EXISTENCE AND UNIQUENESS OF SOLUTIONS TO PARABOLIC EQUATIONS WITH SUPERLINEAR HAMILTONIANS

ANDREA DAVINI

Abstract. We give a proof of existence and uniqueness of viscosity solutions to parabolic quasilinear equations for a fairly general class of nonconvex Hamiltonians with superlinear growth in the gradient variable. The approach is mainly based on classical techniques for uniformly parabolic quasilinear equations and on the Lipschitz estimates provided in [1], as well as on viscosity solution arguments.

Introduction

In this paper we prove existence and uniqueness of viscosity solutions to a parabolic quasilinear equation of the form

$$\partial_t u - \mathrm{tr}(A(x)D_x^2 u) + H(x, D_x u) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d \tag{1}$$

subject to bounded uniformly continuous initial data. Here A is a $d\times d$ symmetric and positive semi-definite matrix with Lipschitz and bounded coefficients, and the Hamiltonian H is a locally Lipschitz function on $\mathbb{R}^d\times\mathbb{R}^d$, which has superlinear growth in the gradient variable but is not necessarily convex. The precise conditions we assume on H will be discussed later. Our interest for this issue originates from our recent work [10], where this type of results are needed for the study of related homogenization problems.

Existence and uniqueness results for equations of this kind are usually derived either via the classical approach to quasilinear parabolic equations, or from suitable comparison principles for semicontinuous viscosity sub and supersolutions through a standard application of Perron's method.

The classical parabolic theory yields existence and uniqueness of classical solutions provided the diffusion matrix A is regular enough and uniformly positive definite, and the nonlinearity H grows at most quadratically with respect to the gradient variable, see [13, Chapter V, §8].

The second approach is, on the other hand, more flexible, but the comparison results available in literature are usually proved under a uniform continuity condition on H of the form

$$|H(x,p)-H(y,p)| \leqslant \omega\big((1+|p|)|x-y|\big) \qquad \text{for all } x,y,p\in\mathbb{R}^d,$$

for some continuity modulus ω, see for instance [7, hypothesis (3.14)], [3, hypothesis (H2)], [4, hypothesis (H1)]. Such a condition is typically not satisfied by Hamiltonians with superlinear growth in p as soon as the dependence in x and p is not decoupled. The case of Hamiltonians with superlinear growth in p of polynomial type has been specifically addressed in [8, 9] for a class of equations and of initial data that includes ours as a special instance. The Hamiltonians therein considered may also depend on t and are not uniformly superlinear with respect to x, but unfortunately the techniques employed allow the authors to treat only the case of H that is either convex in p, as in [9], or the sum of a convex and a concave one, where either one of the two grows at most linearly with respect to p, see [8] and [9, Remark 2.1].

Date: Submitted Version February 6, 2017.

2010 *Mathematics Subject Classification.* 35K59, 35B40, 35B51.

Key words and phrases. quasilinear parabolic equation, nonconvex Hamiltonian, viscosity solution, comparison principle.

Several results holding for viscous Hamilton–Jacobi equations have been recently presented in [1] for a fairly general class of t–independent Hamiltonians with superlinear growth in p. The precise conditions assumed on H are the hypotheses (H3) and (H4) with $\mu = +\infty$ listed in Section 1.2 below. Stationary Hamilton–Jacobi equations are also considered, but we will restrict our discussion here to the parabolic case. The authors prove two kind of results: comparison principle for semicontinuous sub and supersolutions of (1) with, let us say, sublinear growth at infinity, see [1, Theorem 2.3]; and interior Lipschitz estimates for continuous solutions of (1) whose time–derivative satisfies a uniform bound from below, see [1, Proposition 3.5] or Proposition 1.6 in the next section. The comparison principle is proved by employing techniques close to the ones used in [9]. For this, it is crucial to additionally assume H convex in p. On the contrary, the Lipschitz estimates are independent of this convexity condition, which is therefore dropped. Moreover, the authors provide a quantitative estimate of such Lipschitz constants in terms of the parameters that appear in the structural hypotheses (H3)–(H4) below. This is very convenient when one is, for instance, interested in approximating a given Hamiltonian in this class.

The present work is aimed at removing the convexity condition on H from the existence and uniqueness part of the quoted results of [1]. The existence results are herein established under the regime of conditions (H3)–(H4), while the uniqueness is obtained by proving suitable comparison principle for semicontinuous sub and supersolutions to (1) with sublinear growth at infinity. In the case of uniformly continuous Hamiltonians, i.e. when (H4) holds with constants a_r, M_r independent of r, such a comparison principle follows rather easily from the existence part. In this instance, in fact, the solutions constructed in the first part are globally Lipschitz in $[0,T] \times \mathbb{R}^d$ whenever the initial datum belongs to $\mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ and in order to compare them with a semicontinuous sub or supersolution we just need a mild uniform continuity property on H, which holds true in view of conditions (H3) and (1.5) in (H4), see Proposition 1.4. By exploiting the density of such initial data in the class of bounded uniformly continuous functions, a general comparison principle for semicontinuous sub and supersolutions is finally derived, see Theorem 3.1.

When the Hamiltonian is not uniformly superlinear, this idea can no longer be applied since the solutions are, in the best case scenario, only locally Lipschitz in $[0,T] \times \mathbb{R}^d$. To deal with this case, we revisit the arguments employed in [1, Section 2] and propose a minor generalization of [1, Theorem 2.3] for Hamiltonians that satisfy (H3)–(H4) with $\mu = +\infty$ and that can be written as the pointwise infimum of a collection of convex Hamiltonians $\{H_i\}_{i \in \mathcal{I}}$ of same type, where the constants that appear in the structural conditions do not depend on the index i, see Theorem 3.3. Actually, we allow the associated exponents m to possibly depend on i, and we remark that we do not need to assume neither condition (1.5) nor even continuity with respect to x for such H_i. This can be useful for applications, see Example 3.7.

The existence part is the core of this work. Our approach mimic the classical one for uniformly parabolic quasilinear equations, based on the use of the Schauder fixed point Theorem and on suitable *a priori* L^∞ and Hölder estimates on the gradient of the solutions, with the difference that, in order to have the necessary compactness to apply these tools, we approximate (1) with a sequence of periodic parabolic equations of the same type with diverging size of periodicity. The advantage is that, in this way, we just need *a priori interior* L^∞ and Hölder estimates on the gradient of the solutions for an equation of the form (1). For the former we directly apply [1, Proposition 3.5], while for the latter we use more classical results, see [13, Chapter VI, Theorem 1.1]. The fact that we have an explicit expression for such L^∞ bounds is crucial for the remainder of the proof. We stress that conditions (H3)–(H4) could be replaced by any other set of assumptions yielding similar L^∞ bounds, but it is important to have an explicit expression for them in order to be able to control the local Lipschtiz constants of the approximating solutions that intervene in the limiting procedures we bring into play.

The arguments we employ are not new and are certainly known to some experts, see for instance [5, Section 4] or [15, Section 3], however we could not locate in literature any reference where the issues herein considered have been proved in this generality, at least as far as the case of uniformly superlinear Hamiltonians is concerned. Our main motivation to write this note was to provide a reference for this kind of results. We hope this work could be useful for other researchers working in this domain.

Plan of the paper. Section 1 contains some preliminary material. In Section 1.1 we fix notation and define the functional spaces we use in the paper. In particular, we define the Hölder and parabolic Hölder spaces and their norms, and recall an interpolation inequality and a compact immersion result we will need for the existence part. Section 1.2 contains our standing assumptions on the diffusion matrix A and on the Hamiltonian H and some viscosity solution preliminaries. The existence results are derived in Section 2. In Section 2.1 we deal with the uniformly parabolic case, while in Section 2.2 we derive the existence result in the general case. The uniqueness part is treated in Section 3. In Section 3.1 we deal with the uniformly superlinear case, while Section 3.2 is devoted to the case of non–uniformly superlinear Hamiltonians. The proof of the comparison principle stated in Theorem 3.3 is postponed to the Appendix. In Section 3.3 we give some examples of non–uniformly superlinear Hamiltonians covered by our study.

Acknowledgements. – I am grateful to Panagiotis Souganidis for a brief but illuminating discussion we had at the conference *Hamilton-Jacobi Equations: new trends and applications* held at Rennes (France), 30 May-3 June, 2016. I would also like to thank Annalisa Cesaroni and Fabio Punzo for references and suggestions concerning the classical theory of parabolic equations, Elena Kosygina for her useful comments on earlier versions of this paper, Scott Armstrong and Hung Tran for their prompt replies to my queries about their joint work [1] and related issues.

1. Preliminaries

1.1. **Notation and functional spaces.** Throughout the paper, we will denote by $\mathbb{N}$ and $\mathbb{N}_0$ the set of positive and nonnegative integer numbers, respectively. We will denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the scalar product and the Euclidean norm on $\mathbb{R}^d$, where d is a positive integer number. We will denote by $B_r(x_0)$ and B_r the open balls in $\mathbb{R}^d$ of radius r centered at x_0 and 0, respectively. For a given a subset E of $\mathbb{R}^d$ or of $\mathbb{R}^{d+1}$, we will denote by $\overline{E}$ its closure.

Given a metric space X, we will write $\varphi_n \rightrightarrows_{\text{loc}} \varphi$ on X to mean that the sequence of functions $(\varphi_n)_n$ uniformly converges to φ on compact subsets of X. We will denote by $\mathrm{C}(X)$, $\mathrm{UC}(X)$, $\mathrm{LSC}(X)$, $\mathrm{USC}(X)$ the space of continuous, uniformly continuous, lower semicontinuous, upper semicontinuous real functions on the metric space X, respectively. We will add the subscript b to those spaces to mean that we are considering functions that are also bounded on X.

Given an open subset Ω of either $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$ and a measurable function $g : \Omega \to \mathbb{R}$, we will denote by $\|g\|_{L^\infty(\Omega)}$ its usual L^∞–norms. We will denote $L^\infty(\Omega)$ the space of essentially bounded functions on Ω, and by $W^{k,\infty}(\Omega)$ the space of functions $u \in L^\infty(\Omega)$ having essentially bounded distributional derivatives up to order $k \in \mathbb{N}$, inclusively.

Let D be a smooth domain of $\mathbb{R}^d$ and $k \in \mathbb{N}$. We will denote by $\mathrm{C}^k(D)$ the space of continuous functions $u : D \to \mathbb{R}$ that are differentiable in D with continuous derivatives up to order k inclusively, and by $\mathrm{C}^\infty(D) := \bigcap_{k\in\mathbb{N}} \mathrm{C}^k(D)$. We will denote by $\mathrm{C}^k(\overline{D})$ the space of continuous functions $u : \overline{D} \to \mathbb{R}$ that are differentiable in D with continuous derivatives on $\overline{D}$ up to order k inclusively. In what follows, the letter s refers to a multi–index, namely $s = (s_1, \dots, s_d) \in (\mathbb{N}_0)^d$, the symbol $|s|$ refers to the quantity $s_1 + \cdots + s_d$, and with the symbol $D^s u$ or $D^s_x u$ we mean $\partial^{s_1}_{x_1} \dots \partial^{s_d}_{x_d} u$.

Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. For $u \in \mathrm{C}^k(\overline{D})$ we set

$$\|u\|_{H^{k+\alpha}(D)} := \sum_{|s|\leqslant k} \|D^s u\|_{L^\infty(D)} + \sum_{|s|=k} [D^s u]_D^{(\alpha)}, \tag{1.2}$$

with

$$[\varphi]_D^{(\alpha)} := \sup_{\substack{x,y\in D\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}.$$

We define

$$H^{k+\alpha}(\overline{D}) := \{u \in \mathrm{C}^k(\overline{D}) \,:\, \|u\|_{H^{k+\alpha}(D)} < +\infty\}.$$

The Hólder space $H^{k+\alpha}(\overline{D})$, endowed with the norm (1.2), is a Banach space, see [13].

We record here for later use the following density result.

Lemma 1.1. *The space of functions* $\mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ *is dense in* $\mathrm{UC}_b(\mathbb{R}^d)$ *with respect to the* $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$ *norm.*

Proof. Since $W^{1,\infty}(\mathbb{R}^d)$ is dense in $\mathrm{UC}_b(\mathbb{R}^d)$ with respect to the $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$ norm, see for instance [11, Theorem 1], it is enough to show that any Lipschitz and bounded function $g : \mathbb{R}^d \to \mathbb{R}$ can be uniformly approximated in $\mathbb{R}^d$ by functions in $\mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. But this readily follows by regularizing g via a convolution with a standard mollification kernel. □

For a given $T > 0$ and a smooth domain D of $\mathbb{R}^d$, we will denote by D_T the set $(0,T)\times D$. We will denote by $\mathrm{C}^{k/2,k}(\overline{D}_T)$ the space of functions $u : \overline{D}_T \to \mathbb{R}$ that are continuous in $\overline{D}_T$ together with all derivatives of the form $\partial_t^r D_x^s u$ for $2r + |s| \leqslant k$.

Let $\alpha \in (0,1)$. For $\psi \in \mathrm{C}(\overline{D}_T)$, we set $[\psi]_{D_T}^{(\alpha)} := [\psi]_{t,D_T}^{(\alpha/2)} + [\psi]_{x,D_T}^{(\alpha)}$, where

$$[\psi]_{t,D_T}^{(\alpha/2)} := \sup_{x\in D} \|\psi(\cdot,x)\|_{H^{\alpha/2}((0,T))}, \qquad [\psi]_{x,D_T}^{(\alpha)} := \sup_{0<t<T} \|\psi(t,\cdot)\|_{H^\alpha(D)}.$$

We introduce the following norms:

$$\|u\|_{H^{\alpha/2,\alpha}(D_T)} := \|u\|_{L^\infty(D_T)} + [u]_{D_T}^{(\alpha)},$$

$$\|u\|_{H^{(1+\alpha)/2,1+\alpha}(D_T)} := \|u\|_{L^\infty(D_T)} + \sum_{i=1}^d \|\partial_{x_i} u\|_{H^{\alpha/2,\alpha}(D_T)} + [u]_{t,D_T}^{(\frac{1+\alpha}{2})},$$

$$\|u\|_{H^{(2+\alpha)/2,2+\alpha}(D_T)} := \|u\|_{L^\infty(D_T)} + \sum_{i=1}^d \|\partial_{x_i} u\|_{H^{(1+\alpha)/2,1+\alpha}(D_T)} + \|\partial_t u\|_{H^{\alpha/2,\alpha}(D_T)}.$$

For $k \in \{0,1,2\}$, we define

$$H^{(k+\alpha)/2,k+\alpha}(\overline{D}_T) := \{u \in \mathrm{C}^{k/2,k}(\overline{D}_T) \,:\, \|u\|_{H^{(k+\alpha)/2,k+\alpha}(D_T)} < +\infty\}.$$

The parabolic Hölder space $H^{k+\alpha/2,k+\alpha}(\overline{D}_T)$, endowed with the norm $\|\cdot\|_{H^{(k+\alpha)/2,k+\alpha}(D_T)}$, is a Banach space, see [13].

In the sequel we will often write

$$\|D_x u\| := \sum_{i=1}^d \|\partial_{x_i} u\|, \qquad \|D_x^2 u\| := \sum_{i,j=1}^d \|\partial^2_{x_i x_j} u\|,$$

where u is a real function defined either on D or on D_T and $\|\cdot\|$ is a norm.

We record the following result for further use:

Proposition 1.2. *Let D be an open and convex subset of $\mathbb{R}^d$, $T > 0$ and $\alpha \in (0,1)$. There exists a constant $N = N(d,D)$ such that for any $\varepsilon > 0$ and $u \in H^{(2+\alpha)/2,2+\alpha}(\overline{D}_T)$ we have*

$$\|D_x u\|_{H^{\alpha/2,\alpha}(D_T)} \leqslant 3\varepsilon \|u\|_{H^{(2+\alpha)/2,2+\alpha}(D_T)} + N \max\{\varepsilon^{-1/(1+\alpha)}, \varepsilon^{-(1+\alpha)}\} \|u\|_{L^\infty(D_T)}.$$

Proof. We apply [12, §8.8, Theorem 8.8.1]. The assertion follows by summing the inequalities (8.8.3) and (8.8.4) and by noticing that

$$[\partial_{x_i} u]_{\alpha/2,\alpha;D_T} \geqslant \frac{1}{2}[\partial_{x_i} u]^{(\alpha)}_{D_T}, \qquad [u]_{1+\alpha/2,2+\alpha;D_T} \leqslant \|u\|_{H^{(2+\alpha)/2,2+\alpha}(D_T)}.$$

□

For $n \in \mathbb{N}$, we will denote by $\mathrm{C}^k_n(\mathbb{R}^d)$, $\mathrm{C}^{k/2,k}_n([0,T]\times\mathbb{R}^d)$, $H^{(k+\alpha)/2,k+\alpha}_n([0,T]\times\mathbb{R}^d)$ the subspace of $\mathrm{C}^k(\mathbb{R}^d)$, $\mathrm{C}^{k/2,k}([0,T]\times\mathbb{R}^d)$, $H^{(k+\alpha)/2,k+\alpha}([0,T]\times\mathbb{R}^d)$, respectively, made up of functions that are $n\mathbb{Z}^d$–periodic in $\mathbb{R}^d$ with respect to the x–variable. We record for later use the following result, that can be easily proved with the aid of Ascoli–Arzelà Theorem.

Proposition 1.3. *Let $n \in \mathbb{N}$, $\alpha \in (0,1)$ and $T > 0$. The bounded subsets of the space $H^{(1+\alpha)/2,1+\alpha}_n([0,T]\times\mathbb{R}^d)$ are precompact in $H^{\alpha/2,\alpha}_n([0,T]\times\mathbb{R}^d)$.*

1.2. **Viscosity solution theory.** In this paper we will consider parabolic quasilinear equations of the form

$$\partial_t u - \mathrm{tr}(A(x)D^2_x u) + H(x, D_x u) = 0 \quad \text{in } (0,T)\times U, \tag{1.3}$$

where $T > 0$ and U is an open subset of $\mathbb{R}^d$. The diffusion matrix $A(x)$ is a positive semidefinite symmetric $d\times d$ matrix, depending on $x \in \mathbb{R}^d$, with bounded and Lipschitz square root, namely $A = \sigma\sigma^T$ for some $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$, where σ satisfies the following hypotheses fos some fixed constant $\Lambda_A > 0$:

(A1) $|\sigma(x)| \leqslant \Lambda_A$ for every $x \in \mathbb{R}^d$;

(A2) $|\sigma(x) - \sigma(y)| \leqslant \Lambda_A|x-y|$ for every $x, y \in \mathbb{R}^d$.

We emphasize that the diffusion matrix can be degenerate, in general.

The nonlinearity H, henceforth called *Hamiltonian*, is a function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following basic assumptions:

(H1) there exist a continuous, coercive and nondecreasing functions $\Theta : \mathbb{R}_+ \to \mathbb{R}$ and a constant $\mu \in \mathbb{R}$ such that

$$-\mu \leqslant H(x,p) \leqslant \Theta(|p|) \qquad \text{for every } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d;$$

(H2) $H \in \mathrm{UC}(\mathbb{R}^d \times B_r)$ for every $r > 0$.

By coercive, we mean that $\lim\limits_{h\to+\infty} \Theta(h) = +\infty$. The second inequality in (H1) amounts to saying that the Hamiltonian is locally bounded in p, uniformly with respect to x.

In order to obtain Lipschitz estimates for solutions to (1.3), we introduce another set of assumptions on H, holding for constants $m > 1$ and $\mu > 0$:

(H3) $|H(x,p) - H(x,q)| \leqslant \Lambda\left(|p| + |q| + 1\right)^{m-1}|p-q|$ for all $x, p, q \in \mathbb{R}^d$;

(H4) for every $r > 0$, there exist constants $a_r \in (0,1]$ and $M_r \geqslant 1$ such that

$$\max\{-\mu, a_r|p|^m - M_r\} \leqslant H(x,p) \leqslant \Lambda(|p|^m + 1) \tag{1.4}$$

$$|H(x,p) - H(y,p)| \leqslant (\Lambda|p|^m + M_r)|x-y| \tag{1.5}$$

for all $x, y \in B_r$ and $p \in \mathbb{R}^d$.

When the above constants α_r, M_r can be chosen independently of r, we will say that the Hamiltonian is *uniformly superlinear.* Note that, in this instance, one can choose $\mu = +\infty$ in (1.4), as in [1], and that condition (H2) is fulfilled. When on the other hand H is not uniformly superlinear, condition (H2) needs not hold.

Unless otherwise specified, all the differential inequalities in the paper are to be interpreted in the *viscosity* sense, which is the usual notion of weak solution for Hamilton–Jacobi equations. We briefly recall some basic definitions and refer to [2, 7] for further details.

We will say that a function $v \in \mathrm{USC}((0,T)\times U)$ is an (upper semicontinuous) *viscosity subsolution* of (1.3) if, for every $\phi \in \mathrm{C}^2((0,T)\times U)$ such that $v-\phi$ attains a local maximum at $(t_0,x_0)\in(0,+\infty)\times U$, we have

$$\partial_t\phi(t_0,x_0) - \mathrm{tr}\big(A(x_0)D_x^2\phi(t_0,x_0)\big) + H(x_0, D_x\phi(t_0,x_0)) \leqslant 0.$$

Any such test function ϕ will be called *supertangent* to v at (t_0,x_0).

We will say that $w \in \mathrm{LSC}((0,+\infty)\times U)$ is a (lower semicontinuous) *viscosity supersolution* of (1.3) if, for every $\phi \in \mathrm{C}^2((0,T)\times U)$ such that $w-\phi$ attains a local minimum at $(t_0,x_0)\in(0,+\infty)\times\mathbb{R}^d$, we have

$$\partial_t\phi(t_0,x_0) - \mathrm{tr}\big(A(x_0)D_x^2\phi(t_0,x_0)\big) + H(x_0, D_x\phi(t_0,x_0)) \geqslant 0.$$

Any such test function ϕ will be called *subtangent* to w at (t_0,x_0). It is well known, see for instance [2,7], that the notion of sub or supertangent is local, in the sense that the test function ϕ needs to be defined only in a neighborhood of the point (t_0,x_0). A continuous function on $(0,+\infty)\times\mathbb{R}^d$ is a *viscosity solution* of (1.3) if it is both a viscosity sub and supersolution.

The following comparison principle holds:

Proposition 1.4. *Assume that A satisfy (A1)–(A2) and $H \in \mathrm{UC}(U\times B_r)$ for every $r>0$, where U is an open subset of $\mathbb{R}^d$. Let $v \in \mathrm{USC}([0,T]\times\overline{U})$ and $w\in\mathrm{LSC}([0,T]\times\overline{U})$ be, respectively, a sub and a supersolution of (1.3) satisfying*

$$\limsup_{\substack{|x|\to+\infty\\ x\in U}}\ \sup_{t\in[0,T]}\frac{v(t,x)}{1+|x|} \leqslant 0 \leqslant \liminf_{\substack{|x|\to+\infty\\ x\in U}}\ \inf_{t\in[0,T]}\frac{w(t,x)}{1+|x|}. \tag{1.6}$$

Let us furthermore assume that either D_xv or D_xw belongs to $\big(L^\infty((0,T)\times U)\big)^d$. Then

$$v(t,x)-w(t,x) \leqslant \sup_{\partial_P((0,T)\times U)}\big(v-w\big) \qquad \text{for every } (t,x)\in(0,T)\times U,$$

where $\partial_P((0,T)\times U) := \{0\}\times U \cup [0,T)\times\partial U$ is the parabolic boundary of $(0,T)\times U$.

The proof is standard, however we provide it in the Appendix for the reader's convenience.

A first application of the above comparison principle is the following.

Proposition 1.5. *Assume that A satisfy (A1)–(A2) and H satisfies (H1)–(H2). Let $u\in \mathrm{C}_b([0,T]\times\mathbb{R}^d)$ be a solution of (1.3) with $U:=\mathbb{R}^d$ satisfying the initial condition $u(0,\cdot)=g$ for some $g\in\mathrm{W}^{2,\infty}(\mathbb{R}^d)$. Let us furthermore assume that $D_xu\in\big(L^\infty((0,T)\times\mathbb{R}^d)\big)^d$. Then there exists a constant κ, only depending on $\|Dg\|_{L^\infty(\mathbb{R}^d)}$, $\|D^2g\|_{L^\infty(\mathbb{R}^d)}$, μ, Λ_A and on the function Θ, such that*

$$|u(t,x)-u(s,x)| \leqslant \kappa|t-s| \qquad \text{for all } (t,x),\ (s,x)\in[0,T]\times\mathbb{R}^d.$$

Proof. Take a constant κ large enough so that

$$\kappa > d\,\Lambda_A^2\|D^2g\|_{L^\infty(\mathbb{R}^d)} + \max\left\{\mu, \Theta\left(\|Dg\|_{L^\infty(\mathbb{R}^d)}\right)\right\}.$$

Then the functions $u_-(t,x) := g(x)-\kappa t$ and $u_+(t,x) := g(x)+\kappa t$ are, respectively, a bounded Lipschitz continuous sub and supersolution of (1.3) with $U:=\mathbb{R}^d$. By Proposition 1.4, we infer that $u_-(t,x)\leqslant u(t,x)\leqslant u_+(t,x)$ for every $(t,x)\in[0,T]\times\mathbb{R}^d$. For any fixed $h\in(0,T)$, the function $v(t,x):=u(t+h,x)$ is a bounded continuous solution to (1.3) in $(0,T-h)\times\mathbb{R}^d$ with initial datum $v(0,\cdot)=u(h,\cdot)$. Furthermore, it is Lipschitz in $(0,T)\times\mathbb{R}^d$ with respect to x, so by Proposition 1.4 we infer

$$\|u(t+h,\cdot)-u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u(h,\cdot)-u(0,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leqslant \kappa\, h,$$

yielding the claimed Lipschitz continuity of u in t. ◻

We recall the following crucial Lipschitz estimates for solutions to (1.3) proved in [1].

Proposition 1.6. *Assume that A satisfy (A1)–(A2) and H satisfies (H3)–(H4) with $\mu = +\infty$. Let $u \in \mathrm{C}([0,T] \times \overline{B}_{r+1})$ be a solution of* (1.3) *with $U := B_{r+1}$ for some $r > 0$, satisfying $u(0,\cdot) = g \in W^{1,\infty}(\overline{B}_{r+1})$ and*

$$\partial_t u \geqslant -\kappa \qquad \text{in } (0,T) \times B_{r+1}$$

for some positive constant $\kappa > 0$. Then

$$|u(t,x) - u(t,y)| \leqslant K_r |x-y| \qquad \text{for all } (t,x),(t,y) \in (0,T) \times B_r,$$

with $K_r > 0$ given by

$$K_r := C \left\{ \left(\frac{(1+\Lambda_A)^{1/2}\Lambda}{a_{r+1}} \right)^{1/(m-1)} + \left(\frac{M_{r+1}+\kappa}{a_{r+1}} \right)^{1/m} \right\}, \tag{1.7}$$

where C is a positive constant only depending on d and m.

2. Existence of solutions

The purpose of this section is to establish existence of solutions $u \in \mathrm{C}_b([0,T] \times \mathbb{R}^d)$ to the equation

$$\partial_t u - \mathrm{tr}(A(x) D_x^2 u) + H(x, D_x u) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{2.1}$$

subject to the initial condition $u(0,\cdot) = g \in \mathrm{UC}_b(\mathbb{R}^d)$. We first deal with the uniformly parabolic case and show existence of classical solutions to (2.1) when the initial datum is smooth enough, and then proceed to show the result in full generality.

2.1. The uniformly parabolic case: existence of classical solutions.

In this subsection we will show the existence of a solution $u \in \mathrm{C}^{1,2}((0,T) \times \mathbb{R}^d) \cap \mathrm{C}_b([0,T] \times \mathbb{R}^d)$ to (2.1) subject to the initial condition $u(0,\cdot) = g \in \mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ when the diffusion matrix is regular and uniformly positive definite. More precisely, throughout this subsection we will assume, besides (A1)–(A2), the following further assumptions on A:

(A3) $A \in \mathrm{C}^1(\mathbb{R}^d)$;

(A4) there exists a constant $\lambda > 0$ such that

$$\langle A(x)\xi, \xi \rangle \geqslant \lambda |\xi|^2 \qquad \text{for every } x, \xi \in \mathbb{R}^d.$$

For the Hamiltonian, we will assume conditions (H3)–(H4).

The strategy we are going to implement is the following: we will approximate A, H and g with a sequence of diffusion matrices A_n, of Hamiltonians H_n and of initial data g_n, that are $n\mathbb{Z}^d$–periodic in the x–variable and coincide with A, H, g, respectively, for x belonging to a ball of radius $n/2$. The gain in compactness obtained in this way allows us to prove the existence of classical solutions u_n for the approximating Cauchy problems. This is essentially achieved by following the classical approach to parabolic quasilinear equations, based on the use of Schauder fixed point theorem and on suitable *a priori* L^∞ and Hölder estimates on the gradient of the solutions, see Proposition 2.1. For the L^∞ estimate, we will exploit Proposition 1.6, while the Hölder estimates follow from more classical results. Then we will send $n \to +\infty$: since the functions $(u_n)_n$ are equi–bounded and locally equi–Lipschitz in $[0,T] \times \mathbb{R}^d$, Ascoli–Arzelà Theorem, together with the stability properties of the notion of viscosity solution, implies that any accumulation point u of the $(u_n)_n$ is a locally Lipschitz solution of (2.1) satisfying the initial condition $u(0,\cdot) = g$ on $\mathbb{R}^d$. The classical parabolic regularity theory (and Proposition 1.4) finally yields that such a u is in $\mathrm{C}^{1,2}((0,T) \times \mathbb{R}^d)$, hence a classical solution to (2.1).

We proceed to implement the strategy outlined above. To this aim, choose $\chi \in \mathrm{C}^\infty(\mathbb{R}^d)$ so that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $B_{1/2}$ and $\chi \equiv 0$ on $\mathbb{R}^d \setminus B_{3/4}$. For every $n \in \mathbb{N}$, we set

$$
\begin{aligned}
&g_n(x) := g(x)\chi(x/n) && \text{for } x \in [-n,n]^d, \qquad (2.2)\\
&A_n(x) := A(x)\,\chi(x/n) + \mathrm{Id}\,(1-\chi(x/n)) && \text{for } x \in [-n,n]^d,\\
&H_n(x,p) := H(x,p)\,\chi(x/n) + \Lambda\big(|p|^m+1\big)\,(1-\chi(x/n)) && \text{for } (x,p) \in [-n,n]^d \times \mathbb{R}^d,
\end{aligned}
$$

and we extend them by periodicity to $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$, respectively. Note that

$$
g_n = g, \ \ A_n = A \quad \text{in } B_{n/2}, \qquad H_n = H \quad \text{in } B_{n/2} \times \mathbb{R}^d \tag{2.3}
$$

It is easily seen that the Hamiltonians H_n satisfy (H3)–(H4), where the constants a_r, M_r and Λ can be chosen independent of n. Also note that, by periodicity, each H_n is uniformly superlinear, i.e. (H4) holds with $\alpha_r = \alpha_n$, $M_r = M_n$ for every $r > 0$. For each $n \in \mathbb{N}$, we define the quasilinear parabolic operator

$$
P_n u := \partial_t u - \mathrm{tr}(A(x)D_x^2 u) + H_n(x, D_x u)
$$

We start by deriving the *a priori* Lipschitz and Hölder estimates.

Proposition 2.1. *Suppose $u \in \mathrm{C}^{1,2}_n([0,\tau] \times \mathbb{R}^d)$ satisfies $P_n u = 0$ in $(0,\tau) \times \mathbb{R}^d$, $u(0,\cdot) = \phi$ on $\mathbb{R}^d$ with $\phi \in \mathrm{C}^2_n(\mathbb{R}^d)$. Let $L > \|D\phi\|_{L^\infty(\mathbb{R}^d)} + \|D^2\phi\|_{L^\infty(\mathbb{R}^d)}$. Then*

$$
\|\partial_t u\|_{L^\infty((0,\tau)\times\mathbb{R}^d)} \leqslant \kappa, \qquad \|D_x u\|_{L^\infty((0,\tau)\times\mathbb{R}^d)} \leqslant K_n,
$$

where κ is a constant only depending on L, μ, Λ_A, Λ, m, and K_r is the constant given by (1.7) with $r := n$. Moreover, there exist constants $\tilde{C}$ and $\alpha \in (0,1)$, only depending on K_n, Λ, λ, Λ_A and L (and independent of $\tau > 0$, in particular), such that $\sum_i [\partial_{x_i} u]^{(\alpha)}_{(0,\tau)\times\mathbb{R}^d} \leqslant \tilde{C}$. In particular,

$$
\|u\|_{H^{(1+\alpha)/2,1+\alpha}((0,\tau)\times\mathbb{R}^d)} \leqslant \kappa\tau + \|\phi\|_{L^\infty(\mathbb{R}^d)} + K_n + \tilde{C} + \kappa\tau^{\frac{1-\alpha}{2}}. \tag{2.4}
$$

Proof. By periodicity, the solution u is clearly Lipschitz continuous in $(0,\tau) \times \mathbb{R}^d$. The Lipschitz estimates follow at once by Propositions 1.5 and 1.6. The Hölder estimates on $D_x u$ can be derived by applying [13, Chapter VI, Theorem 1.1] with $\Omega := [-2n, 2n]^d$, $\Omega' := [-n,n]^d$. The inequality (2.4) is a trivial consequence of these estimates. □

We proceed by showing existence of a classical solution for the approximating parabolic Cauchy problems.

Proposition 2.2. *There exists a function $u \in \mathrm{C}^{1,2}_n([0,T] \times \mathbb{R}^d)$ that solves the problem*

$$
P_n u = 0 \quad \text{in } [0,T] \times \mathbb{R}^d, \qquad u(0,\cdot) = g_n \quad \text{on } \mathbb{R}^d. \tag{2.5}
$$

Proof. The proof is divided in two steps: we will first prove the local existence, i.e. the existence of a classical solution to (2.5) in $[0,\tau] \times \mathbb{R}^d$ for some $\tau \in (0,T]$; then we will prove that the maximal τ for which such a solution exists is equal to T. For notational brevity, throughout the proof we will write Q_τ in place of $(0,\tau) \times \mathbb{R}^d$.

Step 1: let $\tau \in (0,T]$ to be chosen and denote by $\alpha \in (0,1)$ the exponent provided by Proposition 2.1 with g_n in place of ϕ and by C the corresponding constant appearing at the right hand–side of (2.4). Set

$$
\mathcal{S} := \left\{ v \in H_n^{(1+\alpha)/2,1+\alpha}(\overline{Q}_\tau) \ : \ \|v\|_{H^{(1+\alpha)/2,1+\alpha}(Q_\tau)} \leqslant 2C \right\}.
$$

Then we define a map $J : \mathcal{S} \to H_n^{(1+\alpha)/2,1+\alpha}(\overline{Q}_\tau)$ by $u = Jv$, where u solves the Cauchy problem

$$
\partial_t u - \mathrm{tr}(A_n(x)D_x^2 u) + H_n(x, D_x v) = 0 \ \text{ in } \overline{Q}_\tau, \quad u(0,\cdot) = g_n \ \text{ on } \mathbb{R}^d.
$$

Note that, by [13, Chapter IV, Theorem 5.1], for each $v \in \mathcal{S}$, this problem has a unique solution $u \in H_n^{(2+\alpha)/2,2+\alpha}(\overline{Q}_\tau)$ satisfying

$$\|u\|_{H^{(2+\alpha)/2,1+\alpha}(Q_\tau)} \leqslant c\left(\|g_n\|_{H^{2+\alpha}(Q_\tau)} + \|H_n(x, D_x v)\|_{H^{\alpha/2,\alpha}(Q_\tau)}\right), \tag{2.6}$$

where c is a constant independent of $\tau \in (0,T]$, g_n and v. Therefore

$$\|Jv\|_{H^{(2+\alpha)/2,2+\alpha}(Q_\tau)} \leqslant \Gamma(C) \qquad \text{for all } v \in \mathcal{S}, \tag{2.7}$$

for some continuous nondecreasing function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, only depending on H_n. We proceed to show that we can choose $\tau \in (0,T]$ small enough so that $\|Jv\|_{H^{(1+\alpha)/2,1+\alpha}(Q_\tau)} \leqslant 2C$ for all $v \in \mathcal{S}$. To this aim, first note that, for $u = Jv$, we have

$$\|u\|_{H^{(1+\alpha)/2,1+\alpha}(Q_\tau)} \leqslant C + \|u - g_n\|_{H^{(1+\alpha)/2,1+\alpha}(Q_\tau)}.$$

Set $\tilde{u} := u - g_n$ and let us estimate the term

$$\|\tilde{u}\|_{H^{(1+\alpha)/2,1+\alpha}(Q_\tau)} = \|\tilde{u}\|_{L^\infty(Q_\tau)} + \|D_x\tilde{u}\|_{H^{\alpha/2,\alpha}(Q_\tau)} + [\tilde{u}]^{(\frac{1+\alpha}{2})}_{t,Q_\tau}.$$

From the fact that $\tilde{u}(0,\cdot) = 0$ on $\mathbb{R}^d$ and $\|\partial_t\tilde{u}\|_{L^\infty(Q_\tau)} \leqslant \Gamma(C)$, we get

$$\|\tilde{u}\|_{L^\infty(Q_\tau)} \leqslant \Gamma(C)\tau, \qquad [\tilde{u}]^{(\frac{1+\alpha}{2})}_{t,Q_\tau} \leqslant \Gamma(C)\tau^{\frac{1-\alpha}{2}},$$

while the term $\|D_x\tilde{u}\|_{H^{\alpha/2,\alpha}(Q_\tau)}$ can be controlled with the aid of Proposition 1.2. We conclude that we can choose $\varepsilon > 0$ and a sufficiently small $\tau \in (0,T]$ so that

$$\|D_x\tilde{u}\|_{H^{\alpha/2,\alpha}(Q_\tau)} \leqslant \frac{C}{2}, \qquad \|\tilde{u}\|_{L^\infty(Q_\tau)} + [\tilde{u}]^{(\frac{1+\alpha}{2})}_{t,Q_\tau} \leqslant \frac{C}{2}.$$

In particular, J maps $\mathcal{S}$ into itself, for such a τ. Since $\mathcal{S}$ is a convex and compact subset of the Banach space $H_n^{\alpha/2,\alpha}(\overline{Q}_\tau)$, see Proposition 1.3, we can apply the Schauder fixed point Theorem, see for instance [14, Theorem 8.1], and derive the existence of a fixed point u of J, which is clearly in $H_n^{(2+\alpha)/2,2+\alpha}(\overline{Q}_\tau)$ and hence solves the Cauchy problem (2.5) in $[0,\tau] \times \mathbb{R}^d$.

Step 2: let us set

$$T^* := \sup\{\tau \in (0,T] \,:\, (2.5) \text{ admits a solution in } \mathrm{C}_n^{1,2}(\overline{Q}_\tau)\,\}.$$

By the step 1, we know that the above set is nonempty. We want to show that $T^* = T$. To this aim, take a sequence $\big((\tau_k, u_k)\big)_k$ in $(0,T^*) \times \mathrm{C}_n^{1,2}(\overline{Q}_{\tau_k})$, where $(\tau_k)_k$ converges increasingly to T^* and u_k solves (2.5) in $\overline{Q}_{\tau_k}$. From Proposition 2.1 we derive that each u_k belongs to $H_n^{(1+\alpha)/2,1+\alpha}(\overline{Q}_{\tau_k})$ and that there exist a constant $C > 0$ and an exponent $\alpha \in (0,1)$, independent of $k \in \mathbb{N}$, such that $\|u_k\|_{H^{(1+\alpha)/2,1+\alpha}(Q_{\tau_k})} \leqslant C$ for every $k \in \mathbb{N}$. By applying [13, Chapter IV, Theorem 5.1] with $f(t,x) := H_n(x, D_x u_k)$ we infer that u_k satisfies (2.6) with Q_{τ_k} in place of Q_τ and u_k in place of v, where c is a constant independent of k. We derive

$$\|u_k\|_{H^{(2+\alpha)/2,2+\alpha}(Q_{\tau_k})} \leqslant \Gamma(C) \qquad \text{for every } k \in \mathbb{N}. \tag{2.8}$$

Also notice that, by Proposition 1.4,

$$u_k = u_h \quad \text{on } \overline{Q}_{\tau_h} \text{ for every } k \geqslant h. \tag{2.9}$$

We define a function $u : [0,T^*) \times \mathbb{R}^d \to \mathbb{R}$ by setting $u = u_k$ on $\overline{Q}_{\tau_k}$ for every $k \in \mathbb{N}$, and then by taking its continuous extension to $[0,T^*] \times \mathbb{R}^d$. According to (2.9) and (2.8), u is well defined and belongs to $H_n^{(2+\alpha)/2,2+\alpha}(\overline{Q}_{T^*})$. Moreover, it solves the Cauchy problem (2.5) in $[0,T^*) \times \mathbb{R}^d$ by construction, and also on $[0,T^*] \times \mathbb{R}^d$ by continuity of u, $\partial_t u$, $D_x u$, $D_x^2 u$. If, by contradiction, $T^* < T$, we could argue as in step 1 to find $\beta \in (0,1)$, $\tau \in (0, T - T^*)$ and a function $w \in H_n^{(1+\beta)/2,1+\beta}\left([0,\tau] \times \mathbb{R}^d\right)$ such that

$$P_n w = 0 \quad \text{in } [0,\tau] \times \mathbb{R}^d, \qquad w(0,\cdot) = u(T^*,\cdot) \quad \text{on } \mathbb{R}^d.$$

It is easy to check that the function u^* defined as

$$u^*(t,x) := \begin{cases} u(t,x) & \text{if } (t,x) \in [0,T^*] \times \mathbb{R}^d, \\ w(t-T^*,x) & \text{if } (t,x) \in [T^*, T^*+\tau] \times \mathbb{R}^d, \end{cases}$$

belongs to $\mathrm{C}^{1,2}_n(\overline{Q}_{T^*+\tau})$ and solves the Cauchy problem (2.5) in $\overline{Q}_{T^*+\tau}$, thus contradicting the maximality of T^*. □

We now proceed to prove the announced result.

Theorem 2.3. *Let A satisfy (A1)–(A4) and let H satisfy (H3)–(H4). Then, for every $g \in \mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$, there exists a classical solution $u \in \mathrm{C}^{1,2}((0,T) \times \mathbb{R}^d) \cap \mathrm{C}_b([0,T] \times \mathbb{R}^d)$ to* (2.1) *subject to the initial condition $u(0,\cdot) = g$ on $\mathbb{R}^d$. Moreover*

$$\|\partial_t u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \kappa, \qquad \|D_x u\|_{L^\infty([0,T]\times B_r)} \leqslant K_r, \tag{2.10}$$

where κ is a constant only depending on $\|Dg\|_{L^\infty(\mathbb{R}^d)}$, $\|D^2 g\|_{L^\infty(\mathbb{R}^d)}$, μ, Λ_A, Λ, m, and K_r is the constant defined in (1.7).

Proof. In view of Proposition 2.2, for each $n \in \mathbb{N}$ there exists a solution $u_n \in \mathrm{C}^{1,2}_n([0,T]\times\mathbb{R}^d)$ to the problem

$$P_n u = 0 \quad \text{in } [0,T] \times \mathbb{R}^d, \qquad u(0,\cdot) = g_n \quad \text{on } \mathbb{R}^d.$$

By combining Proposition 1.5 and Proposition 1.6, we get that the functions u_n satisfy the Lipschitz estimates (2.10), at least eventually for every fixed $r > 0$. By Ascoli–Arzelà Theorem and by possibly extracting a subsequence, we infer that there exists a function $u \in \mathrm{C}([0,T] \times \mathbb{R}^d)$ such that $u_n \rightrightarrows_{\mathrm{loc}} u$ on $[0,T] \times \mathbb{R}^d$. Since $H_n \rightrightarrows_{\mathrm{loc}} H$ on $\mathbb{R}^d \times \mathbb{R}^d$ and $A_n \rightrightarrows_{\mathrm{loc}} A$, $g_n \rightrightarrows_{\mathrm{loc}} g$ on $\mathbb{R}^d$, we infer that u is a viscosity solution of (2.1) subject to the initial condition $u(0,\cdot) = g$ on $\mathbb{R}^d$. It is clear that u satisfies (2.10). Being g bounded on $\mathbb{R}^d$, we get in particular that u in bounded on $[0,T] \times \mathbb{R}^d$.

Let us prove the asserted regularity of u. Let us fix $r > 0$ and choose a smooth and bounded function $f_r : \mathbb{R} \to \mathbb{R}$ with $f_r(h) \equiv h$ on $[-z_h, z_h]$, with z_h big enough so that $f_r(H(x,p)) = H(x,p)$ for every $x \in B_r$ and $|p| \leqslant K_r$. Then u is a viscosity solution of

$$\partial_t u - \operatorname{tr}(A(x) D^2_x u) + f_r\big(H(x, D_x u)\big) = 0 \quad \text{in } (0,T) \times B_r.$$

On the other hand, [14, Theorem 12.22] guarantees the existence of a solution $v \in \mathrm{C}([0,T] \times \overline{B}_r) \cap \mathrm{C}^{1,2}((0,T) \times B_r)$ satisfying the boundary condition $v = u$ on $\partial_P\big((0,T) \times B_r\big)$. In view of the Comparison Principle stated in Proposition 1.4 we infer that $u = v$ on $[0,T] \times \overline{B}_r$. The proof is complete. □

We end this subsection proving a comparison–type result for solutions to (2.1) obtained via approximation through periodic parabolic problems, as described above.

Proposition 2.4. *Let A satisfy (A1)–(A4), H satisfy (H3)–(H4) and g^1, $g^2 \in \mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. Then there exists a pair u^1, $u^2 \in \mathrm{C}^{1,2}((0,T) \times \mathbb{R}^d) \cap \mathrm{C}_b([0,T] \times \mathbb{R}^d)$ of classical solutions to* (2.1) *subject to the initial condition $u^i(0,\cdot) = g^i$ on $\mathbb{R}^d$, $i \in \{1,2\}$, satisfying*

$$\|u^1 - u^2\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \|g^1 - g^2\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. For $i \in \{1,2\}$, let g^i_n be the $n\mathbb{Z}^d$–periodic function on $\mathbb{R}^d$ defined via (2.2) and denote by $u^i_n \in \mathrm{C}^{1,2}_n([0,T] \times \mathbb{R}^d)$ the solution to the problem

$$P_n u = 0 \quad \text{in } [0,T] \times \mathbb{R}^d, \qquad u(0,\cdot) = g_n \quad \text{on } \mathbb{R}^d$$

obtained according to Proposition 2.2. The functions u^i_n are Lipschitz continuous on $[0,T] \times \mathbb{R}^d$, hence, in view of Proposition 1.4, we infer

$$\|u^1_n - u^2_n\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \|g^1_n - g^2_n\|_{L^\infty(\mathbb{R}^d)} \leqslant \|g^1 - g^2\|_{L^\infty(\mathbb{R}^d)} \quad \text{for each } n \in \mathbb{N}. \tag{2.11}$$

According to the proof of Theorem 2.3, there exists a pair u^1, u^2 of bounded and continuous classical solutions to (2.1) subject to the initial condition $u^i(0,\cdot) = g^i$ on $\mathbb{R}^d$ such that, up to subsequences, $u_n^i \rightrightarrows_{\rm loc} u^i$ in $[0,T]\times\mathbb{R}^d$ for $i\in\{1,2\}$. The assertion follows by passing to the limit with respect to n in (2.11). □

2.2. **General existence results.** In this subsection we will prove existence of solutions to (2.1), where we drop the regularity and uniform positivity conditions on the diffusion matrix, i.e. we will assume conditions (A1)–(A2) only.

Theorem 2.5. *Let A satisfy (A1)–(A2) and let H satisfy (H3)–(H4). Then, for every $g\in \mathrm{UC}_b(\mathbb{R}^d)$, there exists a solution $u\in \mathrm{C}_b([0,T]\times\mathbb{R}^d)$ to the equation (2.1) subject to the initial condition $u(0,\cdot)=g$ on $\mathbb{R}^d$. If $g\in \mathrm{C}^\infty(\mathbb{R}^d)\cap W^{2,\infty}(\mathbb{R}^d)$, u also satisfies*

$$\|\partial_t u\|_{L^\infty([0,T]\times\mathbb{R}^d)}\leqslant \kappa,\qquad \|D_x u\|_{L^\infty([0,T]\times B_r)}\leqslant K_r, \tag{2.12}$$

where κ is a constant only depending on $\|Dg\|_{L^\infty(\mathbb{R}^d)}$, $\|D^2 g\|_{L^\infty(\mathbb{R}^d)}$, μ, Λ_A, Λ, m, and K_r is the constant defined in (1.7).

Proof. Let us first assume that $g\in \mathrm{C}^\infty(\mathbb{R}^d)\cap W^{2,\infty}(\mathbb{R}^d)$. We introduce a sequence $(\rho_n)_n$ of standard mollifiers and for each $n\in\mathbb{N}$ we set

$$\widetilde{A}_n(x):=\frac{1}{n^2}\mathrm{Id}+(\rho_n * A)(x),\qquad x\in\mathbb{R}^d.$$

In view of Theorem 2.3, for every $n\in\mathbb{N}$ there exists a classical solution $u_n\in \mathrm{C}^{1,2}((0,T)\times\mathbb{R}^d)\cap \mathrm{C}_b([0,T]\times\mathbb{R}^d)$ to the equation (2.1) with $\widetilde{A}_n$ in place of A and subject to the initial condition $u_n(0,\cdot)=g$ on $\mathbb{R}^d$. Moreover this family of solutions satisfy (2.12) for some constants κ and K_r independent of n (notice that $\Lambda_{\widetilde{A}_n}\leqslant\Lambda_A+1/n$). By Ascoli–Arzelà Theorem and by possibly extracting a subsequence, we infer that there exists a function $u\in \mathrm{C}_b([0,T]\times\mathbb{R}^d)$ such that $u_n\rightrightarrows_{\rm loc} u$ on $[0,T]\times\mathbb{R}^d$. Since $\widetilde{A}_n\rightrightarrows_{\rm loc} A$ on $\mathbb{R}^d$, we infer that u is a viscosity solution of (2.1) subject to the initial condition $u(0,\cdot)=g$ on $\mathbb{R}^d$. It is clear that u satisfies (2.12).

Let us now assume that $g\in \mathrm{UC}_b(\mathbb{R}^d)$. Choose a sequence $(g^k)_k$ of initial data in $\mathrm{C}^\infty(\mathbb{R}^d)\cap W^{2,\infty}(\mathbb{R}^d)$ such that $\|g-g^k\|_{L^\infty(\mathbb{R}^d)}\to 0$ as $k\to+\infty$ and let us denote by u^k a solution to (2.1) with initial datum g^k obtained via the procedure described in the previous step. According to Proposition 2.4 and by using a diagonal argument, this can be done in such a way that

$$\|u^k-u^h\|_{L^\infty([0,T]\times\mathbb{R}^d)}\leqslant\|g^k-g^h\|_{L^\infty(\mathbb{R}^d)}\qquad\text{for every } k,h\in\mathbb{N}.$$

From the fact that $(g^k)_k$ is a converging sequence in $\mathrm{C}_b(\mathbb{R}^d)$, we infer that $(u^k)_k$ is a Cauchy sequence in $\mathrm{C}_b([0,T]\times\mathbb{R}^d)$. Therefore the solutions u^k converge to some u in $\mathrm{C}_b([0,T]\times\mathbb{R}^d)$ and by stability we conclude that u is a solution of (2.1) with initial datum g. □

3. Comparison Principles

In this section we are concerned with uniqueness properties of the solutions provided in the previous section, at least in the class of continuous bounded functions in cylinders of the form $[0,T]\times\mathbb{R}^d$. This will be obtained as a consequence of the comparison principles we will prove below.

3.1. **Uniformly superlinear Hamiltonians.** In this subsection, we will deal with Hamiltonians satisfying (H3)–(H4) that are uniformly superlinear, i.e. for which (H4) holds with constants a_r, M_r independent of $r>0$. In this case, the solutions to (2.1) with initial datum in $\mathrm{C}^\infty(\mathbb{R}^d)\cap W^{2,\infty}(\mathbb{R}^d)$ provided by Theorem 2.5 are globally Lipschitz in $[0,T]\times\mathbb{R}^d$. Moreover, such Hamiltonians satisfy condition (H2) as well. We can therefore apply Proposition 1.4 and, by exploiting the density of such initial data in $\mathrm{UC}_b(\mathbb{R}^d)$, we can easily derive the following general Comparison Principle:

Theorem 3.1. *Assume A satisfies (A1)–(A2) and H satisfies (H3)–(H4), with constants a_r, M_r independent of $r > 0$. Let $v \in \mathrm{USC}([0,T] \times \mathbb{R}^d)$ and $w \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$ be, respectively, a sub and a supersolution of* (2.1) *satisfying*

$$\limsup_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{v(t,x)}{1+|x|} \leqslant 0 \leqslant \liminf_{|x|\to+\infty} \inf_{t\in[0,T]} \frac{w(t,x)}{1+|x|}.$$

Let us furthermore assume that $v(0,\cdot) \leqslant g \leqslant w(0,\cdot)$ for some $g \in \mathrm{UC}_b(\mathbb{R}^d)$. Then

$$v(t,x) \leqslant w(t,x) \qquad \textit{for every } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Proof. Fix $\varepsilon > 0$ and set $w_\varepsilon := w + \varepsilon$. Since $v(0,\cdot) \leqslant g < g + \varepsilon \leqslant w_\varepsilon(0,\cdot)$, in view of Lemma 1.1 we can find a function $g_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ such that $v(0,\cdot) \leqslant g_\varepsilon \leqslant w_\varepsilon(0,\cdot)$ on $\mathbb{R}^d$. By Theorem 2.5 and by taking into account that H is uniformly superlinear, there exists a solution $u_\varepsilon \in \mathrm{C}([0,T] \times \mathbb{R}^d) \cap W^{1,\infty}([0,T] \times \mathbb{R}^d)$ of (2.1) with initial datum g_ε. Since H satisfies (H2), we can apply the Comparison Principle stated in Proposition 1.4 with $U := \mathbb{R}^d$ to infer that $v \leqslant u_\varepsilon \leqslant w_\varepsilon = w + \varepsilon$ in $[0,T] \times \mathbb{R}^d$. The assertion follows since $\varepsilon > 0$ was arbitrarily chosen. □

As a simple consequence of Theorems 2.3 and 3.1 we derive the following result:

Theorem 3.2. *Let A satisfy (A1)–(A2) and let H satisfy (H3)–(H4), with constants a_r, M_r independent of $r > 0$. Then, for every $g \in \mathrm{UC}_b(\mathbb{R}^d)$, there exists a unique function $u \in \mathrm{UC}_b([0,T] \times \mathbb{R}^d)$ that solves the equation* (2.1) *subject to the initial condition $u(0,\cdot) = g$ on $\mathbb{R}^d$. If $g \in W^{2,\infty}(\mathbb{R}^d)$, u is Lipschitz continuous in $[0,T] \times \mathbb{R}^d$ and satisfies*

$$\|\partial_t u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \kappa, \qquad \|D_x u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant K,$$

where κ is a constant only depending on $\|Dg\|_{L^\infty(\mathbb{R}^d)}$, $\|D^2 g\|_{L^\infty(\mathbb{R}^d)}$, μ, Λ_A, Λ, m, and K is the constant, independent of $r > 0$, defined in (1.7).

Proof. The uniqueness part is obvious in view of Theorem 3.1. Let $g \in W^{2,\infty}(\mathbb{R}^d)$ and denote by u the unique function in $\mathrm{C}_b([0,T]\times\mathbb{R}^d)$ that solves (2.1). The fact that $\|\partial_t u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \kappa$ for a constant κ only depending on $\|Dg\|_{L^\infty(\mathbb{R}^d)}$, $\|D^2 g\|_{L^\infty(\mathbb{R}^d)}$, μ, Λ_A, Λ, m is derived by arguing as in the proof of Proposition 1.5 and by using Theorem 3.1 in place of Proposition 1.4. The L^∞–bound on $D_x u$ follows by applying Proposition 1.6.

Let us now assume that $g \in \mathrm{UC}_b(\mathbb{R}^d)$. Choose a sequence $(g^k)_k$ of initial data in $W^{2,\infty}(\mathbb{R}^d)$ such that $\|g - g^k\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $k \to +\infty$ and denote by u, u^k the unique solution to (2.1) in $\mathrm{C}_b([0,T] \times \mathbb{R}^d)$ with initial datum g, g^k, respectively. By Theorem 3.1

$$\|u - u^k\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \|g - g^k\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for every } k \in \mathbb{N}.$$

As a uniform limit of a sequence of Lipschitz functions, we conclude that $u \in \mathrm{UC}_b([0,T] \times \mathbb{R}^d)$. □

3.2. **Non–uniformly superlinear Hamiltonians.** When the Hamiltonian H is not uniformly superlinear, i.e. the constants a_r, M_r in (H4) actually depend on $r > 0$, Theorem 2.5 provides us with solutions to (2.1) that are, in the best case scenario, only locally Lipschitz in $[0,T] \times \mathbb{R}^d$ and the idea exploited in the previous subsection can no longer be used. We will therefore restrict our analysis to Hamiltonians of special form, by slightly relaxing the convexity condition in p assumed in [1]. The results of this subsection are based on a technical refinement of the arguments therein employed.

It is convenient to introduce a piece of notation first. Let $m > 1$, $\Lambda > 0$, $(a_r)_{r>0}$ in $(0,1]$ and $(M_r)_{r>0}$ in $[1,+\infty)$ be fixed constants. We will denote by $\mathfrak{B}\,(m,\Lambda,(a_r)_{r>0},(M_r)_{r>0})$ the family of Borel functions $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that are convex in p and satisfy (H3) and condition (1.4) in (H4) with $\mu = +\infty$, and by $\mathcal{H}\,(m,\Lambda,(a_r)_{r>0},(M_r)_{r>0})$ the family of Hamiltonians $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying conditions (H3) and (H4) with $\mu = +\infty$. Note that we are not assuming neither condition (H2) nor that H is bounded from below.

We consider a Hamiltonian $H \in \mathcal{H}\left(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0}\right)$ of the form

$$H(x,p) = \inf_{i\in\mathcal{I}} H_i(x,p), \qquad \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{3.1}$$

where $\mathcal{I}$ is a set of indexes and each H_i belongs to $\mathcal{B}\left(m_i, \Lambda, (a_r)_{r>0}, (M_r)_{r>0}\right)$, with exponent $m_i > 1$ possibly depending on i. Notice that $m = \inf_i m_i$.

Theorem 3.3. *Let A satisfy (A1)–(A2) and H as above. Let U an open subset of $\mathbb{R}^d$ and let $v \in \mathrm{USC}([0,T]\times\overline{U})$ and $w \in \mathrm{LSC}([0,T]\times\overline{U})$ be, respectively, a sub and a supersolution of*

$$\partial_t u - \mathrm{tr}(A(x)D_x^2 u) + H(x, D_x u) = 0 \quad \text{in } (0,T)\times U, \tag{3.2}$$

satisfying

$$\limsup_{\substack{|x|\to+\infty\\ x\in U}} \sup_{t\in[0,T]} \frac{v(t,x)}{1+|x|} \leqslant 0 \leqslant \liminf_{\substack{|x|\to+\infty\\ x\in U}} \inf_{t\in[0,T]} \frac{w(t,x)}{1+|x|}. \tag{3.3}$$

Then

$$v(t,x) - w(t,x) \leqslant \sup_{\partial_P((0,T)\times U)} \big(v - w\big) \qquad \text{for every } (t,x) \in (0,T)\times U,$$

where $\partial_P\left((0,T)\times U\right) := \{0\}\times U \cup [0,T)\times\partial U$ is the parabolic boundary of $(0,T)\times U$.

For the proof of Theorem 3.3, we will use in a crucial way the following estimate, that we will prove separately:

Lemma 3.4. *Let H be as above. For fixed $\eta \in (0,1/8)$ and $R_\eta > 1$, let $x_\varepsilon, y_\varepsilon, q_\varepsilon \in \mathbb{R}^d$ such that $|x_\varepsilon|$, $|y_\varepsilon| \leqslant R_\eta - 1$, $|q_\varepsilon| \leqslant \eta$ for every $\varepsilon \in (0,1)$, and*

$$\lim_{\varepsilon\to0^+} |x_\varepsilon - y_\varepsilon| = 0. \tag{3.4}$$

Then there exist $\varepsilon(\eta) > 0$, $C > 0$ and a constant $C_\eta > 0$, depending on η, such that, for every $\varepsilon < \varepsilon(\eta)$ and for $s = 1 - 4\eta$ we have

$$sH\left(x_\varepsilon, \frac{p_\varepsilon + q_\varepsilon}{s}\right) - H\left(y_\varepsilon, p_\varepsilon\right) \geqslant -C(1-s) - C_\eta|x_\varepsilon - y_\varepsilon|, \tag{3.5}$$

where $p_\varepsilon := (x_\varepsilon - y_\varepsilon)/\varepsilon$.

Proof. The proof relies on the arguments used in [1], up to some technical modifications that we detail below. Let us denote by I the left–hand side term of (3.5). We have

$$I = \underbrace{\left(sH\big(x_\varepsilon, \frac{p_\varepsilon + q_\varepsilon}{s}\big) - H\left(x_\varepsilon, p_\varepsilon\right)\right)}_{I_1} + \underbrace{\Big(H\left(x_\varepsilon, p_\varepsilon\right) - H\left(y_\varepsilon, p_\varepsilon\right)\Big)}_{I_2}.$$

By the fact that H satisfies hypothesis (1.5) and $m \leqslant m_i$, we get

$$I_2 \geqslant -\Lambda\left(|p_\varepsilon|^m + M_{R_\eta}\right)|x_\varepsilon - y_\varepsilon| \geqslant -\Lambda\left(|p_\varepsilon|^{m_i} + 2M_{R_\eta}\right)|x_\varepsilon - y_\varepsilon| \quad \text{for all } i \in \mathcal{I}. \tag{3.6}$$

As for the term I_1, we obviously have $I_1 \geqslant \inf_{i\in\mathcal{I}} J_i$ with

$$J_i := sH_i\big(x_\varepsilon, \frac{p_\varepsilon + q_\varepsilon}{s}\big) - H_i\left(x_\varepsilon, p_\varepsilon\right).$$

Let us estimate J_i, for each fixed i. Set $r := (1+s)/2 < 1$. We exploit the convexity of H_i in p: by arguing as in [1], we get

$$J_i \geqslant \left(\frac{s}{r}H_i\left(x_\varepsilon, \frac{r}{s}p_\varepsilon\right) - H_i(x_\varepsilon, p_\varepsilon)\right) - \frac{\Lambda}{2}(1-s)\left(1 + \frac{(4\eta)^{m_i}}{(1-s)^{m_i}}\right).$$

Using the fact that $1 - s = 4\eta$, this inequality can be restated as

$$J_i \geqslant \underbrace{\left(\frac{s}{r}H_i\left(x_\varepsilon, \frac{r}{s}p_\varepsilon\right) - H_i(x_\varepsilon, p_\varepsilon)\right)}_{G_\varepsilon} - \Lambda(1-s). \tag{3.7}$$

We proceed to estimate G_ε: by arguing as in [1] we get

$$G_\varepsilon \geqslant \frac{1-s}{2}\left(\left(\gamma\, a_{R_\eta} - \Lambda\gamma^{m_i}\right)|p_\varepsilon|^{m_i} - \gamma M_{R_\eta} - \Lambda\right),$$

where γ is any fixed parameter in $(0,1/2)$. Notice that, in view of the fact that $m_i \geqslant m$, we have $\gamma^{m_i} \leqslant \gamma^m$. We conclude that we can choose γ sufficiently small in such a way that the term in front of $|p_\varepsilon|^{m_i}$ can be estimated from below by $2C_\eta$, where C_η is a positive constant only depending on η (and, in particular, independent of i), and the term γM_{R_η} can be estimated from above by Λ. We get

$$J_i \geqslant G_\varepsilon - \Lambda(1-s) \geqslant C_\eta(1-s)|p_\varepsilon|^{m_i} - 2\Lambda(1-s).$$

By taking into account (3.6), we infer

$$J_i + I_2 \geqslant \big(C_\eta(1-s) - \Lambda|x_\varepsilon - y_\varepsilon|\big)|p_\varepsilon|^{m_i} - 2\Lambda(1-s) - 2M_{R_\eta}|x_\varepsilon - y_\varepsilon|. \tag{3.8}$$

Since $|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0^+$ and $1-s = 4\eta$, we infer that, for every fixed $\eta > 0$, we can find $\varepsilon(\eta) > 0$, independent of i, such that, for every $0 < \varepsilon < \varepsilon(\eta)$, the term in front of $|p_\varepsilon|^{m_i}$ is positive. By discarding it from the right–hand side of (3.8) and by recalling that $I_1 + I_2 \geqslant \inf_{i\in\mathcal{I}} J_i + I_2$, we finally obtain (3.5) with $C := 2\Lambda$ and $C_\eta := 2M_{R_\eta}$. □

With the aid of Lemma 3.4, the proof of Theorem 3.3 can be carried on by reasoning as in [1]. For the reader's convenience, we give it in the Appendix. We will furnish more details than in [1] and also correct a misleading misprint therein contained, see (A.8).

We end this subsection by providing a slight generalization of Theorem 3.3.

Proposition 3.5. *Let $H \in \mathcal{H}\left(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0}\right)$ and assume there exists $\rho > 0$ such that*

$$H(x,p) = \inf_{i\in\mathcal{I}} H_i(x,p), \qquad \text{for all } (x,p) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus B_\rho), \tag{3.9}$$

where $\mathcal{I}$ is a set of indexes and each H_i belongs to $\mathcal{B}\left(m_i, \Lambda, (a_r)_{r>0}, (M_r)_{r>0}\right)$, with exponent $m_i > 1$ possibly depending on i. Then, for such a H, the statement of Theorem 3.3 holds.

Proof. Looking at the proof of Theorem 3.3, it is clear that it suffices to prove that, for every fixed $\eta \in (0,1/8)$, there is an infinitesimal sequence $(\varepsilon_k)_k$ such that H satisfies (3.5) in Lemma 3.4 for every $\varepsilon \in \{\varepsilon_k : k \in \mathbb{N}\}$. Therefore, let us fix $\eta \in (0,1/8)$, set $s := 1-4\eta$, and let $x_\varepsilon, y_\varepsilon, p_\varepsilon, q_\varepsilon$ as in the statement of that lemma. Then there exists an infinitesimal sequence $(\varepsilon_k)_k$ such that either $|p_{\varepsilon_k}| \leqslant \rho+1$ for all $k \in \mathbb{N}$, or $|p_{\varepsilon_k}| > \rho+1$ for all $k \in \mathbb{N}$. Let $\varepsilon = \varepsilon_k$ with $k \in \mathbb{N}$. We follow the notation used in the proof of Lemma 3.4.
In the first case, first notice that $|(p_\varepsilon + q_\varepsilon)/s| < 2\rho + 3$. From (H4) and (H3) we get

$$I_1 \geqslant -(1-s)\Lambda\big(1 + (2\rho+3)^m\big) - \omega\left(\Big|\frac{p_\varepsilon + q_\varepsilon}{s} - p_\varepsilon\Big|\right),$$

where ω is a continuity modulus of $H(x_\varepsilon,\cdot)$ in $B_{2\rho+3}$. In view of (H3) and of the relation $s = 1-4\eta$, we infer that there exists a constant C, only depending on m, Λ and ρ, such that

$$I_1 \geqslant -C\,(1-s).$$

As for I_2, from (3.6) we infer

$$I_2 \geqslant -\Lambda\left((\rho+1)^m + M_{R_\eta}\right)|x_\varepsilon - y_\varepsilon|.$$

In the second case, notice that $|(p_\varepsilon + q_\varepsilon)/s| > \rho$. We set $\tilde{H}(x,p) := \max\{H(x,p), \mu(x)\}$ for every $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$, with $\mu(x) := \inf_{|p|\geqslant\rho} H(x,p)$. Now remark that such $\tilde{H}$ belongs to $\mathcal{H}\big(m, \tilde{\Lambda}, (\tilde{a}_r)_{r>0}, (\tilde{M}_r)_{r>0}\big)$ for suitable constants $\tilde{\Lambda} > 0$, $(\tilde{a}_r)_{r>0}$ in $(0,1]$, $(\tilde{M}_r)_{r>0}$ in $[1,+\infty)$, and it can be written as in (3.1) with $\max\{H_i(x,p), \mu(x)\}$ in place of H_i, for each $i \in \mathcal{I}$. We can therefore apply Lemma 3.4 to $\tilde{H}$ and conclude that H satisfies (3.5) since $H = \tilde{H}$ on $\mathbb{R}^d \times \big(\mathbb{R}^d \setminus B_\rho\big)$, by definition of $\tilde{H}$. The proof is complete. □

3.3. **Examples.** In this subsection we give a couple of examples of Hamiltonians in the class $\mathcal{H}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ that can be written in the form (3.1) for some functions $H_i \in \mathcal{B}(m_i, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$.

Example 3.6. Let $H \in \mathcal{H}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ be of the form

$$H(x,p) := K(x,p) + G(x,p), \qquad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where K is a convex Hamiltonian belonging to $\mathcal{H}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$, while

$$G(x,p) := \inf_{i\in\mathcal{I}} \{\langle g_i(x), p\rangle + f_i(x)\}, \qquad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where the functions $g_i : \mathbb{R}^d \to \mathbb{R}^d$ and $f_i : \mathbb{R}^d \to \mathbb{R}$ are Borel measurable and equibounded. Then $H = \inf_{i\in\mathcal{I}} H_i$, where the functions

$$H_i(x,p) := K(x,p) + \langle g_i(x), p\rangle + f_i(x)$$

belong to $\mathcal{B}\big(m, \tilde\Lambda, (\tilde a_r)_{r>0}, (\tilde M_r)_{r>0}\big)$, for suitable constants $\tilde\Lambda > 0$, $(\tilde a_r)_{r>0}$ in $(0,1]$, $(\tilde M_r)_{r>0}$ in $[1,+\infty)$.

Our second example consists in considering $H \in \mathcal{H}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ such to satisfy a semiconcavity–type condition in p inside a compact set of momenta and a convexity condition in p in the complement. This example is, of course, already covered by Proposition 3.5. We include it nevertheless to show why it is useful to drop continuity with respect to x for the approximating functions H_i and to allow the associated exponents to possibly depend on the index. We remark that it is natural to expect some kind of semi–concavity property in p for a Hamiltonian of the form (3.1). Indeed, the fact that the H_i are convex in p and are trapped between two paraboloids, according to condition (1.4) in (H4), should entail, loosely speaking, a form of equi–semiconcavity in p for the approximating functions H_i, locally with respect to x.

Example 3.7. Let $H \in \mathcal{H}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ and assume there exist $\rho > 0$ and $K \in \mathcal{B}(\ell, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ for some $\ell \geqslant m$ such that

(a) $H(x,\cdot) - K(x,\cdot)$ is concave in B_ρ for every $x \in \mathbb{R}^d$;

(b) the function $\max\{H(x,p), \mu(x)\}$ is convex in p for every fixed $x \in \mathbb{R}^d$, with $\mu(x) := \inf_{|p|\geqslant\rho} H(x,p)$.

Then there exists a family of functions $H_i \in \mathcal{B}(m_i, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$ such that H can be written in the form (3.1).

Let us prove the assertion. We first remark that μ is locally bounded on $\mathbb{R}^d$ since

$$-M_r \leqslant \mu(x) \leqslant \min_{|p|=\rho} H(x,p) \leqslant \Lambda(|\rho|^m + 1) \quad \text{for all } x \in B_r \text{ and } r > 0,$$

in view of the fact that H satisfies (1.4). This readily implies that the function $H_\flat(x,p) := \max\{H(x,p), \mu(x)\}$ belongs to $\mathfrak{B}(m, \Lambda, (a_r)_{r>0}, (M_r)_{r>0})$, for possibly different constants Λ, $(a_r)_{r>0}$, $(M_r)_{r>0}$. Note that $H(x,p) \geqslant \mu(x)$ for every $x \in \mathbb{R}^d$ and $|p| \geqslant \rho$, by definition of $\mu(x)$, in particular

$$H_\flat(x,p) = H(x,p) \qquad \text{for } x \in \mathbb{R}^d \text{ and } |p| \geqslant \rho. \tag{3.10}$$

By assumption, the function $F(x,p) := H(x,p) - K(x,p)$ is concave in B_ρ with respect to p, for every fixed $x \in \mathbb{R}^d$, and Borel–measurable with respect to (x,p). By well known result of convex analysis, for every fixed $q \in B_\rho$ we know that

$$F(x,p) \leqslant \langle \xi, p-q\rangle + F(x,q) \qquad \text{for all } p \in B_\rho, \tag{3.11}$$

with equality holding at $p = q$, where ξ is any vector in the superdifferential $\partial_p^+ F(x,q)$, in the sense of convex analysis, of $F(x,\cdot)$ at q. By the measurable selection Theorem, see [6, Theorem III.30], we infer that there exists a Borel–measurable map $\xi : \mathbb{R}^d \times B_\rho \to \mathbb{R}^d$ such that $\xi(x,q) \in \partial_p^+ F(x,q)$ for every $(x,q) \in \mathbb{R}^d \times B_\rho$. Since the function F satisfies condition (H3), it is easily seen that there exists a constant $C > 0$ such that $|\xi(x,q)| \leqslant C$

for every $(x,q) \in \mathbb{R}^d \times B_\rho$. We introduce the set of indexes $\mathcal{I} := B_\rho$ and for every $q \in \mathcal{I}$ we set

$$G_q(x,p) := K(x,p) + \langle \xi(x,q), p-q \rangle + F(x,q) \qquad \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Notice that, in view of (3.11) and of the continuity of H and G_q in p, we have

$$H(x,p) \leqslant G_q(x,p) \qquad \text{for all } (x,p) \in \mathbb{R}^d \times \overline{B}_\rho, \tag{3.12}$$

with equality holding at $p = q$. For every $q \in \mathcal{I}$ we set

$$H_q(x,p) := \begin{cases} G_q(x,p) & \text{if } |p| \leqslant \rho, \\ \max\{G_q(x,p), H_\flat(x,p)\} & \text{if } |p| > \rho. \end{cases}$$

Note that, for every fixed $x \in \mathbb{R}^d$, the function $H_q(x,\cdot)$ thus defined is continuous since $G_q(x,p) \geqslant H(x,p) = H_\flat(x,p)$ for every $|p| = \rho$, in view of (3.12) and (3.10). Furthermore, by construction,

$$H \leqslant H_q \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d \quad \text{for each } q \in \mathcal{I}, \qquad H = \inf_{q \in \mathcal{I}} H_q \quad \text{in } \mathbb{R}^d \times \overline{B}_\rho, \tag{3.13}$$

An easy check shows that each H_q satisfies (1.4) and (H3) with ℓ in place of m and for suitable constants $\tilde{\Lambda}$, $(\tilde{a}_r)_{r>0}$, $(\tilde{M}_r)_{r>0}$, independent of $q \in \mathcal{I}$. We claim that $H_q(x,\cdot)$ is convex on $\mathbb{R}^d$, for every fixed $x \in \mathbb{R}^d$ and $q \in \mathcal{I}$. To prove this, we will show that the function $H_q(x,\cdot)$ possesses a subdifferential at each point $p_0 \in \mathbb{R}^d$. If p_0 is such that $H_q(x,p_0) = G_q(x,p_0)$, it suffices to take a subdifferential of the convex function $G_q(x,\cdot)$ at p_0. Let us then assume $G_q(x,p_0) < H_q(x,p_0) = H_\flat(x,p_0)$, implying in particular that $|p_0| > \rho$. Let η be a subdifferential of the convex function $H_\flat(x,\cdot)$ at p_0, i.e.

$$f(p) := H_\flat(x,p_0) + \langle \eta, p - p_0 \rangle \leqslant H_\flat(x,p) \qquad \text{for every } p \in \mathbb{R}^d.$$

To prove that η is a subdifferential of $H_q(x,\cdot)$ on $\mathbb{R}^d$, it suffices to show that the function $\varphi(p) := G_q(x,p) - f(p)$ is nonnegative on $\overline{B}_\rho$. But this is clearly true since $\varphi \geqslant 0$ on ∂B_ρ, $\varphi(p_0) < 0$ and φ is convex on $\mathbb{R}^d$. We conclude that $H_q \in \mathcal{B}\big(\ell, \tilde{\Lambda}, (\tilde{a}_r)_{r>0}, (\tilde{M}_r)_{r>0}\big)$ for every $q \in \mathcal{I}$.

The asserted representation formula for H is finally obtained by remarking that

$$H(x,p) = \inf_{q \in \mathcal{I} \cup \{\flat\}} H_q(x,p) \qquad \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d$$

in view of (3.10) and (3.13).

It would be very interesting, in Example 3.6, to take as G a concave function of p of more general form, for instance such that $-G \in \mathcal{H}\left(\ell, \Lambda, (a_r)_{r>0}, (M_r)_{r>0}\right)$ for some $\ell < m$; or to allow $\rho = +\infty$ in Example 3.7, which is basically an equivalent fact. Such an extension seems out of reach with the techniques we have employed. We remark that an analogous question was raised in [9, Remark 2.1].

Appendix A

In this section we give a proof of Theorem 3.3 and Proposition 1.4.

Proof of Theorem 3.3. We assume $\sup_{\partial_P((0,T)\times U)} \big(v - w\big) < +\infty$, being the statement otherwise trivial. We set $\phi(x) := \sqrt{1 + |x|^2}$ and remark that, due to hypothesis (3.3), the linear growth of ϕ at infinity and the upper semicontinuity of v and $-w$, for every $\varrho > 0$ there exists $\mu_\varrho > 0$ such that

$$v(t,x) \leqslant \varrho\phi(x) + \mu_\varrho, \quad -w(t,x) \leqslant \varrho\phi(x) + \mu_\varrho \qquad \text{for all } (t,x) \in (0,T) \times U. \tag{A.1}$$

Fix $b > 0$ and first observe that $\tilde{w} := w + b/(T-t)$ satisfies

$$\partial_t \tilde{w} - \operatorname{tr}(A(x) D_x^2 \tilde{w}) + H(x, D_x \tilde{w}) \geqslant \frac{b}{T^2} =: c \quad \text{in } (0,T) \times U. \tag{A.2}$$

Clearly, it is enough to prove the assertion for v and $\tilde{w}$ for any fixed $b > 0$. We will thus prove the comparison principle under the additional assumption that w solves (A.2) and that, for every $\varrho > 0$, there exists $\mu_\varrho > 0$ such that

$$-w(t,x) \leqslant \varrho\phi(x) + \mu_\varrho - \frac{b}{T-t} \qquad \text{for all } (t,x) \in (0,T) \times U. \tag{A.3}$$

Moreover, up to adding to v a suitable constant, we will also assume, without any loss of generality, that $\sup_{\partial_P((0,T)\times U)} \big(v - w\big) = 0$. The assertion is thus reduced to proving that $v \leqslant w$ in $(0,T) \times U$. We argue by contradiction: suppose that $v > w$ at some point of $(0,T) \times U$, which, up to translations, we can assume to be of the form $(\bar{t}, 0)$ for some $\bar{t} \in (0,T)$, and set $\theta := v(\bar{t},0) - w(\bar{t},0) > 0$. Fix $\eta \in (0, \theta/4)$, $s \in (0,1)$ and $\varepsilon \in (0,1)$, and consider the auxiliary function $\Phi : [0,T] \times \overline{U} \times \overline{U} \to \mathbb{R}$ defined by

$$\Phi(t,x,y) := sv(t,x) - w(t,y) - \frac{|x-y|^2}{2\varepsilon} - \eta\phi(x), \quad (t,x,y) \in [0,T] \times \overline{U} \times \overline{U}.$$

Choose $s_0 \in (1/2, 1)$ sufficiently close to 1 so that

$$\Phi(\bar{t},0,0) = sv(\bar{t},0) - w(\bar{t},0) - \eta\phi(0) > \frac{\theta}{2} \quad \text{for all } \eta \in (0,\theta/4) \text{ and } s \in (s_0, 1).$$

By using (A.1) and (A.3), a tedious but standard computation shows that there exists $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0,T] \times \overline{U} \times \overline{U}$ such that

$$\Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = \sup_{(0,T)\times U \times U} \Phi \geqslant \Phi(\bar{t},0,0) > \frac{\theta}{2}. \tag{A.4}$$

By [7, Lemma 3.1], up to subsequences,

$$\lim_{\varepsilon\to 0} (t_\varepsilon, x_\varepsilon, y_\varepsilon) = (t_0, x_0, x_0) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0 \tag{A.5}$$

for some $(t_0, x_0) \in [0,T] \times \overline{U}$ satisfying

$$sv(t_0,x_0) - w(t_0,x_0) - \eta\phi(x_0) = \sup_{(t,x)\in(0,T)\times U} \Phi(t,x,x) > \frac{\theta}{2}. \tag{A.6}$$

By exploiting inequalities (A.1) and (A.3) with $\varrho := \eta/4$ in (A.6), we easily get that any point $(t_0,x_0) \in [0,T] \times \overline{U}$ enjoying (A.6) satisfies

$$\eta\phi(x_0) + \frac{2b}{T-t_0} \leqslant 4\mu_{\eta/4} - \theta.$$

We infer that there exist a constant $R_\eta > 1$, only depending on $\eta > 0$, and a constant $T_{b,\eta} \in (0,T)$, depending on $b > 0$ and $\eta > 0$, such that $|x_0| \leqslant R_\eta - 1$ and $t_0 \leqslant T_{b,\eta}$. Furthermore, any such point (t_0, x_0) actually lies in $(0,T) \times U$ provided

$$(1-s) \leqslant \min\left\{4\eta, \frac{\theta}{2\mu_{1/4}}\right\}, \tag{A.7}$$

where $\mu_{1/4}$ is the positive constant appearing in (A.1) and (A.3) with $\varrho = 1/4$. Indeed, if $(t_0,x_0) \in \partial_P((0,T) \times U)$, by exploiting the parabolic boundary condition $v \leqslant w$ on $\partial_P((0,T)\times U)$, we get

$$\frac{\theta}{2} < (1-s)\big(-w(t_0,x_0)\big) - \eta\phi(x_0) \leqslant \left(\frac{1-s}{4} - \eta\right)\phi(x_0) + (1-s)\mu_{1/4},$$

which is never satisfied as soon as s is chosen as in (A.7).

Let us hereafter choose $s = 1 - 4\eta$ and $0 < \eta < \min\big\{1/8, \theta/4, \theta/(8\mu_{1/4})\big\}$, so that $(t_0,x_0) \in (0,T) \times U$. In particular, $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in (0,T) \times U \times \mathbb{R}^d$ for sufficiently small $\varepsilon > 0$. Now we use (A.4), the fact that v is a subsolution of (3.2) and w is a supersolution of (A.2), and [7, Theorem 8.3] to infer that there exist $\tau_\varepsilon \in \mathbb{R}$ and symmetric $d \times d$ matrices $X_\varepsilon, Y_\varepsilon$ satisfying

$$-\frac{3}{\varepsilon}\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \leqslant \begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leqslant \frac{3}{\varepsilon}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}$$

such that

$$\tau_\varepsilon - \operatorname{tr}\Big(A(x_\varepsilon)\Big(X_\varepsilon + \eta D\phi(x_\varepsilon)\Big)\Big) + sH\left(x_\varepsilon, \frac{p_\varepsilon + q_\varepsilon}{s}\right) \leqslant 0, \tag{A.8}$$

$$\tau_\varepsilon - \operatorname{tr}\left(A(y_\varepsilon)Y_\varepsilon\right) + H\left(y_\varepsilon, p_\varepsilon\right) \geqslant c, \tag{A.9}$$

where we have set

$$p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \quad \text{and} \quad q_\varepsilon := \eta D\phi(x_\varepsilon).$$

As usual, the idea is to derive a contradiction by showing that the difference between (A.9) and (A.8) must be negative, after sending first $\varepsilon \to 0^+$ and then $\eta \to 0^+$ (and consequently $s = 1 - 4\eta \to 1^-$).

To estimate difference between the terms involving A in (A.9) and (A.8), we argue as in the proof of Theorem 2.1 in [1] to get

$$\operatorname{tr}\Big(A(x_\varepsilon)\Big(X_\varepsilon + \eta D^2\psi(x_\varepsilon)\Big) - A(y_\varepsilon)Y_\varepsilon\Big) \leqslant \tilde{C}\left(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + \eta\right) \tag{A.10}$$

for some constant $\tilde{C} > 0$ independent of ε and η. Therefore, by subtracting (A.8) from (A.9) and by taking into account (A.10) and Lemma 3.4, we end up with

$$0 < c \leqslant \tilde{C}\left(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + \eta\right) + C(1-s) + C_\eta|x_\varepsilon - y_\varepsilon|. \tag{A.11}$$

Now we send $\varepsilon \to 0^+$ and then $\eta \to 0^+$ (and consequently $s = 1 - 4\eta \to 1^-$) in (A.11) and we obtain the sought contradiction, in view of (A.5). □

We now proceed to give a proof of Proposition 1.4, that, as we will see, is derived via a minor modification from the one just presented. In what follows, we will denote by $D_x^+ v(t_\varepsilon, x_\varepsilon)$ the set of *superdifferentials* of the function $v(t_\varepsilon, \cdot)$ at the point x_ε, and by $D_x^- w(t_\varepsilon, y_\varepsilon)$ the set of *subdifferentials* of the function $w(t_\varepsilon, \cdot)$ at the point y_ε.

Proof of Proposition 1.4. We argue as in the proof of Theorem 3.3 choosing now $s = 1$. The only difference consists in the estimate of the term $H\left(x_\varepsilon, p_\varepsilon + q_\varepsilon\right) - H\left(y_\varepsilon, p_\varepsilon\right)$. Notice that $p_\varepsilon + q_\varepsilon \in D_x^+ v(t_\varepsilon, x_\varepsilon)$ and $p_\varepsilon \in D_x^- w(t_\varepsilon, y_\varepsilon)$. From the fact that either $\|D_x v\|_{L^\infty((0,T)\times U)}$ or $\|D_x w\|_{L^\infty((0,T)\times U)}$ is finite, let us say less than a positive constant κ, and that $|q_\varepsilon| < \eta$, we infer

$$|p_\varepsilon + q_\varepsilon| \leqslant \kappa + \eta, \qquad |p_\varepsilon| \leqslant \kappa + \eta.$$

Let us choose $\eta < 1$ and let ω be a continuity modulus of H in $U \times B_{\kappa+1}$. We have

$$\left|H\left(x_\varepsilon, p_\varepsilon + q_\varepsilon\right) - H\left(y_\varepsilon, p_\varepsilon\right)\right| \leqslant \omega\left(|x_\varepsilon - y_\varepsilon| + \eta\right). \tag{A.12}$$

The assertion follows by arguing as in the proof of Theorem 3.3 and by using (A.12) in place of the inequality (3.5) stated in Lemma 3.4. □

References

[1] S. N. Armstrong and H. V. Tran, *Viscosity solutions of general viscous Hamilton-Jacobi equations*, Math. Ann., 361 (2015), pp. 647–687.

[2] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, vol. 17 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Paris, 1994.

[3] G. Barles, S. Biton, M. Bourgoing, and O. Ley, *Uniqueness results for quasilinear parabolic equations through viscosity solutions' methods*, Calc. Var. Partial Differential Equations, 18 (2003), pp. 159–179.

[4] G. Barles, S. Biton, and O. Ley, *Uniqueness for parabolic equations without growth condition and applications to the mean curvature flow in* $\mathbb{R}^2$, J. Differential Equations, 187 (2003), pp. 456–472.

[5] G. Barles and P. E. Souganidis, *Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations*, SIAM J. Math. Anal., 32 (2001), pp. 1311–1323 (electronic).
[6] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin-New York, 1977.
[7] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.
[8] F. Da Lio and O. Ley, *Uniqueness results for second-order Bellman-Isaacs equations under quadratic growth assumptions and applications*, SIAM J. Control Optim., 45 (2006), pp. 74–106 (electronic).
[9] ———, *Convex Hamilton-Jacobi equations under superlinear growth conditions on data*, Appl. Math. Optim., 63 (2011), pp. 309–339.
[10] A. Davini and E. Kosygina, *Homogenization of viscous and non-viscous HJ equations: a remark and an application*, ArXiv e-prints, (2016).
[11] M. I. Garrido and J. A. Jaramillo, *Lipschitz-type functions on metric spaces*, J. Math. Anal. Appl., 340 (2008), pp. 282–290.
[12] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, vol. 12 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1996.
[13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.
[14] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[15] P.-L. Lions and P. E. Souganidis, *Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media*, Comm. Partial Differential Equations, 30 (2005), pp. 335–375.

Dip. di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, 00185 Roma, Italy
E-mail address: davini@mat.uniroma1.it