System of fermions with zero-range interactions

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We discuss the stability problem for a system of N identical fermions with unit mass interacting with a different particle of mass m via zero-range interactions in dimension three. We find a stability parameter $m^*(N) > 0$, increasing with N, such that the Hamiltonian of the system is self-adjoint and bounded from below for $m > m^*(N)$.

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1. Introduction

A system of n quantum particles in $\mathbb{R}^3$ with two-body zero-range interactions is described by the formal Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{n} \frac{1}{2m_i} \Delta_{\mathbf{x}_i} + \sum_{\substack{i,j=1\\ i<j}}^{n} \mu_{ij}\, \delta(\mathbf{x}_i - \mathbf{x}_j), \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^3$, $i = 1, \ldots, n$, m_i is the mass, $\Delta_{\mathbf{x}_i}$ is the Laplacian relative to $\mathbf{x}_i$, and $\mu_{ij} \in \mathbb{R}$ are coupling constants. We set $\hbar = 1$. In recent years these class of Hamiltonians have been extensively used in the physical literature to describe the dynamics of ultra-cold quantum gases (see e.g. Ref. 2). From the mathematical point of view, the definition of $\mathcal{H}$ as a self-adjoint operator in $L^2(\mathbb{R}^{3n})$ is usually given in the following way. Let us consider the operator

$$\dot{\mathcal{H}}_0 = -\sum_{i=1}^{n} \frac{1}{2m_i} \Delta_{\mathbf{x}_i}, \qquad D(\dot{\mathcal{H}}_0) = C_0^\infty(\mathbb{R}^{3n} \setminus \cup_{i<j} \{\mathbf{x}_i = \mathbf{x}_j\}) \tag{2}$$

where $\{\mathbf{x}_i = \mathbf{x}_j\}$ denotes the hyperplane characterized by the coincidence of the coordinates $\mathbf{x}_i$ and $\mathbf{x}_j$. The operator (2) is symmetric but not self-adjoint. Then, any self-adjoint extension of $\dot{\mathcal{H}}_0$, different from the free Hamiltonian, is by definition a Hamiltonian of n quantum particles in $\mathbb{R}^3$ with two-body zero-range interactions. Roughly speaking, such Hamiltonians act as the free Hamiltonian in $\mathbb{R}^3 \setminus \cup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\}$ and the elements of the domain satisfy a sort of generalized boundary condition on each hyperplane $\{\mathbf{x}_i = \mathbf{x}_j\}$. The explicit construction of the self-adjoint extensions is not trivial and a general characterization is not known, except in the simpler case $n = 2$. Indeed, for $n = 2$ and extracting the center of mass motion, the domain of each extension consists of $\psi \in L^2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3 \setminus \{0\})$

satisfying the boundary condition at the origin

$$\psi(\mathbf{x}) = \frac{q}{|\mathbf{x}|} + \alpha q + o(1) \tag{3}$$

for $|\mathbf{x}| \to 0$, where $\mathbf{x}$ is the relative coordinate, $q \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ is the strength of the interaction characterizing each extension (see Ref. 1 for details). In the general case $n > 2$, proceeding by analogy, one introduces the so-called Skornyakov-Ter-Martirosyan (STM) extension H_α, defined on elements of $L^2(\mathbb{R}^{3n}) \cap H^2(\mathbb{R}^{3n} \setminus \cup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\})$ satisfying the boundary condition

$$\psi(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{q_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|} + \alpha q_{ij} + o(1) \tag{4}$$

for $|\mathbf{x}_i - \mathbf{x}_j| \to 0$, where q_{ij} are functions defined on $\{\mathbf{x}_i = \mathbf{x}_j\}$ and α is the strength of the interaction. As a matter of fact, the STM extension H_α is a symmetric operator but, in general, it is not self-adjoint. This happens, for instance, in the cases of three identical bosons and three different particles. In such cases, it is known that any self-adjoint extension of H_α is unbounded from below (see Refs. 8 and 10). Such instability effect is known in the literature as Thomas effect.

On the other hand, there is an important physical situation where one can expect absence of the Thomas effect, i.e., the case of a two groups of particles made of identical fermions. This is due to the fact that the antisymmetry constraint makes the zero-range interaction ineffective for fermions of the same type. In this generality the stability problem is open. Here we consider the particular case of N identical fermions, with unit mass, interacting via zero-range interactions with a different particle of mass m.

In the physical literature (see e.g. Refs. 2, 3, 4, 5) the case $N = 2$ and $N = 3$ has been studied using analytical and numerical arguments. In both cases it is found that for m larger than a critical mass, depending on N, the STM extension H_α is self-adjoint and bounded from below, while for m smaller the Thomas effect occurs. Rigorous proofs of the above results for $N = 2$ have been obtained in Refs. 13, 14, 12, while in Ref. 11 a stability result for $N < 5$ and m sufficiently large is found.

In the next section we discuss a stability result recently obtained in Ref. 6 (see also Ref. 9) valid for any N. In particular we find a stability parameter $m^*(N) > 0$ (see definition below) such that stability holds for $m > m^*(N)$. The result has been obtained exploiting a quadratic form method along the line developed in Ref. 7.

2. Result

In order to explain the result, we introduce the following function

$$\Lambda(m, N) = \frac{2}{\pi}(N-1)(m+1)^2 \left[\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right) \right] \tag{5}$$

It is easy to see that, for each N, the function $\Lambda(m, N)$ is positive, decreasing with m and satisfies $\lim_{m\to 0} \Lambda(m, N) = \infty$, $\lim_{m\to\infty} \Lambda(m, N) = 0$. Therefore we can introduce the following definition.

Definition 2.1. (Stability parameter $m^*(N)$). For any N fixed, $m^*(N)$ is the unique solution of the equation

$$\Lambda(m, N) = 1 \tag{6}$$

Notice that $m^*(N)$ is increasing with N. Moreover the condition $\Lambda(m, N) < 1$ is equivalent to $m > m^*(N)$, which is precisely our stability condition for the system expressed in the following theorem.

Theorem 2.1. *For $m > m^*(N)$ the STM extension H_α is self-adjoint and bounded from below. In particular H_α is positive for $\alpha \geq 0$ and*

$$\inf \sigma(H_\alpha) \geq -\frac{\alpha^2}{4\pi^4(1 - \Lambda(m, N))} \tag{7}$$

for $\alpha < 0$.

We remark that the result is optimal for $N = 2$, in the sense that, according to the previously mentioned results, for $m < m^*(2)$ the three-particle system exhibits the Thomas effect. On the other hand, for $N > 2$ our stability condition is only sufficient and, in order to improve the result, the role of the antisymmetry must be more carefully taken into account. The numerical value of $m^*(N)$ can also be easily computed. For example
$m^*(2) = 0.0735$, $m^*(3) = 0.1890$, ... , $m^*(8) = 0.9473$, $m^*(9) = 0.1215$, ...
We notice that in the special case of equal masses we find stability up to $N = 8$.
The proof of the theorem can be found in Ref. 6. Here we only outline the method based on the quadratic form naturally associated to H_α. Limiting ourselves to the simpler case $N = 2$, we introduce relatives coordinates $\mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{x}_0$, where $\mathbf{x}_1$, $\mathbf{x}_2$ are the coordinates of the two fermions and $\mathbf{x}_0$ the coordinate of the different particle. Moreover we denote by $\mathbf{k}_1$, $\mathbf{k}_2$ the corresponding coordinates in the Fourier space. Neglecting the center of mass motion, the Hilbert space is $L^2_f(\mathbb{R}^6)$, where the subscript f denotes the restriction to antisymmetric functions, and the free Hamiltonian in the Fourier space reads

$$\widehat{H_0} = \mathbf{k}_1^2 + \mathbf{k}_2^2 + \frac{2}{m+1}\mathbf{k}_1 \cdot \mathbf{k}_2 \tag{8}$$

The quadratic form associated to H_0 is $F_0(u) = (u, H_0 u)$ and it is closed and positive on the domain $D(F_0) = H^1_f(\mathbb{R}^6)$. The quadratic form associated to H_α turns out to be a singular perturbation of F_0, defined on a domain larger than $H^1_f(\mathbb{R}^6)$, which can be derived using a renormalization procedure (explained in Ref. 6). The procedure leads to the following definition

Definition 2.2. For any $\alpha \in \mathbb{R}$, the quadratic form $\mathcal{F}_\alpha$, $D(\mathcal{F}_\alpha)$ is given by

$$D(\mathcal{F}_\alpha) = \left\{ u \in L^2_f(\mathbb{R}^6) \,|\, u = w + \mathcal{G}\xi,\ |\nabla w| \in L^2_f(\mathbb{R}^6),\ \xi \in H^{1/2}(\mathbb{R}^3) \right\} \tag{9}$$

$$\mathcal{F}_\alpha(u) = F_0(w) + 2\left(\Phi(\xi) + \alpha \|\xi\|^2\right) \tag{10}$$

where $\mathcal{G}\xi$ in the Fourier space is

$$\widehat{\mathcal{G}\xi}(\mathbf{k}_1,\mathbf{k}_2) = \frac{\hat{\xi}(\mathbf{k}_1) - \hat{\xi}(\mathbf{k}_2)}{\mathbf{k}_1^2 + \mathbf{k}_2^2 + \frac{2}{m+1}\,\mathbf{k}_1\cdot\mathbf{k}_2} \tag{11}$$

and

$$\Phi(\xi) = 2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}}\int\! d\mathbf{p}\,|\mathbf{p}||\hat{\xi}(\mathbf{p})|^2 + \int\! d\mathbf{p}d\mathbf{q}\,\frac{\overline{\hat{\xi}(\mathbf{p})}\hat{\xi}(\mathbf{q})}{\mathbf{p}^2+\mathbf{q}^2+\frac{2}{m+1}\mathbf{p}\cdot\mathbf{q}} \tag{12}$$

We remark that $\mathcal{G}\xi$ is locally in $L^2_f(\mathbb{R}^6)$ and $|\nabla\mathcal{G}\xi| \notin L^2_f(\mathbb{R}^6)$. We also notice that in Ref. 6 the definition of the quadratic form is given in a slightly different form, but it is easily seen that the two definitions are equivalent.
The proof of the theorem is essentially based on the analysis of the above quadratic form. In particular, we show that for $m > m^*(2)$ the form is closed and bounded from below and that the associated self-adjoint and bounded from below operator coincides with the STM extension H_α.
The key point of the proof is to show that $\Phi(\xi) \geq 0$ for $m > m^*(2)$. Since such a result can be obtained using elementary methods, in the remaining few lines we summarize the main steps.
- We introduce the expansion in spherical harmonics $\hat{\xi}(\mathbf{p}) = \sum_{lm}\xi_{lm}(p)Y_l^m(\theta,\phi)$ and we have

$$\begin{aligned}\Phi(\xi) &= 2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}}\sum_{lm}\int_0^\infty\! dp\,p^3|\xi_{lm}(p)|^2 \\ &+2\pi\sum_{lm}\int_0^\infty\! dp\int_0^\infty\! dq\,p^2\overline{\xi_{lm}(p)}\,q^2\xi_{lm}(q)\int_{-1}^1\! dy\frac{P_l(y)}{p^2+q^2+\frac{2}{m+1}pqy}\end{aligned} \tag{13}$$

- Using the map $\mathcal{M}\,:\,\xi_{lm}\to\xi^\sharp_{lm}$ defined as

$$\xi^\sharp_{lm}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\! dx\,e^{-ikx}e^{2x}\xi_{lm}(e^x) \tag{14}$$

and performing explicit integrations, one obtains

$$\Phi(\xi) = \sum_{lm}\int_{\mathbb{R}}\! dk\left(2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}} + S_l(k)\right)|\xi^\sharp_{lm}(k)|^2 \tag{15}$$

where

$$S_l(k) = 2\pi^2\int_{-1}^1 dy\,P_l(y)\frac{\sinh\left(k\arccos\frac{y}{m+1}\right)}{\sin\left(\arccos\frac{y}{m+1}\right)\,\sinh(\pi k)} \tag{16}$$

- By an explicit study of the function S_l one sees that

$$S_l(k) \geq 0 \qquad \text{for } l \text{ even} \tag{17}$$

$$S_l(k) \geq S_1(k) \geq S_1(0) \qquad \text{for } l \text{ odd} \tag{18}$$

for any $k \in \mathbb{R}$, where

$$S_1(0) \equiv -4\pi(m+1)\sqrt{m(m+2)}\left[\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right)\right] < 0 \tag{19}$$

- Taking into account of the above bounds one has

$$\begin{aligned}\Phi(\xi) &\geq \sum_{lm}\int_{\mathbb{R}} dk \left(2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}} + S_1(0)\right)|\xi^\sharp_{lm}(k)|^2 \\ &= 2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}}\sum_{lm}\int_{\mathbb{R}} dk\,\big(1-\Lambda(m,2)\big)|\xi^\sharp_{lm}(k)|^2\end{aligned} \tag{20}$$

where $\Lambda(m, 2)$ has been defined in (5). Since $\Lambda(m, 2) < 1$ is equivalent to $m > m^*(2)$, formula (20) implies positivity of $\Phi(\xi)$ for $m > m^*(2)$.

References

1. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, New-York, 1988).
2. E. Braaten, H.W. Hammer, Universality in few-body systems with large scattering length, *Phys. Rep.* **428**, 259 (2006).
3. Y. Castin, C. Mora, L. Pricoupenko, Four-Body Efimov Effect for Three Fermions and a Lighter Particle, *Phys. Rev. Lett.* **105**, 223201 (2010).
4. Y. Castin, F. Werner, Unitary gas in an isotropic harmonic trap: symmetry properties and applications, *Phys. Rev. A* **74**, 053604 (2006).
5. Y. Castin, F. Werner, Unitary quantum three-body problem in a harmonic trap, *Phys. Rev. Lett.* **97**, 150401 (2006).
6. M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, A. Teta, Stability for a system of N fermions plus a different particle with zero-range interactions, *Rev. in Math. Phys.* **24**, no. 5, 1250017 (2012).
7. G. Dell'Antonio, R. Figari, A. Teta, Hamiltonians for Systems of N Particles Interacting through Point Interactions, *Ann. Inst. H. Poincaré Phys. Théor.* **60**, 253 (1994).
8. L. Faddeev, R.A. Minlos, On the point interaction for a three-particle system in Quantum Mechanics, *Soviet Phys. Dokl.* **6**, 1072 (1962).
9. D. Finco, A. Teta, Quadratic Forms for the Fermionic Unitary Gas Model, *Reports on Math. Phys.* **69**, no. 2, 131 (2012).
10. A.M. Melnikov, R.A. Minlos, On the Pointlike Interaction of Three Different Particles, *Adv. Soviet Math.* **5**, 99 (1991).
11. R.A. Minlos, On point-like interaction between n fermions and another particle, *Moscow Math. Journal* **11**, 113 (2011).
12. R.A. Minlos, On point-like interaction between three particles (two fermions and another particle), *ISRN Math. Phys.* **2012**, 230245 (2012).

13. R.A. Minlos, M.K. Shermatov, On Pointlike Interaction of Three Particles, *Vestnik Mosk. Univ. Ser. Math. Mekh.* **6**, 7 (1989).
14. M.K. Shermatov, Point Interaction between Two Fermions and a Particle of Different Nature, *Theo. Math. Phys.* **136**, 1119 (2003).