Classical-like trajectories of a quantum particle in a cloud chamber

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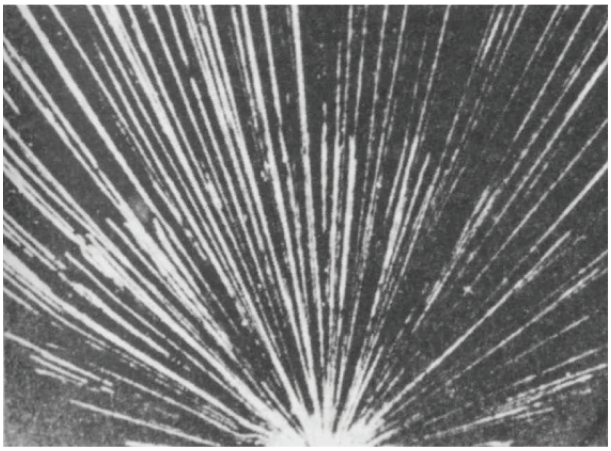
Abstract

We review old and new explanations of a case of quantum particle evolution with an intriguing phenomenology: the formation of particle tracks in a tracking chamber. We recall the early debate on the subject at the fifth Solvay Conference and we present the out of the mainstream point of view assumed by Mott to examine the problem in 1929. We then give some details on recent results clarifying, making rigorous and generalizing Mott's analysis. In the conclusions, we propose some new directions of research and some open problems in the topic.

1 The Wilson cloud chamber

The cloud chamber, *"the most original and wonderful instrument in scientific history"* according to Ernest Rutherford, was devised and made available by Charles Thomson Rees Wilson at the end of the first decade of last century.
The main idea of Wilson was to make visible the "ionizing radiation" by condensing water on the ions produced by α and β rays in air supersaturated with water-vapor. His aim was to give a conclusive experimental verification of the particle nature of the α and β-radiations.
On one hand, some experimental problems were overwhelmingly complicate to be confronted by. A perfect synchronization of the sudden cooling, of the illumination of the chamber and of the photographic capture was required.
On the other end, the overall intuitive picture of what was happening in the chamber was perfectly depicted: the α-particle, emitted by a radioactive source (in an unpredictable although specific direction), ionizes molecules of the gas filling the chamber. In turn, ions become condensation nuclei for water-vapor giving rise to a sequence of small drops of water along the trajectory of the α-particle.



In the early days of quantum mechanics, it was immediately noticed that the explanation of the phenomenon, as it was given by physicists at the time of Wilson, seemed at variance with some of the cornerstones of the new theory.
In fact, inside the framework of the orthodox theory, a classical trajectory of a quantum particle could only be the result of repeated "collapses" of its wave function due to repeated measurements. Pushing to the limit, each scattering event should be considered a measurement process. On the other end, in the orthodox approach, a measurement apparatus must be considered as a classical object whose evolution has to be described and predicted using classical kinematics and dynamics. Classical observables like position or trajectory can only be properties emerging as a consequence of the interaction of the microscopic system with the classical measurement apparatus. By no means, they are properties possessed by the system before the measurement.

It is clear that this point of view is hardly compatible with the interpretation of a simple scattering event as a measurement process.
The apparent contradiction shows that the issue of describing what "really" happens in a cloud chamber highlights the most subtle problems of the Copenhagen interpretation:

- Where has to be placed the frontier between the quantum and the classical realm? In particular,
- how large should be an array of atoms to behave as a classical system?
- When and where does the collapse process take place?

It is well known that the final formalization of quantum mechanics, given by John von Neumann in 1932, assumes two different evolution laws for a microscopic system: the continuous unitary evolution given by the Schrödinger equation, as long as the system remains isolated, and a stochastic and/or non-linear sudden change driving to the collapse of the wave function, during the measurement process. As is clear, von Neumann's dynamical assumptions leave the issues listed above unanswered.
In 1929, Charles Galton Darwin proposed a completely new way to look at the formation of classical tracks in a cloud chamber. In retrospect, one can say that the following simple Darwin's reflection was extremely far-reaching: the Schrödinger equation describes the evolution in the configuration space of the whole system. Taking into account the electrons of the gas atoms together with the particle, no contradiction exists in principle with the existence of solutions which show the α-particle moving in a small cone with apex in the radioactive source and only atoms in the cone ionized.
In the same year, Sir Nevill Francis Mott concretized Darwin's proposal analyzing the long time behavior of the Schrödinger equation solutions relative to the α-particle and the electrons of two hydrogen atoms playing the role of the gas in the cloud chamber.
Mott's paper did not receive the attention that it would have deserved and it remained almost unknown also during the '60s when the interest in probing the frontier between classical and quantum behavior had a new start.
By now, a significant amount of experimental and theoretical work in the field is available. More and more sophisticated experiments have pushed the border quantum-classical towards larger and larger system sizes. At the same time, new theoretical investigations analyzed the loss of quantum coherence due to the interaction of a microscopic system with a large environment (see, e.g., [JZKGKS], [BGJKS], [Z], [JZ], [HS], [H], [AFFT1], [AFFT2] and references therein).
Main purpose of this report is to present further attempts to investigate whether a purely quantum mechanical treatment can justify the (experimentally verified) statement that in in a cloud chamber filled with supersaturated gas an α-decay produces at most one sequence of liquid droplets (track) and that this track is compatible with the trajectory of a classical particle. In summary, we want to test the compatibility of the experimental outputs with the following rough statement: the α-wave is turned, by the interaction with the environment, into

an α-particle of the same energy as the initial wave and with a momentum direction having a definite orientation.

2 The earliest theoretical investigations

In 1928 Gamow ([Ga]) and Condon and Gurney ([CG]) made the first attempt to approach the α-decay phenomenon according to quantum mechanics. A crucial point of their analysis was that, at the time of the emission, the α-particle state had to be a spherical wave centered in the radioactive nucleus, with a highly isotropic average momentum. As pointed out before, it was immediately manifest that the assumption of a spherical wave as initial state makes the explanation of the tracks observed in a cloud chamber rather problematic.
Such a difficulty had already been mentioned by Born who, during the general discussion at the Solvay Conference in 1927 ([BV]), noted: *"Mr. Einstein has considered the following problem: A radioactive sample emits α-particles in all directions; these are made visible by the method of the Wilson cloud chamber. Now, if one associates a spherical wave with each emission process, how can one understand that the track of each α-particle appears as a (very nearly) straight line? In other words: how can the corpuscular character of the phenomenon be reconciled here with the representation by waves?"*
According to Born, the question could be answered using the notion of *"reduction of the probability packet"* induced by "observation" by means of light, discussed by Heisenberg in [He1]. Indeed, Born claimed: *"As soon as such ionization is shown by the appearance of cloud droplets, in order to describe what happens afterwards one must reduce the wave packet in the immediate vicinity of the drops. One thus obtains a wave packet in the form of a ray, which corresponds to the corpuscular character of the phenomenon"*. It should be stressed that, according to this point of view, the evolution of an α-particle in a cloud chamber can be described as the result of the interaction of a quantum system (the α-particle) with a classical measurement apparatus (the atoms of the vapor). Such interaction is responsible for the "reduction" of the spherical wave to a wave packet with definite position and momentum.
As a possible alternative description of the process, Born also considered an approach where both the α-particle and the atoms of the vapor are considered part of a unique quantum system, described by a wave function depending on the coordinates of all the particles of the system. In particular, he proposed a simplified one-dimensional model consisting of a test particle (the α-particle) and two harmonic oscillators placed at fixed positions (the atoms of the vapor). At initial time, the test particle is described by a superposition state of two wave packets with opposite momentum and position close to the origin, while the harmonic oscillators are in their ground states. A qualitative discussion of such a model led Born to the claim that the test particle has a very low probability of exciting both oscillators unless they are located on the same half-line starting from the origin (for a quantitative analysis see e.g. [DFT1], [FinT]).
The analysis of the model is then completed with a statement involving once again the re-

duction postulate: *"To the reduction of the wave packets corresponds the choice of one of the two directions of propagations"*, and the choice is done when the excitation of an oscillator is observed. Only after the observation, the evolution of the test particle can be assimilated to that of a classical particle propagating along a definite trajectory.
In conclusion, Born conceded that an analysis of the quantum evolution of the α-particle in interaction with the (quantum) environment made of the vapor atoms is possible, *"but this does not lead us further as regards the fundamental questions"*, in the sense that the reduction postulate is anyway required for a complete description.
A further fundamental contribution to the theoretical analysis of the cloud chamber problem was given by Heisenberg in his lectures at the University of Chicago in 1929, published in [He2]. He pointed out that a quantum description of an experimental situation always requires to fix an arbitrary border between the quantum system under consideration and the (classical) measuring apparatus. For any fixed border one has different, but equivalent, descriptions of the phenomenon. In the case of the cloud chamber Heisenberg considered the following two reasonable choices: a) the quantum system consists of the α-particle alone (and then the molecules of the vapor play the role of the measurement apparatus); b) the quantum system consists of the α-particle and of the molecules of the vapor.
In case a) the analysis proceeds as follows. The α-particle, evolving in the chamber as a spherical wave, collides with a molecule of the vapor which acts as a measuring device of the particle position. Therefore, immediately after the collision, the state of the α-particle is reduced to a narrow wave packet concentrated around the position of the molecule. Furthermore, one knows that at time zero the α-particle starts from the position of the radioactive nucleus and that the momentum is conserved. This implies that the wave packet has an average momentum along the direction γ joining the radioactive nucleus and the position of the molecule (notice that in this way Heisenberg states that the measurement actualize a posteriori the momentum possessed by the particle before the measurement itself).
Such wave packet emerging from the molecule propagates in the chamber according to the free Schrödinger dynamics, with an inevitable spreading in position. However, the α-particle collides with the next molecule placed along γ and a new position measurement takes place, determining a refocusing of the wave packet along γ. The process is repeated a large number of times and the result is that the wave packet remains concentrated along γ, which is identified as the observed "trajectory" of the α-particle.
In case b) the molecules of the vapor and the α-particle form a many-particle quantum system, whose dynamics is governed by the Schrödinger equation. Heisenberg notes that in this case the physical description *"is more complicated than the preceding method, but has the advantage that the discontinuous change of the probability function recedes one step and seems less in conflict with intuitive ideas."*
In order to give a qualitative idea of the behavior of the whole system, Heisenberg considers a three-particle model, made of the α-particle and two molecules with centers of mass fixed in the positions a_I, a_{II}. At initial time the α-particle is described by a plane wave with momentum p

and the molecules are in their ground states.
The problem is now reduced to find an approximate solution of the Schrödinger equation and to compute the probability that both molecules are excited. The procedure is only briefly outlined and many mathematical details are neglected, but the result, based on a deep physical intuition, is clearly stated: the probability that both molecules are excited is significantly different from zero only if the momentum p is parallel to the line joining a_I and a_{II}. Such a result can be considered a satisfactory explanation of the observed trajectory of an α-particle in the chamber, since the trajectory can only be observed through the excitations of the molecules.
However Heisenberg stresses that also in case b) the reduction postulate is used when one "observes" the excitation of the molecules. This means that in case b) the border between system and apparatus has only been moved to include the molecules in the system.
The conclusion is that, according to Heisenberg, the approaches in cases a) and b) are conceptually equivalent since in both cases the reduction postulate must be invoked for a complete description of the physical situation.
In 1929, C.G. Darwin ([Da]) presented a different interesting inspection of the cloud chamber problem. The approach proposed by Darwin is entirely based on the Schrödinger equation and it is surprisingly close to the one of decoherence theory developed in the last decades of the last century.
More precisely, he stresses that a satisfactory description of a quantum system $\mathcal{S}$ (like the α-particle) is achieved only if one takes into account its interaction with (part of) the environment $\mathcal{E}$. As a consequence, one has to compute the evolution of a wave function ψ depending on the coordinates of $\mathcal{S}$ and of $\mathcal{E}$. Given such ψ, the probabilistic predictions on the system $\mathcal{S}$ can be obtained by taking an average over all possible final configurations of the environment $\mathcal{E}$. Such a strategy is surely *"discouragingly complicated"*, but it can provide an explanation of the particle behavior of $\mathcal{S}$ without any reference to the reduction postulate.
In the case of the cloud chamber, the wave function ψ is a function of the coordinates of the α-particle and of the atoms in the chamber. At the initial time, such ψ can be reasonably assumed to be a product of the spherical wave for the α-particle and of the ground states for the atoms. *"But the first collision changes this product into a function in which the two types of coordinates are inextricably mixed, and every subsequent collision makes it worse."* A detailed computation of such ψ is impossible but one can obtain for ψ an integral representation containing a phase factor and *"without in the least seeing the details, it looks quite natural to expect that this phase factor will have some special character, such as vanishing, when the various co-ordinates satisfy a condition of collinearity."* It should be noted that in these words it is clearly outlined the stationary phase method as the correct technical tool to prove the emergence of the particle behavior.
Darwin's view can be summarized by saying that the wave function ψ is the only crucial object of quantum theory and all the particle and the wave properties of a system can be derived from an accurate analysis of ψ.

3 Mott's analysis

Darwin's program was concretely realized by Mott in his seminal paper [Mo]. Despite its importance, this work of Mott's does not seem to be sufficiently well known and therefore, in the following, we shall describe it in some details (we also refer the reader to [Fa], [CK], [FigT1], [FigT2] for further critical considerations on the paper).
In the introduction, Mott acknowledges having been inspired by Darwin's paper. He admits that the perspective outlined by Darwin seems counterintuitive at first, since *"it is a little difficult to picture how it is that an outgoing spherical wave can produce a straight track; we think intuitively that it should ionise atoms at random throughout space"*. Like Heisenberg, Mott points out that the crucial point is to establish the border line between the system under consideration and the measuring device. In a first possible approach (corresponding to case a) in Heisenberg's approach), the α-ray is the system and the gas of the chamber is the measurement device by which we observe the particle. Here, the α-ray must be considered a particle immediately after the disintegration process, since at that moment the gas (i.e., the device) reduces the initial spherical wave to a narrow wave packet with a definite momentum. In the other approach (case b) in Heisenberg), the α-particle and the gas are considered together as the system under consideration. In this case, are the ionized atoms to be observed and the wave function ψ of the system should provide the ionization probability. Only after the ionization has been observed are we allowed to consider the α-ray as a particle.
According to this point of view, this mentioned intuitive difficulty can be overcome, since it arises from our erroneous *"tendency to picture the wave as existing in ordinary three dimensional space, whereas we are really dealing with wave functions in multispace formed by the co-ordinates both of the α-particle and of every atom in the Wilson chamber."*
In the rest of his paper, Mott discusses a simple model showing how this second approach actually works. The model is essentially the same as the one considered by Heisenberg and it consists of the α-particle, initially described by a spherical wave centered at the origin, and only two hydrogen atoms. The nuclei of the atoms are supposed as at rest in the fixed positions $\mathbf{a_1}$, $\mathbf{a_2}$, with $|\boldsymbol{a}_1| < |\boldsymbol{a}_2|$. It is assumed that the α-particle does not interact with the nuclei, and the interaction between the two electrons is also neglected. Moreover, the interaction between the α-particle and the electrons is treated as a small perturbation. The main result of the paper can be summarized in the following statement:
the two hydrogen atoms cannot both be excited (or ionized) unless $\mathbf{a_1}$, $\mathbf{a_2}$ and the origin lie on the same straight line.
We shall outline the way Mott derives the result under suitable assumptions, trying to follow his original notation and line of reasoning. We suggest the reader who is not interested in mathematical details to skip the next few pages and proceed directly to the final remarks at the end of this section.
The main object of the investigation are periodic solutions $F(\boldsymbol{R}, \boldsymbol{r_1}, \boldsymbol{r_2})e^{iEt/\hbar}$ of the Schrödinger equation for the three-particle system, where $\boldsymbol{R}$, $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ denote the coordinates of the α-particle

and of the two electrons of the hydrogen atoms respectively. Such F is the solution of the stationary Schrödinger equation

$$-\frac{\hbar^2}{2M}\Delta_{\boldsymbol{R}}F+\Big(-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{r}_1}-\frac{e^2}{|\boldsymbol{r_1}-\boldsymbol{a_1}|}\Big)F+\Big(-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{r}_2}-\frac{e^2}{|\boldsymbol{r_2}-\boldsymbol{a_2}|}\Big)F \\ -\Big(\frac{2e^2}{|\boldsymbol{R}-\boldsymbol{r_1}|}+\frac{2e^2}{|\boldsymbol{R}-\boldsymbol{r_2}|}\Big)F=E\,F \quad (3.1)$$

where $\Delta_{\boldsymbol{x}}$ is the laplacian with respect to the coordinate $\boldsymbol{x}$, M is the mass of the α-particle, m is the mass of the electron, $-e$ is the charge of the electron and $2e$ is the charge of the α-particle.
The solution of equation (3.1) can be conveniently expanded in series of the eigenfunctions of the two hydrogen atoms. More precisely, let ψ_j be the j-th eigenfunction of a hydrogen atom centered in the origin, with ψ_0 denoting the ground state. Then the corresponding eigenfunctions of the atoms in $\mathbf{a_1}$, $\mathbf{a_2}$ are

$$\Psi^I_j(\boldsymbol{r_1})=\psi_j(\boldsymbol{r_1}-\boldsymbol{a_1}),\qquad \Psi^{II}_j(\boldsymbol{r_2})=\psi_j(\boldsymbol{r_2}-\boldsymbol{a_2}) \quad (3.2)$$

Note that here it seems tacitly assumed that the index j can be an integer or a real positive number (and, correspondingly, ψ_j is a proper eigenfunction or a generalized eigenfunction).
Taking advantage of completeness of the system of the eigenfunctions, we have the following representation for F

$$F(\boldsymbol{R},\boldsymbol{r_1},\boldsymbol{r_2})=\sum_{j_1,j_2} f_{j_1j_2}(\boldsymbol{R})\Psi^I_{j_1}(\boldsymbol{r_1})\Psi^{II}_{j_2}(\boldsymbol{r_2}) \quad (3.3)$$

The Fourier coefficients $f_{j_1j_2}(\mathbf{R})$ of the expansion have a direct physical interpretation. Indeed, using Born's rule, the probability for finding the first atom in the state labeled by j_1 and the second atom in the state labeled by j_2 is

$$\int d\boldsymbol{R}\,|f_{j_1j_2}(\boldsymbol{R})|^2 \quad (3.4)$$

According to this interpretation, one might loosely say that the "wave function" of the α-particle is $f_{00}(\mathbf{R})$ if both atoms remain in the ground state, $f_{j_10}(\mathbf{R})$, $j_1\neq 0$, if the first atom is in the j_1-th excited (or ionized) state and the second in the ground state, $f_{j_1j_2}(\mathbf{R})$, $j_1,j_2\neq 0$, if both atoms are excited (or ionized).
The analysis shows that $f_{00}(\mathbf{R})$ is a (slightly deformed) spherical wave and $f_{j_10}(\mathbf{R})$, $j_1\neq 0$, is a wave packet emerging from $\mathbf{a_1}$ with a momentum along the line $\overline{O\mathbf{a_1}}$. This means that the second atom can be excited by such wave packet only if $\mathbf{a_2}$ lies on the line $\overline{O\mathbf{a_1}}$. Thus the desired result will follow, i.e. $f_{j_1j_2}(\mathbf{R})$, $j_1,j_2\neq 0$, is approximately zero unless the condition of collinearity is satisfied.

The computation is carried out using second order perturbation theory and treating the interaction of the α-particle with the two electrons as a small perturbation. Then one writes

$$F = F^{(0)} + F^{(1)} + F^{(2)} + \cdots \tag{3.5}$$

and each term of the series is computed by the method of successive approximation, starting from the unperturbed zero-th order term given in the form of a diverging spherical wave multiplied the ground state of the two atoms

$$F^{(0)}(\boldsymbol{R}, \boldsymbol{r_1}, \boldsymbol{r_2}) = f_{00}^{(0)}(\boldsymbol{R})\Psi_0^I(\boldsymbol{r_1})\Psi_0^{II}(\boldsymbol{r_2})\,, \qquad f_{00}^{(0)}(\boldsymbol{R}) = \frac{e^{ik|\boldsymbol{R}|}}{|\boldsymbol{R}|}\,, \qquad k = \frac{\sqrt{2M(E-2E_0)}}{\hbar} \tag{3.6}$$

where E_j denotes the j-th eigenvalue of the hydrogen atom. We see that the context of the stationary Schrödinger equation forces Mott to choose a solution not in L^2, which, strictly speaking, is not legitimate. In particular the probabilistic interpretation (3.4) fails for $f_{00}^{(0)}$.
For the first order term

$$F^{(1)}(\boldsymbol{R}, \boldsymbol{r_1}, \boldsymbol{r_2}) = \sum_{i_1, i_2} f_{i_1 i_2}^{(1)}(\boldsymbol{R})\Psi_{i_1}^I(\boldsymbol{r_1})\Psi_{i_2}^{II}(\boldsymbol{r_2}) \tag{3.7}$$

one finds

$$f_{j_1 j_2}^{(1)}(\boldsymbol{R}) = \frac{M}{2\pi\hbar^2}\int d\boldsymbol{R}'\, K_{j_1 j_2}(\boldsymbol{R}')\,\frac{e^{\pm ik'|\boldsymbol{R}-\boldsymbol{R}'|}}{|\boldsymbol{R}-\boldsymbol{R}'|}\,, \qquad k' = \frac{\sqrt{2M(E-E_{j_1}-E_{j_2})}}{\hbar} \tag{3.8}$$

where

$$K_{j_1 j_2}(\boldsymbol{R}) = f_{00}^{(0)}(\boldsymbol{R})\Big(\delta_{0 j_2} V_{j_1 0}(\boldsymbol{R}-\boldsymbol{a}_1) + \delta_{j_1 0} V_{0 j_2}(\boldsymbol{R}-\boldsymbol{a}_2)\Big) \tag{3.9}$$

$$V_{ij}(\boldsymbol{x}) = -\int d\boldsymbol{y}\,\frac{2e^2}{|\boldsymbol{x}-\boldsymbol{y}|}\psi_i(\boldsymbol{y})\psi_j(\boldsymbol{y}) \tag{3.10}$$

Note that, from (3.9), one has $K_{j_1 j_2}(\boldsymbol{R}) = 0$ if both j_1 and j_2 are different from zero and therefore, by (3.8), one also has $f_{j_1 j_2}^{(1)} = 0$ if both j_1 and j_2 are different from zero.
From these preliminary considerations a first conclusion can be drawn:
At first order in perturbation theory the probability that both atoms are excited is always zero. The result is not surprising since, as Mott remarks, in perturbation theory the probability that one atom is excited is a first order quantity and the probability that both atoms are excited is a second order quantity. This explains why the second order term $F^{(2)}$ is required in order to obtain an estimate of the double excitation occurrence.
The further point is to give an approximate expression for $f_{j_1 0}^{(1)}$ and $f_{0 j_2}^{(1)}$. From (3.8), and (3.9), for $f_{j_1 0}^{(1)}$ one has

$$f_{j_1 0}^{(1)}(\boldsymbol{R}) = \frac{M}{2\pi\hbar^2}\int d\boldsymbol{y}\, f_{00}^{(0)}(\boldsymbol{y}+\boldsymbol{a}_1) V_{j_1 0}(\boldsymbol{y})\,\frac{e^{ik'|\boldsymbol{R}-\boldsymbol{a}_1-\boldsymbol{y}|}}{|\boldsymbol{R}-\boldsymbol{a}_1-\boldsymbol{y}|}\,, \qquad j_1 \neq 0 \tag{3.11}$$

and analogously for $f^{(1)}_{0j_2}$. In order to find the required approximate expression Mott introduces the following assumptions:
a) the "observation point" $\boldsymbol{R}$ is far away from the origin and the atom, i.e. $|\boldsymbol{a_1}| \ll |\boldsymbol{R}|$;
b) the collision for the α-particle is almost elastic, i.e. $k - k' \ll k$;
c) the α-particle has a high momentum k.
Using assumption a) one obtains the asymptotic formula

$$f^{(1)}_{j_1 0}(\boldsymbol{R}) \simeq \frac{e^{ik'|\boldsymbol{R}-\boldsymbol{a}_1|}}{|\boldsymbol{R}-\boldsymbol{a}_1|}\,\frac{M}{2\pi\hbar^2}\int d\boldsymbol{y}\, f^{(0)}_{00}(\boldsymbol{y}+\boldsymbol{a}_1)V_{j_1 0}(\boldsymbol{y})\, e^{-ik'\boldsymbol{u}_1(\boldsymbol{R})\cdot\boldsymbol{y}} \tag{3.12}$$

where

$$\boldsymbol{u}_1(\boldsymbol{R}) = \frac{\boldsymbol{R}-\boldsymbol{a}_1}{|\boldsymbol{R}-\boldsymbol{a}_1|} \tag{3.13}$$

Using the explicit expression of $f^{(0)}_{00}$ (see (3.6)) and assumption b) one can write

$$f^{(1)}_{j_1 0}(\boldsymbol{R}) \simeq \frac{e^{ik'|\boldsymbol{R}-\boldsymbol{a}_1|}}{|\boldsymbol{R}-\boldsymbol{a}_1|}\,\mathcal{I}(\boldsymbol{u}_1(\boldsymbol{R})) \tag{3.14}$$

$$\mathcal{I}(\boldsymbol{u}_1(\boldsymbol{R})) = \frac{M}{2\pi\hbar^2}\int d\boldsymbol{y}\,\frac{V_{j_1 0}(\boldsymbol{y})}{|\boldsymbol{y}+\boldsymbol{a}_1|}\, e^{ik(|\boldsymbol{y}+\boldsymbol{a}_1|-\boldsymbol{u}_1(\boldsymbol{R})\cdot\boldsymbol{y})} \tag{3.15}$$

One sees that $f^{(1)}_{j_1 0}(\boldsymbol{R})$ has the form of a wave diverging from $\boldsymbol{a}_1$, whose amplitude $\mathcal{I}$ is given by the integral in (3.15) and it is explicitly dependent on the direction $\boldsymbol{u}_1(\boldsymbol{R})$.
The crucial point is now to evaluate such amplitude. By assumptions c), the integral in (3.15) is a highly oscillatory integral and then stationary phase arguments can be used. The leading term of the asymptotic expansion for $k \to \infty$ is determined by the value of the integrand at the critical points of the phase, i.e. for points $\boldsymbol{y}$ such that

$$\nabla_{\boldsymbol{y}}\Big(|\boldsymbol{y}+\boldsymbol{a}_1| - \boldsymbol{u}_1(\boldsymbol{R})\cdot\boldsymbol{y}\Big) = \frac{\boldsymbol{y}+\boldsymbol{a}_1}{|\boldsymbol{y}+\boldsymbol{a}_1|} - \boldsymbol{u}_1(\boldsymbol{R}) = 0 \tag{3.16}$$

On the other hand, the integrand in (3.15) is very small except in a neighborhood of $\boldsymbol{y} = 0$. Therefore one obtains the condition

$$\boldsymbol{u}_1(\boldsymbol{R}) \simeq \frac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|} \tag{3.17}$$

Using condition (3.17) in (3.13) one can deduce that the amplitude $\mathcal{I}$ is significantly different from zero only for those $\boldsymbol{R}$ such that $\boldsymbol{R} - \boldsymbol{a}_1$ is (almost) parallel to $\boldsymbol{a}_1$, i.e. the observation point $\boldsymbol{R}$ must be (almost) aligned with the first atom and the origin.
From the above argument one concludes that $f^{(1)}_{j_1 0}(\boldsymbol{R})$ is approximately given by a wave diverging from $\boldsymbol{a}_1$ with an amplitude very small except for $\boldsymbol{R}$ given by (3.17), i.e. except in a small cone

with vertex in $\boldsymbol{a}_1$ and pointing away from the origin. A completely analogous analysis is valid for $f^{(1)}_{0j_2}(\boldsymbol{R})$.
We will not give here details of Mott's estimate of the second order term. The final result is a straightforward consequence of the "focusing" of the first order term in the direction connecting the origin to the first atom.
If one agrees that the (amplified) effect of the excitations of the atoms is the true observed phenomenon in a cloud chamber then the result can be rephrased to say that one can only observe straight tracks. In this sense, Mott provides a satisfactory explanation of the straight tracks observed in the chamber based entirely on the Schrödinger equation.
It is worth emphasizing that the analysis Mott developed is based on a deep physical intuition. Indeed, the three-body problem discussed in his paper is an extremely simple but non-trivial model and it is especially well suited for highlighting the emergence of the qualitatively behavior of the α-particle without unnecessary complications.
Even though it is not particularly stressed in Mott's paper, another important aspect is the fact that the result is valid under specific physical assumptions (large value of k and quasi-elastic interaction). In other words, the observed behavior of the α-particle in a cloud chamber is far from being universal.
In this sense, Mott's analysis can be considered the original prototype of the modern approach to the theory of environment-induced decoherence. In fact, the classical behavior (the trajectory) of the system (α-particle) emerges as an effect of the interaction with the environment (vapor atoms in the chamber) under suitable assumptions on the physical parameters of the model.
It should also be noted that there is surely a gap in the mathematical rigor of Mott's paper. For instance, the stationary phase theorem is used without an accurate control of the conditions of applicability. Another unsatisfactory aspect is the use of the stationary Schrödinger equation, which prevents a more transparent time-dependent description of the evolution of the whole system.

4 The role of semi-classical analysis

In the study of the classical limit of quantum mechanics, the use of stationary phase techniques is suggested by analogy with geometrical optics since the initial wave has a very high frequency (momentum).
In fact, from the technical point of view the situation is similar to the case of optics, since we compare a theory based on point particles propagating along trajectories (classical mechanics corresponding to geometrical optics) with a theory based on wave solutions of the Schrödinger equation (quantum mechanics in analogy with "physical" optics). Instead, it at once becomes clear that the situation is quite different from the point of view of the physical interpretation. In quantum mechanics, the wave representing the quantum state does not describe a physical

object distributed in ordinary space, as in optics. Rather, it is a probability amplitude in the (classical) configuration space associated to the (whole) quantum system under consideration. Its role is to provide the statistical distribution of the outputs of repeated experiments. Moreover, the superposition principle introduces a crucial difficulty since no definite meaning can be given to the configuration of a quantum system in a superposition state. In other words, the standard formulation of quantum mechanics does not provide a space-time description of the behavior of a quantum object easily comparable with the classical one. For these reasons, the classical limit of quantum mechanics is, both technically and conceptually, hard to ascertain.
The traditional approach is essentially based on the analysis of the solutions of the Schrödinger equation in the limit "$\hbar \to 0$" for a suitable choice of the initial state. We recall that in this context the limit "$\hbar \to 0$" simply means that the typical action of the system is large with respect to the Planck's constant.
Usually one considers two possible kinds of initial states, chosen by analogy with the case of optics: WKB states and coherent states.
The former are defined by an amplitude independent of $\hbar$ and a highly oscillating phase for $\hbar$ small. In this case one can show, for $\hbar$ small and for short times, that the corresponding solution of the Schrödinger equation has the same form, with amplitude and phase governed by the classical transport and Hamilton-Jacobi equations respectively. This means that in the limit the quantum state propagates like a classical fluid and in this sense the classical description is recovered.
Coherent states are wave packets well concentrated in position and momentum around a point (x_0, p_0) in classical phase space for $\hbar$ small. One can prove that the time evolution, for $\hbar$ small and a time interval not too long, is again described by a wave packet well concentrated in position and momentum around the classical trajectory starting from (x_0, p_0).
Making available precise statements and mathematical proofs to detail the above qualitative pictures has required a great deal of technical work. Many refined and detailed results and summaries of the theory are at one's disposal in the literature (see, e.g. [Ro] and [**?**] and references therein).
Despite their mathematical elegance, such kind of results cannot be considered conclusive for a complete understanding of the classical limit of quantum mechanics. The reason is that the approach is crucially based on the choice of specific, essentially classical, initial states. Given the characteristic of the quantum system and of its environment, we expect that a classical behavior should emerge also starting from a genuine quantum state, like a superposition state. In such a case the usual procedure $\hbar \to 0$ turns out to be insufficient.
The idea behind this theoretical analysis is that quantum coherence between the components of a superposition state is very fragile. Even a weak interaction of the system with the environment can significantly reduce coherence and a classical behavior of the system can emerge.
In the case of a cloud chamber, the initial state of the α-particle is a spherical wave, i.e., a (continuous) superposition of semi-classical states of the type mentioned above.
To explain the classical trajectories that are observed one has to analyze in detail the decoher-

ence effect induced by the environment. The intuitive picture is clear-cut: each semi-classical component of the α-particle initial state evolves, according to semi-classical theory, along an almost straight line interacting in the meanwhile with a small section of the environment. The crucial point is that different semi-classical components interact with different parts of the environment. As a consequence, the state of entire system becomes an almost incoherent sum of states supported in distant regions of the classical configuration space.
Our attempt is to give a rigorous version of this picture. To achieve this goal we have to quantify the response of a model quantum environment to the particle passage.
It is worth emphasizing that a modification of the environment is the only experimental output one can observe. Contrary to what is often stated, one should not "trace out" the environment degrees of freedom, but rather those of the particle. That is exactly what Mott did when trying to estimate the multiple ionization probability of atoms in a cloud chamber.
The results we obtained during the last decade [DFT1], [DFT2], [FinT], [RT], [CCF] suggest an effective strategy for dealing with the study of quantum mechanical microscopic systems in interaction with quantum environments. It consists in building up simple models of quantum environments and analyzing their evolution under specific hypotheses on the physical parameters of the models.

5 Mott's analysis revisited

In [DFT2] we propose a rigorous, time-dependent version of the original Mott's result. We consider a three-particle quantum system consisting of the α-particle, initially described by a spherical wave centered in the origin, and two model atoms placed at fixed positions $\boldsymbol{a}_1$, $\boldsymbol{a}_2 \in \mathbb{R}^3$, with $0 < |\boldsymbol{a}_1| < |\boldsymbol{a}_2|$. For the sake of simplicity, each model atom is described by a particle subject to an attractive point interaction placed in $\boldsymbol{a}_i$, $i = 1, 2$. For a detailed study of an Hamiltonian with a point interaction we refer to [AGH-KH]. Here we only recall that it is a solvable model, i.e., spectrum and eigenfunctions can be explicitly computed. In particular the absolutely continuous spectrum is the positive real semi-axis, the singular continuous spectrum is empty and, in the attractive case, there is one negative, non degenerate eigenvalue. We also recall that a Hamiltonian with a given smooth interaction potential V can be reconstructed as the limit of Hamiltonians with many randomly distributed point interactions ([FHT]).
At initial time we assume that both atoms are in their ground state (the eigenvector corresponding to the negative eigenvalue). Furthermore, we assume that the α-particle and the atoms interact via a smooth two-body potential.
A crucial step for the analysis is a precise specification of the assumptions on the physical parameters of the model. We assume that:
i) the wavelength associated to the α-particle at time zero is much smaller than the spatial localization of the spherical wave (semi-classical regime);
ii) the spatial localization of the spherical wave, the "diameter" of the atoms and the effective

range of the interaction between α-particle and atoms are much smaller than the macroscopic distance $|\boldsymbol{a}_1|$;
iii) the ionization energy of the atoms and the strength of the interaction between α-particle and atoms are much smaller than the initial kinetic energy of the α-particle (quasi-elastic regime).
We also introduce the time τ_j, $j = 1, 2$, as the time spent by a classical particle, starting from the origin with velocity equal to the mean isotropic velocity of the spherical wave, to reach the atom in $\boldsymbol{a}_j$. We remark that, under the above assumptions, it is reasonable to consider τ_1 and τ_2 as the collision times of the spherical wave emerging from the origin with the first atom in $\boldsymbol{a}_1$ and the second atom in $\boldsymbol{a}_2$.
Our aim is the consider the time evolution of the three-particle system up to second order in perturbation theory and to compute the probability $\mathcal{P}_2(t)$ that both atoms are ionized for $t > \tau_2$.
The result we find is in agreement with the original Mott's analysis and it can be roughly summarized as follows:

$\mathcal{P}_2(t)$ is negligible unless the positions $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ of the atoms are aligned with the origin.

For a more precise formulation of the result and for the proof we refer to [DFT2], [FigT2]. Here we only add some remarks.
- The assumptions i), ii), iii) are crucial for the validity of the result, in the sense that a different qualitative behavior of the system must be expected if the assumptions on the physical parameters of the model are modified.
- Since in a cloud chamber one observes the (amplified) effect of the ionization of the atoms, the result shows that one can only observe straight trajectories.
- The method of the proof is essentially based on a representation formula for $\mathcal{P}_2(t)$ in terms of highly oscillatory integrals and on the asymptotic analysis of such integrals using stationary and non-stationary phase methods.
In the rest of this section we shall give some technical details of a result obtained in [RT] (see also [FigT2]) which, in our opinion, clarifies the dynamical mechanism that underlies Mott's result. Once more, the material presented in the rest of this section can be passed over by the reader who wants to avoid mathematical details.
Let us consider a simpler model of a non relativistic quantum system made of only two spinless particles in dimension three of masses M and m. The latter is bound by an harmonic potential of frequency ω around the equilibrium position $\boldsymbol{a}$. The first particle plays the role of the α-particle while the harmonically bounded particle plays the role of an electron in a very simplified version of model-atom with fixed nucleus. The interaction between the test particle and the harmonic oscillator is described by a smooth two-body potential V.
Denoting by $\boldsymbol{R}$ the position coordinate of the α-particle and by $\boldsymbol{r}$ the position coordinate of the harmonic oscillator, the Hamiltonian of the system in $L^2(\mathbb{R}^6)$ is given by

$$H = H_0 + \lambda\, V(\delta^{-1}(\boldsymbol{R} - \boldsymbol{r})), \qquad \lambda, \delta > 0 \tag{5.1}$$

where H_0 is the free Hamiltonian of the system

$$H_0 = -\frac{\hbar^2}{2M}\Delta_{\boldsymbol{R}} - \frac{\hbar^2}{2m}\Delta_{\boldsymbol{r}} + \frac{1}{2}m\omega^2(\boldsymbol{r}-\boldsymbol{a})^2 \tag{5.2}$$

We recall that the eigenfunctions of the harmonic oscillator are

$$\varphi_{\underline{n}}(\boldsymbol{r}) = \gamma^{-3/2}\phi_{\underline{n}}(\gamma^{-1}(\boldsymbol{r}-\boldsymbol{a})), \qquad \gamma = \sqrt{\frac{\hbar}{m\omega}}\,, \qquad \phi_{\underline{n}}(\boldsymbol{x}) \equiv \phi_{n_1}(x_1)\phi_{n_2}(x_2)\phi_{n_3}(x_3) \tag{5.3}$$

where $\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$ and ϕ_{n_k} is the Hermite function of order n_k. In particular the ground state corresponds to $\underline{n} = \underline{0} = (0,0,0)$.
Under the same kind of assumptions i), ii), iii) made above, we analyze the evolution of this system when the initial state is a product state of a spherical wave for the α-particle and the ground state for the oscillator. In order to satisfy the assumptions, it is convenient to introduce a small parameter $\varepsilon > 0$ and to fix

$$\hbar = \varepsilon^2 \qquad M = 1 \qquad \sigma = \varepsilon \qquad m = \varepsilon \qquad \omega = \varepsilon^{-1} \qquad \delta = \varepsilon \qquad \lambda = \varepsilon^2 \tag{5.4}$$

where σ is the spatial localization of the spherical wave. Under this scaling the Hamiltonian becomes

$$H^\varepsilon = H_0^\varepsilon + \varepsilon^2\, V\left(\varepsilon^{-1}(\boldsymbol{R}-\boldsymbol{r})\right) \tag{5.5}$$

where

$$H_0^\varepsilon = -\frac{\varepsilon^4}{2}\Delta_{\boldsymbol{R}} + \frac{1}{\varepsilon}\left[-\frac{\varepsilon^4}{2}\Delta_{\boldsymbol{r}} + \frac{1}{2}(\boldsymbol{r}-\boldsymbol{a})^2\right] \tag{5.6}$$

The rescaled initial state takes the form

$$\Psi_0^\varepsilon(\boldsymbol{R},\boldsymbol{r}) = \psi^\varepsilon(\boldsymbol{R})\varphi_{\underline{0}}^\varepsilon(\boldsymbol{r}) \tag{5.7}$$

$$\varphi_{\underline{n}}^\varepsilon(\boldsymbol{r}) = \frac{1}{\varepsilon^{3/2}}\phi_{\underline{n}}\left(\varepsilon^{-1}(\boldsymbol{r}-\boldsymbol{a})\right) \qquad \underline{n} \in \mathbb{N}^3 \tag{5.8}$$

In (5.7) the spherical wave ψ^ε is explicitly given by

$$\psi^\varepsilon(\boldsymbol{R}) = \frac{\mathcal{N}_\varepsilon}{\varepsilon^{5/2}\pi^{3/4}}\, e^{-\frac{|\boldsymbol{R}|^2}{2\varepsilon^2}} \int_{S^2} d\hat{\boldsymbol{u}}\, e^{\frac{i}{\varepsilon^2}v_0\hat{\boldsymbol{u}}\cdot\boldsymbol{R}} \tag{5.9}$$

where $v_0 > 0$ is the mean isotropic velocity of the spherical wave and $\mathcal{N}_\varepsilon$ is a normalization constant, with

$$\lim_{\varepsilon\to 0}\mathcal{N}_\varepsilon = \mathcal{N}_0 \;\equiv\; \frac{v_0}{4\pi} \tag{5.10}$$

Notice that the spherical wave is obtained considering a wave packet localized in momentum around $v_0\hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}}$ is a generic unit vector, and then taking an average over all possible unit vectors of the sphere S^2.
We are interested in asymptotic behavior for $\varepsilon \to 0$ of the solution of the Schrödinger equation of the system

$$\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon\,, \qquad \mathcal{U}^\varepsilon(t) = e^{-i\frac{t}{\varepsilon^2}H^\varepsilon} \tag{5.11}$$

for $t > \tau$, where

$$\tau = \frac{|\boldsymbol{a}|}{v_0} \tag{5.12}$$

is the (classical) collision time of the α-particle with the oscillator.
In order to formulate the result, we fix a reference frame such that

$$\hat{\boldsymbol{a}} = (0,0,1), \qquad \hat{\boldsymbol{a}} \equiv \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \tag{5.13}$$

and we introduce the following definition.

Let $P^\varepsilon = P^\varepsilon(\boldsymbol{R},\boldsymbol{r})$ be the function

$$P^\varepsilon(\boldsymbol{R},\boldsymbol{r}) = \sum_{\underline{n}} P^\varepsilon_{\underline{n}}(\boldsymbol{R})\,\varphi^\varepsilon_{\underline{n}}(\boldsymbol{r}) \tag{5.14}$$

where $P^\varepsilon_{\underline{n}}$ is the wave packet for the α-particle given by

$$P^\varepsilon_{\underline{n}}(\boldsymbol{R}) \equiv P^\varepsilon_{\underline{n}}(R_1,R_2,R_3) = \frac{C^\varepsilon_{\underline{n}}}{\varepsilon^{3/2}}\,\mathcal{F}_{\underline{n}}\left(\frac{R_1}{\varepsilon},\frac{R_2}{\varepsilon},0\right)\,e^{-\frac{1}{2\varepsilon^2}\left(R_3-\mathcal{Z}^\varepsilon_{\underline{n}}\right)^2+\frac{i}{\varepsilon^2}v^\varepsilon_{\underline{n}}R_3} \tag{5.15}$$

$$C^\varepsilon_{\underline{n}} = \frac{2\pi^{5/4}}{i|\boldsymbol{a}|^2}\,e^{\frac{i}{\varepsilon}|n|\tau+i\frac{|n|^2\tau}{2v_0^2}} \tag{5.16}$$

$$\mathcal{F}_{\underline{n}}(\boldsymbol{y}) \equiv \mathcal{F}_{\underline{n}}(y_1,y_2,y_3) = e^{-i\frac{|\boldsymbol{y}|^2}{2\tau}}\left(\widetilde{\phi_{\underline{n}}\phi_{\underline{0}}}\cdot\widetilde{V}\right)\left(-\frac{y_1}{\tau},-\frac{y_2}{\tau},-\frac{y_3}{\tau}-\frac{|n|}{v_0}\right) \tag{5.17}$$

$$\mathcal{Z}^\varepsilon_{\underline{n}} = \frac{|n|\tau}{v_0}\varepsilon \tag{5.18}$$

$$v^\varepsilon_{\underline{n}} = v_0 - \frac{|n|}{v_0}\varepsilon \tag{5.19}$$

In (5.17) we have used the notation $\widetilde{f}$ for the Fourier transform of a function f.
Let us briefly comment on the above definition. The function P^ε is an infinite linear combination of product states, made of stationary states of the harmonic oscillator and wave packets $P^\varepsilon_{\underline{n}}$ of the α-particle. Each $P^\varepsilon_{\underline{n}}$ is well concentrated, for ε small, in position around

$$\boldsymbol{R}(0) = (0,0,\mathcal{Z}^\varepsilon_{\underline{n}}) \;\equiv\; \mathcal{Z}^\varepsilon_{\underline{n}}\,\hat{\boldsymbol{a}} \tag{5.20}$$

and in momentum around

$$\boldsymbol{P}(0) = (0, 0, v_{\underline{n}}^{\varepsilon}) \;\equiv\; v_{\underline{n}}^{\varepsilon}\,\hat{\boldsymbol{a}} \tag{5.21}$$

The free evolution at time t of $P_{\underline{n}}^{\varepsilon}$, for ε small, is again a wave packet well concentrated in position around

$$\boldsymbol{R}(t) = \mathcal{Z}_{\underline{n}}^{\varepsilon}\,\hat{\boldsymbol{a}} + v_{\underline{n}}^{\varepsilon}\,\hat{\boldsymbol{a}}\,t \tag{5.22}$$

and in momentum around $\boldsymbol{P}(0)$ (the momentum is conserved). In particular, at time $t = \tau$ the wave packet is well localized in position around

$$\boldsymbol{R}(\tau) = \mathcal{Z}_{\underline{n}}^{\varepsilon}\,\hat{\boldsymbol{a}} + v_{\underline{n}}^{\varepsilon}\,\hat{\boldsymbol{a}}\,\tau \;=\; \boldsymbol{a} \tag{5.23}$$

i.e., around the position of the oscillator.
The wave packets $P_{\underline{n}}^{\varepsilon}$ play a crucial role in the asymptotic expression of the wave function of the system for ε small. More precisely, the following result holds.

Theorem. *Let us fix $t > \tau$. Then there exists $C(t) > 0$, independent of ε, such that*

$$\mathcal{U}^{\varepsilon}(t)\Psi_0^{\varepsilon} \;=\; \mathcal{U}_0^{\varepsilon}(t)\Psi_0^{\varepsilon} \;+\; \varepsilon^2\,\mathcal{U}_0^{\varepsilon}(t)P^{\varepsilon} \;+\; \mathcal{E}^{\varepsilon}(t) \tag{5.24}$$

where

$$\mathcal{U}_0^{\varepsilon}(t) = e^{-i\frac{t}{\varepsilon^2}H_0^{\varepsilon}} \tag{5.25}$$

and

$$\|\mathcal{E}^{\varepsilon}(t)\| \le C(t)\,\varepsilon^3 \tag{5.26}$$

For the proof (in the more general case of $N \ge 1$ harmonic oscillators) we refer to [RT]. Here we only comment on the result.
Using the expressions for the free propagator $\mathcal{U}_0^{\varepsilon}(t)$, the initial state Ψ_0^{ε} and the function P^{ε}, from (5.24) one has

$$\begin{aligned}\left(\mathcal{U}^{\varepsilon}(t)\Psi_0^{\varepsilon}\right)(\boldsymbol{R},\boldsymbol{r}) \;\simeq\; & e^{-i\frac{t}{\varepsilon^2}E_{\underline{0}}^{\varepsilon}}\left[\left(e^{-i\frac{t}{\varepsilon^2}h_0^{\varepsilon}}\psi^{\varepsilon}\right)(\boldsymbol{R}) + \varepsilon^2\left(e^{-i\frac{t}{\varepsilon^2}h_0^{\varepsilon}}P_{\underline{0}}^{\varepsilon}\right)(\boldsymbol{R})\right]\varphi_{\underline{0}}^{\varepsilon}(\boldsymbol{r}) \\ & +\; \varepsilon^2\sum_{\underline{n}\neq\underline{0}} e^{-i\frac{t}{\varepsilon^2}E_{\underline{n}}^{\varepsilon}}\left(e^{-i\frac{t}{\varepsilon^2}h_0^{\varepsilon}}P_{\underline{n}}^{\varepsilon}\right)(\boldsymbol{R})\,\varphi_{\underline{n}}^{\varepsilon}(\boldsymbol{r})\end{aligned} \tag{5.27}$$

where $E_{\underline{n}}^{\varepsilon}$ denotes the energy level of the oscillator

$$E_{\underline{n}}^{\varepsilon} = \varepsilon\left(|n| + \frac{3}{2}\right), \qquad |n| = n_1 + n_2 + n_3 \tag{5.28}$$

and h_0^{ε} is the free Hamiltonian for the α-particle

$$h_0^{\varepsilon} = -\frac{\varepsilon^4}{2}\Delta_{\boldsymbol{R}} \tag{5.29}$$

In (5.27) the approximate wave function for $t > \tau$ has been written as the sum of two terms, distinguished for the different behavior of the oscillator (unperturbed or excited). In the first one, the oscillator remains in its ground state and the α-particle evolves according to

$$\left(e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}\psi^\varepsilon\right)(\boldsymbol{R}) + \varepsilon^2\left(e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}P_{\underline{0}}^\varepsilon\right)(\boldsymbol{R}) \tag{5.30}$$

i.e., the free evolution of the initial spherical wave slightly deformed by the free evolution of the small wave packet $P_{\underline{0}}^\varepsilon$, emerging from the oscillator. The second term is a sum over all possible excited states of the oscillator. In each term of the sum, the evolution of the α-particle is given by ε^2 times

$$\left(e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}P_{\underline{n}}^\varepsilon\right)(\boldsymbol{R}) \tag{5.31}$$

i.e., the free evolution of the wave packet $P_{\underline{n}}^\varepsilon$, $\underline{n} \neq \underline{0}$. As we already remarked, each wave packet emerges at $t = \tau$ from the excited oscillator with momentum $v_{\underline{n}}^\varepsilon\,\hat{\boldsymbol{a}}$ and, for $t > \tau$, it will be concentrated around the uniform classical motion (5.22), which can be more conveniently rewritten as

$$\boldsymbol{R}(t) = \boldsymbol{a} + \left(v_0 - \frac{|n|}{v_0}\,\varepsilon\right)(t-\tau)\,\hat{\boldsymbol{a}} \tag{5.32}$$

We emphasize that the result expressed in the theorem provides a physical explanation of Mott's result in the three-particle model (α-particle plus two atoms in $\boldsymbol{a}_1$, $\boldsymbol{a}_2$). In fact, if the collision with the first atom in $\boldsymbol{a}_1$ produces excitation of the atom then, according to the above result, the α-particle is described by a localized wave packet emerging from $\boldsymbol{a}_1$, with momentum along the direction $\hat{\boldsymbol{a}}_1$. As a consequence, the requirement that also the second atom in $\boldsymbol{a}_2$ is excited can be satisfied only if such atom is hit by the wave packet, and this happens only if $\boldsymbol{a}_2$ lies on the direction $\hat{\boldsymbol{a}}_1$.
This explains why excitation of both atoms can occur only if their positions are aligned with the origin.

6 Different approaches and open problems

The simple models we analyzed should be considered as first steps in a branch of research that we consider relevant and promising. In our opinion, there are different strategies that might be exploited and open problems that would be well worth the study.
As we mentioned above the initial spherical wave packet can be seen as a continuous superposition of coherent states pointing toward all possible radial directions. In fact, (5.9) is the rigorous formulation of this claim.
The partition (slicing) of the incoming α-wave in fuzzy coherent slices that move as semiclassical waves (keeping their coherence until one of them interacts with a real atom) can be deemed to be artificial. But the wave itself is a probability wave and therefore has no objective reality. We are slicing something that does not exist!

Indeed we are manipulating mathematical objects that enter into the mathematical framework by which quantum mechanics describes outcomes of experimental observations; only part of this mathematical framework is given a direct physical correspondence with experiments. The remaining part is there in order to give a meaningful dynamics (meaningful = as close as possible to the dynamics of material bodies). What counts is that the mathematical description we give be consistent and our results indicate that the family of coherent states we suggest as a "basis" in order to analyze the initial state of the α-particle is in fact the right "pointer basis" for the problem under investigation.

In our opinion, this qualitative description can be turned into an effective technical tool to examine details of the asymptotic evolution of the α-particle and of the environment in models of cloud chamber.

We recall that semiclassical waves propagate, under the Schrödinger equation, keeping their barycenter on a classical path. Their shape changes slightly and their dispersion is of order $\sqrt{\hbar}$. As soon as we write the initial state as a superposition of such states, the problem is reduced to the interaction of a semiclassical wave with an atom, leading to ionization. It should be noted that the semiclassical wave packet remains such after the interaction with an atom only if the momentum transfer within the process is a little percentage of the initial momentum (almost zero-angle scattering). In the physical problem we are considering, this implies that after the interaction either the atom is in an excited state or the atom is ionized and low energy electrons are emitted. In the real Wilson chamber, this last condition is needed in order that the ionized atoms, acting as condensation seeds, may cause the formation of liquid droplets (once more we stress that all we see and measure is the formation of tracks of droplets in the cloud chamber). It is therefore reasonable to restrict attention to the case in which the semiclassical wave packet retains its identity during interaction, changing perhaps slightly its shape.

While the evolution of a semiclassical wave packet interacting with an external potential has been extensively investigated ([Hep1], [Y], [Ro], [HJ]), semiclassical inelastic scattering has not received, at the best of our knowledge, comparable attention and any result in this direction would be welcome. We are planning to examine this approach to the Mott's problem in further work. Here, we want to focus briefly on the connection of the strategy outlined above with the analysis of Michel Bauer and Denis Bernard on the wave function collapse in repeated quantum non demolition measurements ([BB]). In the paper the authors investigate the evolution of a microscopic system together with a probe. The latter is meant to perform a sequence of "nondemolition" measurements on the system.

The nondemolition character of the measurement is turned into the main assumption of authors' analysis: let $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_P$ (S standing for system and P for probe) be the Hilbert space of the states of the whole system. The assumption reads: there exist an orthonormal basis ϕ_n in $\mathcal{H}_S$ and unitary operators U_n in the Hilbert space of the probe $\mathcal{H}_P$ such that the evolution of the whole system, starting from an initial state $\phi_n \otimes \Psi$, is given by the unitary operator $\mathbb{I} \otimes U_n$. In short, each state in the chosen basis of the microscopic system evolves unaltered, whereas the state of the probe evolves according to a Schrödinger dynamics whose generator depends

on the microscopic system state.
From the assumption stated above, the authors proceed to prove a list of interesting results making use only of classical probability tools. In summary, after a large number of repeated measurements processes, obeying at each time the main assumption,

- the state of the system tends to one of the states ϕ_n with probability $|(\phi_n, \phi)|^2$ if ϕ is the initial state of the system.
- A limit probability measure on the states of the probe is uniquely associated to each state ϕ_n of the system.

The authors give also results on the rate of convergence of the sequences to the final states.
Let us associate the chosen basis with a "suitable slicing" of the radial initial condition in a cloud chamber model and each U_n with the response of a model atom in the chamber to each scattering event with a particle initially in the state ϕ_n. It is reasonable to predict (and it should be feasible to prove) that, in the limit of quasi-elastic, small angle scattering, the system α-particle plus model-atoms fulfill the assumption of Bauer and Bernard. In this scenario, the α-particle is submitted to a sequence of non demolition measurement by the atoms of the gas in the chamber. The coherent states in which the initial state is analyzed are the "pointer basis" of the alpha particle, whereas the tracks are the pointer states of the probe.

For a given model of environment, it would be interesting to extend the results to more general initial conditions of the microscopic system and to cases in which external fields act on the microscopic system.
Another crucial step ahead would be to consider more realistic models of quantum environments. As was already pointed out by K. Hepp in [Hep2], complete decoherence requires an infinite time in models where a quantum particle evolves in an environment made up of non-interacting quantum subsystems. An alternative way to reach complete decoherence would be to consider an ever larger number of environment constituents in a finite region.
This idea brought us to consider models of environment made up of multi-channel point interactions (one can think either of point atoms with a finite number of energetic levels or of localized spins). There are many advantages in working with such kind of solvable models. To mention the most important, a non-perturbative theory is practicable (the environment is not an unmanageable multi-particle system) and it is possible to investigate the asymptotic limit of infinitely many scattering centers in a finite volume. A quite detailed introduction to these models is given in chapter 3 of the book [FigT2].
A more realistic choice would be to model the environment with self-interacting fields (e.g., spins ferromagnetically interacting among them), initially in a genuine meta-stable state. The non-linear self-interaction would enhance the response of the environment, which might show macroscopic modifications in finite time.
In conclusion a complete description of the mechanism of production of the tracks in a cloud chamber still escapes us, but we have outlined the role of quantum mechanics, of semi-classical

analysis and of stationary phase techniques within the time-dependent Schrödinger equation in shedding some light on the investigation of the classical-quantum border.

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