B(-e)) < 0$ and, obviously, $H(e)$ is not a symmetry hyperplane for u (see [25] for more details).

By Theorem 1.2, at least for convex nonlinearities f, the condition $\lambda_1^+(\mathcal{L}_u, B(e)) \geq 0$ for some $e \in S^{n-1}$ is sufficient for u to be foliated Schwarz symmetric. Concerning necessary conditions, we have the following result.

Theorem 4.6. *Assume that problem* (4.1) *has a solution u which is not radial but foliated Schwarz symmetric with respect to $p \in S^{n-1}$. Then, for all $e \in S^{n-1}$ such that $e \cdot p = 0$, one has*

$$\lambda_1^-(\mathcal{L}_u, B(e)) \geq 0\,.$$

Proof. For $e \in S^{n-1}$ orthogonal to p, let us denote by θ the polar angle coordinate $\theta_{p,e}$ defined as the angle formed by p and the orthogonal projection of x in the plane $\Pi(p, e)$. By Proposition 4.3 and by Lemma 3.1 (ii), u_θ satisfies

$$\begin{cases} -\mathcal{L}_u[u_\theta] \leq 0\,, & \text{in } B(e) \\ u_\theta \leq 0 \ \text{ in } B(e)\,,\ u_\theta = 0 & \text{on } \partial B(e) \end{cases}$$

The strong maximum principle implies that either $u_\theta < 0$ or $u_\theta \equiv 0$ in $B(e)$. Since u is not radially symmetric, we deduce $u_\theta < 0$ in $B(e)$ and therefore, by its very definition, $\lambda_1^-(\mathcal{L}_u, B(e)) \geq 0$.

□

Remark 4.7. We notice that, if u is not radial but foliated Schwarz symmetric with respect to p, then, for any $e \in S^{n-1}$ such that $e \cdot p = 0$, we have $\lambda_1^+(\mathcal{L}_u, B(e)) \leq 0$ by Theorem 4.4. Thus, in the semilinear case for which $\alpha = \beta$ and $\lambda_1^+(\mathcal{L}_u, B(e)) = \lambda_1^-(\mathcal{L}_u, B(e))$, Theorem 4.6 yields that if u is a not radial foliated Schwarz symmetric solution, then necessarily

$$\lambda_1(\Delta + f'(|x|, u), B(e)) = 0$$

for all $e \in S^{n-1}$ orthogonal to the symmetry axes of u.

5. Applications and spectral properties.

The main symmetry result of Theorem 1.2 was based on the assumption that there exists some direction $e \in S^{n-1}$ such that $\lambda_1^+(\mathcal{L}_u, B(e)) \geq 0$. We wish to comment on this eigenvalue and its role in providing bounds for the eigenvalues of fully nonlinear operators.

Let us start by recalling that in the introduction, for any bounded domain, we defined in (1.6) the value $\mu_2^+ = \mu_2^+(\mathcal{L}_u, \Omega)$, and we showed that its non negativity, when $\Omega = B$ is a radial domain, easily implies, by Theorem 1.2, that u is foliated Schwarz symmetric. It would be very interesting to study the sign of μ_2^+ for positive solutions of (4.1), in particular for those found in [28].

Let us observe that when $\alpha = \beta$, i.e. when F is the Laplace operator, μ_2^+ is the second eigenvalue of $\mathcal{L}_u$, hence the inequality $\mu_2^+ \geq 0$ just means that u has Morse index less than or equal to one. On the contrary, in the fully nonlinear case, the following proposition holds.

Proposition 5.1. *If $\alpha < \beta$, then $\mu_2^+(\mathcal{L}_u, \Omega)$ is not an eigenvalue for $\mathcal{L}_u$ in Ω with corresponding sign changing eigenfunctions having exactly two nodal regions.*

Proof. Suppose by contradiction that $\mu_2^+ = \mu_2^+(\mathcal{L}_u, \Omega)$ is such an eigenvalue. Hence, there exists a sign changing function ψ solution of

$$\begin{cases} \mathcal{M}^+(D^2\psi) + (f'(|x|, u) + \mu_2^+)\psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega\,, \end{cases}$$

such that $\Omega^- = \{x \in \Omega : \psi(x) < 0\}$ and $\Omega^+ = \{x \in \Omega : \psi(x) > 0\} = \Omega \setminus \overline{\Omega^-}$ are subdomains of Ω. Since $\alpha < \beta$, Proposition 2.2 yields

$$\lambda_1^+(\mathcal{L}_u, \Omega^-) < \lambda_1^-(\mathcal{L}_u, \Omega^-) = \mu_2^+ = \lambda_1^+(\mathcal{L}_u, \Omega^+).$$

By these inequalities and using again Proposition 2.2, one can choose D containing Ω^+ but sufficiently close to it so that

$$\lambda_1^+(\mathcal{L}_u, \Omega^-) < \lambda_1^+(\mathcal{L}_u, \Omega \setminus \overline{D}) \leq \lambda_1^+(\mathcal{L}_u, D) < \lambda_1^+(\mathcal{L}_u, \Omega^+) = \mu_2^+\,,$$

and this contradicts the fact that, by the definition (1.6), we have $\mu_2^+ \leq \lambda_1^+(\mathcal{L}_u, D)$. □

Remark 5.2. The proof of Proposition 5.1 leads to believe that a natural candidate for being the second eigenvalue of $\mathcal{L}_u$ could be

$$\gamma_2^+(\mathcal{L}_u, B) = \inf_{D \subset B} \max\left\{\lambda_1^+(\mathcal{L}_u, D), \lambda_1^-(\mathcal{L}_u, B \setminus \overline{D})\right\} \geq \mu_2^+(\mathcal{L}_u, B)\,.$$

It would be also interesting to know whether the non negativity of $\gamma_2^+(\mathcal{L}_u, B)$ would imply that u is foliated Schwarz symmetric.

We now prove Theorem 1.3 which concerns the estimates on $\Lambda_2(F)$ defined in (1.5).

Proof of Theorem 1.3. Remark first that Corollary 3.6 implies that if λ is any nodal radial eigenvalue of F in B , then, for any $e \in S^{n-1}$,

$$\lambda_1^+(\mathcal{M}^+ + \lambda, B(e)) < 0\,. \tag{5.1}$$

But

$$\lambda_1^+(\mathcal{M}^+ + \lambda, B(e)) = \lambda_1^+(\mathcal{M}^+, B(e)) - \lambda\,,$$

so that the first inequality of the statement follows.

Next, in order to prove the second inequality, suppose by contradiction that for some $\lambda < \lambda_1^+(\mathcal{M}^+, B(e))$ there exists $\psi \neq 0$ sign changing solution of

$$\begin{cases} F(x, D^2\psi) + \lambda\psi = 0 & \text{in } B \\ \psi = 0 & \text{on } \partial B. \end{cases}$$

Then

$$\lambda_1^+(\mathcal{L}_\psi, B(e)) = \lambda_1^+(\mathcal{M}^+ + \lambda, B(e))) = \lambda_1^+(\mathcal{M}^+, B(e)) - \lambda > 0\,.$$

By Proposition 3.3 it follows that ψ is radially symmetric and then (5.1) holds true, a contradiction.

□

Let us observe that if it happens that $\max\{\lambda_1^-(F), \lambda_1^+(F)\}$ is larger than $\lambda_1^+(\mathcal{M}^+, B(e))$, then the estimate $\Lambda_2(F) \geq \lambda_1^+(\mathcal{M}^+, B(e))$ provided by Theorem 1.3 is not relevant. However, when the ellipticity constants α and β are sufficiently close to each other, this is not the case. It would be interesting to investigate the relationship between $\max\{\lambda_1^-(F), \lambda_1^+(F)\}$ and $\lambda_1^+(\mathcal{M}^+, B(e))$.

In the two dimensional case, Theorem 1.3 can be extended to a larger class of domains, precisely to domains Ω which are symmetric and convex with respect to two orthogonal directions, say $e_1 = (1, 0)$ and $e_2 = (0, 1)$, i.e. the same kind of domains considered in Section 2. Following the same notation, we consider the eigenvalues $\lambda_1^+(\mathcal{M}^+, \Omega(e_1))$ and $\lambda_1^+(\mathcal{M}^+, \Omega(e_2))$.

By using Proposition 3.3 and Corollary 3.8, the analogous result to Theorem 1.3 is

Theorem 5.3. *Let Ω be as in Lemma 3.7 and let λ be a nodal eigenvalue of F in Ω associated with an eigenfunction ψ having two nodal regions, with F as in Corollary 3.8. Then:*

(i) $\lambda \geq \min\left\{\lambda_1^+(\mathcal{M}^+, \Omega(e_1)), \lambda_1^+(\mathcal{M}^+, \Omega(e_2))\right\}$;

(ii) *if ψ is doubly symmetric, then*

$$\lambda > \max\left\{\lambda_1^+(\mathcal{M}^+, \Omega(e_1)), \lambda_1^+(\mathcal{M}^+, \Omega(e_2))\right\}\,.$$

The proof proceeds as the one of Theorem 1.3.

To conclude, we observe that an important question which remains open is whether $\lambda_1^+(\mathcal{M}^+, B(e))$ is a nodal eigenvalue for $\mathcal{M}^+$ in B, as for the laplacian, or not. Note that if this was the case, then, by Theorem (1.3), $\lambda_1^+(\mathcal{M}^+, B(e))$ would be the smallest nodal eigenvalue of $\mathcal{M}^+$ in B. Next we describe some qualitative properties that a corresponding eigenfunction should have.

Proposition 5.4. *Assume that $\lambda_1^+(\mathcal{M}^+, B(e))$ is a nodal eigenvalue for $\mathcal{M}^+$ in B and that ψ_2 is a corresponding eigenfunction, i.e.*

$$\begin{cases} \mathcal{M}^+(D^2\psi_2) + \lambda_1^+(\mathcal{M}^+, B(e))\psi_2 = 0 & \text{in } B \\ \psi_2 = 0 & \text{on } \partial B \end{cases}$$

Then

(i) *ψ_2 is not radial;*
(ii) *ψ_2 is foliated Schwarz symmetric;*
(iii) *the nodal set of $\mathcal{N}(\psi_2)$ does intersect the boundary;*
(iv) *if $\alpha < \beta$, then, for any $e \in S^{n-1}$, $B^+ := \{x \in B \,:\, \psi_2 > 0\} \neq B(e)$.*

Proof. (i) is just the first inequality in Theorem 1.3, and (ii) follows directly from Theorem 1.2. Then, (ii) and Proposition 3.5 yield (iii).

Finally, in order to prove (iv), suppose by contradiction that, for some e, $B(e) = B^+$. This implies that $B^- = B(-e)$. Hence, $\lambda_1^-(\mathcal{M}^+, B(-e)) = \lambda_1^+(\mathcal{M}^+, B(e))$. On the other hand, the symmetry of the domain implies $\lambda_1^-(\mathcal{M}^+, B(-e)) = \lambda_1^-(\mathcal{M}^+, B(e)) > \lambda_1^+(\mathcal{M}^+, B(e))$, if $\alpha < \beta$. The contradiction proves the claim.

□

Let us denote by ψ_1^+ a positive eigenfunction in $B(e)$ corresponding to $\lambda_1^+(\mathcal{M}^+, B(e))$. Then, statement (iv) of Proposition 5.4 implies that ψ_1, the sign changing function constructed by odd reflection of ψ_1^+, is not an eigenfunction for $\mathcal{M}^+$ provided that $\alpha < \beta$, contrarily to the case when $\alpha = \beta$. The same argument shows that, if $\alpha < \beta$, then $\mathcal{M}^+$ cannot have a nodal eigenfunction in B antisymmetric with respect to $H(e)$ for some $e \in S^{n-1}$ and such that $B^+ = B(e)$.

References

[1] A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, *C. R. Math. Acad. Sci. Paris* **339** (2004), 339–344.

[2] A. Alexandrov, A characteristic property of spheres, *Ann. Mat. Pura e Appl.* **58** (1962), 303–315.

[3] S. N. Armstrong, Principal eigenvalues and an anti-maximum principle for homogeneous fully nonlinear elliptic equations, *J. Differential Equations* **246**, (2009), 2958–2987.

[4] T. Bartsch, T. Weth, M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, *J. Anal. Math.* **96** (2005), 1–18.

[5] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, *Comm. Pure Appl. Math.* **47** (1994), 47–92.

[6] I. Birindelli, F. Demengel, First eigenvalue and maximum principle for fully nonlinear singular operators, *Adv. Differential Equations* **11** (2006), 91–119.

[7] I. Birindelli, F. Demengel, Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. *Commun. Pure Appl. Anal.* **6** (2007), no. 2, 335–366.

[8] I. Birindelli, F. Demengel, Overdetermined problems for some fully non linear perators, *Comm. Partial Differential Equations* **38** (2013), 608–628.

[9] I. Birindelli, F. Leoni, Symmetry minimizes the principal eigenvalue: an example for the Pucci's sup operator, *Mathematical Research Letter* **21** (2014), 953–967

[10] F. Brock, Symmetry and monotonicity of solutions to some variational problems in cylinders and annuli , *Electron. J. Differ. Equ.* **108** (2003).

[11] J. Busca, M.J. Esteban, A. Quaas, Nonlinear eigenvalues and bifurcation problems for Pucci's operator, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), 187–206.

[12] L. Caffarelli, X. Cabré, *Fully Nonlinear Elliptic Equations*, A.M.S., Providence, 1995.

[13] M. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations., *Bull. Amer. Math. Soc.* **27** (1992), 1–67.

[14] F. Da Lio, B. Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations, *J. Eur. Math. Soc.* **9**, (2007), 317–330.

[15] L. Damascelli, F. Gladiali, F. Pacella, Symmetry results for cooperative elliptic systems in unbounded domains, *Indiana Univ. Math. J.* **63** (2014), 615–649.
[16] F. Demengel, Generalized eigenvalues for fully nonlinear singular or degenerate operators in the radial case, *Adv. Differential Equations* **14** (2009), no. 11-12, 1127–1154.
[17] M.J. Esteban, P. Felmer, A. Quaas, Eigenvalues for radially symmetric fully nonlinear operators, *Comm. Partial Differential Equations* **35** (2010), no. 9, 1716–1737.
[18] P. Felmer, A. Quaas, Positive radial solutions to a "semilinear" equation involving the Pucci's operator, *J. Differential Equations* **199** (2004), 376–393.
[19] B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209-243.
[20] F. Gladiali, F. Pacella, T. Weth, Symmetry and nonexistence of low Morse index solutions in unbounded domains, *J. Math. Pures Appl.* **93** , (2010), 536–558.
[21] N. Ikoma, H. Ishii, Eigenvalue problem for fully nonlinear second-order elliptic PDE on balls, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29** (2012), 783–812.
[22] N. Ikoma, H. Ishii, Eigenvalue problem for fully nonlinear second-order elliptic PDE on balls, II, *Bull. Math. Sci.* **5** (2015), no. 3, 451–510.
[23] N. Nadirashvili, S. Vlăduţ, On axially symmetric solutions of fully nonlinear elliptic equations, *Math. Z.* **270** (2012), 331–336.
[24] F. Pacella, Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities, *J. Funct. Analysis* **192** (2002), 271–282.
[25] F. Pacella, M. Ramaswamy, Symmetry of solutions of elliptic equations via maximum principles, *Handbook of Differential Equations* **6** (2008).
[26] F. Pacella, T. Weth, Symmetry of solutions to semilinear elliptic equations via Morse index, *Proc. Amer. Math. Soc.* **135** (2007), 1753–1762.
[27] S. Patrizi, Principal eigenvalues for Isaacs operators with Neumann boundary conditions, *Nonlinear Differential Equations Appl.* **16** (2009), 79–107.
[28] A. Quaas, B. Sirakov, Existence results for nonproper elliptic equations involving the Pucci operator, *Comm. Partial Differential Equations* **31** (2006), 987–1003.
[29] A. Quaas, B. Sirakov, Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators, *Adv. Math.* **218** (2008), no. 1, 105–135.
[30] J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.* **43** (1971), 304– 318.
[31] L. Silvestre, B. Sirakov, Overdetermined problems for fullly non linear equations , *Calc. Var. Partial Differential Equations*, **54** (2015), 989–1007.
[32] D. Smets, M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems, *Calc. Var. Partial Differential Equations* **18** (2003), 57–75.
[33] A. Swiech, $W^{1,p}$-interior estimates for solutions of fully nonlinear, uniformly elliptic equations, *Adv. Differential Equations* **2** (1997), 1005–1027.
[34] T. Weth, Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods, *Jahresber Dtsch Math-Ver* **112** (2010), 119–158.

Dipartimento di Matematica
Sapienza Università di Roma
P.le Aldo Moro 2, I–00185 Roma, Italy.

E-mail address: `isabeau@mat.uniroma1.it`

E-mail address: `leoni@mat.uniroma1.it`

E-mail address: `pacella@mat.uniroma1.it`