MINIMAL SUPERVARIETIES WITH FACTORABLE IDEAL OF GRADED POLYNOMIAL IDENTITIES

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Abstract. Given a minimal superalgebra $A = A_{ss} \oplus J(A)$, any subsequence of the graded simple summands of A_{ss} determines a homogeneous subalgebra of A which is still a minimal superalgebra. In the present paper we provide a sufficient condition so that the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities satisfied by A factorizes as the product of the $T_{\mathbb{Z}_2}$-ideals associated to its suitable homogeneous subalgebras of such a type. We use this fact to show that in this event A generates a minimal supervariety of fixed superexponent.

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1. Introduction

A topic of increasing interest in PI theory is the study of group graded algebras. Apart from their own interesting features, they may provide significative information on quite general questions. For instance, this is the case for the solution of the Specht problem provided by Kemer (see [13]) in which $\mathbb{Z}_2$-gradings play a key role. Here we aim to explore the structure and the ideal of graded polynomial identities of a special class of superalgebras, namely that of minimal superalgebras, in order to contribute to the classification of minimal supervarieties of fixed graded exponent.

Let F be a field of characteristic zero. An associative F-algebra A is a $\mathbb{Z}_2$-*graded algebra* or a *superalgebra* if it has a vector space decomposition $A = A^{(0)} \oplus A^{(1)}$ such that $A^{(i)}A^{(j)} \subseteq A^{(i+j)}$ (where, obviously, the indices are intendeed modulo 2). The elements of $A^{(0)}$ are called *homogeneous of degree* 0 and those of $A^{(1)}$ *homogeneous of degree* 1. Let $F\langle Y \cup Z\rangle$ be the free associative F-algebra on the disjoint countable sets of variables $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$. It has a natural superalgebra structure if we require that the variables from Y have degree 0 and those from Z have degree 1.

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The superalgebra $F\langle Y \cup Z\rangle$ is said to be the *free superalgebra* over F. An element $f(y_1, \dots, y_m, z_1, \dots, z_n)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$*-graded polynomial identity* for an F-superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \dots, a_m, b_1, \dots, b_n) = 0_A$ for every $a_1, \dots, a_m \in A^{(0)}$ and $b_1, \dots, b_n \in A^{(1)}$. Let $T_{\mathbb{Z}_2}(A)$ be the set of all the $\mathbb{Z}_2$-graded polynomial identities satisfied by A, which is easily seen to be a $T_{\mathbb{Z}_2}$-ideal of $F\langle Y \cup Z\rangle$, namely a two-sided ideal of the free superalgebra invariant under every endomorphism of $F\langle Y \cup Z\rangle$ preserving the grading. Given a $T_{\mathbb{Z}_2}$-ideal I of $F\langle Y \cup Z\rangle$, the *variety of superalgebras* or *supervariety* $\mathcal{V}^{sup}$ associated to I is the class of all F-superalgebras whose $T_{\mathbb{Z}_2}$-ideals of graded polynomial identities contain I. The $T_{\mathbb{Z}_2}$-ideal I is denoted by $T_{\mathbb{Z}_2}(\mathcal{V}^{sup})$. The supervariety $\mathcal{V}^{sup}$ is generated by the superalgebra A if $T_{\mathbb{Z}_2}(\mathcal{V}^{sup}) = T_{\mathbb{Z}_2}(A)$, and in this case we write $\mathcal{V}^{sup} = \mathrm{supvar}(A)$.

Extending into the setting of $\mathbb{Z}_2$-graded algebras the approach introduced by Regev in [15], one considers a numerical sequence that can be attached to the graded polynomial identities of a supervariety $\mathcal{V}^{sup}$ (or of a superalgebra A), that of $\mathbb{Z}_2$*-graded codimensions* of $\mathcal{V}^{sup}$. In details, for every $n \geq 1$, let us define the *n-th* $\mathbb{Z}_2$*-graded codimension* $c_n^{\mathbb{Z}_2}(\mathcal{V}^{sup})$ of $\mathcal{V}^{sup}$ as the dimension of the vector space $\frac{P_n^{sup}}{P_n^{sup} \cap T_{\mathbb{Z}_2}(\mathcal{V}^{sup})}$, where P_n^{sup} is the space of multilinear polynomials of degree n of $F\langle Y \cup Z\rangle$ in the variables $y_1, \dots, y_n, z_1, \dots, z_n$. Since F has characteristic zero, $T_{\mathbb{Z}_2}(\mathcal{V}^{sup})$ is completely determined by multilinear polynomials it contains and hence $\{c_n^{\mathbb{Z}_2}(\mathcal{V}^{sup})\}_{n \geq 1}$ in some sense measures the rate of growth of the graded polynomial identities of the variety $\mathcal{V}^{sup}$. In [7] it was proved that this sequence is exponentially bounded if, and only if, $\mathcal{V}^{sup}$ is generated by a superalgebra A satisfying an ordinary polynomial identity. Under the extra assumption that A is also finitely generated, in [3] the authors stated that

$$\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup}) := \lim_{m \to +\infty} \sqrt[m]{c_m^{\mathbb{Z}_2}(\mathcal{V}^{sup})}$$

exists and is a non-negative integer, which is called the $\mathbb{Z}_2$*-graded exponent* or *superexponent* of $\mathcal{V}^{sup}$. In this case set $\exp_{\mathbb{Z}_2}(A) := \exp_{\mathbb{Z}_2}(\mathcal{V}^{sup})$, the *superexponent* of the superalgebra A. This result was already established for varieties of PI algebras by Giambruno and Zaicev ([8] and [9]), whereas more recently it has been extended by Aljadeff, Giambruno and La Mattina to varieties of G-graded PI associative algebras. Namely, in a series of papers ([2], [6] and [1]) they have captured the exponential growth of the corresponding codimension sequence for varieties generated by a G-graded algebra A when G is a finite group and A satisfies a polynomial identity.

The existence of the exponent provides an integral scale allowing to measure the growth of any variety of such a type and in a natural manner addresses the research towards a classification of varieties according to the asymptotic behaviour of their corresponding codimensions. In this framework, among varieties of some fixed exponent a prominent role is played by the *minimal* ones, namely those varieties of exponent d such that every proper subvariety has exponent strictly less than d.

In [11] it has been proved that in the ungraded case a variety of exponential growth is minimal if, and only if, it is generated by the Grassmann envelope, $G(A)$, of a so-called *minimal superalgebra* A. More recently, motivated by the result of [3], in [5] varieties of PI associative superalgebras of *finite basic rank* (that is, generated by a finitely generated superalgebra satisfying an ordinary polynomial identity) which are minimal of fixed superexponent $d \geq 2$ have been investigated. In particular, it has been stated that any such a supervariety is generated by one of the above mentioned minimal superalgebras introduced by Giambruno and Zaicev in [11]. But the question of which minimal superalgebras generate a minimal supervariety of fixed graded exponent is still open. Unfortunately, its possible solution seems to be more involved than that of the ungraded case. In fact, in [10] Giambruno and Zaicev proved that a variety $\mathcal{V}$ of PI algebras of finite basic rank is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(m_1, \dots, m_n)$. Moreover, as an application of Lewin's Theorem [14], it was shown that its ideal of polynomial identities, $\mathrm{Id}(\mathcal{V})$, has the nice property to be factored as the product of the T-ideals of polynomial identities of the blocks along the main diagonal, that is $\mathrm{Id}(M_{m_1}(F)) \cdots \mathrm{Id}(M_{m_n}(F))$.

In the superalgebras setting, one replaces upper block triangular matrix algebras with minimal superalgebras but the factorability property for their $T_{\mathbb{Z}_2}$-ideals fails. In more details, if A is a minimal superalgebra and $A_1, \dots, A_n$ are the graded simple algebras appearing in the semisimple component of its Wedderburn-Malcev decomposition, in general it is untrue that the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of A factorizes as $T_{\mathbb{Z}_2}(A_1) \cdots T_{\mathbb{Z}_2}(A_n)$ (see Section 4 of [5]). But this naturally leads to investigate the question of when such a factorization holds. We provide here a positive answer in the case in which all the algebras $A_1, \dots, A_n$ are non-simple graded simple except for at most one between A_1 and A_n. Furthermore we show that if there exists an integer $h \notin \{1, n\}$ such that $A_1, \dots, A_h$ are non-simple graded simple and $A_{h+1}, \dots, A_n$ are simple graded simple (or conversely), then $T_{\mathbb{Z}_2}(A)$ factorizes in a weaker sense. We use these results to prove that in any of these cases the minimal superalgebra A generates a minimal supervariety of fixed superexponent.

2. Minimal Superalgebras with (weakly) factorable ideal of graded polynomial identities and $\mathbb{Z}_2$-regular algebras

Throughout the rest of the paper, unless otherwise stated, F is a field of characteristic zero and all the algebras are assumed to be associative and to have the same ground field F. For any pair of positive integers s and t the symbol $M_{s\times t}$ means the space of all rectangular matrices with s rows and t columns over F and set $M_s := M_{s\times s}$; whereas, if $m_1, \dots, m_n$ is a sequence of integers, let $UT(m_1, \dots, m_n)$ be the upper block triangular matrix algebra of size $m_1, \dots, m_n$.

Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra. An element w of A is *homogeneous* if it is homogeneous of degree 0 or 1, whereas a subalgebra or an ideal $V \subseteq A$ is *homogeneous* if $V = (V \cap A^{(0)}) \oplus (V \cap A^{(1)})$. The superalgebra A is called *simple* (or $\mathbb{Z}_2$-*simple*) if the multiplication is non-trivial and it has no non-trivial homogeneous ideals. In this case, we shall also refer to A as a *graded simple* algebra.

Assume that A is a finite-dimensional superalgebra and $J = J(A)$ is its Jacobson radical. Then J is homogeneous and set $J^{(i)} := J \cap A^{(i)}$ for $i = 0, 1$. Moreover, by the generalization of the Wedderburn-Malcev Theorem we can write $A = A_{ss} + J$, where A_{ss} is a maximal semisimple subalgebra of A having an induced $\mathbb{Z}_2$-grading. Also A_{ss} can be written as the direct sum of graded simple algebras whose structure is well-known, at least when the ground field is algebraically closed. In fact, they are one of the following types:

(a) $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $M_{k,l}^{(0)} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $M_{k,l}^{(1)} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;

(b) $M_m(F \oplus tF)$, where $t^2 = 1$, with grading (M_m, tM_m).

Giambruno and Zaicev in [11] introduced the definition of *minimal* superalgebra in the following manner.

Definition 2.1. *Let F be an algebraically closed field. A superalgebra A is called* minimal *if it is finite-dimensional and $A = A_{ss} + J$ where*

(i) *$A_{ss} = A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n$ graded simple algebras;*

(ii) *there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J^{(0)} \cup J^{(1)}$ and minimal homogeneous idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that*

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \qquad 1 \leq i \leq n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;$$

(iii) *$w_{12}, \ldots, w_{n-1,n}$ generate J as a two-sided ideal of A.*

We observe that, when $n = 1$, A is nothing but a graded simple algebra. Since we are interested to the case in which the Jacobson radical of A is non-zero we assume throughout (unless explicitly mentioned) that $n > 1$.

In Lemma 3.5 of [11] it was shown that the minimal superalgebra A has the following vector space decomposition

$$A = \bigoplus_{1\leq i\leq j\leq n} A_{ij}, \tag{1}$$

where $A_{11} := A_1, \ldots, A_{nn} := A_n$ and, for all $i < j$,

$$A_{ij} := A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j.$$

Moreover $J = \oplus_{i<j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where δ_{jk} is the Kronecker delta.

As stressed in Chapter 8 of [12], the order of the components $A_1, \ldots, A_n$ of the semisimple part A_{ss} of a minimal superalgebra A is important. For this reason in the sequel we shall tacitly agree that if $A = A_{ss} + J$ is a minimal superalgebra with semisimple part $A_{ss} = A_1 \oplus \cdots \oplus A_n$, then $A_1 J A_2 J \cdots J A_n \neq 0_A$. According to the main result of [3], $\exp_{\mathbb{Z}_2}(A) = \dim_F(A_{ss})$. Furthermore, if $(A_{i_1}, \ldots, A_{i_t})$ is a subsequence of $(A_1, \ldots, A_n)$ set, for every $1 \leq j \leq t-1$, $u_{i_j, i_{j+1}} := w_{i_j, i_j+1} \cdots w_{i_{j+1}-1, i_{j+1}}$, the subalgebra of A generated by $A_{i_1}, \ldots, A_{i_t}$ and the homogeneous radical elements $u_{i_1,i_2}, \ldots, u_{i_{t-1},i_t}$ is a minimal superalgebra as well (of superexponent $\dim_F(A_{i_1} \oplus \ldots \oplus A_{i_t})$). In particular, for every $1 \leq k \leq l \leq n$ let us denote by $A^{(k,l)}$ the homogeneous subalgebra of A corresponding to the sequence $(A_k, A_{k+1}, \ldots, A_l)$, namely

$$A^{(k,l)} = \bigoplus_{k \leq i \leq j \leq l} A_{ij}.$$

We premise now an easy (but crucial for our aims) result explaining the relation among the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of A and that of its subalgebras $A^{(k,l)}$. We recall that a finite subset I of $\mathbb{N}$ is said to be an *interval* if there exist $1 \leq i \leq j$ such that $I = \{i, i+1, \ldots, j\}$.

Lemma 2.2. *Let $A = A_{ss} + J$ be a minimal superalgebra with $A_{ss} = A_1 \oplus \cdots \oplus A_n$, $t \geq 1$ be an integer and, for every $1 \leq i \leq t$, $S_i := \{k_i, \ldots, l_i\}$ be (non-necessarily disjoint) intervals of $\mathbb{N}$ such that $k_1 \leq k_2 \leq \ldots \leq k_t$ and $\{1, \ldots, n\} = \cup_{i=1}^t S_i$. Then $T_{\mathbb{Z}_2}(A^{(k_1,l_1)}) \cdots T_{\mathbb{Z}_2}(A^{(k_t,l_t)}) \subseteq T_{\mathbb{Z}_2}(A)$.*

Proof. Assume that $t > 1$ (otherwise the statement is trivial) and take $f_1 \in T_{\mathbb{Z}_2}(A^{(k_1,l_1)}), \ldots, f_t \in T_{\mathbb{Z}_2}(A^{(k_t,l_t)})$. For any $1 \leq i \leq t$ and graded evaluation σ_i of the polynomial f_i in A one has that $\sigma_i(f_i) = \sum_{1 \leq p \leq q \leq n} b_{pq}^{(i)}$, where $b_{pq}^{(i)}$ are elements of A_{pq} such that $b_{pq}^{(i)} = 0_A$ for every $k_i \leq p \leq q \leq l_i$. By recalling the multiplication roles among the subspaces A_{pq} of A, it follows that any graded evaluation of the polynomial $f_1 \cdots f_t$ in A must be zero, and this concludes the proof. □

By virtue of the above lemma, it has sense to introduce the following

Definition 2.3. *The $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of a minimal superalgebra $A = A_{ss} + J$ with $A_{ss} = A_1 \oplus \cdots \oplus A_n$ is said to be* weakly factorable *if there exist $1 \leq l_1 < l_2 < \ldots < l_t < n$ such that*

$$T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A^{(1,l_1)}) \cdot T_{\mathbb{Z}_2}(A^{(l_1+1,l_2)}) \cdots T_{\mathbb{Z}_2}(A^{(l_t+1,n)}).$$

It is called factorable *if*

$$T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A_1) \cdots T_{\mathbb{Z}_2}(A_n).$$

In the next section we shall provide sufficient conditions on the sequence $(A_1, \ldots, A_n)$ of the summands of A_{ss} so that $T_{\mathbb{Z}_2}(A)$ is weakly factorable in the product of the $T_{\mathbb{Z}_2}$-ideals of graded polynomial identities of some "nice" homogeneous subalgebras of A. At this aim, another important tool we shall use is the concept of $\mathbb{Z}_2$-*regular* algebra introduced in [4]. Let us consider the complete matrix algebra M_m. A $\mathbb{Z}_2$-grading on M_m is called *elementary* if there exists an m-tuple $(g_1, \ldots, g_m) \in \mathbb{Z}_2^m$ such that the matrix units E_{ij} of M_m are homogeneous and $E_{ij} \in M_m^{(k)}$ if, and only if, $k =$

$g_j - g_i$. In an equivalent manner, we can say that it is defined a map $|\ | : \{1, \dots, m\} \longrightarrow \mathbb{Z}_2$ inducing a grading on M_m by setting the degree of E_{ij} equal to $|j| - |i|$. Obviously the algebra of upper block triangular matrices also admits an elementary grading. In fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra.

Let A be any homogeneous subalgebra of $(M_m, |\ |)$. Denote by $P(A)$ the polynomial ring associated to A, namely $P(A) := F[t_i^{(h)} \,|\, 1 \leq i \leq \dim_F A,\, h \geq 1]$ is the polynomial ring in the countable set of commuting variables $t_i^{(h)}$. It is well known that there exists a standard method to realize the superalgebra $F\langle Y \cup Z\rangle / T_{\mathbb{Z}_2}(A)$ as a subalgebra of $M_m \otimes P(A)$. For every $g \in \mathbb{Z}_2$ we consider the F-linear map

$$\pi_g : M_m \otimes P(A) \longrightarrow M_m \otimes P(A), \qquad \sum_{1 \leq i,j \leq m} a_{ij} E_{ij} \longmapsto \sum_{\substack{1 \leq i,j \leq m, \\ |i| = g}} a_{ij} E_{ij},$$

and the restrictions $\hat{\pi}_g$ of π_g to $F\langle Y \cup Z\rangle / T_{\mathbb{Z}_2}(A)$.

Definition 2.4 (4.3 of [4]). *A homogeneous subalgebra A of $(M_m, |\ |)$ is said to be* $\mathbb{Z}_2$-regular *if, for every $g \in \mathbb{Z}_2$, the maps $\hat{\pi}_g$ are injective.*

As proved in [4], the simple superalgebra $M_{k,l}$ is $\mathbb{Z}_2$-regular if, and only if, $k = l$ and, for any integer m, $M_m(F \oplus tF)$ is a homogeneous $\mathbb{Z}_2$-regular subalgebra of M_{2m}.

A key result that we need to borrow from [4] is the following.

Lemma 2.5 (Theorem 4.5 of [4]). *Let $(M_m, |\ |_m)$ and $(M_n, |\ |_n)$ be matrix algebras endowed with an elementary $\mathbb{Z}_2$-grading, $A \subseteq M_m$ and $B \subseteq M_n$ their homogeneous subalgebras, respectively, and $U := M_{m \times n}$. Define the map $|\ | : \{1, \dots, m+n\} \longrightarrow \mathbb{Z}_2$ setting $|i| = |i|_m$ if $i \leq m$ and $|i| = |i - m|_n$ otherwise. If one of A and B is $\mathbb{Z}_2$-regular, then the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of the homogeneous subalgebra of $(M_{m+n}, |\ |)$*

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix}$$

factorizes as $T_{\mathbb{Z}_2}(R) = T_{\mathbb{Z}_2}(A) \cdot T_{\mathbb{Z}_2}(B)$.

3. Factorization of the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of a minimal superalgebra and minimal supervarieties

We aim to prove the results announced in the Introduction on the decomposition of the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities of a minimal superalgebra upon certain constraints on the sequence of the graded simple summands of its maximal semisimple homogeneous subalgebra. Namely, we shall state the following

Theorem 3.1. *Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \dots, A_h$ are non-simple graded simple and $A_{h+1}, \dots, A_n$ are simple graded simple algebras,* *then*

$$T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A_1) \cdots T_{\mathbb{Z}_2}(A_h) \cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}) = T_{\mathbb{Z}_2}(A^{(1,h)}) \cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}).$$

On the other hand, if $h < n$ and $A_1, \ldots, A_h$ are simple graded simple and $A_{h+1}, \ldots, A_n$ are non-simple graded simple algebras, then

$$T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A^{(1,h)}) \cdot T_{\mathbb{Z}_2}(A_{h+1}) \cdots T_{\mathbb{Z}_2}(A_n) = T_{\mathbb{Z}_2}(A^{(1,h)}) \cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}).$$

An immediate consequence of the above theorem is the following

Corollary 3.2. *Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and at most one between A_1 and A_n is a simple graded simple algebra whereas A_i is a non-simple graded simple algebra for the remaining indices i, then $T_{\mathbb{Z}_2}(A)$ is factorable.*

As in the ordinary case, we shall see in the last part of the paper that the factorization property has an important role in the classification of minimal supervarieties of fixed superexponent.

The strategy of the proof of Theorem 3.1 consists in constructing a homogeneous subalgebra A' of A such that $T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A')$ and proving that it is isomorphic to a suitable homogeneous subalgebra $\mathcal{C}(A')$ of an upper block triangular matrix algebra with action of an automorphism of order 2. Finally we shall show that $\mathcal{C}(A')$ is isomorphic to a homogeneous subalgebra $\mathcal{R}(A')$ of the same upper block triangular matrix algebra endowed with an elementary grading. Hence we shall study the graded identities of $\mathcal{R}(A')$ (and hence of A') to get information on those of the original superalgebra A. In order to do this, as mentioned above, we shall often use the language of actions of automorphisms. In fact, it is well-known that any superalgebra A can be viewed as an algebra with action of an automorphism ϕ of A of order at most 2. Indeed, the homomorphism ϕ of $A = A^{(0)} \oplus A^{(1)}$ defined by $\phi(a_0) := a_0$ and $\phi(a_1) := -a_1$ for any $a_0 \in A^{(0)}$ and $a_1 \in A^{(1)}$ is an automorphism of A of order at most 2. Conversely, if A is an algebra with an automorphism ϕ of order at most 2, then, setting $A^{(0)} := \{a|\, a \in A,\ \phi(a) = a\}$ and $A^{(1)} := \{a|\, a \in A,\ \phi(a) = -a\}$, A is a superalgebra with grading $(A^{(0)}, A^{(1)})$.

STEP I: Construction of A'. Let $A = A_{ss} + J$ be a minimal superalgebra as in Theorem 3.1 and suppose that, for every $1 \leq i \leq h$, $A_i = M_{m_i}(F \oplus t_i F)$ whereas $A_i = M_{k_i,l_i}$ when $h+1 \leq i \leq n$. By regarding A as a ϕ-algebra, we can write the non-simple graded simple algebras A_i as $A_i = I_i \oplus \phi(I_i)$, where I_i is a minimal two-sided ideal of A_i, and the corresponding homogeneous idempotents (of degree zero) e_i appearing in the Definition 2.1 as $e_i = \rho_i + \phi(\rho_i)$ with ρ_i a non-homogeneous minimal idempotent of I_i. For simplicity, set $\bar{\rho}_i := \phi(\rho_i)$ and $\bar{I}_i := \phi(I_i)$.

Let us consider the element $w_{1n} := w_{12} \cdots w_{n-1,n}$. As for the homogeneous radical elements $w_{i,i+1}$ defining A the equality

$$e_i w_{i,i+1} e_{i+1} = e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}$$

is satisfied, one has that

$$\begin{aligned}
w_{1n} &= (\rho_1 + \bar{\rho}_1) w_{12} (\rho_2 + \bar{\rho}_2) \cdots (\rho_{h-1} + \bar{\rho}_{h-1}) w_{h-1,h} (\rho_h + \bar{\rho}_h) \cdot \\
&\quad \cdot w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n \\
&= \sum_{\epsilon_i \in \{\rho_i, \bar{\rho}_i\}} \epsilon_1 w_{12} \epsilon_2 \cdots \epsilon_{h-1} w_{h-1,h} \epsilon_h w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n.
\end{aligned}$$

According to the definition of minimal superalgebra, $w_{1n} \neq 0_A$. This implies that at least one of the 2^{h-1} homogeneous elements appearing in the above sum is non-zero. Let us pick a sequence $(\epsilon_1, \ldots, \epsilon_h)$ such that the corresponding term occurring in w_{1n} is non-zero, i.e. we are assuming that

$$\begin{aligned}
&\epsilon_1 w_{12} \epsilon_2 \cdots \epsilon_{h-1} w_{h-1,h} \epsilon_h w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n + \\
&+ \phi(\epsilon_1) w_{12} \phi(\epsilon_2) \cdots \phi(\epsilon_{h-1}) w_{h-1,h} \phi(\epsilon_h) w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n \neq 0_A.
\end{aligned}$$

Call $\mathcal{I}_i$ the minimal two-sided ideal of A_i such that ϵ_i belongs to (namely, $\mathcal{I}_i = I_i$ if $\epsilon_i \in I_i$ and $\mathcal{I}_i = \bar{I}_i$ otherwise) and, as before, set $\bar{\epsilon}_i := \phi(\epsilon_i)$ and $\bar{\mathcal{I}}_i := \phi(\mathcal{I}_i)$.

For every $1 \leq i \leq n$, let us define

$$v_{i,i+1} := \begin{cases}
\epsilon_i w_{i,i+1} \epsilon_{i+1} + \bar{\epsilon}_i w_{i,i+1} \bar{\epsilon}_{i+1} & 1 \leq i \leq h-1 \text{ and } \deg(w_{i,i+1}) = 0; \\
\epsilon_i w_{i,i+1} \epsilon_{i+1} - \bar{\epsilon}_i w_{i,i+1} \bar{\epsilon}_{i+1} & 1 \leq i \leq h-1 \text{ and } \deg(w_{i,i+1}) = 1; \\
\epsilon_h w_{h,h+1} + \bar{\epsilon}_h w_{h,h+1} & i = h \text{ and } \deg(w_{h,h+1}) = 0; \\
\epsilon_h w_{h,h+1} - \bar{\epsilon}_h w_{h,h+1} & i = h \text{ and } \deg(w_{h,h+1}) = 1; \\
w_{i,i+1} & h+1 \leq i \leq n
\end{cases}$$

and observe that $v_{i,i+1}$ is a homogeneous element of $J(A)$ (of degree zero if $1 \leq i \leq h$) such that

$$e_i v_{i,i+1} = v_{i,i+1} = v_{i,i+1} e_{i+1}.$$

Moreover, using the fact that for any $1 \leq i \leq h-1$

$$\epsilon_i v_{i,i+1} \bar{\epsilon}_{i+1} = 0_A = \bar{\epsilon}_i v_{i,i+1} \epsilon_{i+1},$$

$$\epsilon_i v_{i,i+1} \epsilon_{i+1} = \epsilon_i w_{i,i+1} \epsilon_{i+1}, \qquad \bar{\epsilon}_i v_{i,i+1} \bar{\epsilon}_{i+1} = (-1)^{\deg(w_{i,i+1})} \bar{\epsilon}_i w_{i,i+1} \bar{\epsilon}_{i+1}$$

and

$$\epsilon_h v_{h,h+1} = \epsilon_h w_{h,h+1}, \qquad \bar{\epsilon}_h v_{h,h+1} = (-1)^{\deg(w_{h,h+1})} \bar{\epsilon}_h w_{h,h+1},$$

we get that

$$\begin{aligned}
v_{1n} &:= v_{12} \cdots v_{n-1,n} = e_1 v_{12} e_2 \cdots e_{n-1} v_{n-1,n} e_n \\
&= (\epsilon_1 + \bar{\epsilon}_1) v_{12} (\epsilon_2 + \bar{\epsilon}_2) \cdots (\epsilon_h + \bar{\epsilon}_h) v_{h,h+1} e_{h+1} \cdots e_{n-1} v_{n-1,n} e_n \\
&= \epsilon_1 v_{12} \epsilon_2 \cdots \epsilon_h v_{h,h+1} e_{h+1} \cdots e_{n-1} v_{n-1,n} e_n + \\
&\quad + \bar{\epsilon}_1 v_{12} \bar{\epsilon}_2 \cdots \bar{\epsilon}_h v_{h,h+1} e_{h+1} \cdots e_{n-1} v_{n-1,n} e_n \\
&= \epsilon_1 w_{12} \epsilon_2 \cdots \epsilon_h w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n + \\
&\quad + (-1)^m \bar{\epsilon}_1 w_{12} \bar{\epsilon}_2 \cdots \bar{\epsilon}_h w_{h,h+1} e_{h+1} \cdots e_{n-1} w_{n-1,n} e_n,
\end{aligned}$$

where m is the number of indices $1 \leq i \leq h$ so that $\deg(w_{i,i+1}) = 1$. If $v_{1n} = 0_A$, also $\epsilon_1 v_{1n}$ must be zero. This implies that the first summand in the last term of the above equality is zero, which is in contradiction with the choice of the sequence $(\epsilon_1, \ldots, \epsilon_h)$. Actually we notice that both the summands appearing in v_{1n} are non-zero.

Therefore the subalgebra A' of A generated by $A_1, \ldots, A_n$ and the homogeneous elements $v_{12}, \ldots, v_{n-1,n}$ is a minimal superalgebra as well.

Our goal now is to describe the subspaces A'_{ij} appearing in the decomposition (1) of the superalgebra A'. To this end, assume first that $1 \le i < j \le h$. Since for every $1 \le k \le j-1$

$$\begin{aligned} v_{k,k+1} A_{k+1} v_{k+1,k+2} &= v_{k,k+1}(\mathcal{I}_{k+1} \oplus \bar{\mathcal{I}}_{k+1}) v_{k+1,k+2} \\ &= v_{k,k+1}(\epsilon_{k+1} + \bar{\epsilon}_{k+1})(\mathcal{I}_{k+1} \oplus \bar{\mathcal{I}}_{k+1})(\epsilon_{k+1} + \bar{\epsilon}_{k+1}) v_{k+1,k+2} \\ &= \operatorname{span}_F \langle v_{k,k+1} \epsilon_{k,k+1} v_{k+1,k+2}, v_{k,k+1} \bar{\epsilon}_{k,k+1} v_{k+1,k+2} \rangle, \end{aligned}$$

we obtain

$$\begin{aligned} A'_{ij} &= A_i v_{i,i+1} A_{i+1} v_{i+1,i+2} \cdots v_{j-1,j} A_j \\ &= A_i \epsilon_i v_{i,i+1} \epsilon_{i+1} v_{i+1,i+2} \epsilon_{i+2} \cdots \epsilon_{j-1} v_{j-1,j} \epsilon_j A_j \oplus \\ &\quad \oplus A_i \bar{\epsilon}_i v_{i,i+1} \bar{\epsilon}_{i+1} v_{i+1,i+2} \bar{\epsilon}_{i+2} \cdots \bar{\epsilon}_{j-1} u_{j-1,j} \bar{\epsilon}_j A_j \\ &= \mathcal{I}_i \epsilon_i v_{i,i+1} \epsilon_{i+1} \cdots \epsilon_{j-1} v_{j-1,j} \epsilon_j \mathcal{I}_j \oplus \bar{\mathcal{I}}_i \bar{\epsilon}_i v_{i,i+1} \bar{\epsilon}_{i+1} \cdots \bar{\epsilon}_{j-1} v_{j-1,j} \bar{\epsilon}_j \bar{\mathcal{I}}_j. \end{aligned}$$

In particular, $\mathcal{I}_i \epsilon_i v_{i,i+1} \epsilon_{i+1} \cdots \epsilon_{j-1} v_{j-1,j} \epsilon_j \mathcal{I}_j$ is a non-zero irreducible $(\mathcal{I}_i, \mathcal{I}_j)$-bimodule isomorphic to $\mathcal{I}_i \epsilon_i \otimes \epsilon_j \mathcal{I}_j$, whereas the other subspace appearing as a direct summand of A'_{ij} is a non-zero irreducible $(\bar{\mathcal{I}}_i, \bar{\mathcal{I}}_j)$-bimodule isomorphic to $\bar{\mathcal{I}}_i \bar{\epsilon}_i \otimes \bar{\epsilon}_j \bar{\mathcal{I}}_j$.

When $h+1 \le i < j \le n$,

$$\begin{aligned} A'_{ij} = A_{ij} &= A_i e_i v_{i,i+1} e_{i+1} A_{i+1} e_{i+1} v_{i+1,i+2} e_{i+2} \cdots e_{j-1} A_{j-1} e_{j-1} v_{j-1,j} e_j A_j \\ &= A_i e_i v_{i,i+1} \cdots v_{j-1,j} e_j A_j, \end{aligned}$$

which is a non-zero irreducible (A_i, A_j)-bimodule isomorphic to $A_i e_i \otimes e_j A_j$.

Finally, if $i \le h < j$ proceeding in the same way we conclude that

$$\begin{aligned} A'_{ij} = &\mathcal{I}_i \epsilon_i v_{i,i+1} \epsilon_{i+1} \cdots \epsilon_{h-1} v_{h-1,h} \epsilon_h v_{h,h+1} e_{h+1} \cdots e_{j-1} v_{j-1,j} e_j A_j \oplus \\ &\oplus \bar{\mathcal{I}}_i \bar{\epsilon}_i v_{i,i+1} \bar{\epsilon}_{i+1} \cdots \bar{\epsilon}_{h-1} v_{h-1,h} \bar{\epsilon}_h v_{h,h+1} e_{h+1} \cdots e_{j-1} v_{j-1,j} e_j A_j. \end{aligned}$$

In this case the first summand of the above equality is a non-zero irreducible $(\mathcal{I}_i, A_j)$-bimodule isomorphic to $\mathcal{I}_i \epsilon_i \otimes e_j A_j$ and the second one is a non-zero irreducible $(\bar{\mathcal{I}}_i, A_j)$-bimodule isomorphic to $\bar{\mathcal{I}}_i \bar{\epsilon}_i \otimes e_j A_j$.

STEP II: Construction of $\mathcal{C}(A')$. Set $m_i := k_i + l_i$ when $h+1 \le i \le n$ and $d_0 := 0$, for every $1 \le i \le n$ let us define

$$d_i := \begin{cases} \sum_{j=1}^{i} 2m_j & 1 \le i \le h; \\ \sum_{j=1}^{h} 2m_j + \sum_{j=h+1}^{i} m_j & h+1 \le i \le n, \end{cases}$$

$\mathrm{Bl}_i := \{d_{i-1}+1, \ldots, d_i\}$ if $i \ge h+1$ and $\mathrm{Bl}_i := \mathrm{Bl}_i^{(1)} \cup \mathrm{Bl}_i^{(2)}$ with $\mathrm{Bl}_i^{(1)} := \{d_{i-1}+1, \ldots, d_{i-1}+m_i\}$ and $\mathrm{Bl}_i^{(2)} := \{d_{i-1}+m_i+1, \ldots, d_i\}$ in the case in which $i \le h$. Furthermore let $\Gamma := \cup_{i=1}^{n} \mathrm{Bl}_i$ and, when $h+1 \le i \le n$,

$$\alpha_s^{(i)} : \mathrm{Bl}_i \longrightarrow \mathbb{Z}_2, \quad x \longmapsto \begin{cases} 0 & x \in \{d_{i-1}+1, \ldots, d_{i-1}+k_i\}; \\ 1 & x \in \{d_{i-1}+k_i+1, \ldots, d_i\} \end{cases}$$

and

$$\alpha_c^{(i)} : \mathrm{Bl}_i \longrightarrow \mathbb{Z}_2, \quad x \longmapsto \begin{cases} 1 & x \in \{d_{i-1}+1, \ldots, d_{i-1}+k_i\}; \\ 0 & x \in \{d_{i-1}+k_i+1, \ldots, d_i\}. \end{cases}$$

We claim that the minimal superalgebra A' is isomorphic (as a superalgebra) to a homogeneous subalgebra of $R := UT(2m_1, \ldots, 2m_h, m_{h+1}, \ldots, m_n)$ endowed with the grading induced by the automorphism ϑ of order 2 defined on the matrix units E_{ij} of R by

$$\vartheta(E_{ij}) := (-1)^{\alpha(i)+\alpha(j)} E_{\sigma(i),\sigma(j)},$$

where

$$\sigma : \Gamma \longrightarrow \Gamma, \qquad x \longmapsto \begin{cases} x + m_i & 1 \leq i \leq h \text{ and } x \in \mathrm{Bl}_i^{(1)}; \\ x - m_i & 1 \leq i \leq h \text{ and } x \in \mathrm{Bl}_i^{(2)}; \\ x & h+1 \leq i \leq n \text{ and } x \in \mathrm{Bl}_i \end{cases}$$

is a bijection so that $\sigma^2 = \mathrm{id}_\Gamma$ and $\alpha : \Gamma \longrightarrow \mathbb{Z}_2$ is the map such that

$$\alpha(x) := 0 \qquad x \in \cup_{i=1}^{h} \mathrm{Bl}_i,$$

$$\alpha_{|\,\mathrm{Bl}_{h+1}} := \alpha_s^{(h+1)}$$

and, inductively, for every $h+2 \leq i \leq n$

$$\alpha_{|\,\mathrm{Bl}_i} := \begin{cases} \alpha_s^{(i)} & \alpha_{|\,\mathrm{Bl}_{i-1}} = \alpha_s^{(i-1)} \text{ and } \deg(v_{i-1,i}) = 0 \text{ or } \alpha_{|\,\mathrm{Bl}_{i-1}} = \alpha_c^{(i-1)} \text{ and } \deg(v_{i-1,i}) = 1; \\ \alpha_c^{(i)} & \text{otherwise.} \end{cases}$$

In order to prove the claim, for convenience put

$$E_{rs}^{(p,q)} := E_{rs}$$

if $r \in \mathrm{Bl}_p$ and $s \in \mathrm{Bl}_q$ and consider the subspace $\mathcal{C}(A')$ generated by the matrix units $E_{rs}^{(p,q)}$ of R such that, when $1 \leq p \leq q \leq h$, $r \in \mathrm{Bl}_p^{(1)}$ and $s \in \mathrm{Bl}_q^{(1)}$ or $r \in \mathrm{Bl}_p^{(2)}$ and $s \in \mathrm{Bl}_q^{(2)}$. By taking in account that, for any $1 \leq i \leq n$, $\sigma(\mathrm{Bl}_i) = \mathrm{Bl}_i$ and in particular, when $1 \leq i \leq h$, $\sigma(\mathrm{Bl}_i^{(j)}) = \mathrm{Bl}_i^{(j+1)}$ (where obviously the indices j and $j+1$ are intendeed modulo 2), it is straightforward to check that this is actually a homogeneous subalgebra of R. A homogeneous basis for $\mathcal{C}(A')$ is given by the elements

$$E_{rs}^{(p,q)} \pm E_{\sigma(r),\sigma(s)}^{(p,q)} \quad 1 \leq p \leq q \leq h \text{ and } r \in \mathrm{Bl}_p^{(1)}, s \in \mathrm{Bl}_q^{(1)},$$

$$E_{rs}^{(p,q)} \quad h+1 \leq p \leq q \leq n \text{ and } r \in \mathrm{Bl}_p, s \in \mathrm{Bl}_q$$

and

$$E_{rs}^{(p,q)} \pm E_{\sigma(r),s}^{(p,q)} \quad 1 \leq p \leq h < q \leq n \text{ and } r \in \mathrm{Bl}_p^{(1)}, s \in \mathrm{Bl}_q.$$

We want to construct now an isomorphism $\Psi : \mathcal{C}(A') \longrightarrow A'$ of algebras with actions (of automorphisms of order 2). At this regard, we observe that for every $h+1 \leq p < q \leq n$ and $r \in \mathrm{Bl}_p$ and $s \in \mathrm{Bl}_q$

$$E_{rs}^{(p,q)} = E_{r,d_{p-1}+1}^{(p,p)} E_{d_{p-1}+1,d_{q-1}+1}^{(p,q)} E_{d_{q-1}+1,s}^{(q,q)}, \tag{2}$$

whereas, in the case in which $1 \leq p < q \leq h$, $E_{rs}^{(p,q)}$ is exactly as in (2) if $r \in \mathrm{Bl}_p^{(1)}$ and $s \in \mathrm{Bl}_q^{(1)}$ and

$$E_{rs}^{(p,q)} = E_{r,d_{p-1}+m_p+1}^{(p,p)} E_{d_{p-1}+m_p+1,d_{q-1}+m_q+1}^{(p,q)} E_{d_{q-1}+m_q+1,s}^{(q,q)}$$

if $r \in \mathrm{Bl}_p^{(2)}$ and $s \in \mathrm{Bl}_q^{(2)}$ and finally, when $1 \leq p \leq h < q \leq n$, it is again as in (2) if $r \in \mathrm{Bl}_p^{(1)}$ and $s \in \mathrm{Bl}_q$ and

$$E_{rs}^{(p,q)} = E_{r,d_{p-1}+m_p+1}^{(p,p)} E_{d_{p-1}+m_p+1,d_{q-1}+1}^{(p,q)} E_{d_{q-1}+1,s}^{(q,q)}$$

if $r \in \mathrm{Bl}_p^{(2)}$ and $s \in \mathrm{Bl}_q$. Hence in order to define Ψ it is sufficient to define the images of the elements of the diagonal blocks $\Delta_1, \ldots, \Delta_n$ of $\mathcal{C}(A')$ with the induced grading and those of the form $E_{d_{p-1}+1,d_{q-1}+1}^{(p,q)}$ and (when they exist) $E_{d_{p-1}+m_p+1,d_{q-1}+1}^{(p,q)}$ and $E_{d_{p-1}+m_p+1,d_{q-1}+m_q+1}^{(p,q)}$ for $p < q$. But, by the definition of $\mathcal{C}(A') \subseteq R$, for every $1 \leq i \leq n$ there exists an isomorphism of algebras with action

$$\eta_i : \Delta_i \longrightarrow A_i.$$

Furthermore we can assume that these isomorphisms are such that

$$\eta_i(E_{d_{i-1}+1,d_{i-1}+1}^{(i,i)}) = \epsilon_i, \quad \eta_i(E_{d_{i-1}+m_i+1,d_{i-1}+m_i+1}^{(i,i)}) = \bar{\epsilon}_i \qquad 1 \leq i \leq h$$

and

$$\eta_i(E_{d_{i-1}+1,d_{i-1}+1}^{(i,i)}) = e_i \qquad h+1 \leq i \leq n.$$

At this stage, standard computations show that if we set

$$\Psi(E_{rs}^{(i,i)}) := \eta_i(E_{rs}^{(i,i)}) \qquad 1 \leq i \leq n,$$

$$\Psi(E_{d_{p-1}+1,d_{q-1}+1}^{(p,q)}) := \begin{cases} \epsilon_p v_{p,p+1} \epsilon_{p+1} \cdots \epsilon_{q-1} v_{q-1,q} \epsilon_q & 1 \leq p < q \leq h; \\ e_p v_{p,p+1} e_{p+1} \cdots e_{q-1} v_{q-1,q} e_q & h+1 \leq p < q \leq n; \\ \epsilon_p v_{p,p+1} \epsilon_{p+1} \cdots \epsilon_h v_{h,h+1} e_{h+1} \cdots e_{q-1} v_{q-1,q} e_q & 1 \leq p \leq h < q, \end{cases}$$

$$\Psi(E_{d_{p-1}+m_p+1,d_{q-1}+m_q+1}^{(p,q)}) := \bar{\epsilon}_p v_{p,p+1} \bar{\epsilon}_{p+1} \cdots \bar{\epsilon}_{q-1} v_{q-1,q} \bar{\epsilon}_q \qquad 1 \leq p < q \leq h$$

and

$$\Psi(E_{d_{p-1}+m_p+1,d_{q-1}+1}^{(p,q)}) := \bar{\epsilon}_p v_{p,p+1} \bar{\epsilon}_{p+1} \cdots \bar{\epsilon}_h v_{h,h+1} e_{h+1} \cdots e_{q-1} v_{q-1,q} e_q$$

for all $1 \leq p \leq h < q$, then Ψ is actually an isomorphism of algebras with action.

STEP III: Construction of $\mathcal{R}(A')$. The last step which completes our preliminary work consists into proving that $\mathcal{C}(A')$ is isomorphic (as a superalgebra) to a homogeneous subalgebra $\mathcal{R}(A')$ of R endowed with the elementary grading induced by the map $\beta : \Gamma \longrightarrow \mathbb{Z}_2$ such that

$$\beta(x) := \alpha(x) \qquad x \in \cup_{i=h+1}^n \mathrm{Bl}_i$$

and

$$\beta(x) := \begin{cases} 0 & x \in \mathrm{Bl}_i^{(1)}; \\ 1 & x \in \mathrm{Bl}_i^{(2)} \end{cases}$$

for every $1 \leq i \leq h$. To this end, take the linear map $\eta : \mathcal{C}(A') \longrightarrow R$ which is defined on the homogeneous basis of $\mathcal{C}(A')$ presented at Step II in the following manner:

$$\eta(E_{rs}^{(p,q)} + E_{\sigma(r),\sigma(s)}^{(p,q)}) := E_{rs}^{(p,q)} + E_{\sigma(r),\sigma(s)}^{(p,q)},$$

$$\eta(E_{rs}^{(p,q)} - E_{\sigma(r),\sigma(s)}^{(p,q)}) := E_{r,\sigma(s)}^{(p,q)} + E_{\sigma(r),s}^{(p,q)}$$

when $1 \leq p \leq q \leq h$ and $r \in \mathrm{Bl}_p^{(1)}$ and $s \in \mathrm{Bl}_q^{(1)}$,

$$\eta(E_{rs}^{(p,q)}) := E_{rs}^{(p,q)} \qquad h+1 \leq p \leq q \leq n \text{ and } r \in \mathrm{Bl}_p, s \in \mathrm{Bl}_q$$

and

$$\eta(E_{rs}^{(p,q)} + E_{\sigma(r),s}^{(p,q)}) := E_{rs}^{(p,q)}, \qquad \eta(E_{rs}^{(p,q)} - E_{\sigma(r),s}^{(p,q)}) := E_{\sigma(r),s}^{(p,q)}$$

if $1 \leq p \leq h < q \leq n$ and $r \in \mathrm{Bl}_p^{(1)}$ and $s \in \mathrm{Bl}_q$. It is straightforward to verify that η actually defines an embedding of $\mathcal{C}(A')$ into (R, β).

Before attacking the proof of Theorem 3.1, we need to explore in more details the properties of the superalgebra $\mathcal{R}(A')$. To this end, let us denote by $\mathcal{R}(A')^{(1,h)}$ the homogeneous subalgebra of $\mathcal{R}(A')$ consisting of the elements of $\mathcal{R}(A')$ which are linear combination of the matrices $E_{rs}^{(p,q)}$ with $1 \leq p \leq q \leq h$ endowed with the grading induced by the restriction of the map β to the set $\mathrm{Bl}_1 \cup \ldots \cup \mathrm{Bl}_h$. Obviously $\mathcal{R}(A')^{(1,h)}$ is isomorphic as a superalgebra to $A'^{(1,h)}$. In analogous manner let us define the homogeneous subalgebras $\mathcal{R}(A')^{(h+1,n)}$ of $\mathcal{R}(A')$ (which is clearly isomorphic to $A'^{(h+1,n)} = A^{(h+1,n)}$) and $\mathcal{C}(A')^{(1,h)}$ of $\mathcal{C}(A')$.

Lemma 3.3. *The superalgebra $\mathcal{R}(A')^{(1,h)}$ is a $\mathbb{Z}_2$-regular subalgebra of M_{d_h}.*

Proof. Using the notations introduced before of Definition 2.4, it is clear that $F\langle Y \cup Z\rangle / T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)})$ is the subalgebra of $M_{d_h} \otimes P(\mathcal{R}(A')^{(1,h)})$ generated by the matrices

$$u_k := \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} (E_{ij}^{(p,q)} + E_{\sigma(i),\sigma(j)}^{(p,q)}) u_{ij}^{(k)}$$

and

$$v_k := \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} (E_{i,\sigma(j)}^{(p,q)} + E_{\sigma(i),j}^{(p,q)}) v_{ij}^{(k)},$$

where $u_{ij}^{(k)}$ and $v_{ij}^{(k)}$ are variables of $P(\mathcal{R}(A')^{(1,h)})$ and $k \in \mathbb{N}$.

At this stage, it is immediate to see that

$$\hat{\pi}_0(u_k) = \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} E_{ij}^{(p,q)} u_{ij}^{(k)} \quad \text{and} \quad \hat{\pi}_0(v_k) = \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} E_{i,\sigma(j)}^{(p,q)} v_{ij}^{(k)},$$

whereas

$$\hat{\pi}_1(u_k) = \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} E_{\sigma(i),\sigma(j)}^{(p,q)} u_{ij}^{(k)} \quad \text{and} \quad \hat{\pi}_1(v_k) = \sum_{i \in \mathrm{Bl}_p^{(1)}\, j \in \mathrm{Bl}_q^{(1)}} E_{\sigma(i),j}^{(p,q)} v_{ij}^{(k)}$$

and, consequently, that $\hat{\pi}_0$ and $\hat{\pi}_1$ are injective. □

Lemma 3.4. $T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)}) = T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h)$.

Proof. It is easily seen that $\mathcal{C}(A')^{(1,h)}$ (and hence $\mathcal{R}(A')^{(1,h)}$) is isomorphich as an algebra with action (of an automorphism of order 2) to

$$S := UT(m_1,\ldots,m_h) \oplus UT(m_1,\ldots,m_h)$$

with action given by

$$\phi(a,b) = (b,a).$$

For this graded algebra one clearly has that

$$S^{(0)} = \{(a,a) \mid a \in UT(m_1,\ldots,m_h)\}$$

and

$$S^{(1)} = \{(a,-a) \mid a \in UT(m_1,\ldots,m_h)\}.$$

Now, consider a polynomial $f(y_1,\ldots,y_r,z_1,\ldots,z_s) \in T_{\mathbb{Z}_2}(\mathcal{C}(A')^{(1,h)}) = T_{\mathbb{Z}_2}(S)$ and the element $f(x_1,\ldots,x_r,x_{r+1},\ldots,x_{r+s})$ of the free associative F-algebra $F\langle X\rangle$ freely generated by the countable set $X := \{x_1,x_2,\ldots\}$ (roughly speaking, we are replacing graded variables with ungraded ones). Let $\sigma : F\langle X\rangle \longrightarrow UT(m_1,\ldots,m_h)$ be an evaluation of the polynomial $f(x_1,\ldots,x_{r+s})$. In particular, say $\sigma(x_i) := t_i$ for every integer i. Then the graded evaluation $\bar{\sigma} : F\langle Y \cup Z\rangle \longrightarrow S$ defined by

$$\bar{\sigma}(y_i) := (t_i,t_i), \qquad \bar{\sigma}(z_j) := (t_{r+j},-t_{r+j})$$

for every $1 \le i \le r$ and $1 \le j \le s$ is such that

$$\bar{\sigma}(f) = (f(t_1,\ldots,t_r,t_{r+1},\ldots,t_{r+s}), f(t_1,\ldots,t_r,-t_{r+1},\ldots,-t_{r+s})).$$

But $\bar{\sigma}(f(y_1,\ldots,y_r,z_1,\ldots,z_s)) = 0_S$ since $f(y_1,\ldots,y_r,z_1,\ldots,z_s) \in T_{\mathbb{Z}_2}(S)$. This implies that $f(t_1,\ldots,t_r,t_{r+1},\ldots,t_{r+s})$ is zero. Hence $f(x_1,\ldots,x_{r+s})$ is a polynomial identity for $UT(m_1,\ldots,m_h)$. According to Theorem 2 of [10], $f(x_1,\ldots,x_{r+s}) \in \mathrm{Id}(M_{m_1})\cdots \mathrm{Id}(M_{m_h})$ and, by invoking Remark 5.2 of [5], we get

$$f(y_1,\ldots,y_r,z_1,\ldots,z_s) \in T_{\mathbb{Z}_2}(\Delta_1)\cdots T_{\mathbb{Z}_2}(\Delta_h) = T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h).$$

Hence $T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)}) = T_{\mathbb{Z}_2}(\mathcal{C}(A')^{(1,h)}) \subseteq T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h)$. But by Lemma 2.2 one has that

$$T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h) \subseteq T_{\mathbb{Z}_2}({A'}^{(1,h)}) = T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)}),$$

and, consequently, that $T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)}) = T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h)$. □

We are now in a position to prove the first of our main results.

Proof of Theorem 3.1. Suppose that $A_1,\ldots,A_h$ are non-simple graded simple. From the above discussion (and using exactly the same notations) it follows that there exists a homogeneous subalgebra A' of A isomorphic to a homogeneous subalgebra $\mathcal{R}(A')$ of (R,β). In particular, $\mathcal{R}(A')$ can be written as

$$\begin{pmatrix} V & U \\ 0 & W \end{pmatrix}$$

where $V = \mathcal{R}(A')^{(1,h)}$, $W = \mathcal{R}(A')^{(h+1,n)}$ and $U = M_{d_h \times (d_n - d_h)}$. According to Lemma 3.3, V is $\mathbb{Z}_2$-regular. Hence we can apply Lemma 2.5 and conclude that $T_{\mathbb{Z}_2}(\mathcal{R}(A'))$ is factored as $T_{\mathbb{Z}_2}(V)\cdot T_{\mathbb{Z}_2}(W)$. As $A'\subseteq A$, the combination of Lemmas 3.4 and 2.2 with the fact that $A'^{(h+1,n)}=A^{(h+1,n)}$ yields

$$\begin{aligned}T_{\mathbb{Z}_2}(A)\subseteq T_{\mathbb{Z}_2}(A')=T_{\mathbb{Z}_2}(\mathcal{R}(A'))&=T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(1,h)})\cdot T_{\mathbb{Z}_2}(\mathcal{R}(A')^{(h+1,n)})\\&=T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h)\cdot T_{\mathbb{Z}_2}(A'^{(h+1,n)})\\&=T_{\mathbb{Z}_2}(A_1)\cdots T_{\mathbb{Z}_2}(A_h)\cdot T_{\mathbb{Z}_2}(A^{(h+1,n)})\\&\subseteq T_{\mathbb{Z}_2}(A^{(1,h)})\cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}).\end{aligned}$$

The reverse containment is a direct consequence of Lemma 2.2. In particular, $T_{\mathbb{Z}_2}(A)=T_{\mathbb{Z}_2}(A')$.

The case in which $A_1,\dots,A_h$ are simple graded simple uses the same above arguments and its discussion is left to the reader. $\square$

We want to apply Theorem 3.1 to the classification of minimal supervarieties of fixed superexponent. We recall the definition.

Definition 3.5. *A variety $\mathcal{V}^{sup}$ of PI associative superalgebras is said to be* minimal of superexponent d *if* $\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup})=d$ *and* $\exp_{\mathbb{Z}_2}(\mathcal{U}^{sup})<d$ *for every proper subvariety $\mathcal{U}^{sup}$ of $\mathcal{V}^{sup}$.*

In the case in which $\mathcal{V}^{sup}$ is a minimal supervariety of finite basic rank (that is, generated by a finitely generated PI superalgebra), it has been proved in Proposition 3.2 of [5] that $\mathcal{V}^{sup}=\mathrm{supvar}(A)$ for a suitable minimal superalgebra A. The question which remains still open is to decide which minimal superalgebras generate minimal supervarieties of fixed superexponent. In this direction it has been proved that $\mathrm{supvar}(A)$ is minimal when either all the summands of the maximal semisimple homogeneous subalgebra A_{ss} of A are simple graded simple (Theorem 4.7 of [5]) or A_{ss} has exactly two graded simple components (Theorem 5.4 of [5]). Here we show that the same occurs when A satisfies the conditions of Theorem 3.1.

Theorem 3.6. *Let $A=A_{ss}+J$ be a minimal superalgebra. If $A_{ss}=A_1\oplus\cdots\oplus A_n$ and there exists $1\le h\le n$ such that $A_1,\dots,A_h$ are non-simple graded simple and $A_{h+1},\dots,A_n$ are simple graded simple algebras (or conversely), then the supervariety generated by A is minimal of superexponent* $\dim_F(A_1\oplus\cdots\oplus A_n)$.

Proof. Set $\mathcal{V}^{sup}:=\mathrm{supvar}(A)$ and let us consider a subvariety $\mathcal{U}^{sup}\subseteq\mathcal{V}^{sup}$ such that $\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup})=\exp_{\mathbb{Z}_2}(\mathcal{U}^{sup})$. Since $\mathcal{V}^{sup}$ satisfies some Capelli identities, $\mathcal{U}^{sup}$ has finite basic rank (see Theorem 11.4.3 of [12]). Hence, by a result of Kemer, $\mathcal{U}^{sup}$ is generated by a finite-dimensional superalgebra $\hat{B}$. According to Lemma 8.1.4 of [12], there exists a minimal superalgebra B such that $T_{\mathbb{Z}_2}(\hat{B})\subseteq T_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(\hat{B})=\exp_{\mathbb{Z}_2}(B)$. Therefore $T_{\mathbb{Z}_2}(A)\subseteq T_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(A)=\exp_{\mathbb{Z}_2}(B)$ as well. Furthermore from Lemma 3.3 of [5] we know that $B_{ss}=A_1\oplus\cdots\oplus A_n$.

Assume that $A_1,\dots,A_h$ are non-simple graded simple and take the homogeneous subalgebra B' of B constructed in the same way we did in order to prove Theorem 3.1. Using the same notations adopted there, B' is the subalgebra generated by $A_1, \dots, A_n$ and the homogeneous radical elements $v_{12}, \dots, v_{n-1,n}$. For this minimal superalgebra one has that

$$T_{\mathbb{Z}_2}(B) = T_{\mathbb{Z}_2}(B') \quad \text{and} \quad A_{ss} = B'_{ss}.$$

Therefore $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B')$ and $\exp_{\mathbb{Z}_2}(A) = \exp_{\mathbb{Z}_2}(B')$.

We claim that $T_{\mathbb{Z}_2}(A^{(h+1,n)}) \subseteq T_{\mathbb{Z}_2}(B'^{(h+1,n)})$. Assume, if possible, that the inclusion does not hold and take a graded polynomial $f \in T_{\mathbb{Z}_2}(A^{(h+1,n)}) \setminus T_{\mathbb{Z}_2}(B'^{(h+1,n)})$. Hence, for every $h+1 \le i \le j \le n$, there exist elements $b_{ij} \in B'_{ij}$ and a graded evaluation σ of f in $B'^{(h+1,n)}$ such that $\sigma(f) = \sum_{i,j} b_{ij} \neq 0_{B'^{(h+1,n)}}$. Let us pick a pair of indices $i \le j$ such that $b_{ij} \neq 0_{B'_{ij}}$. Then $1_{A_i} \cdot \sigma(f) \cdot 1_{A_j} = b_{ij}$ and $b_{ij} \in A_i e_i v_{i,i+1} \cdots v_{j-1,j} e_j A_j$. By multiplying b_{ij} for suitable homogeneous elements $c_l^{(ij)}$ and $d_l^{(ij)}$ (with $(c_l^{(ij)}, d_l^{(ij)}) \in (A_i, A_j)$), we get

$$\sum_l c_l^{(ij)} b_{ij} d_l^{(ij)} = e_i v_{i,i+1} \cdots v_{j-1,j} e_j. \tag{3}$$

Let us consider the graded polynomial

$$g := \sum_l v_l^{(ij)} x' f x'' t_l^{(ij)},$$

where $x' := y' + z'$ and $x'' = y'' + z''$ are sum of a homogeneous variable of degree 0 and one of degree 1 not appearing in f and $v_l^{(ij)}$ and $t_l^{(ij)}$ are distinct graded variables pairwise different from those appearing in f, x' and x'' and of the same degree of $c_l^{(ij)}$ and $d_l^{(ij)}$, respectively. Then g is an element of $T_{\mathbb{Z}_2}(A^{(h+1,n)})$ (because $f \in T_{\mathbb{Z}_2}(A^{(h+1,n)})$) which has a graded evaluation in $B'^{(h+1,n)}$ coinciding with (3).

Now, the set $T_{\mathbb{Z}_2}(A^{(1,h)}) \setminus T_{\mathbb{Z}_2}(B'^{(1,h+1)})$ is non-empty (otherwise $\dim_F(A_1 \oplus \cdots \oplus A_h) = \exp_{\mathbb{Z}_2}(A^{(1,h)}) \ge \exp_{\mathbb{Z}_2}(B'^{(1,h+1)}) = \dim_F(A_1 \oplus \cdots \oplus A_h \oplus A_{h+1})$, which is clearly false). Hence there exists a graded polynomial $p \in T_{\mathbb{Z}_2}(A^{(1,h)})$ which has a non-zero graded evaluation in $B'^{(1,h+1)}$ (and whose variables are pairwise different from those involved in g). By using similar arguments to the above, we can assume that such a non-zero graded evaluation coincides with $\epsilon_r v_{r,r+1} \cdots v_{s-1,s} \epsilon_s$ or $\bar{\epsilon}_r v_{r,r+1} \cdots v_{s-1,s} \bar{\epsilon}_s$ or $\epsilon_r v_{r,r+1} \cdots v_{h,h+1} e_{h+1}$ or $\bar{\epsilon}_r v_{r,r+1} \cdots v_{h,h+1} e_{h+1}$ or with e_{h+1} $(1 \le r \le s \le h)$.

At this stage, let us consider the graded polynomial $q := pxg$ where $x := y + z$ is sum of a variable of degree 0 and one of degree 1 appearing neither in g nor in p. According to Lemma 2.2,

$$q \in T_{\mathbb{Z}_2}(A^{(1,h)}) \cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}) \subseteq T_{\mathbb{Z}_2}(A),$$

but it is not a graded polynomial identity for B', which contradicts the original assumption. Therefore $T_{\mathbb{Z}_2}(A^{(h+1,n)}) \subseteq T_{\mathbb{Z}_2}(B'^{(h+1,n)})$.

On the other hand, the minimal superalgebras $A^{(h+1,n)}$ and $B'^{(h+1,n)}$ have the same superexponent, namely $\dim_F(A_{h+1} \oplus \cdots \oplus A_n)$. Since $A_{h+1}, \dots, A_n$ are simple graded simple, we can directly apply Theorem 4.7 of [5] to conclude that $T_{\mathbb{Z}_2}(A^{(h+1,n)}) = T_{\mathbb{Z}_2}(B'^{(h+1,n)})$ (actually $A^{(h+1,n)}$ and $B'^{(h+1,n)}$

are isomorphic). Therefore it follows from Theorem 3.1 that

$$T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(A_1) \cdots T_{\mathbb{Z}_2}(A_h) \cdot T_{\mathbb{Z}_2}(A^{(h+1,n)}) = T_{\mathbb{Z}_2}(B') = T_{\mathbb{Z}_2}(B).$$

The case in which $A_1, \ldots, A_h$ are simple graded simple can be proved using the same arguments. □

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