On the nonexistence of Green's function and failure of the strong maximum principle

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Abstract

Given any Borel function $V : \Omega \to [0, +\infty]$ on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, we establish that the strong maximum principle for the Schrödinger operator $-\Delta + V$ in Ω holds in each Sobolev-connected component of $\Omega \setminus Z$, where $Z \subset \Omega$ is the set of points which cannot carry a Green's function for $-\Delta + V$. More generally, we show that the equation $-\Delta u + V u = \mu$ has a distributional solution in $W_0^{1,1}(\Omega)$ for a nonnegative finite Borel measure μ if and only if $\mu(Z) = 0$.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded connected open set and let $V : \Omega \to [0, +\infty]$ be a Borel function. The *weak* maximum principle for the Schrödinger operator $-\Delta + V$ ensures that if $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a function such that $Vw \in L^1(\Omega)$ and

$$-\Delta w + V w = f \quad \text{in the sense of distributions in } \Omega, \tag{1.1}$$

where $f \in L^\infty(\Omega)$ is nonnegative, then w must be nonnegative in Ω. In this paper, we are interested in the mechanism that guarantees the validity or the failure of the *strong* maximum principle for $-\Delta + V$ and whether there is some unifying property that holds regardless of the potential V. More precisely, for a fixed potential V, we want to understand whether the alternative holds:

$$\text{either} \quad w > 0 \text{ in } \Omega \quad \text{or} \quad w \equiv 0 \text{ in } \Omega, \tag{1.2}$$

and, when it fails, to identify the location of the zero-set $\{w = 0\}$ in Ω and decide whether there is some form of the strong maximum principle that survives in each component of $\Omega \setminus \{w = 0\}$.

To clarify the pointwise meaning of w in Ω, we reformulate (1.2) using the precise representative $\widehat{w}$. We recall that, by the Lebesgue differentiation theorem, $\widehat{w} = w$ almost everywhere in Ω. Since in

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our case w is almost everywhere the difference between a continuous and a bounded superharmonic function, every $x \in \Omega$ is a Lebesgue point of w. Hence, the precise representative can be computed pointwise using the limit

$$\widehat{w}(x) = \lim_{r \to 0} \fint_{B_r(x)} w,$$

where $\fint_{B_r(x)} := \frac{1}{|B_r|} \int_{B_r(x)}$ denotes the average integral over the ball.

Observe that (1.2) classically holds for potentials V in $L^\infty(\Omega)$ and, more generally, in the Lorentz space $L^{\frac{N}{2},1}(\Omega)$ or in the Kato class $K(\Omega)$; see [24], [33], [13, Section 3] or Example 1.5 below. Such a conclusion is no longer true when V merely belongs to $L^p(\Omega)$ for some $p \le N/2$:

Example 1.1. Given $1 \le p \le N/2$ and any compact set $K \subset \Omega$ with finite $\mathcal{H}^{N-2p}$ Hausdorff measure, we construct in [26, Section 6] a nonnegative potential $V \in L^p(\Omega)$, depending on K, such that every nontrivial solution w associated to the Schrödinger operator $-\Delta + V$ with nonnegative bounded datum f satisfies

$$\{\widehat{w} = 0\} = K. \tag{1.3}$$

We prove in [26] that the $W^{2,p}$ capacity is the correct way of quantifying the smallness of $\{\widehat{w} = 0\}$ when one considers the full class of L^p potentials V for $p > 1$; the counterpart for $p = 1$ involves the $W^{1,2}$ capacity, as first identified by Ancona [2]. By looking at a specific potential V one may have a zero-set of dimension strictly smaller than $N - 2p$:

Example 1.2. Given $a \in \Omega$ and $\alpha \in \mathbb{R}$, let

$$V(x) = \frac{1}{|x-a|^\alpha} \ .$$

When $\alpha \ge 2$, every nontrivial solution w with nonnegative bounded datum f satisfies

$$\{\widehat{w} = 0\} = \{a\}. \tag{1.4}$$

To see why $\widehat{w}(a) = 0$, one relies on the fact that

$$\int_{B_r(a)} Vw = o(r^{N-2}) \quad \text{as } r \to 0,$$

which follows from a scaling argument in the equation (1.1) by means of test functions of the form $\varphi(\frac{x-a}{r})$. For $\alpha \ge N$, one may argue differently by observing that V is not summable in any neighborhood of a but $Vw \in L^1(\Omega)$.

These examples are particular cases of the general principle implied by our Corollary 1.2 below that *the zero-set is independent of the solution* whenever $V \in L^1(\Omega)$ or, more generally, when the set where V fails to be locally summable has $\mathcal{H}^{N-1}$ Hausdorff measure zero; see also Proposition 12.2. When the singular set of V is large enough, a splitting of the domain in connected components of *analytic* type may occur:

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Example 1.3. In the unit ball $\Omega = B_1(0)$, take

$$V(x) = \frac{1}{|x_1 - a|^\alpha} + \frac{1}{|x_1 - b|^\beta},$$

where $-1 < a < b < 1$ and x_1 denotes the first component of $x = (x_1, \ldots, x_N)$. Here, the *strength of the singularity modifies the geometric configuration* of the zero-set, even inside the range $\alpha \geq 1$ and $\beta \geq 1$ where $V \notin L^1(B_1(0))$:

(a) For $\alpha \geq 2$ and $1 \leq \beta < 2$, every nontrivial solution w satisfies

$$\{\widehat{w} = 0\} = \{x_1 \geq a\} \cap B_1(0), \tag{1.5}$$

and in particular vanishes on a non-empty open set.

(b) For a stronger singularity with $\alpha \geq 2$ and $\beta \geq 2$, the zero-set of w depends on f. The reason is that the Dirichlet problem splits in three independent regions inside $B_1(0)$, identified by the conditions

$$x_1 < a, \quad a < x_1 < b \quad \text{and} \quad x_1 > b.$$

In particular, the choice $f \equiv 1$ yields a smaller zero-set, namely

$$\{\widehat{w} = 0\} = \big(\{x_1 = a\} \cup \{x_1 = b\}\big) \cap B_1(0). \tag{1.6}$$

These assertions can be established using [25, Section 9] and are related to the failure of the Hopf boundary lemma.

The previous example illustrates in (b) the fact that strong singularities may be used to confine physical particles in prescribed regions; see [14, 15]. Potentials V which are $+\infty$ in some large parts of Ω are also of interest and model the presence of impurities or coolers in the domain; see [30]. This is intended to prescribe regions where solutions must vanish:

Example 1.4. Take

$$V(x) = \frac{1}{d(x, \overline{\omega})^\alpha},$$

where $\omega \Subset \Omega$ is a smooth open set and $d(x, \overline{\omega})$ denotes the distance from x to $\overline{\omega}$. The strong maximum principle depends on the exponent α:

(a) When $1 \leq \alpha < 2$, there is only the trivial solution $w \equiv 0$ in Ω, as an application of the Hopf lemma.

(b) When $\alpha \geq 2$, nontrivial supersolutions do exist since the Hopf lemma fails pointwise on $\partial\omega$, see [27, Proposition 2.7], and they all satisfy

$$\{\widehat{w} = 0\} = \overline{\omega}.$$

To understand the unifying idea behind the strong maximum principle for an arbitrary Borel function $V : \Omega \to [0, +\infty]$, we first select the subset of points in Ω where distributional solutions of the Schrödinger equation must vanish:

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Definition 1.1. *Given a Borel function $V : \Omega \to [0, +\infty]$, the* universal zero-set Z *associated to $-\Delta + V$ is the set of points $x \in \Omega$ characterized by the property that, for every nonnegative $f \in L^\infty(\Omega)$ such that (1.1) has a solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, such a solution satisfies*

$$\widehat{w}(x) = 0.$$

The universal zero-set depends on V, but to simplify the notation we do not explicit such a dependence. In our Examples 1.1 to 1.3, the sets Z are given by (1.3) to (1.6). In the latter example, $\Omega \backslash Z$ has three connected components; a variant of this case using singularities on infinitely many hyperplanes $\{x_1 = a_i\}$ with exponents $\alpha_i \geq 2$ yields a set $\Omega \setminus Z$ with an infinite number of components. Finally, for V as in Example 1.4 one has $Z = \Omega$ when $1 \leq \alpha < 2$ and $Z = \overline{\omega}$ when $\alpha \geq 2$.

We prove later on that Z is, topologically speaking, a *Sobolev-closed* set in the sense that there exists a nonnegative function $\xi \in W_0^{1,2}(\Omega)$ such that every $x \in \Omega$ is a Lebesgue point of ξ and

$$Z = \big\{ x \in \Omega : \widehat{\xi}(x) = 0 \big\}. \tag{1.7}$$

For example, applying Proposition 5.1 and Corollary 6.3 below one deduces that the solution of (1.1) with the characteristic function $f = \chi_{\Omega \backslash Z}$ exists and satisfies (1.7). From the concept of Sobolev-closed set, one defines *Sobolev-open* and *Sobolev-connected* sets by analogy with their classical topological counterparts; see Definitions 10.1 and 11.1. In Remark 10.4 below, we relate Sobolev-open sets with other notions of open sets that have been extensively investigated in Potential theory.

Our main result provides one with a quantization property for the strong maximum principle that can be detected using the Sobolev quasitopology above and shows that the relevant singularities of the potential V for $-\Delta + V$ are encoded in the universal zero-set Z. More precisely,

Theorem 1.1. *For every Borel function $V : \Omega \to [0, +\infty]$, the Sobolev-open set $\Omega \setminus Z$ can be uniquely decomposed as a finite or countably infinite union of disjoint Sobolev-connected-open sets $(D_j)_{j \in J}$ and any solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the Schrödinger equation (1.1) for nonnegative $f \in L^\infty(\Omega)$ satisfies, in each component D_j,*

$$either \quad \widehat{w} > 0 \ \ in \ D_j \quad or \quad \widehat{w} \equiv 0 \ \ in \ D_j.$$

As we explain in Section 12, the Dirichlet problem in Ω uncouples to the various Sobolev-connected components of $\Omega \setminus Z$ provided by Theorem 1.1 and each alternative can effectively happen in D_j without interaction with the other parts of $\Omega \setminus Z$. In particular, for every subset of indices $L \subset J$, there exists a solution with

$$\widehat{w} > 0 \ \ \text{in} \ \bigcup_{j \in L} D_j \quad \text{and} \quad \widehat{w} = 0 \ \ \text{otherwise.}$$

In dimension $N = 1$, solutions are continuous and the picture that comes from Theorem 1.1 is rather simple when $Z \neq \emptyset$: the universal zero-set Z is relatively closed in Ω for the Euclidean topology and then $\Omega \setminus Z$ is a finite or countable union of disjoint open intervals $D_j = (a_j, b_j)$ where

$$V \in L^1_{\text{loc}}(D_j) \quad \text{and} \quad \int_{D_j} V(x) d(x, \partial D_j) \, \mathrm{d}x = +\infty.$$

Indeed, at an endpoint $c_j = a_j$ or b_j inside Ω, the Hopf lemma in D_j must fail at c_j, which is the case if and only if

$$\int_{c_j}^{\frac{a_j+b_j}{2}} V(x)(x - c_j)\,\mathrm{d}x = +\infty.$$

One thus recovers [4, Theorem 2.1] by Bertsch, Smarrazzo and Tesei; see also [27].

In dimension $N \geq 2$, one deduces that $\Omega \setminus Z$ has only one Sobolev-connected component for small Z using the Intermediate value theorem for Sobolev functions by Van Schaftingen and Willem [32]:

Corollary 1.2. *If $\mathcal{H}^{N-1}(Z) = 0$, then $\Omega \setminus Z$ is Sobolev-connected. Hence, the zero-set of any solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the Schrödinger equation (1.1) with nonnegative $f \in L^\infty(\Omega)$ and $\int_\Omega f > 0$ does not depend on w, and*

$$\widehat{w}(x) = 0 \quad \text{if and only if} \quad x \in Z.$$

The proof of Theorem 1.1 relies on the fact that the universal zero-set Z is the set of points where the Schrödinger operator $-\Delta + V$ is unable to have a Green's function in the sense of distributions. For example, in the spirit of the seminal work of Bénilan and Brezis [3] one verifies that when V is the potential in Example 1.2 with exponent $\alpha \geq 2$ the equation

$$-\Delta u + Vu = \delta_a$$

involving a Dirac mass δ_a does not have a distributional solution in Ω, see [29, Section 9], and as we have observed in this case, $Z = \{a\}$. More generally, we establish that

Theorem 1.3. *Let $V : \Omega \to [0, +\infty]$ be a Borel function. Given $x \in \Omega$, there exists $G_x \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V\,\mathrm{d}x)$ such that*

$$-\Delta G_x + VG_x = \delta_x \quad \text{in the sense of distributions in } \Omega$$

if and only if

$$x \notin Z.$$

Moreover, one has Green's representation formula

$$\widehat{w}(x) = \int_\Omega G_x f \quad \text{at each } x \in \Omega \setminus Z,$$

for every function $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ that satisfies (1.1) with $f \in L^\infty(\Omega)$.

In dimension $N \geq 3$ and at a point $x \in \Omega$ where the Newtonian potential

$$\mathcal{N}V : z \in \mathbb{R}^N \longmapsto \int_\Omega \frac{V(y)}{|z - y|^{N-2}}\,\mathrm{d}y$$

is finite, the Green's function G_x exists and thus $x \notin Z$. The reason is that the fundamental solution of the Laplacian yields a supersolution for $-\Delta + V$ with Dirac mass δ_x and then one can apply the method of sub- and supersolutions from Section 2 below. Here are some consequences of this observation:

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Example 1.5. If $V \in L^{\frac{N}{2},1}(\Omega)$, then by $(L^{\frac{N}{2},1}, L^{\frac{N}{N-2},\infty})$ duality in Lorentz spaces the Newtonian potential $\mathcal{N}V$ is a bounded function in Ω. Thus,

$$Z = \emptyset$$

and the classical alternative (1.2) is satisfied.

Example 1.6. If $V \in L^1(\Omega)$, then $\mathcal{N}V$ satisfies the Poisson equation

$$- \Delta(\mathcal{N}V) = \gamma_N V \quad \text{in the sense of distributions in } \Omega,$$

where $\gamma_N > 0$. From classical Potential theory, we have in particular that $\mathcal{N}V$ can only be infinite on a set of $W^{1,2}$ capacity zero. Hence,

$$\mathrm{cap}_{W^{1,2}}(Z) = 0$$

and Corollary 1.2 applies since in this case the Hausdorff dimension of Z is at most $N - 2$. While Ancona's maximum principle from [2] already asserts that $\{\widehat{w} = 0\}$ has $W^{1,2}$ capacity zero for every nontrivial solution of (1.1) with nonnegative f, we now have the stronger new property that $\{\widehat{w} = 0\}$ is actually independent of the solution.

Example 1.7. Assume that $V \in L^p(\Omega)$ for some $1 < p \leq N/2$, which is an intermediate case between the two previous examples. We now have $\Delta(\mathcal{N}V) \in L^p(\Omega)$ and then, by singular-integral estimates, $\mathcal{N}V \in W^{2,p}_{\mathrm{loc}}(\Omega)$. As the exceptional set of $W^{2,p}$ functions has $W^{2,p}$ capacity zero, we deduce that

$$\mathrm{cap}_{W^{2,p}}(Z) = 0,$$

which combined with Corollary 1.2 above implies Theorem 1 from our previous work [26].

The universal zero-set Z identifies not only the Dirac masses, but in fact all nonnegative finite Borel measures μ for which the Dirichlet problem

$$\begin{cases} -\Delta u + Vu = \mu & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

has a *distributional solution*, where by a solution we mean a function $u \in W^{1,1}_0(\Omega) \cap L^1(\Omega; V \, \mathrm{d}x)$ which verifies the equation in the sense of distributions in Ω. Observe that the Green's function G_x arises as a special case of this setting with $\mu = \delta_x$. The zero-boundary value of u in (1.8) is encoded by the requirement that $u \in W^{1,1}_0(\Omega)$. An equivalent formulation, without relying on Sobolev spaces, consists of using test functions in the larger class $C_0^\infty(\overline{\Omega})$ of smooth functions in $\overline{\Omega}$ that vanish on $\partial\Omega$, not necessarily with compact support in Ω; see Section 2.

Our next theorem fully characterizes the nonnegative finite measures for which (1.8) has a solution:

Theorem 1.4. *For every Borel function* $V : \Omega \to [0, +\infty]$, *the Dirichlet problem* (1.8) *has a distributional solution with a nonnegative finite Borel measure* μ *in* Ω *if and only if*

$$\mu(Z) = 0.$$

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Observe in particular that (1.8) has a distributional solution with $\mu = \chi_{\Omega \setminus Z} \, \mathrm{d}x$, since

$$\mu(Z) = \int_Z \chi_{\Omega \setminus Z} \, \mathrm{d}x = 0.$$

When Z is negligible with respect to the Lebesgue measure, we also deduce the existence of a distributional solution with $\mu = f \, \mathrm{d}x$ for every $f \in L^1(\Omega)$. Then, for $f \in L^\infty(\Omega)$, such a solution belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and the representation formula in Theorem 1.3 holds. Although Theorem 1.1 only applies to bounded data, the tools we use can be adapted to get its counterpart for general solutions of (1.8) with nonnegative measures; see Theorem 13.1.

We rely in this paper on the notion of duality solution of (1.8) by Malusa and Orsina [22], which was inspired from the fundamental work of Littman, Stampacchia and Weinberger [21]:

Definition 1.2. *Given $\mu \in \mathcal{M}(\Omega)$, we say that $u \in L^1(\Omega)$ is a* duality solution *of (1.8) whenever*

$$\int_\Omega u f = \int_\Omega \widehat{\zeta_f} \, \mathrm{d}\mu \quad \textit{for every } f \in L^\infty(\Omega),$$

where ζ_f is the unique minimizer of the energy functional

$$E(z) = \frac{1}{2} \int_\Omega \left(|\nabla z|^2 + V z^2 \right) - \int_\Omega f z \quad \textit{in } W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x). \tag{1.9}$$

In contrast with distributional solutions, duality solutions exist for any finite measure μ regardless of the potential V. One reason is that they typically require less test functions, just enough to ensure uniqueness. In Sections 3 and 4, we compare both concepts. A defect of the duality formulation is that the same function can solve the Schrödinger equation for different measures. It may happen that $u \equiv 0$ is the duality solution associated to the Dirac mass δ_x when $x \in Z$; see Section 9. Duality solutions are nevertheless a convenient tool to apply Perron's method and find distributional solutions of (1.8). Such an approach is pursued in Section 5, where we first prove Theorem 1.4 for $\mu = \chi_{\Omega \setminus Z} \, \mathrm{d}x$. This is used in Section 6 to establish an orthogonality principle between the sets Z and $\Omega \setminus Z$ which is later applied in Section 8 to prove the existence of distributional solutions of (1.8) in full generality.

In Section 7, we develop another fundamental tool: A comparison principle which relates a solution of (1.8) with nonnegative *measure* datum to another one with nonnegative *bounded* datum. Namely, we prove that

$$u \geq w \quad \text{almost everywhere in } \Omega, \tag{1.10}$$

where $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is the solution of (1.1) with right-hand side $f = H(u)$ for some fixed bounded nondecreasing continuous function H that is positive on $(0, +\infty)$. Such a function H can be chosen of the form

$$H(t) = \epsilon \min \{ t^\alpha, 1 \},$$

for any $\alpha > 1$ and any $\epsilon > 0$ small enough, independently of u and V. Estimate (1.10) relates the zero-sets of u and w, and works as a replacement of the Harnack inequality, which is false for singular potentials V.

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In Section 8, we prove Theorems 1.3 and 1.4, where the comparison principle (1.10) is used to prove that $\mu(Z) = 0$ is necessary for the existence of a distributional solution of (1.8). We apply again (1.10) in Section 9 to show that $\Omega \setminus Z$ is a disjoint union of superlevel sets of Green's functions of $-\Delta + V$.

The topological properties of the Sobolev-components of $\Omega \setminus Z$ are investigated in Sections 10 and 11. We show for example that they are Sobolev-connected using a variant of Poincaré's balayage method on Sobolev-open sets. We then prove Theorem 1.1 and Corollary 1.2 in Section 12 using the decomposition of $\Omega \setminus Z$ and Green's representation formula. In Section 13, we present a weaker version of this formula for solutions of (1.8), which entitles us to adapt the proof of Theorem 1.1 and get its counterpart for general nonnegative measures.

2. Method of sub- and supersolutions

We denote by $\mathcal{M}(\Omega)$ the vector space of finite Borel measures in Ω, which we equip with the total variation norm

$$\|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\Omega) = \int_\Omega \mathrm{d}|\mu|.$$

We recall that a function $u \in L^1(\Omega)$ satisfies the equation

$$-\Delta u + V u = \mu \quad \text{in the sense of distributions in } \Omega$$

for some $\mu \in \mathcal{M}(\Omega)$ whenever one has $u \in L^1(\Omega; V \, \mathrm{d}x)$ and

$$\int_\Omega u \left(-\Delta \varphi + V \varphi\right) = \int_\Omega \varphi \, \mathrm{d}\mu \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

We prove in this section the following form of the method of sub- and supersolutions for distributional solutions of the Dirichlet problem (1.8) involving the Schrödinger operator:

Proposition 2.1. *If* (1.8) *has a distributional solution with a nonnegative measure* $\mu \in \mathcal{M}(\Omega)$, *then* (1.8) *also has a distributional solution for every datum* $\nu \in \mathcal{M}(\Omega)$ *such that* $|\nu| \leq \mu$.

Although this statement is already proved in [28], we present a different argument based on the truncation of the potential. This will be the occasion for us to recall several properties of solutions involving measures that are used throughout the paper.

In view of the linearity of the equation, a natural approach would be to rely on a duality argument based on the estimate

$$\left| \int_\Omega \varphi \, \mathrm{d}\mu \right| \leq C \|-\Delta \varphi + V \varphi\|_{L^p(\Omega)} \quad \text{for every } \varphi \in C_c^\infty(\Omega), \tag{2.1}$$

where $p > N/2$, that follows from the Sobolev imbedding of solutions of the Schrödinger equation with measure data; see (2.3) below. However, since V is merely a Borel function, such an estimate is useless as the right-hand side may be infinite for various choices of $\varphi \in C_c^\infty(\Omega)$.

We begin instead by proving that a solution of (1.8) can be obtained as the limit of solutions of (1.8) involving the operator $-\Delta + T_k(V)$, with the bounded potential $T_k(V)$. Here, $T_k : \mathbb{R} \to \mathbb{R}$ denotes the truncation at levels $\pm k$:

$$T_k(s) := \begin{cases} -k & \text{if } s < -k, \\ s & \text{if } -k \leq s \leq k, \\ k & \text{if } s > k. \end{cases}$$

We recall that, for nonnegative bounded potentials, (1.8) has a solution u for every $\mu \in \mathcal{M}(\Omega)$; see [31]. Independently of the fact that V is bounded or not, we have the absorption estimate

$$\|Vu\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)}, \tag{2.2}$$

which can be obtained using as test function a suitable approximation of $\operatorname{sgn} u$; see [7, Proposition 4.B.3] or [28, Proposition 21.5]. Moreover, a solution of (1.8) belongs to $W_0^{1,q}(\Omega)$ for every $1 \leq q < \frac{N}{N-1}$ and satisfies

$$\|u\|_{W^{1,q}(\Omega)} \leq C\|\mu\|_{\mathcal{M}(\Omega)}, \tag{2.3}$$

for some constant $C > 0$ depending on q and Ω, but not on the potential V.

To see why C in (2.3) can be chosen independently of V, one observes that, by the equation satisfied by u and by the absorption estimate (2.2),

$$\|\Delta u\|_{\mathcal{M}(\Omega)} \leq \|Vu\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} \leq 2\|\mu\|_{\mathcal{M}(\Omega)}.$$

Since (2.3) is true for $-\Delta$, see [21, Lemma 7.3] or [28, Proposition 5.1], we get

$$\|u\|_{W^{1,q}(\Omega)} \leq C'\|\Delta u\|_{\mathcal{M}(\Omega)} \leq 2C'\|\mu\|_{\mathcal{M}(\Omega)}.$$

This argument thus shows that the estimate (2.3) for $-\Delta$ implies its counterpart for every $-\Delta + V$ with a nonnegative V.

The approximation scheme that is used in the proof of Proposition 2.1 is given by the following

Lemma 2.2. *Let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative measure and, for every $k \in \mathbb{N}$, let $u_k \in W_0^{1,1}(\Omega)$ be such that*

$$-\Delta u_k + T_k(V)u_k = \mu \quad \text{in the sense of distributions in } \Omega.$$

If (1.8) has a distributional solution u, then $(u_k)_{k \in \mathbb{N}}$ converges to u and $(T_k(V)u_k)_{k \in \mathbb{N}}$ converges to Vu, both in $L^1(\Omega)$.

Proof of Lemma 2.2. We first observe that u also satisfies the Dirichlet problem with the operator $-\Delta + T_k(V)$ and datum $\mu - (V - T_k(V))u$. Thus, subtracting the equations satisfied by u_k and u we find

$$-\Delta(u_k - u) + T_k(V)(u_k - u) = (V - T_k(V))u$$

in the sense of distributions in Ω. Using the absorption estimate for the operator $-\Delta + T_k(V)$, we get

$$\|T_k(V)(u_k - u)\|_{L^1(\Omega)} \leq \|(V - T_k(V))u\|_{L^1(\Omega)}. \tag{2.4}$$

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Since the constant in (2.3) does not depend on the potential, we also have

$$\|u_k - u\|_{L^1(\Omega)} \leq C\|(V - T_k(V))u\|_{L^1(\Omega)}. \tag{2.5}$$

Observing that $Vu \in L^1(\Omega)$,

$$\lim_{k \to \infty} \|(V - T_k(V))u\|_{L^1(\Omega)} = 0.$$

Hence, the conclusion follows from (2.4) and (2.5). □

To prove a weak maximum principle for (1.8), it is convenient to reformulate the definition of distributional solution as: $u \in L^1(\Omega)$ is such that $u \in L^1(\Omega; V \, dx)$ and

$$-\Delta u + Vu = \mu \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))',$$

that is,

$$\int_\Omega u \, (-\Delta\psi + V\psi) = \int_\Omega \psi \, d\mu \quad \text{for every } \psi \in C_0^\infty(\overline{\Omega}). \tag{2.6}$$

The fact that we can use a larger class of smooth test functions comes from the assumption that $u \in W_0^{1,1}(\Omega)$, which encodes the zero boundary value of u; see [28, Proposition 6.3].

Lemma 2.3. *Let u be the distributional solution of (1.8) with $\mu \in \mathcal{M}(\Omega)$. If $\mu \leq 0$ in Ω, then $u \leq 0$ almost everywhere in Ω.*

Proof of Lemma 2.3. By assumption on μ,

$$-\Delta u + Vu \leq 0 \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'.$$

Applying the formulation of Kato's inequality up to the boundary from [7, Proposition 4.B.5], see also [28, Lemma 20.8], we have

$$-\Delta u^+ + \chi_{\{u>0\}} Vu \leq 0 \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'.$$

Thus, for every *nonnegative* $\psi \in C_0^\infty(\overline{\Omega})$,

$$-\int_\Omega u^+ \Delta\psi \leq -\int_{\{u>0\}} Vu\psi \leq 0.$$

Taking any such a ψ with $-\Delta\psi > 0$ in Ω, we deduce that $u^+ = 0$ almost everywhere in Ω. □

Proof of Proposition 2.1. For every $k \in \mathbb{N}$, let u_k be as in Lemma 2.2 and let v_k be also a solution of (1.8) for $-\Delta + T_k(V)$, but with datum ν. Observe that v_k exists in this case since $T_k(V)$ is bounded. Thus,

$$\int_\Omega v_k \, (-\Delta\varphi + T_k(V)\varphi) = \int_\Omega \varphi \, d\nu \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

By (2.3), the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$ for $1 \leq q < \frac{N}{N-1}$. Hence, there exists a subsequence $(v_{k_j})_{j\in\mathbb{N}}$ which converges in $L^1(\Omega)$ and almost everywhere to some function $v \in$

$W_0^{1,1}(\Omega)$. Since $|\nu| \leq \mu$ and $T_k(V)$ is nonnegative, by linearity of the equation and the weak maximum principle above we have

$$|v_k| \leq u_k \quad \text{almost everywhere in } \Omega.$$

By Lemma 2.2, the sequence $(T_k(V)u_k)_{k\in\mathbb{N}}$ converges to Vu in $L^1(\Omega)$. Thus, by the Dominated convergence theorem the sequence $(T_{k_j}(V)v_{k_j})_{j\in\mathbb{N}}$ converges to Vv in $L^1(\Omega)$. Therefore, letting $k = k_j \to \infty$ in the integral identity above, we deduce that v satisfies the equation involving $-\Delta + V$ with datum ν. $\qquad\square$

3. Distributional solutions are duality solutions

Given $f \in L^2(\Omega)$, the unique minimizer $\zeta_f \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, dx)$ of the energy functional (1.9) is the (variational) solution of the Euler-Lagrange equation

$$\int_\Omega \left(\nabla\zeta_f \cdot \nabla z + V\zeta_f \, z\right) = \int_\Omega fz \quad \text{for every } z \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, dx). \tag{3.1}$$

Under the additional assumption that $f \in L^\infty(\Omega)$, which is used in the definition of duality solution of (1.8), we have $\zeta_f \in L^\infty(\Omega)$ and

$$\|\zeta_f\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}\|\theta\|_{L^\infty(\Omega)},$$

where θ is the classical solution of

$$\begin{cases} -\Delta\theta = 1 & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We also observe that, for $f \in L^\infty(\Omega)$, the precise representative $\widehat{\zeta}_f$ is well-defined everywhere in Ω. To see why this is true, we first recall that $x \in \Omega$ is a Lebesgue point of a function $v \in L^1(\Omega)$ whenever there exists $c \in \mathbb{R}$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |v - c| = 0 \, ;$$

the precise representative of v at x is then defined as $\widehat{v}(x) := c$. Observe that by linearity of (3.1) it suffices to consider the case where f is nonnegative. One then shows that $\zeta_f \in L^1(\Omega; V \, dx)$ and

$$-\Delta\zeta_f + V\zeta_f \leq f \quad \text{in the sense of distributions in } \Omega, \tag{3.3}$$

which implies that ζ_f is almost everywhere the difference between a continuous and a bounded superharmonic function, and then every $x \in \Omega$ is a Lebesgue point of ζ_f as claimed; see [28, Lemma 8.10]. Inequality in (3.3) comes from an application of Fatou's lemma; see [22] or [27, Proposition 8.1]. When V is bounded, one can apply the Dominated convergence theorem instead to get equality; see [26, Proposition 3.1].

These functions $\widehat{\zeta}_f$ can be used as test functions for distributional solutions of (1.8):

Proposition 3.1. *If u is a distributional solution of* (1.8) *for some $\mu \in \mathcal{M}(\Omega)$, then*

$$\int_\Omega uf = \int_\Omega \widehat{\zeta}_f \, \mathrm{d}\mu \quad \text{for every } f \in L^\infty(\Omega).$$

Hence, u is a duality solution of (1.8).

This result is proved in [22] using an approximation of μ of the form $\rho_k * \mu$ where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of mollifiers in $C_c^\infty(\mathbb{R}^N)$; see also Proposition 3.5 below. For the convenience of the reader, we present an alternative approximation based on the truncation of the potential V, without changing the measure μ.

Lemma 3.2. *Given $f \in L^\infty(\Omega)$ and $k \in \mathbb{N}$, let $\zeta_{f,k}$ be the minimizer of*

$$E_k(z) = \frac{1}{2} \int_\Omega (|\nabla z|^2 + T_k(V)z^2) - \int_\Omega fz \quad \text{in } W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x).$$

Then,

$$\lim_{k\to\infty} \widehat{\zeta_{f,k}}(x) = \widehat{\zeta}_f(x) \quad \text{for every } x \in \Omega.$$

As the function $\zeta_{f,k}$ satisfies

$$-\Delta\zeta_{f,k} + T_k(V)\zeta_{f,k} = f \quad \text{in the sense of distributions in } \Omega,$$

we have that $\Delta\zeta_{f,k} \in L^\infty(\Omega)$ and then $\widehat{\zeta_{f,k}}$ is continuous (and even C^1) in Ω. The proof of the lemma relies on the property that for a uniformly bounded and nondecreasing sequence $(v_k)_{k\in\mathbb{N}}$ of nonnegative superharmonic functions converging almost everywhere to v, the sequence of precise representatives $(\widehat{v_k})_{k\in\mathbb{N}}$ converges *everywhere* to $\widehat{v}$; see [22, Lemma 4.12] or [28, Exercise 8.4].

Proof of Lemma 3.2. We first prove that the sequence $(\zeta_{f,k})_{k\in\mathbb{N}}$ converges weakly in $W_0^{1,2}(\Omega)$ to ζ_f. We begin by observing that

$$E_k(\zeta_{f,k}) \leq E_k(\zeta_f) \leq E(\zeta_f) \quad \text{for every } k \in \mathbb{N}. \tag{3.4}$$

This implies that $(\zeta_{f,k})_{k\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$. Thus, there exists a subsequence $(\zeta_{f,k_j})_{j\in\mathbb{N}}$ which converges weakly in $W_0^{1,2}(\Omega)$ and almost everywhere in Ω to some function z. In particular, by Fatou's lemma,

$$\int_\Omega V z^2 \leq \liminf_{j\to\infty} \int_\Omega T_{k_j}(V)\zeta_{f,k_j}^2.$$

Taking $k = k_j$ in (3.4) and letting $j \to \infty$, we get

$$E(z) \leq E(\zeta_f).$$

Since ζ_f is the unique minimizer of the functional E, we deduce that $z = \zeta_f$ almost everywhere in Ω. By uniqueness of the limit, the entire sequence $(\zeta_{f,k})_{k\in\mathbb{N}}$ converges weakly to ζ_f.

By linearity of the Euler-Lagrange equation, we may proceed with the proof of the lemma assuming that f is nonnegative. In this case, by the weak maximum principle the sequence $(\zeta_{f,k})_{k\in\mathbb{N}}$

is non-increasing in Ω and then, by the first part of the proof, converges almost everywhere to ζ_f. Let v_k and w be such that

$$\begin{cases} -\Delta v_k = T_k(V)\zeta_{f,k} & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

We then have $v_k = w - \zeta_{f,k}$ almost everywhere in Ω, which implies that

$$\widehat{v_k}(x) = \widehat{w}(x) - \widehat{\zeta_{f,k}}(x) \quad \text{for every } x \in \Omega.$$

Observe that $(\widehat{v_k})_{k\in\mathbb{N}}$ is a uniformly bounded and nondecreasing sequence of nonnegative superharmonic functions. Thus, its pointwise limit v coincides with the precise representative $\widehat{v}$ in Ω. We then get

$$\widehat{v}(x) = v(x) = \widehat{w}(x) - \lim_{k\to\infty} \widehat{\zeta_{f,k}}(x) \quad \text{for every } x \in \Omega. \tag{3.5}$$

Since $v = w - \zeta_f$ almost everywhere in Ω, we also have

$$\widehat{v}(x) = \widehat{w}(x) - \widehat{\zeta_f}(x) \quad \text{for every } x \in \Omega. \tag{3.6}$$

The conclusion follows from comparison between (3.5) and (3.6) and the boundedness of w. $\qquad\square$

Proof of Proposition 3.1. Let us first assume that V is bounded. In this case, for every $f \in L^\infty(\Omega)$, $\widehat{\zeta_f}$ is continuous, $\Delta\zeta_f$ is bounded and satisfies

$$-\Delta\zeta_f = f - V\zeta_f \quad \text{in the sense of distributions in } \Omega.$$

One can thus approximate $\widehat{\zeta_f}$ uniformly by a sequence $(\widehat{\zeta_{f_k}})_{k\in\mathbb{N}}$ in $C_0^\infty(\overline{\Omega})$ such that $(f_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and converges almost everywhere to f. To construct such an example, one can take $g_k = \rho_k * (f - V\zeta_f)$, where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of mollifiers, and $v_k \in C_0^\infty(\overline{\Omega})$ as the classical solution of

$$\begin{cases} -\Delta v_k = g_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We then have the desired approximation of $\widehat{\zeta_f}$ by observing that $v_k = \widehat{\zeta_{f_k}}$ with $f_k = g_k + Vv_k$. For every $k \in \mathbb{N}$, we get from (2.6) that

$$\int_\Omega uf_k = \int_\Omega u\,(-\Delta v_k + Vv_k) = \int_\Omega v_k\,\mathrm{d}\mu = \int_\Omega \widehat{\zeta_{f_k}}\,\mathrm{d}\mu$$

and the conclusion for V bounded follows as $k \to \infty$.

We now assume that V is merely a Borel function and denote by u_k the distributional solution of the Dirichlet problem associated to $-\Delta + T_k(V)$ and datum μ. By Lemma 2.2, $(u_k)_{k\in\mathbb{N}}$ converges to u in $L^1(\Omega)$. On the other hand, from the first part of this proof and using the notation of Lemma 3.2,

$$\int_\Omega u_k f = \int_\Omega \widehat{\zeta_{f,k}}\,\mathrm{d}\mu \quad \text{for every } f \in L^\infty(\Omega).$$

By uniform boundedness and pointwise convergence of $(\widehat{\zeta_{f,k}})_{k\in\mathbb{N}}$, the proposition follows. $\qquad\square$

The concept of duality solution of the Dirichlet problem (1.8) is a useful tool in establishing the connection between the failure of the strong maximum principle and the nonexistence of distributional solutions of (1.8). Uniqueness of the duality solution is a straightforward consequence of the fact that $u \equiv 0$ is the only solution with $\mu = 0$. More generally, the weak maximum principle also holds in the duality setting. While Proposition 3.1 states that distributional solutions (whenever they exist) are duality solutions, the latter exist for any given finite measure:

Proposition 3.3. *The Dirichlet problem* (1.8) *has a unique duality solution for every* $\mu \in \mathcal{M}(\Omega)$.

This proposition is proved in [22, Theorem 5.6], in the spirit of [21]. The proof is based on Stampacchia's estimate:

$$\|\zeta_f\|_{L^\infty(\Omega)} \leq C\|f\|_{(W_0^{1,q}(\Omega))'} \quad \text{for every } f \in L^\infty(\Omega),$$

where $q < \frac{N}{N-1}$ and $C > 0$ depends on q and Ω, which implies that any duality solution belongs to $W_0^{1,q}(\Omega)$ and satisfies

$$\|u\|_{W^{1,q}(\Omega)} \leq C\|\mu\|_{\mathcal{M}(\Omega)}. \tag{3.7}$$

Remark 3.4. From the Euler-Lagrange equation (3.1), minimizers of E are also duality solutions. Indeed, for any $h \in L^2(\Omega)$, one can apply (3.1) with $z = \zeta_h$ to get

$$\int_\Omega \zeta_h f = \int_\Omega \zeta_f h \quad \text{for every } f \in L^\infty(\Omega).$$

Since $\zeta_f = \widehat{\zeta_f}$ almost everywhere in Ω, ζ_h is then a duality solution of (1.8) with $\mu = h \, dx$. Assuming that (1.8) has a distributional solution w with $\mu = h \, dx$, then w is also a duality solution by Proposition 3.1. Hence, by uniqueness, one has $w = \zeta_h$.

Denoting by G_x the duality solution associated to the Dirac mass $\mu = \delta_x$ at any point $x \in \Omega$, we have the following representation formula:

$$\widehat{\zeta_f}(x) = \int_\Omega G_x f \quad \text{for every } f \in L^\infty(\Omega). \tag{3.8}$$

By Proposition 3.4, this formula also applies to distributional solutions with bounded data. It thus seems that we have already fulfilled part of our goals we set in the introduction, more specifically in Theorem 1.3. However, we do not know whether G_x is a distributional solution of (1.8) with $\mu = \delta_x$, and we still have to prove that this is the case if and only if $x \in \Omega \setminus Z$. In addition, we would like to identify all nonnegative functions $f \in L^\infty(\Omega)$ such that ζ_f is indeed a distributional solution with datum f. We eventually prove that this is true if and only if $\int_Z f = 0$.

To conclude this section we explain why duality solutions enjoy good approximation properties in the sense that reasonable approximation schemes of the measure μ yield sequences of solutions that converge to the duality solution u associated to μ.

Proposition 3.5. *Let* $N \geq 2$ *and let* $(\rho_k)_{k \in \mathbb{N}}$ *be a sequence of mollifiers of the form* $\rho_k(x) = \frac{1}{r_k^N}\varphi(\frac{x}{r_k})$ *for a fixed* $\varphi \in C_c^\infty(\mathbb{R}^N)$ *and a sequence* $(r_k)_{k \in \mathbb{N}}$ *of positive numbers converging to zero. Given* $\mu \in \mathcal{M}(\Omega)$, *the sequence* $(\zeta_{\rho_k * \mu})_{k \in \mathbb{N}}$ *converges in* $L^p(\Omega)$ *to the duality solution of* (1.8) *for every* $1 \leq p < \frac{N}{N-2}$.

Proof. The assumption on $(\rho_k)_{k\in\mathbb{N}}$ implies that, for every $f \in L^\infty(\Omega)$ and $x \in \Omega$,

$$\lim_{k\to\infty} \breve{\rho}_k * \zeta_f(x) = \widehat{\zeta}_f(x),$$

where $\breve{\rho}_k(y) = \rho_k(-y)$. Then, by Fubini's theorem and the Dominated convergence theorem,

$$\int_\Omega \zeta_f\, \rho_k * \mu = \int_\Omega \breve{\rho}_k * \zeta_f\, \mathrm{d}\mu \to \int_\Omega \widehat{\zeta}_f\, \mathrm{d}\mu.$$

Since $\zeta_{\rho_k*\mu}$ is the duality solution with datum $\rho_k * \mu$, the latter convergence can be rewritten as

$$\lim_{k\to\infty} \int_\Omega \zeta_{\rho_k*\mu}\, f = \int_\Omega uf,$$

where u is the duality solution with datum μ As a result, $(\zeta_{\rho_k*\mu})_{k\in\mathbb{N}}$ converges to u with respect to the $L^\infty(\Omega)$-weak* topology. By boundedness of $(\zeta_{\rho_k*\mu})_{k\in\mathbb{N}}$ in $W_0^{1,q}(\Omega)$ for every $1 \le q < \frac{N}{N-1}$, see (3.7), we have the strong convergence to u in L^p spaces. $\qquad\square$

4. Duality solutions as distributional solutions

We henceforth denote by S the subset of Ω defined as the zero-set of the *torsion function* ζ_1, namely

$$S = \big\{x \in \Omega : \widehat{\zeta}_1(x) = 0\big\}. \tag{4.1}$$

We recall that ζ_1 is the minimizer of the energy functional E with constant $f \equiv 1$, and so S is a Sobolev-closed set and depends on the potential V. By the weak maximum principle for variational solutions, for every $f \in L^\infty(\Omega)$ the torsion function dominates ζ_f in the sense that

$$|\zeta_f| \le \|f\|_{L^\infty(\Omega)}\zeta_1 \quad \text{almost everywhere in } \Omega. \tag{4.2}$$

The same estimate is then satisfied by the precise representatives, this time at every point in Ω, and we deduce that

$$S = \big\{x \in \Omega : \widehat{\zeta}_f(x) = 0 \text{ for every } f \in L^\infty(\Omega)\big\}. \tag{4.3}$$

By Proposition 3.4, this characterization of S involves more functions than in the definition of the universal zero-set Z. Therefore,

$$S \subset Z.$$

For example, when V is bounded, the notions of distributional and duality solution coincide and the strong maximum principle holds everywhere in Ω. We thus have in this case

$$S = Z = \emptyset.$$

For unbounded potentials V, the inclusion can be strict:

Example 4.1. The Dirichlet problem

$$\begin{cases} -\Delta u + \dfrac{1}{|x_1|^\alpha}\, u = \mu & \text{in } B_1(0), \\ \qquad\qquad u = 0 & \text{on } \partial B_1(0), \end{cases}$$

has no distributional solution with μ nonnegative, $\mu \neq 0$, for any exponent $1 \leq \alpha < 2$; see [25, Theorem 9.1]. In this case, ζ_1 solves two independent Dirichlet problems, one on each side of the hyperplane $\{x_1 = 0\}$, and then

$$S = \{x_1 = 0\} \cap B_1(0) \quad \text{and} \quad Z = \Omega.$$

For $\alpha \geq 2$, the singularity of V is even stronger and, nevertheless, one has the equality

$$S = Z = \{x_1 = 0\} \cap B_1(0),$$

since ζ_1 now satisfies (1.1) with $f \equiv 1$.

We prove in this section that duality solutions can be seen as distributional solutions of the Dirichlet problem, but when $S \neq \emptyset$ they need not solve the equation in the sense of distributions with the same datum μ.

Proposition 4.1. *If $u \in L^1(\Omega)$ is a duality solution of (1.8) for some nonnegative measure $\mu \in \mathcal{M}(\Omega)$, then $u \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V\,\mathrm{d}x)$ and*

$$-\Delta u + Vu = \mu\lfloor_{\Omega \setminus S} - \lambda \quad \text{in the sense of distributions in } \Omega,$$

where $\lambda \in \mathcal{M}(\Omega)$ is nonnegative, diffuse with respect to the $W^{1,2}$ capacity and carried by S, that is,

$$\lambda(\Omega \setminus S) = 0.$$

Here, $\mu\lfloor_A$ denotes the contraction of μ with respect to a Borel set A, defined by

$$\mu\lfloor_A(B) = \mu(B \cap A).$$

By a diffuse measure we mean that $\lambda(B) = 0$ for every Borel subset $B \subset \Omega$ having $W^{1,2}$ capacity zero. We finally recall that the $W^{1,2}$ capacity of a compact subset $K \subset \mathbb{R}^N$ is defined as

$$\mathrm{cap}_{W^{1,2}}(K) = \inf\left\{ \|\varphi\|_{W^{1,2}(\mathbb{R}^N)} : \varphi \in C_c^\infty(\mathbb{R}^N),\ \varphi \geq 0 \text{ in } \mathbb{R}^N \text{ and } \varphi > 1 \text{ on } K \right\}.$$

It is then extended to open sets by inner regularity and then to arbitrary sets by outer regularity.

By Proposition 4.1, a duality solution thus fails from being a distributional one for the same nonnegative datum μ for two possible reasons: The existence of some nontrivial mass carried by μ on S or the appearance of a nonpositive measure carried by S. This latter phenomenon always happens in the case of Example 4.1 when $1 \leq \alpha < 2$ since there are simply no distributional supersolutions, other than the trivial one. One also shows that the measure λ is always singular with respect to the Lebesgue measure; see Proposition 4.3.

We begin with the following approximation procedure, where in contrast with Lemma 2.2 we do not assume that (1.8) has a distributional solution.

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Lemma 4.2. *Let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative measure and, for every $k \in \mathbb{N}$, let $u_k \in W_0^{1,1}(\Omega)$ be such that*

$$-\Delta u_k + T_k(V)u_k = \mu \quad \text{in the sense of distributions in } \Omega.$$

Then, the sequence $(u_k)_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the duality solution u of (1.8). Moreover, $u \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V\,\mathrm{d}x)$ and there exists a nonnegative measure $\lambda \in \mathcal{M}(\Omega)$ such that

$$-\Delta u + Vu = \mu - \lambda \quad \text{in the sense of distributions in } \Omega.$$

Proof of Lemma 4.2. By the weak maximum principle, the sequence $(u_k)_{k\in\mathbb{N}}$ is non-increasing and nonnegative, hence it converges in $L^1(\Omega)$ to some function u. Using the notation of Lemma 3.2, we have

$$\int_\Omega u_k f = \int_\Omega \widehat{\zeta_{f,k}}\,\mathrm{d}\mu \quad \text{for every } f \in L^\infty(\Omega).$$

The sequence $(\widehat{\zeta_{f,k}})_{k\in\mathbb{N}}$ is uniformly bounded and, by Lemma 3.2, converges pointwise to $\widehat{\zeta_f}$. By the Dominated convergence theorem, we thus have

$$\int_\Omega uf = \int_\Omega \widehat{\zeta_f}\,\mathrm{d}\mu,$$

so that u is the duality solution of (1.8) and then belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$.

Since the sequence $(T_k(V)u_k)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$ and converges pointwise to Vu, by Fatou's lemma we have $Vu \in L^1(\Omega)$. For every $\varphi \in C_c^\infty(\Omega)$ we also have

$$\left| \int_\Omega u_k \Delta\varphi + \int_\Omega \varphi\,\mathrm{d}\mu \right| = \left| \int_\Omega T_k(V)u_k\varphi \right| \le C\|\varphi\|_{L^\infty(\Omega)},$$

for some constant independent of k. Letting $k \to \infty$, we deduce from the Riesz representation theorem that there exists $\nu \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega \varphi\,\mathrm{d}\nu = \int_\Omega u\Delta\varphi + \int_\Omega \varphi\,\mathrm{d}\mu. \tag{4.4}$$

By Fatou's lemma, for nonnegative test functions φ we also have

$$\int_\Omega Vu\varphi \le \lim_{k\to\infty} \int_\Omega T_k(V)u_k\varphi = \lim_{k\to\infty} \int_\Omega u_k\Delta\varphi + \int_\Omega \varphi\,\mathrm{d}\mu = \int_\Omega u\Delta\varphi + \int_\Omega \varphi\,\mathrm{d}\mu. \tag{4.5}$$

Combining (4.4) and (4.5), we deduce that $Vu\,\mathrm{d}x \le \nu$ in the sense of distributions in Ω. By the regularity of finite Borel measures such an inequality also holds in the sense of measures, that is,

$$\int_A Vu\,\mathrm{d}x \le \nu(A) \quad \text{for every Borel set } A \subset \Omega;$$

see e.g. [28, Proposition 6.12]. The conclusion is then satisfied by the finite measure $\lambda = \nu - Vu\,\mathrm{d}x$. $\qquad\square$

Proof of Proposition 4.1. Let u be the solution of (1.8) with datum μ. By the characterization (4.3) of S we have $\widehat{\zeta_f} = 0$ on S for every $f \in L^\infty(\Omega)$, which implies that

$$\int_\Omega \widehat{\zeta_f}\,\mathrm{d}\mu\llcorner_{\Omega\setminus S} = \int_\Omega \widehat{\zeta_f}\,\mathrm{d}\mu = \int_\Omega uf. \tag{4.6}$$

Hence u is also a duality solution with datum $\mu\lfloor_{\Omega\setminus S}$. By Lemma 4.2 applied to the measure $\mu\lfloor_{\Omega\setminus S}$, there exists a nonnegative measure λ such that u is a distributional solution of (1.8) with datum $\mu\lfloor_{\Omega\setminus S} - \lambda$. Since by Proposition 3.1 a distributional solution is a duality solution with the same datum, for every $f \in L^\infty(\Omega)$ we thus have

$$\int_\Omega uf = \int_\Omega \widehat{\zeta_f}\, \mathrm{d}(\mu\lfloor_{\Omega\setminus S} - \lambda). \tag{4.7}$$

Then, by comparision between (4.6) and (4.7),

$$\int_\Omega \widehat{\zeta_f}\, \mathrm{d}\lambda = 0 \quad \text{for every } f \in L^\infty(\Omega).$$

Apply this identity with $f \equiv 1$. Since $\widehat{\zeta_1} > 0$ on $\Omega \setminus S$, by nonnegativity of λ it follows that $\lambda(\Omega \setminus S) = 0$.

To prove that λ is diffuse, we first recall that λ can be uniquely decomposed as a sum of measures, $\lambda = \lambda_\mathrm{d} + \lambda_\mathrm{c}$, where λ_d is the diffuse part with respect to the $W^{1,2}$ capacity and λ_c is concentrated on a set of $W^{1,2}$ capacity zero. This is analogous to the classical Lebesgue decomposition theorem with respect to a given measure. Although in our case it involves a capacity, the proofs are similar; see [28, Proposition 14.12].

We thus have to check that $\lambda_\mathrm{c} = 0$. For this purpose, we rely on the inverse maximum principle which asserts that, by nonnegativity of u, the concentrated part of Δu satisfies $(\Delta u)_\mathrm{c} \le 0$ in Ω; see [16, Theorem 3] or [28, Proposition 6.13]. Next, the equation

$$-\Delta u + Vu = \mu\lfloor_{\Omega\setminus S} - \lambda$$

holds in the sense of distributions, whence also in the sense of measures in Ω; see [28, Proposition 6.12]. More precisely, for every Borel set $A \subset \Omega$,

$$\int_A (-\Delta u + Vu) = \mu(A \setminus S) - \lambda(A).$$

Restricting such an identity to subsets of S we get $\Delta u = Vu + \lambda$ in S. Then, as the Lebesgue measure is diffuse with respect to the $W^{1,2}$ capacity,

$$(\Delta u)_\mathrm{c} = \lambda_\mathrm{c} \quad \text{in } S.$$

It thus follows from the inverse maximum principle that $\lambda_\mathrm{c} \le 0$ in S. Since $\lambda = 0$ in $\Omega \setminus S$, we conclude that $\lambda_\mathrm{c} \le 0$ in Ω. By nonnegativity of λ, we must have $\lambda_\mathrm{c} = 0$, which means that λ is diffuse. $\qquad\square$

Remark 4.3. The nonnegative measure λ given by Proposition 4.1 is singular with respect to the Lebesgue measure. Indeed, on the one hand, we claim that

$$u = 0 \quad \text{almost everywhere in } S. \tag{4.8}$$

To this end, observe that $G_x = 0$ for every $x \in S$ by an application of the representation formula (3.8) with $f \equiv 1$. As the counterpart of the representation formula is satisfied by u almost everywhere in Ω, see Lemma 13.2, we thus have (4.8). On the other hand, by the Lebesgue decomposition theorem, we can decompose the measure Δu as a sum $\Delta u = (\Delta u)_\mathrm{a} + (\Delta u)_\mathrm{s}$, where

$(\Delta u)_\mathrm{a}$ is absolutely continuous with respect to the Lebesgue measure and $(\Delta u)_\mathrm{s}$ is singular. We now observe that, by [1, Theorem 1.1],

$$(\Delta u)_\mathrm{a} = 0 \quad \text{in } \{u = c\}$$

for every $c \in \mathbb{R}$ and in particular in the level set $\{u = 0\}$. In our case, S is contained in $\{u = 0\}$, except for a set of Lebesgue measure zero, and the equation satisfied by u gives

$$(\Delta u)_\mathrm{a} = V u \, \mathrm{d}x - (\mu_\mathrm{a})\lfloor_{\Omega\setminus S} + \lambda_\mathrm{a} \quad \text{in } \Omega.$$

Restricting this identity to S, we thus have

$$\lambda_\mathrm{a} = (\Delta u)_\mathrm{a} = 0 \quad \text{in } S.$$

As $\lambda = 0$ in $\Omega \setminus S$, we conclude that $\lambda_\mathrm{a} = 0$ in Ω.

The precise pointwise identification of the zero-set S can be obtained using the Wiener test by Dal Maso and Mosco [10, 11], which involves a capacity explicitly defined in terms of the potential V. If one is simply willing to get a rough location of S, up to sets of $W^{1,2}$ capacity zero, then a more elementary approach is to look for nontrivial elements of $W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x)$:

Proposition 4.4. *For every nonnegative function* $v \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x)$, *we have*

$$\mathrm{cap}_{W^{1,2}} \left(\{\widehat{v} > 0\} \cap S \right) = 0.$$

In other words, there exists a set $R \subset \Omega$, possibly depending on v, with $W^{1,2}$ capacity zero and such that

$$S \subset \{\widehat{v} = 0\} \cup R.$$

Observe that R is always negligible with respect to the Lebesgue measure. Therefore, S is negligible whenever there exists *some* $v \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x)$ such that $v > 0$ almost everywhere in Ω.

Proof of Proposition 4.4. Assume by contradiction that the capacity is positive, and take a compact subset $K \subset \{\widehat{v} > 0\} \cap S$ with positive $W^{1,2}$ capacity. Let ν be a finite positive Borel measure supported in K such that $\nu \in (W_0^{1,2}(\Omega))'$. The action of ν as a continuous linear functional in $W_0^{1,2}(\Omega)$ is simply an integration with respect to ν :

$$\nu[\varphi] = \int_\Omega \varphi \, \mathrm{d}\nu \quad \text{for every } \varphi \in C_c^\infty(\Omega)$$

and then, by density,

$$\nu[z] = \int_\Omega \widehat{z} \, \mathrm{d}\nu \quad \text{for every } z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{4.9}$$

Since ν is supported in K and $\widehat{T_1(v)} = T_1(\widehat{v}) > 0$ in K, one then deduces that $\nu[T_1(v)] > 0$. Hence, the minimum of the energy functional

$$E(z) = \frac{1}{2} \int_\Omega (|\nabla z|^2 + V z^2) - \nu[z]$$

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is negative in $W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x)$, as the function $s \in \mathbb{R} \mapsto E(s \, T_1(v))$ is decreasing in a neighborhood of $s = 0$.

We now denote by w the minimizer of E. Using (4.9), one verifies that w is the duality solution of (1.8) with datum ν. Since ν is supported in $K \subset S$ and $\widehat{\zeta_f} = 0$ in S for every $f \in L^\infty(\Omega)$, we then get

$$\int_\Omega wf = \int_\Omega \widehat{\zeta_f} \, \mathrm{d}\nu = 0 \quad \text{for every } f \in L^\infty(\Omega).$$

Hence, $w = 0$ almost everywhere in Ω, which contradicts the fact that $E(w) < 0$. $\qquad \square$

5. Existence of a distributional solution with datum $\chi_{\Omega \setminus Z}$

As a preliminary step towards the proof of Theorem 1.4, we show in this section that

Proposition 5.1. *The set $\Omega \setminus Z$ is such that $\zeta_{\chi_{\Omega \setminus Z}}$ satisfies (1.1) with $f = \chi_{\Omega \setminus Z}$.*

The existence of a largest Borel set with such a property, without identification with $\Omega \setminus Z$ and up to negligible sets, is straightforward:

Lemma 5.2. *There exists a Borel set $A \subset \Omega$ such that ζ_{χ_A} satisfies (1.1) with $f = \chi_A$ and*

$$Z = \big\{ x \in \Omega : \widehat{\zeta_{\chi_A}}(x) = 0 \big\}.$$

In particular, Z is a Sobolev-closed set.

Proof of Lemma 5.2. Let

$$\alpha = \sup \left\{ |B| \, \middle| \, \begin{array}{l} B \subset \Omega \text{ is a Borel set and} \\ (1.8) \text{ has a distributional solution with } \mu = \chi_B \, \mathrm{d}x \end{array} \right\}.$$

We first prove that the supremum is achieved by some Borel set $A \subset \Omega$. To this end, take a maximizing sequence of Borel sets $(B_k)_{k \in \mathbb{N}}$. We observe that $A_n := \bigcup_{k=0}^{n} B_k$ satisfies

$$0 \le \chi_{A_n} \le \sum_{k=0}^{n} \chi_{B_k}.$$

By linearity of the equation, there exists a distributional solution with $\sum_{k=0}^{n} \chi_{B_k} \, \mathrm{d}x$. Hence, by Proposition 2.1, the Dirichlet problem (1.8) also has a distributional solution with datum $\mu = \chi_{A_n} \, \mathrm{d}x$. Since the sequence $(\chi_{A_n})_{n \in \mathbb{N}}$ is nondecreasing and bounded in $L^1(\Omega)$, we deduce using the Monotone convergence theorem and the Sobolev estimate (2.3) that (1.8) has a distributional solution with datum $\mu = \chi_A \, \mathrm{d}x$, where $A = \bigcup_{n=0}^{\infty} A_n$. Since

$$|A| = \lim_{n \to \infty} |A_n| \ge \lim_{n \to \infty} |B_n| = \alpha,$$

the set A achieves the supremum above. As (1.8) has a distributional solution with $\mu = \chi_A \, \mathrm{d}x$, by Proposition 3.4 such a solution must be ζ_{χ_A}.

Claim. If $f \in L^\infty(\Omega)$ is a nonnegative function such that (1.8) has a distributional solution with $\mu = f \, dx$, then $f = 0$ almost everywhere in $\Omega \setminus A$.

Proof of the Claim. We use the maximality of the set A. To this end, given a nonnegative $f \in L^\infty(\Omega)$ such that (1.8) has a distributional solution with measure $f \, dx$, assume by contradiction that the set $B := \{f > \epsilon\} \setminus A$ has positive Lebesgue measure for some $\epsilon > 0$. Since $0 \leq \epsilon \chi_B \leq f$, by Proposition 2.1 the Dirichlet problem also has a distributional solution with measure $\epsilon \chi_B \, dx$ and, by linearity of the equation, with $\chi_B \, dx$, and then also with $\chi_{A \cup B} \, dx = (\chi_A + \chi_B) \, dx$. Since $|A \cup B| > |A|$, we have a contradiction with the maximality of A. $\qquad\square$

From the Claim, we deduce that if w is a distributional solution of (1.8) with $\mu = f \, dx$, for some nonnegative $f \in L^\infty(\Omega)$, then

$$0 \leq f \leq \|f\|_{L^\infty(\Omega)} \chi_A \quad \text{almost everywhere in } \Omega.$$

Applying the weak maximum principle, we thus have

$$0 \leq w \leq \|f\|_{L^\infty(\Omega)} \zeta_{\chi_A} \quad \text{almost everywhere in } \Omega.$$

Hence, the precise representatives satisfy

$$0 \leq \widehat{w} \leq \|f\|_{L^\infty(\Omega)} \widehat{\zeta_{\chi_A}} \quad \text{in } \Omega.$$

Since w is arbitrary, it follows that $Z = \{\widehat{\zeta_{\chi_A}} = 0\}$, and this concludes the proof of the lemma. $\quad\square$

To show that the set A above can be taken at least as large as $\Omega \setminus Z$ we proceed in the spirit of Perron's method. To this end, we introduce a function w which dominates all subsolutions of (1.8) for a fixed measure μ and such that if (1.8) has a distributional solution, then such a solution must be w. More precisely,

Lemma 5.3. *For every nonnegative measure $\mu \in \mathcal{M}(\Omega)$, there exists a nonnegative function $w \in W_0^{1,1}(\Omega)$ such that $\chi_{\Omega \setminus Z} w \in L^1(\Omega; V \, dx)$, $\Delta w \in \mathcal{M}(\Omega)$,*

$$- \Delta w + V \chi_{\Omega \setminus Z} w \leq \mu \quad \text{in the sense of distributions in } \Omega$$

and

$$- \Delta(\chi_{\Omega \setminus Z} w) + V \chi_{\Omega \setminus Z} w \geq \mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z} \quad \text{in the sense of distributions in } \Omega,$$

where μ_{d} is the diffuse part of μ with respect to the $W^{1,2}$ capacity. Moreover, for every $u \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V \, dx)$ such that

$$- \Delta u + V u \leq \mu \quad \text{in the sense of distributions in } \Omega,$$

we have

$$u \leq w \quad \text{almost everywhere in } \Omega.$$

For the sake of proving Proposition 5.1, we could have restricted ourselves to measures of the form $\mu = f \, dx$ with $f \in L^1(\Omega)$, which satisfy in particular $\mu_d = \mu$. The statement for an arbitrary measure is used in the proof of Theorem 1.4 in Section 8 and we also show that

$$w = 0 \quad \text{almost everywhere in } Z.$$

To prove Lemma 5.3 we rely on a truncation strategy where the truncation level depends on $x \in \Omega$. We begin with the following observation:

Lemma 5.4. *Let $v \in L^1(\Omega; V \, dx)$ be a nonnegative function. For every nonnegative measure $\mu \in \mathcal{M}(\Omega)$, there exists $u \in W_0^{1,1}(\Omega)$ such that*

$$-\Delta u + V \min\{v, u\} = \mu \quad \text{in the sense of distributions in } \Omega.$$

Proof of Lemma 5.4. We proceed by approximation by taking a nonnegative sequence $(\mu_k)_{k \in \mathbb{N}}$ in $L^2(\Omega)$ which is bounded in $L^1(\Omega)$ and converges to μ in the sense of measures in Ω; an example is $\mu_k = \rho_k * \mu$ where $(\rho_k)_{k \in \mathbb{N}}$ is a sequence of mollifiers. Consider the energy functional

$$\widetilde{E}_k(z) = \frac{1}{2} \int_\Omega |\nabla z|^2 + \int_\Omega g(\cdot, z) - \int_\Omega \mu_k z,$$

where

$$g(x, t) := V(x) \int_0^t T_{v(x)}(s) \, ds \quad \text{for every } (x, t) \in \Omega \times \mathbb{R}.$$

We take $\widetilde{E}_k$ defined on

$$\mathcal{V} := \left\{ z \in W_0^{1,2}(\Omega) : g(\cdot, z) \in L^1(\Omega) \right\}.$$

Existence of a solution of the Euler-Lagrange equation associated to $\widetilde{E}_k$ follows from [6, Theorem 1] by Brezis and Browder, based on a truncation of g. Here we prove directly that the minimizer satisfies the equation. We first claim that in our case $\mathcal{V}$ is a vector subspace of $W_0^{1,2}(\Omega)$. To this end, observe that g is even, nondecreasing in $[0, +\infty)$, and satisfies the Δ_2 condition

$$0 \leq g(\cdot, 2t) \leq C g(\cdot, t) \quad \text{in } \Omega$$

for every $t \in \mathbb{R}$ and some constant $C > 0$. Thus, for every $t_1, t_2 \in \mathbb{R}$,

$$0 \leq g(\cdot, t_1 + t_2) \leq C \big(g(\cdot, t_1) + g(\cdot, t_2) \big) \quad \text{in } \Omega.$$

These properties of g imply that the condition $g(\cdot, z) \in L^1(\Omega)$ is stable under linear combinations of $z \in L^1(\Omega)$, and then $\mathcal{V}$ is a vector subspace of $W_0^{1,2}(\Omega)$ as claimed.

Since g is nonnegative, $\widetilde{E}_k$ is bounded from below in $\mathcal{V}$. Moreover, by the Rellich-Kondrashov compactness theorem and Fatou's lemma, any minimizing sequence of $\widetilde{E}_k$ in $\mathcal{V}$ has a subsequence that converges weakly in $W_0^{1,2}(\Omega)$ to a minimizer $u_k \in \mathcal{V}$. We now observe that

$$W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{V},$$

which follows from the assumption $v \in L^1(\Omega; V \, dx)$ and the fact that

$$0 \leq g(\cdot, t) \leq V v \, |t| \quad \text{for every } t \in \mathbb{R}.$$

Since $\mathcal{V}$ is a vector space that contains $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, the minimizer u_k satisfies the Euler-Lagrange equation

$$\int_\Omega \left(\nabla u_k \cdot \nabla z + V T_v(u_k) z \right) = \int_\Omega \mu_k z \quad \text{for every } z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Since μ_k is nonnegative, one deduces that u_k is also nonnegative. Hence, $T_v(u_k) = \min\{v, u_k\}$ and

$$-\Delta u_k + V \min\{v, u_k\} = \mu_k \quad \text{in the sense of distributions in } \Omega. \tag{5.1}$$

We next observe that

$$0 \le V \min\{v, u_k\} \le V v \quad \text{for every } k \in \mathbb{N}.$$

From equation (5.1) and the assumption on v, the sequence $(\Delta u_k)_{k \in \mathbb{N}}$ is then bounded in $L^1(\Omega)$. By Sobolev imbedding of solutions of the Dirichlet problem, we can extract a subsequence from $(u_k)_{k \in \mathbb{N}}$ which converges in $L^1(\Omega)$ to some function $u \in W_0^{1,1}(\Omega)$. We then have the conclusion using the Dominated convergence theorem. $\qquad\square$

Proof of Lemma 5.3. Let $v := \zeta_{\chi_A}$, where A is the set given by Lemma 5.2. For each $k \in \mathbb{N}$, let $w_k \in W_0^{1,1}(\Omega)$ be such that

$$-\Delta w_k + V \min\{kv, w_k\} = \mu \quad \text{in the sense of distributions in } \Omega. \tag{5.2}$$

The existence of w_k follows from Lemma 5.4 applied to the nonnegative function kv. Since the function $t \in \mathbb{R} \mapsto \min\{kv, t\}$ is nondecreasing, the weak maximum principle applies; see e.g. [7, Corollary 4.B.2]. The sequence $(w_k)_{k \in \mathbb{N}}$ is then nonnegative and non-increasing, whence converges pointwise and in $L^1(\Omega)$ to some function w. By construction of v, the set $\{v = 0\}$ equals Z, except for a negligible set, so that the sequence $(\min\{kv, w_k\})_{k \in \mathbb{N}}$ converges almost everywhere to $\chi_{\Omega \setminus Z} w$. By the absorption estimate,

$$\left\| V \min\{kv, w_k\} \right\|_{L^1(\Omega)} \le \|\mu\|_{\mathcal{M}(\Omega)},$$

the sequence $(\min\{kv, w_k\})_{k \in \mathbb{N}}$ is bounded in $L^1(\Omega; V \, \mathrm{d}x)$. Hence, by Fatou's lemma we have $\chi_{\Omega \setminus Z} w \in L^1(\Omega; V \, \mathrm{d}x)$ and

$$-\Delta w + V \chi_{\Omega \setminus Z} w \le \mu \quad \text{in the sense of distributions in } \Omega. \tag{5.3}$$

By the boundedness of the sequence $(\Delta w_k)_{k \in \mathbb{N}}$ in $\mathcal{M}(\Omega)$, we also have $w \in W_0^{1,1}(\Omega)$ and $\Delta w \in \mathcal{M}(\Omega)$.

We now suppose that we are given some function $u \in W_0^{1,1}(\Omega)$ such that

$$-\Delta u + V u \le \mu \quad \text{in the sense of distributions in } \Omega.$$

Then, u is a subsolution of equation (5.2) and, by the weak maximum principle, we have $u \le w_k$ almost everywhere in Ω. As $k \to \infty$, we get $u \le w$ almost everywhere in Ω.

We are left with the proof of

$$-\Delta(\chi_{\Omega \setminus Z} w) + V \chi_{\Omega \setminus Z} w \ge \mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z} \quad \text{in the sense of distributions in } \Omega. \tag{5.4}$$

To this end, we begin by writing, for every $a, b \in \mathbb{R}$,

$$\min\{a, b\} = \frac{a + b - (a-b)^+ - (b-a)^+}{2}, \tag{5.5}$$

which we shall apply with $a = \ell v$ and $b = w_k$, where $\ell \in \mathbb{N}$. By Kato's inequality [8, 12], $\Delta(\ell v - w_k)^+$ and $\Delta(w_k - \ell v)^+$ are locally finite measures in Ω that satisfy

$$\left[\Delta(\ell v - w_k)^+\right]_{\mathrm{d}} \geq \chi_{\{\widehat{\ell v} > \widehat{w_k}\}}\left[\Delta(\ell v - w_k)\right]_{\mathrm{d}} \tag{5.6}$$

and

$$\left[\Delta(w_k - \ell v)^+\right]_{\mathrm{d}} \geq \chi_{\{\widehat{\ell v} < \widehat{w_k}\}}\left[\Delta(w_k - \ell v)\right]_{\mathrm{d}} \tag{5.7}$$

in the sense of measures in Ω. Here we recall that, since $\Delta w \in \mathcal{M}(\Omega)$, the precise representative $\widehat{w}$ is defined quasi-everywhere in Ω, i.e. except on a subset of $W^{1,2}$ capacity zero; see [28, Proposition 8.9]. It thus follows from (5.5) to (5.7) that $\Delta \min\{\ell v, w_k\}$ is a locally finite measure in Ω such that

$$\left(\Delta \min\{\ell v, w_k\}\right)_{\mathrm{d}} \leq \chi_{\{\widehat{\ell v} < \widehat{w_k}\}}(\ell \Delta v)_{\mathrm{d}} + \chi_{\{\widehat{\ell v} > \widehat{w_k}\}}(\Delta w_k)_{\mathrm{d}} + \chi_{\{\widehat{\ell v} = \widehat{w_k}\}}\frac{(\ell \Delta v)_{\mathrm{d}} + (\Delta w_k)_{\mathrm{d}}}{2}.$$

Observe that since v is a distributional solution with datum $\chi_A \, \mathrm{d}x$ (and not just a duality solution) we have

$$(\ell \Delta v)_{\mathrm{d}} = \ell \Delta v = \ell V v - \ell \chi_A \leq \ell V v.$$

We also have

$$(\Delta w_k)_{\mathrm{d}} = V \min\{kv, w_k\} - \mu_{\mathrm{d}}.$$

Thus,

$$\left(\Delta \min\{\ell v, w_k\}\right)_{\mathrm{d}} \leq \chi_{\{\widehat{\ell v} < \widehat{w_k}\}}\ell V v + \chi_{\{\widehat{\ell v} > \widehat{w_k}\}}(V \min\{kv, w_k\} - \mu_{\mathrm{d}})$$
$$+ \chi_{\{\widehat{\ell v} = \widehat{w_k}\}}\frac{\ell V v + V \min\{kv, w_k\}}{2}.$$

For $\ell \leq k$, we have

$$\chi_{\{\widehat{\ell v} < \widehat{w_k}\}}\ell V v + \chi_{\{\widehat{\ell v} > \widehat{w_k}\}}V \min\{kv, w_k\} + \chi_{\{\widehat{\ell v} = \widehat{w_k}\}}\frac{\ell V v + V \min\{kv, w_k\}}{2} = V \min\{\ell v, w_k\}$$

almost everywhere in Ω. Hence,

$$\left(\Delta \min\{\ell v, w_k\}\right)_{\mathrm{d}} \leq V \min\{\ell v, w_k\} - \chi_{\{\widehat{\ell v} > \widehat{w_k}\}}\mu_{\mathrm{d}}.$$

Since $w_k \leq w_0$ and μ_{d} is nonnegative, we then have

$$\left(\Delta \min\{\ell v, w_k\}\right)_{\mathrm{d}} \leq V \min\{\ell v, w_k\} - \chi_{\{\widehat{\ell v} > \widehat{w_0}\}}\mu_{\mathrm{d}}. \tag{5.8}$$

Since the function $\min\{\ell v, w_k\}$ is nonnegative, by the inverse maximum principle we also have

$$\left(\Delta \min\{\ell v, w_k\}\right)_{\mathrm{c}} \leq 0. \tag{5.9}$$

Combining (5.8) and (5.9), for every $\ell \leq k$ we get

$$\Delta \min\{\ell v, w_k\} \leq V \min\{\ell v, w_k\} - \chi_{\{\widehat{\ell v} > \widehat{w_0}\}}\mu_{\mathrm{d}}$$

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in the sense of measures and then also in the sense of distributions in Ω. Letting $k \to \infty$ and next $\ell \to \infty$, we deduce that

$$\Delta(\chi_{\Omega \setminus Z} w) \leq V \chi_{\Omega \setminus Z} w - \chi_{\{\widehat{w_0} < \infty\} \setminus \{\widehat{v} = 0\}} \, \mu_{\mathrm{d}} \quad \text{in the sense of distributions in } \Omega.$$

Since $\Delta w_0 \in \mathcal{M}(\Omega)$, the set $\Omega \setminus \{\widehat{w_0} < \infty\}$ has $W^{1,2}$ capacity zero. Moreover, by the choice of v we have $\{\widehat{v} = 0\} = Z$. We thus get

$$\chi_{\{\widehat{w_0} < \infty\} \setminus \{\widehat{v} = 0\}} \, \mu_{\mathrm{d}} = \mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z}$$

and (5.4) follows. □

Remark 5.5. Lemma 5.3 does not provide enough information to conclude that $\chi_{\Omega \setminus Z} w \in W_0^{1,1}(\Omega)$. To encode the zero boundary datum of $\chi_{\Omega \setminus Z} w$, one can rely instead on test functions in the larger class $C_0^\infty(\overline{\Omega})$, which is enough to apply the weak maximum principle. On the one hand, since $w \in W_0^{1,1}(\Omega)$, by [28, Proposition 6.5] the property

$$-\Delta w + V \chi_{\Omega \setminus Z} w \leq \mu \quad \text{in the sense of distributions in } \Omega$$

is equivalent to

$$-\Delta w + V \chi_{\Omega \setminus Z} w \leq \mu \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'.$$

On the other hand, since $w \in W_0^{1,1}(\Omega)$ and $\Delta w \in \mathcal{M}(\Omega)$, by [28, Proposition 20.1] we also have

$$\int_{\{x \in \Omega \,:\, d(x, \partial\Omega) < \epsilon\}} w \leq C \epsilon^2 \|\Delta w\|_{\mathcal{M}(\Omega)} \quad \text{for every } \epsilon > 0.$$

In particular, $\chi_{\Omega \setminus Z} w$ satisfies the following vanishing mean property on the boundary:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}} \chi_{\Omega \setminus Z} w = 0,$$

which combined with an inequality of the type

$$-\Delta(\chi_{\Omega \setminus Z} w) + V \chi_{\Omega \setminus Z} w \geq \mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z} \quad \text{in the sense of distributions in } \Omega$$

entitles us to recover test functions in $C_0^\infty(\overline{\Omega})$ as an application of [28, Proposition 20.2]:

$$-\Delta(\chi_{\Omega \setminus Z} w) + V \chi_{\Omega \setminus Z} w \geq \mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z} \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'.$$

Proof of Proposition 5.1. Let w be the function given by Lemma 5.3 with $\mu = \chi_{\Omega \setminus Z} \, \mathrm{d}x$. In this case, since μ is absolutely continuous with respect to the Lebesgue measure, $\mu_{\mathrm{d}} = \mu$ and then $\mu_{\mathrm{d}} \lfloor_{\Omega \setminus Z} = \mu$. Using the notation $\widetilde{w} := \chi_{\Omega \setminus Z} w$, we thus have

$$-\Delta w + V \chi_{\Omega \setminus Z} w \leq \chi_{\Omega \setminus Z} \quad \text{and} \quad -\Delta \widetilde{w} + V \chi_{\Omega \setminus Z} \widetilde{w} \geq \chi_{\Omega \setminus Z}$$

in the sense of distributions in Ω. By Proposition 5.5, both inequalities hold in the sense of $(C_0^\infty(\overline{\Omega}))'$. Since the potential $V \chi_{\Omega \setminus Z}$ is nonnegative, it thus follows from the weak maximum principle that $\widetilde{w} \geq w$ almost everywhere in Ω. By the nonnegativity of w, the reverse inequality also holds. Hence, $\widetilde{w} = w \in W_0^{1,1}(\Omega)$ and, as $\widetilde{w} = 0$ on Z,

$$-\Delta w + V w = -\Delta w + V \chi_{\Omega \setminus Z} w = \chi_{\Omega \setminus Z} \quad \text{in the sense of distributions in } \Omega.$$

From Proposition 3.4, we thus have $w = \zeta_{\chi_{\Omega \setminus Z}}$. □

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Remark 5.6. As a consequence of Proposition 5.1, one has existence of a solution of (1.1) for every nonnegative function $f \in L^\infty(\Omega)$ such that $f = 0$ almost everywhere in Z. Indeed, observe that a solution of (1.1) with datum $\|f\|_{L^\infty(\Omega)}\chi_{\Omega\setminus Z}$ exists. Since

$$0 \leq f \leq \|f\|_{L^\infty(\Omega)}\chi_{\Omega\setminus Z} \quad \text{almost everywhere in } \Omega,$$

it then suffices to apply the method of sub- and supersolutions (Proposition 2.1).

6. Orthogonality principle

We establish in this section an orthogonality relation between the sets Z and $\Omega \setminus Z$, which is used in the proof of Theorem 1.4:

Proposition 6.1. *The universal zero-set Z satisfies*

$$\int_\Omega \zeta_{\chi_{\Omega\setminus Z}}\chi_Z = \int_\Omega \zeta_{\chi_Z}\chi_{\Omega\setminus Z} = 0.$$

More precisely, we have

$$\widehat{\zeta_{\chi_{\Omega\setminus Z}}}(x) = 0 \quad \text{for every } x \in Z,$$
$$\widehat{\zeta_{\chi_Z}}(x) = 0 \quad \text{for every } x \in \Omega \setminus Z.$$

As a consequence, the function ζ_{χ_Z} satisfies (1.1) with $f = \chi_Z$ if and only if Z is negligible, since one must have $\widehat{\zeta_{\chi_Z}} = 0$ in $(\Omega \setminus Z) \cup Z = \Omega$. The proof of Proposition 6.1 relies on the existence of a solution of (1.1) with $f = \chi_{\Omega\setminus Z}$ that we proved in the previous section. We also need to know that every point of $\Omega \setminus Z$ is a density point of this set, which means that $\Omega \setminus Z$ is open with respect to the density topology [17]. This is a general property of Sobolev-open sets (Proposition 10.1), but here we rely solely on the definition of Z:

Lemma 6.2. *For every $x \in \Omega \setminus Z$, we have*

$$\lim_{r \to 0} \frac{|B_r(x) \setminus Z|}{|B_r(x)|} = 1.$$

Proof of Lemma 6.2. Given $x \in \Omega \setminus Z$, let $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ be a solution of (1.1) for some nonnegative $f \in L^\infty(\Omega)$ with $\widehat{w}(x) > 0$. Since $\widehat{w} = 0$ in Z, by the Lebesgue differentiation theorem we have $w = 0$ almost everywhere in Z. Thus,

$$\fint_{B_r(x)} w = \frac{1}{|B_r(x)|}\int_{B_r(x)\setminus Z} w. \tag{6.1}$$

We now denote $c := \widehat{w}(x)$ and choose $r_1 > 0$ such that

$$c - \epsilon \leq \fint_{B_r(x)} w \quad \text{for every } 0 < r \leq r_1. \tag{6.2}$$

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Observe that $\widehat{w}$, being the difference between a continuous and a superharmonic function, is upper semicontinuous in Ω. Thus,

$$\limsup_{y \to x} \widehat{w}(y) \le \widehat{w}(x) = c.$$

We then take $r_2 > 0$ such that

$$\widehat{w}(y) \le c + \epsilon \quad \text{for every } y \in B_{r_2}(x).$$

In particular, $w \le c + \epsilon$ almost everywhere in $B_{r_2}(x)$, which implies that

$$\frac{1}{|B_r(x)|} \int_{B_r(x) \setminus Z} w \le (c + \epsilon) \frac{|B_r(x) \setminus Z|}{|B_r(x)|} \quad \text{for every } 0 < r \le r_2. \tag{6.3}$$

Combining (6.1) to (6.3), we get

$$c - \epsilon \le (c + \epsilon) \frac{|B_r(x) \setminus Z|}{|B_r(x)|} \quad \text{for every } 0 < r \le \min\{r_1, r_2\}.$$

Therefore, as $r \to 0$,

$$\frac{c - \epsilon}{c + \epsilon} \le \liminf_{r \to 0} \frac{|B_r(x) \setminus Z|}{|B_r(x)|} \le \limsup_{r \to 0} \frac{|B_r(x) \setminus Z|}{|B_r(x)|} \le 1.$$

Since $c > 0$, the conclusion follows as $\epsilon \to 0$. $\qquad\square$

Proof of Proposition 6.1. By Proposition 5.1, the function $\zeta_{\chi_{\Omega \setminus Z}}$ satisfies (1.1) with $f = \chi_{\Omega \setminus Z}$ and, in particular,

$$\widehat{\zeta_{\chi_{\Omega \setminus Z}}}(x) = 0 \quad \text{for every } x \in Z. \tag{6.4}$$

To establish the integral orthogonality relation, we recall that $\zeta_{\chi_{\Omega \setminus Z}}$ is also a duality solution. Using the test function $\chi_Z \in L^\infty(\Omega)$ in the duality formulation, we then have by the Lebesgue differentiation theorem,

$$\int_\Omega \zeta_{\chi_{\Omega \setminus Z}} \chi_Z = \int_\Omega \widehat{\zeta_{\chi_Z}} \chi_{\Omega \setminus Z} = \int_\Omega \zeta_{\chi_Z} \chi_{\Omega \setminus Z} \, . \tag{6.5}$$

By (6.4) and the Lebesgue differentiation theorem, $\zeta_{\chi_{\Omega \setminus Z}} = 0$ almost everywhere in Z. Hence, the integral in the left-hand side of (6.5) vanishes. This establishes the orthogonality identity and then, since ζ_{χ_Z} is nonnegative,

$$\zeta_{\chi_Z} = 0 \quad \text{almost everywhere in } \Omega \setminus Z. \tag{6.6}$$

We now claim that

$$\widehat{\zeta_{\chi_Z}}(x) = 0 \quad \text{for every } x \in \Omega \setminus Z.$$

Indeed, by nonnegativity of ζ_{χ_Z} and (6.6), for every ball $B_r(x) \subset \Omega$ we have

$$0 \le \fint_{B_r(x)} \zeta_{\chi_Z} = \frac{1}{|B_r(x)|} \int_{B_r(x) \cap Z} \zeta_{\chi_Z} \le \|\zeta_{\chi_Z}\|_{L^\infty(\Omega)} \frac{|B_r(x) \cap Z|}{|B_r(x)|}.$$

By Lemma 6.2, the right-hand side converges to zero as $r \to 0$ when $x \in \Omega \setminus Z$ and we conclude that $\widehat{\zeta_{\chi_Z}}(x) = 0$. $\qquad\square$

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From the orthogonality principle, we deduce *a posteriori* that one can take $A = \Omega \setminus Z$ in Lemma 5.2:

Corollary 6.3. *The universal zero-set satisfies*

$$Z = \big\{x \in \Omega : \widehat{\zeta_{\chi_{\Omega\setminus Z}}}(x) = 0\big\}.$$

Proof. We recall that A is defined in the proof of Lemma 5.2 as a maximizer among all Borel sets $B \subset \Omega$ such that (1.8) has a distributional solution with $\mu = \chi_B \, \mathrm{d}x$. Since by Proposition 5.1 a solution with $B = \Omega \setminus Z$ exists, we may assume from the beginning that

$$\Omega \setminus Z \subset A.$$

It thus suffices to verify that $A \cap Z$ is negligible with respect to the Lebesgue measure. To this end, we first observe that by Proposition 2.1 there exists a distributional solution of (1.8) with datum $\mu = \chi_{A\cap Z} \, \mathrm{d}x$, which by Proposition 3.4 can be identified with $\zeta_{\chi_{A\cap Z}}$. On the other hand, by comparison between variational solutions,

$$0 \leq \zeta_{\chi_{A\cap Z}} \leq \zeta_{\chi_Z} \quad \text{almost everywhere in } \Omega.$$

From the orthogonality principle, and in particular (6.6), we thus have $\zeta_{\chi_{A\cap Z}} = 0$ almost everywhere in $\Omega \setminus Z$. But being a distributional solution, the same property holds in Z. Hence, $\zeta_{\chi_{A\cap Z}} = 0$ almost everywhere in the entire domain Ω and then, from the distributional formulation,

$$\int_{A\cap Z} \varphi = \int_\Omega \chi_{A\cap Z} \, \varphi = 0 \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Therefore, $A \cap Z$ is negligible. $\qquad\square$

7. Comparison principle

We investigate a comparison principle which establishes that every solution of the Dirichlet problem (1.8) with positive measure can always be bounded from below by a nontrivial solution involving some nonnegative L^∞ datum. In the proof of Theorem 1.4, it implies that the assumption $\mu(Z) = 0$ is necessary for the existence of distributional solutions. While the naive strategy based on truncation gives a bounded supersolution $T_k(u)$ underneath u, such an approach is unsatisfactory since $\Delta T_k(u)$ typically yields a singular measure on the level set $\{u = k\}$.

Our main result in this direction is the following

Proposition 7.1. *There exists a bounded continuous nondecreasing function $H : [0, +\infty) \to [0, +\infty)$, with $H(t) > 0$ for $t > 0$, such that, for every Borel function $V : \Omega \to [0, +\infty]$, if $u \in L^1(\Omega)$ is a duality solution of the Dirichlet problem (1.8) involving a nonnegative measure $\mu \in \mathcal{M}(\Omega)$, then*

$$u \geq \zeta_{H(u)} \quad \text{almost everywhere in } \Omega.$$

Observe that $\zeta_{H(u)}$ is well defined since $H(u)$ is bounded. We emphasize that H is independent of V, and from the proof one can take $H(t) \sim t^\alpha$ near $t = 0$ for any given $\alpha > 1$; see (7.5) below. The comparison principle above also applies to distributional solutions, as they are also

duality solutions, but the important fact that $\zeta_{H(u)}$ is also a distributional solution with datum $H(u)$ requires some justification; see Proposition 7.3 below.

To prove Proposition 7.1, we rely on a straightforward variant of Kato's inequality for ζ_h in the spirit of [7, Proposition 4.B.5], which formally is

$$- \Delta \zeta_h^+ + V \zeta_h^+ \leq \chi_{\{\zeta_h > 0\}} h,$$

that also takes into account the boundary behavior of ζ_h by allowing ζ_1 as test function.

Lemma 7.2. *For every $h \in L^\infty(\Omega)$, we have*

$$\int_\Omega \zeta_h^+ \leq \int_{\{\zeta_h > 0\}} h \zeta_1.$$

Proof of Lemma 7.2. Since ζ_h and ζ_1 satisfy an Euler-Lagrange equation involving test functions in $W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, dx)$, the proof is implemented by suitable choices of test functions depending on ζ_h and ζ_1 themselves. For example, the equation satisfied by ζ_1 with test function $J(\zeta_h)$ gives

$$\int_\Omega \left(J'(\zeta_h) \nabla \zeta_1 \cdot \nabla \zeta_h + V \zeta_1 J(\zeta_h) \right) = \int_\Omega J(\zeta_h), \tag{7.1}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $J(0) = 0$. Using now the test function $J'(\zeta_h)\zeta_1$ in the equation satisfied by ζ_h, we also have

$$\int_\Omega \left(J''(\zeta_h) |\nabla \zeta_h|^2 \zeta_1 + J'(\zeta_h) \nabla \zeta_h \cdot \nabla \zeta_1 + V \zeta_h J'(\zeta_h) \zeta_1 \right) = \int_\Omega h \, J'(\zeta_h) \zeta_1.$$

Assuming that $J'' \geq 0$, by nonnegativity of ζ_1 we get

$$\int_\Omega \left(J'(\zeta_h) \nabla \zeta_h \cdot \nabla \zeta_1 + V \zeta_h J'(\zeta_h) \zeta_1 \right) \leq \int_\Omega h \, J'(\zeta_h) \zeta_1. \tag{7.2}$$

Subtracting (7.2) from (7.1),

$$\int_\Omega V \zeta_1 [J(\zeta_h) - \zeta_h J'(\zeta_h)] \geq \int_\Omega J(\zeta_h) - \int_\Omega h \, J'(\zeta_h) \zeta_1.$$

We now take J convex such that $J(t) = 0$ for $t \leq 0$ and $0 \leq J(t) \leq t$ for $t \geq 0$. In particular, for every $t \in \mathbb{R}$ we have $J(t) \leq J'(t)t$. Since V and ζ_1 are nonnegative, the integrand in the left-hand side is nonpositive and we deduce that

$$\int_\Omega J(\zeta_h) \leq \int_\Omega h \, J'(\zeta_h) \zeta_1.$$

To conclude, we apply this inequality to a sequence $(J_k)_{k \in \mathbb{N}}$ of convex functions as above that converges pointwise to the function $t \in \mathbb{R} \mapsto t^+$ and such that $(J_k')_{k \in \mathbb{N}}$ converges pointwise to $\chi_{(0, +\infty)}$. As $k \to \infty$, we have the conclusion. $\qquad \square$

Proof of Proposition 7.1. We first assume that μ is a measure of the form $\mu = f \, dx$ with a nonnegative $f \in L^\infty(\Omega)$. We have in this case that $u = \zeta_f$ by uniqueness of duality solutions. For every $\epsilon > 0$, we claim that

$$Cu \geq \epsilon \zeta_{\chi_{\{u > \epsilon\}}} \quad \text{almost everywhere in } \Omega \tag{7.3}$$

for $C := \|\theta\|_{L^\infty(\Omega)}$, where θ is the classical solution of (3.2).

Using the notation $z_\epsilon := \zeta_{\chi_{\{u > \epsilon\}}}$, we have $\epsilon z_\epsilon - Cu = \zeta_h$, where $h = \epsilon \chi_{\{u > \epsilon\}} - Cf$. Thus, by Lemma 7.2,

$$\int_\Omega (\epsilon z_\epsilon - Cu)^+ \leq \int_{\{\epsilon z_\epsilon > Cu\}} (\epsilon \chi_{\{u > \epsilon\}} - Cf) \zeta_1.$$

Since f and ζ_1 are nonnegative,

$$\int_\Omega (\epsilon z_\epsilon - Cu)^+ \leq \int_{\{\epsilon z_\epsilon > Cu\}} \epsilon \chi_{\{u > \epsilon\}} \zeta_1 = \epsilon \int_{\{\epsilon z_\epsilon / C > u > \epsilon\}} \zeta_1 \leq \epsilon \int_{\{z_\epsilon > C\}} \zeta_1. \tag{7.4}$$

The estimate

$$0 \leq z_\epsilon \leq \zeta_1 \leq \theta \quad \text{almost everywhere in } \Omega$$

holds for every $\epsilon > 0$ and is independent of V. In particular, with the choice $C = \|\theta\|_{L^\infty(\Omega)}$, the set $\{z_\epsilon > C\}$ is negligible with respect to the Lebesgue measure. We then deduce from (7.4) that

$$\int_\Omega (\epsilon z_\epsilon - Cu)^+ \leq 0$$

and this implies (7.3).

To obtain H, it now suffices to apply (7.3) using an averaging argument. For this purpose, let $\rho : (0, +\infty) \to \mathbb{R}$ be a summable nonnegative function such that $\int_0^\infty \rho = 1$. Multiplying both sides of (7.3) by $\rho(\epsilon)$ and integrating with respect to ϵ over $(0, +\infty)$, we get

$$Cu(x) \geq \int_0^\infty \epsilon \rho(\epsilon) \zeta_{\chi_{\{u > \epsilon\}}}(x) \, d\epsilon \quad \text{for almost every } x \in \Omega.$$

By linearity of the equation, one identifies the right-hand side as $\zeta_{\widetilde{H}(u)}(x)$, where

$$\widetilde{H}(t) := \int_0^t \epsilon \rho(\epsilon) \, d\epsilon,$$

so that the proposition holds with $H(t) = \widetilde{H}(t)/C$. Given $\alpha > 1$, an explicit admissible choice of ρ is

$$\rho(\epsilon) = \begin{cases} (\alpha - 1)\epsilon^{\alpha - 2} & \text{for } \epsilon < 1, \\ 0 & \text{for } \epsilon \geq 1. \end{cases}$$

In this case,

$$H(t) = \frac{\alpha - 1}{C\alpha} \min\{t^\alpha, 1\} \quad \text{for every } t \geq 0. \tag{7.5}$$

We have assumed so far that $\mu = f \, dx$ with f bounded. For an arbitrary nonnegative measure $\mu \in \mathcal{M}(\Omega)$, we apply the estimate to the function $u_k := \zeta_{\rho_k * \mu}$, where $(\rho_k)_{k \in \mathbb{N}}$ is a suitable sequence of mollifiers; see Proposition 3.5. The sequence $(u_k)_{k \in \mathbb{N}}$ converges to u in $L^1(\Omega)$ and, for every $k \in \mathbb{N}$, we have $u_k \geq \zeta_{H(u_k)}$ almost everywhere in Ω. The conclusion thus follows as $k \to \infty$. $\qquad \square$

We now complement the comparison principle for distributional solutions u by showing that $\zeta_{H(u)}$ is also a distributional solution with datum $H(u)$. More precisely, using the stability of duality solutions under truncation of the potential V and the independence of H with respect to V, we prove

Proposition 7.3. *If u is the distributional solution of* (1.8) *with nonnegative datum $\mu \in \mathcal{M}(\Omega)$, then $\zeta_{H(u)}$ satisfies* (1.1) *with datum $f = H(u) \in L^\infty(\Omega)$, where H is the bounded continuous function given by Proposition 7.1.*

Proof. We use the notations of Lemmas 2.2 and 3.2 for u_k and $\zeta_{f,k}$, respectively. Since $T_k(V)$ is bounded, the function $w_k := \zeta_{H(u_k),k}$ satisfies

$$-\Delta w_k + T_k(V)w_k = H(u_k) \quad \text{in the sense of distributions in } \Omega. \tag{7.6}$$

By nonnegativity of μ, the sequence $(u_k)_{k\in\mathbb{N}}$ is non-increasing and so is $(w_k)_{k\in\mathbb{N}}$. As each w_k is also nonnegative, the sequence $(w_k)_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$ to some function w. Since H is independent of the potential V, by the comparison principle (Proposition 7.1) and the nonnegativity of w_k we also have

$$0 \leq w_k \leq u_k \quad \text{almost everywhere in } \Omega.$$

By Lemma 2.2, the sequence $(T_k(V)u_k)_{k\in\mathbb{N}}$ converges to Vu in $L^1(\Omega)$. Thus, by the Dominated convergence theorem, the sequence $(T_k(V)w_k)_{k\in\mathbb{N}}$ converges to Vw in $L^1(\Omega)$. As $k \to \infty$ in (7.6), we then deduce that w satisfies (1.1) with $f = H(u)$. To conclude, observe that since $(w_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and $(\Delta w_k)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, by interpolation the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$. By the closure property in Sobolev spaces, we then have $w \in W_0^{1,2}(\Omega)$ and $w = \zeta_{H(u)}$. $\qquad\square$

8. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.4. "$\Longrightarrow$". Since $0 \leq \mu\lfloor_Z \leq \mu$, by Proposition 2.1 the Dirichlet problem (1.8) also has a distributional solution v with measure $\mu\lfloor_Z$. As a consequence of the comparison principle from the previous section, we have $v = 0$ almost everywhere in Ω. Indeed, by Proposition 3.1, v is also a duality solution and

$$\int_\Omega vf = \int_\Omega \widehat{\zeta_f}\, \mathrm{d}\mu\lfloor_Z \quad \text{for every } f \in L^\infty(\Omega). \tag{8.1}$$

By Proposition 7.3, the function $\zeta_{H(v)}$ satisfies (1.1) with bounded datum $f = H(v)$ and then, by definition of Z, we have $\widehat{\zeta_{H(v)}} = 0$ in Z. Thus taking $f = H(v)$ in (8.1), we get

$$\int_\Omega vH(v) = \int_\Omega \widehat{\zeta_{H(v)}}\, \mathrm{d}\mu\lfloor_Z = 0.$$

By positivity of H on $(0, +\infty)$ we deduce that $v = 0$ almost everywhere in Ω. Since v solves an equation with $\mu\lfloor_Z$ in the sense of distributions, for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \varphi\, \mathrm{d}\mu\lfloor_Z = \int_\Omega v\,(-\Delta\varphi + V\varphi) = 0.$$

Hence, $\mu\lfloor_Z = 0$ and then $\mu(Z) = 0$. $\qquad\square$

Proof of Theorem 1.4. "⟸". Let w be the function provided by Lemma 5.3: w dominates all distributional subsolutions of (1.8) and, by Proposition 5.5, also satisfies

$$-\Delta w + V\chi_{\Omega\setminus Z}w \leq \mu \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'.$$

Denoting $\widetilde{w} := \chi_{\Omega\setminus Z}w$, we claim that

$$-\Delta\widetilde{w} + V\chi_{\Omega\setminus Z}\widetilde{w} \geq \mu \quad \text{in the sense of } (C_0^\infty(\overline{\Omega}))'. \tag{8.2}$$

Once such a property is established, the weak maximum principle implies that $w \leq \widetilde{w}$ almost everywhere in Ω. By nonnegativity of w, we also have $w \geq \widetilde{w}$. Hence, equality holds and we deduce that

$$w = 0 \quad \text{almost everywhere in } Z$$

and

$$-\Delta w + Vw = \mu \quad \text{in the sense of distributions in } \Omega.$$

It thus suffices to prove (8.2). We perform this task by analyzing separately the diffuse and concentrated parts of $\Delta\widetilde{w}$. Concerning the *diffuse* part, we first observe that the assumption $\mu(Z) = 0$ and the nonnegativity of μ imply that $\mu_{\mathrm{d}}\lfloor_{\Omega\setminus Z} = \mu_{\mathrm{d}}$. Thus, by Lemma 5.3,

$$-\Delta\widetilde{w} + V\chi_{\Omega\setminus Z}\widetilde{w} \geq \mu_{\mathrm{d}} \quad \text{in the sense of distributions in } \Omega,$$

hence also in the sense of measures in Ω. Then, by comparison between the diffuse parts from both sides,

$$(-\Delta\widetilde{w})_{\mathrm{d}} + V\chi_{\Omega\setminus Z}\widetilde{w} \geq \mu_{\mathrm{d}}. \tag{8.3}$$

Concerning the *concentrated* part, we first prove that

$$u \leq \chi_{\Omega\setminus Z}\,w = \widetilde{w} \quad \text{almost everywhere in } \Omega, \tag{8.4}$$

where $u \in L^1(\Omega)$ is the duality solution of (1.8) associated to μ. By Lemma 4.2, we have $u \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V\,\mathrm{d}x)$ and

$$-\Delta u + Vu \leq \mu \quad \text{in the sense of distributions in } \Omega.$$

Thus, by Lemma 5.3,

$$u \leq w \quad \text{almost everywhere in } \Omega. \tag{8.5}$$

To prove that

$$u = 0 \quad \text{almost everywhere in } Z, \tag{8.6}$$

we use χ_Z as test function in the duality formulation:

$$\int_Z u = \int_\Omega u\chi_Z = \int_\Omega \widehat{\zeta_{\chi_Z}}\,\mathrm{d}\mu.$$

By the orthogonality principle (Proposition 6.1), we have $\{\widehat{\zeta_{\chi_Z}} > 0\} \subset Z$. Since $\mu = 0$ on Z, we get $\int_Z u = 0$ which, by nonnegativity of u, implies (8.6). As a consequence of (8.5), (8.6) and the nonnegativity of w, (8.4) follows. Next, from the inverse maximum principle and (8.4), we get

$$(-\Delta\widetilde{w})_{\mathrm{c}} \geq (-\Delta u)_{\mathrm{c}}.$$

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We recall that, by Proposition 4.1, u satisfies

$$-\Delta u + Vu = \mu\lfloor_{\Omega\setminus S} - \lambda \quad \text{in the sense of distributions in } \Omega,$$

where the measure λ is diffuse, that is, $\lambda_{\mathrm{c}} = 0$. Since $S \subset Z$ and $\mu = 0$ on Z, we have $\mu\lfloor_{\Omega\setminus S} = \mu$. Thus,

$$(-\Delta\widetilde{w})_{\mathrm{c}} \geq (-\Delta u)_{\mathrm{c}} = \mu_{\mathrm{c}}. \tag{8.7}$$

Since $\Delta\widetilde{w} = (\Delta\widetilde{w})_{\mathrm{d}} + (\Delta\widetilde{w})_{\mathrm{c}}$, a combination of (8.3) and (8.7) gives (8.2), but only in the sense of distributions in Ω. As explained in Proposition 5.5, the vanishing average property of $\widetilde{w}$ then implies (8.2), which completes the proof. $\square$

Proof of Theorem 1.3. By Theorem 1.4, the Dirichlet problem (1.8) does not have a distributional solution with $\mu = \delta_x$ and $x \in Z$. When $x \notin Z$, again by Theorem 1.4 a distributional solution exists and, by Proposition 3.1 and the uniqueness of the duality solution, it must coincide with the duality solution G_x. In this case, if $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfies (1.1) with $f \in L^\infty(\Omega)$, then by Proposition 3.4 we have $w = \zeta_f$. The representation formula (3.8) satisfied by ζ_f then becomes

$$\widehat{w}(x) = \widehat{\zeta_f}(x) = \int_\Omega G_x f \quad \text{for every } x \in \Omega. \qquad \square$$

9. Green's functions and decomposition of $\Omega \setminus S$

In this section, we utilize Green's function G_x in the duality sense for $x \in \Omega \setminus S$ to identify the various components of $\Omega \setminus S$. The topological properties of these subsets will be investigated in the next two sections. Observe that, when $x \in S$, we have

$$\int_\Omega G_x f = \widehat{\zeta_f}(x) = 0 \quad \text{for every } f \in L^\infty(\Omega),$$

whence

$$G_x = 0 \quad \text{almost everywhere in } \Omega.$$

For $x \in \Omega \setminus S$, the picture is radically different as we know from Proposition 4.1 that G_x satisfies the equation

$$-\Delta G_x + VG_x = \delta_x - \lambda \quad \text{in the sense of distributions in } \Omega \tag{9.1}$$

for some nonnegative diffuse measure $\lambda \in \mathcal{M}(\Omega)$ carried by S, where for simplicity we omit the possible dependence of λ on x. In particular, G_x is a nontrivial locally bounded subharmonic function in $\Omega \setminus \{x\}$. Hence, the Lebesgue set of G_x is $\Omega \setminus \{x\}$ and the precise representative $\widehat{G_x}$ is upper semicontinuous in this set. We can interpret x as the point where G_x diverges to $+\infty$.

Definition 9.1. *For every $x \in \Omega \setminus S$, the* superlevel set U_x *is defined by*

$$U_x = \big\{y \in \Omega : y = x \text{ or } \widehat{G_x}(y) > 0\big\}.$$

By the Lebesgue differentiation theorem and the fact that $G_x \not\equiv 0$ for $x \in \Omega \setminus S$, each superlevel set U_x has positive Lebesgue measure. We also observe that

$$\Omega \setminus U_x = \{\widehat{G_x} = 0\}. \tag{9.2}$$

We now prove that these superlevel sets yield equivalence classes in $\Omega \setminus S$:

Proposition 9.1. *For every $x, y \in \Omega \setminus S$, we have that*

$$either \quad U_x = U_y \quad or \quad U_x \cap U_y = \emptyset.$$

Since each U_x has positive Lebesgue measure, for x running over $\Omega \setminus S$ one then gets a decomposition of $\Omega \setminus S$ as a finite or countably infinite disjoint union of sets U_x. The components D_j that arise in Theorem 1.1 are the superlevel sets that are contained in $\Omega \setminus Z$.

We begin by showing that each point of $\{\widehat{G_x} > 0\}$ is a density point of this set:

Lemma 9.2. *Let $x \in \Omega \setminus S$. For every $z \in \{\widehat{G_x} > 0\}$, we have*

$$\lim_{r \to 0} \frac{\left| B_r(z) \cap \{\widehat{G_x} > 0\} \right|}{|B_r(z)|} = 1.$$

Proof of Lemma 9.2. Let $c = \widehat{G_x}(z) > 0$. Given $\epsilon > 0$, one proceeds as in the proof of Lemma 6.2 using the upper semicontinuity of G_x to find some $\eta > 0$ such that, for every $0 < r \le \eta$,

$$c - \epsilon \le \frac{1}{|B_r(z)|} \int_{B_r(z) \cap \{\widehat{G_x} > 0\}} G_x \le (c + \epsilon) \frac{\left| B_r(z) \cap \{\widehat{G_x} > 0\} \right|}{|B_r(z)|},$$

and then

$$\frac{c - \epsilon}{c + \epsilon} \le \frac{\left| B_r(z) \cap \{\widehat{G_x} > 0\} \right|}{|B_r(z)|} \le 1.$$

The conclusion follows letting $r \to 0$ and then $\epsilon \to 0$. $\qquad\square$

We now prove an orthogonality relation among the superlevel sets U_x:

Lemma 9.3. *Let $x, y \in \Omega \setminus S$ with $x \ne y$. If $\widehat{G_x}(y) = 0$, then $U_x \cap U_y = \emptyset$.*

To prove this property we need the symmetry of the Green's function [22, Theorem 7.4]: For every $x, y \in \Omega$ with $x \ne y$,

$$\widehat{G_x}(y) = \widehat{G_y}(x).$$

Proof of Lemma 9.3. Let $y \in \Omega \setminus \{x\}$ with $\widehat{G_x}(y) = 0$. We first show that

$$\{G_x > 0\} \cap \{G_y > 0\} \quad \text{is negligible.} \tag{9.3}$$

To this end, by the comparison principle (Proposition 7.1) and the representation formula (3.8) we have

$$\widehat{G_x}(y) \ge \widehat{\zeta_{H(G_x)}}(y) = \int_\Omega G_y\, H(G_x).$$

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Since the left-hand side vanishes by assumption and the integrand is nonnegative, we have $G_y H(G_x) = 0$ almost everywhere in Ω, and (9.3) thus holds by positivity of H on $(0, +\infty)$.

It follows from (9.3) and the Lebesgue differentiation theorem that (9.3) is also satisfied by the precise representatives and then, for every $z \in \Omega$,

$$\frac{\left|B_r(z) \cap \{\widehat{G_x} > 0\}\right|}{|B_r(z)|} + \frac{\left|B_r(z) \cap \{\widehat{G_y} > 0\}\right|}{|B_r(z)|} \leq 1.$$

As $r \to 0$, we deduce using Lemma 9.2 that the first quotient converges to 1 for $z \in \{\widehat{G_x} > 0\}$, while the second one also converges to 1 for $z \in \{\widehat{G_y} > 0\}$. Therefore, no point in Ω can belong simultaneously to both sets, and so their intersection must be empty:

$$\{\widehat{G_x} > 0\} \cap \{\widehat{G_y} > 0\} = \emptyset.$$

Since $\widehat{G_y}(x) = \widehat{G_x}(y) = 0$, we also have

$$x \notin \{\widehat{G_y} > 0\} \quad \text{and} \quad y \notin \{\widehat{G_x} > 0\}.$$

Therefore, $U_x \cap U_y = \emptyset$. $\qquad\square$

Proof of Proposition 9.1. Assume that $U_x \cap U_y \neq \emptyset$ and $x \neq y$. We wish to show the equality $U_x = U_y$, which, by (9.2), is equivalent to

$$\{\widehat{G_x} = 0\} = \{\widehat{G_y} = 0\}. \tag{9.4}$$

Let us prove the inclusion "$\subset$" in (9.4). To this end, take $z \in \Omega$ such that $\widehat{G_x}(z) = 0$. Then, by Lemma 9.3,

$$U_x \cap U_z = \emptyset. \tag{9.5}$$

As another application of Lemma 9.3, the assumption $U_x \cap U_y \neq \emptyset$ implies that $\widehat{G_x}(y) > 0$, and then $y \in U_x$ by the definition of U_x. In view of (9.5), we thus have $y \notin U_z$. By symmetry of the Green's function and the definition of U_z, we deduce that $\widehat{G_y}(z) = \widehat{G_z}(y) = 0$. Therefore,

$$\{\widehat{G_x} = 0\} \subset \{\widehat{G_y} = 0\}.$$

We can now interchange the roles of x and y to get the reverse inclusion "$\supset$" and (9.4) then follows. $\qquad\square$

10. Sobolev-openness of U_x

We provide in this section additional properties of the superlevel sets U_x related to the Sobolev-topology induced by the definition below:

Definition 10.1. *A set $O \subset \Omega$ is* Sobolev-open *whenever there exists a nonnegative function $\xi \in W_0^{1,2}(\Omega)$ such that every point in Ω is a Lebesgue point of ξ and*
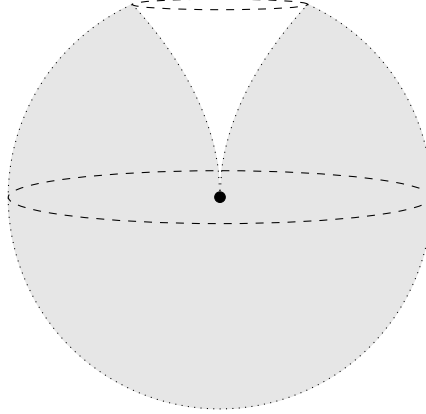
$$O = \{\widehat{\xi} > 0\}.$$

Figure 1: Sobolev-open set which is not open.

Replacing ξ in this definition by the truncated function $T_1(\xi)$, one can assume to start with that $\xi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. One verifies that a set $O \subset \Omega$ is Sobolev-open if and only if $\Omega \setminus O$ is Sobolev-closed, as defined in the Introduction. The family of Sobolev-open subsets of Ω is stable under finite intersections and countably infinite unions.

As a consequence of the Lebesgue differentiation theorem, if O is Sobolev-open and non-empty, then O has positive Lebesgue measure. We now prove that every point of a Sobolev-open set is a density point:

Proposition 10.1. *Let $O \subset \mathbb{R}^N$ be a Sobolev-open set. For every $x \in O$, we have*

$$\lim_{r \to 0} \frac{|B_r(x) \cap O|}{|B_r(x)|} = 1.$$

Proof. For $x \in O$ and $\xi \in W_0^{1,2}(\Omega)$ as in the definition of a Sobolev-open set we have $\widehat{\xi}(x) > 0$. Since $\xi = 0$ almost everywhere in $\Omega \setminus O$,

$$\frac{|B_r(x) \setminus O|}{|B_r(x)|} \widehat{\xi}(x) = \frac{1}{|B_r(x)|} \int_{B_r(x) \setminus O} |\xi - \widehat{\xi}(x)| \leq \fint_{B_r(x)} |\xi - \widehat{\xi}(x)|.$$

As $r \to 0$, the quantity in the right-hand side converges to 0 and then

$$\lim_{r \to 0} \frac{|B_r(x) \setminus O|}{|B_r(x)|} = 0. \qquad \square$$

While every open set in the usual Euclidean topology is Sobolev-open, the converse is not true:

Example 10.1. Let $N \geq 3$. For any given $0 < \alpha < 1$, the set

$$O = \big\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x| < 1 \text{ and } x_N < |x'|^\alpha \big\} \cup \{0\}$$

illustrated in Figure 1 is Sobolev-open but not open in $B_1(0)$. The assumption $\alpha < 1$ ensures that 0 is a density point of O. To verify that O is Sobolev-open, consider the function $\xi : B_1(0) \to \mathbb{R}$ defined for $x' \neq 0$ by

$$\xi(x) = \min \Big\{ \varphi\Big(\frac{x_N}{|x'|^\alpha} \Big), 1 - |x|^2 \Big\}, \tag{10.1}$$

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where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\varphi(t) = 0$ for $t \geq 1$, $\varphi(t) = 1$ for $t \leq 1/2$, and $\varphi(t) > 0$ otherwise. Observe that ξ has a continuous extension to $B_1(0) \setminus \{0\}$, every point in $B_1(0)$ is a Lebesgue point of ξ, and $\widehat{\xi}(0) = 1$. The latter is due to the fact that $\xi(x) = 1 - |x|^2$ on $\{x_N < |x'|^\alpha/2\}$ and the origin is a density point of this set. Moveover,

$$x \in O \quad \text{if and only if} \quad \widehat{\xi}(x) > 0.$$

To verify that $\xi \in W_0^{1,2}(B_1(0))$ it suffices to check that $v(x) = \varphi\big(x_N/|x'|^\alpha\big)$ belongs to $W^{1,2}(B_1(0))$. Observe that

$$|\nabla v(x)| \leq C_1 \Big|\varphi'\Big(\frac{x_N}{|x'|^\alpha}\Big)\Big| \Big(\frac{1}{|x'|^\alpha} + \frac{|x_N|}{|x'|^{\alpha+1}}\Big).$$

As $\varphi' = 0$ outside the interval $(1/2, 1)$ and $0 < \alpha < 1$, we have

$$|\nabla v(x)| \leq C_2 \, \chi_{\{1/2 \leq x_N/|x'|^\alpha \leq 1\}} \Big(\frac{1}{x_N} + \frac{1}{x_N^{1/\alpha}}\Big) \leq 2C_2 \, \chi_{\{|x'|^\alpha \leq 2x_N\}} \frac{1}{x_N^{1/\alpha}}.$$

Thus, by Fubini's theorem,

$$\int_{B_1(0)} |\nabla v|^2 \leq C_3 \int_0^1 \frac{x_N^{(N-1)/\alpha}}{x_N^{2/\alpha}} \, \mathrm{d}x_N = C_3 \int_0^1 x_N^{(N-3)/\alpha} \, \mathrm{d}x_N$$

and the integral in the right-hand side is finite for $\alpha > 0$. This implies that $\xi \in W_0^{1,2}(B_1(0))$ and O is Sobolev-open.

The superlevel sets U_x defined in the previous section are Sobolev-open:

Proposition 10.2. *For every $x \in \Omega \setminus S$, the set U_x is Sobolev-open and contained in $\Omega \setminus S$.*

Proof. Since $\zeta_{\chi U_x}$ belongs to $W_0^{1,2}(\Omega)$ and its Lebesgue set coincides with Ω, it suffices to prove that

$$U_x = \{\widehat{\zeta_{\chi U_x}} > 0\}. \tag{10.2}$$

To this end, recall that for every $y \in \Omega$ the representation formula (3.8) gives

$$\widehat{\zeta_{\chi U_x}}(y) = \int_\Omega G_y \chi_{U_x} = \int_{U_x} G_y.$$

If $y \in U_x$, then by Proposition 9.1 we have $U_x = U_y$ and the integral in the right-hand side is thus positive. When $y \notin U_x$, by Proposition 9.1 we have $U_x \cap U_y = \emptyset$ and the integral equals zero. We conclude that (10.2) holds and, in particular, $U_x \subset \Omega \setminus S$ by (4.3). $\qquad \square$

We recall that

$$S \subset Z \subset \Omega,$$

and then we can decompose Ω as

$$\Omega = S \cup (Z \setminus S) \cup (\Omega \setminus Z). \tag{10.3}$$

The following property implies that $Z \setminus S$ and $\Omega \setminus Z$ can be further decomposed as a disjoint union of sets U_x. In particular, $Z \setminus S$ is also a Sobolev-open set.
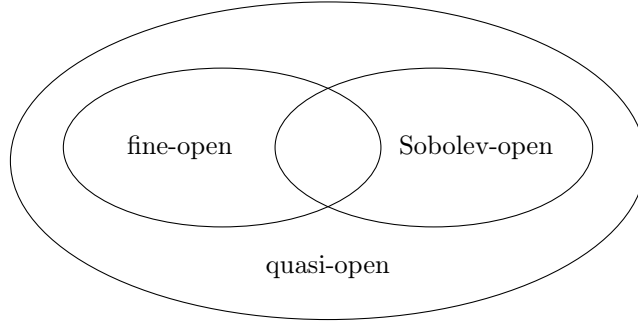
Figure 2: Relation among the classes of quasi-, fine- and Sobolev-open sets.

Proposition 10.3. *For every $x \in \Omega \setminus S$, we have that*

$$either \quad U_x \subset Z \setminus S \quad or \quad U_x \subset \Omega \setminus Z.$$

Proof. Let $x \in \Omega \setminus Z$. By the representation formula (3.8) and the orthogonality principle (Proposition 6.1) we have

$$\int_Z G_x = \widehat{\zeta_{\chi_Z}}(x) = 0.$$

Thus, $G_x = 0$ almost everywhere in Z. Similarly, for $y \in Z$,

$$\int_{\Omega \setminus Z} G_y = \widehat{\zeta_{\chi_{\Omega \setminus Z}}}(y) = 0.$$

Thus, $G_y = 0$ almost everywhere in $\Omega \setminus Z$. It then follows for every $x \in \Omega \setminus Z$ and $y \in Z \setminus S$ that $U_x \cap U_y$ is a Sobolev-open negligible set. Hence, $U_x \cap U_y = \emptyset$. In particular, as the superlevel sets are contained in $\Omega \setminus S$ (Proposition 10.2),

$$U_y \subset (\Omega \setminus S) \setminus \{x\} \quad \text{and} \quad U_x \subset (\Omega \setminus S) \setminus \{y\}.$$

Since both inclusions hold for every $x \in \Omega \setminus Z$ and $y \in Z \setminus S$, we then get

$$U_y \subset Z \setminus S \quad \text{and} \quad U_x \subset \Omega \setminus Z. \qquad \square$$

Remark 10.4. There are in the literature several other definitions of open sets related to classical concepts of Potential theory, like regular point and capacity. For example, fine- and quasi-open sets are of particular interest and we refer the reader to Malý and Ziemer's book [23] for their definitions. It is known that every fine-open set is quasi-open; see [23, Theorem 2.144]. In our case, as Sobolev-open sets are of the form $\{\widehat{\xi} > 0\}$ for some $\xi \in W_0^{1,2}(\Omega)$ and $\widehat{\xi}$ is quasicontinuous [23, Lemma 2.152], every Sobolev-open set is also quasi-open.

The classes of fine- and Sobolev-open sets are nevertheless different and one is not contained in the other, which we summarize in Figure 2. Indeed, any singleton $\{a\}$ in dimension $N \geq 2$ has $W^{1,2}$ capacity zero and thus is fine-open (and also quasi-open), but never Sobolev-open. In dimension $N = 3$, the Sobolev-open set O defined in Example 10.1 is not fine-open as the origin is a regular point of $\mathbb{R}^3 \setminus O$: This observation goes back to Lebesgue and is due to the algebraic behavior of the boundary in the neighborhood of the origin; see [20, Chapter XI, Section 19].

11. Sobolev-connectedness of U_x

One can define Sobolev-connected sets in analogy with their classical topological counterpart:

Definition 11.1. *A set $D \subset \Omega$ is* Sobolev-connected *whenever, for every disjoint Sobolev-open sets $A, B \subset \Omega$ such that $D \subset A \cup B$, one has $D \subset A$ or $D \subset B$.*

Since there are more Sobolev-open sets than open sets, any Sobolev-connected set is connected in the usual Euclidean sense. The converse is false; see Figure 3 (a) that is related to Example 10.1. Using the Intermediate value theorem for Sobolev functions from [32], one verifies that an *open* set is Sobolev-connected if and only if it is connected; see the proof of Proposition 12.2 below. Alternatively, one can rely on the fact that such a property is also true for density-connected sets in the density-topology, see [17], and every Sobolev-open set is density-open by Proposition 10.1 above.

Example 11.1. Let $N \geq 3$. For any given $1 < \alpha < N - 1$, the set

$$D = \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x| < 1 \text{ and } |x_N| > |x'|^\alpha \right\} \cup \{0\}$$

is Sobolev-open and Sobolev-connected in $B_1(0)$, but not open for the Euclidean topology; see Figure 3 (b). One proceeds as in Example 10.1 by taking $\xi : B_1(0) \to \mathbb{R}$ defined by (10.1), where the smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ is now such that $\varphi(t) = 0$ for $|t| \leq 1/2$, $\varphi(t) = 1$ for $|t| \geq 1$, and $\varphi(t) > 0$ otherwise. This new choice of φ ensures that the origin is a Lebesgue point of ξ for $\alpha > 1$. Moreover, for $x \in B_1(0)$ the function $v(x) = \varphi\big(x_N/|x'|^\alpha\big)$ satisfies

$$|\nabla v(x)| \leq C_3 \, \chi_{\{1/2 \leq |x_N|/|x'|^\alpha \leq 1\}} \left(\frac{1}{|x_N|} + \frac{1}{|x_N|^{1/\alpha}} \right) \leq 2C_3 \, \chi_{\{|x'|^\alpha \leq 2|x_N|\}} \frac{1}{|x_N|}.$$

Thus, by Fubini's theorem,

$$\int_{B_1(0)} |\nabla v|^2 \leq C_4 \int_0^1 \frac{x_N^{(N-1)/\alpha}}{x_N^2} \, \mathrm{d}x_N = C_4 \int_0^1 x_N^{-2+(N-1)/\alpha} \, \mathrm{d}x_N.$$

The right-hand side is finite for $\alpha < N - 1$ and then $\xi \in W_0^{1,2}(B_1(0))$.

To prove that D is Sobolev-connected, take disjoint Sobolev-open sets $A, B \subset \Omega$ such that $D \subset A \cup B$ and assume that $0 \in A$. Since 0 is a density point of D but not a density point of $D_+ := D \cap \{x_N > 0\}$ nor of $D_- := D \cap \{x_N < 0\}$, we have that A must intersect both D_+ and D_-. Since both sets are open and connected, they are Sobolev-connected and we deduce that D_+ and D_- are contained in A. Therefore,

$$D = \{0\} \cup D_+ \cup D_- \subset A,$$

which implies that D is Sobolev-connected.

We now show that a Sobolev-connected subset of $\Omega \setminus S$ cannot intersect two different superlevel sets U_x:

Proposition 11.1. *If $D \subset \Omega \setminus S$ is Sobolev-connected, then $D \subset U_x$ for any $x \in D$.*
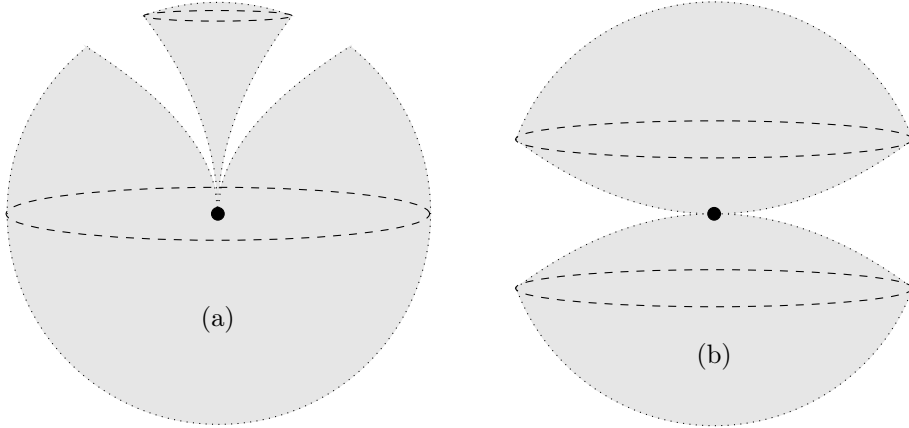
Figure 3: (a) Path-connected set which is not Sobolev-connected; (b) Sobolev-connected set.

Proof. Since $\Omega \setminus S$ is a finite or countably infinite disjoint union of the Sobolev-open sets U_y (Propositions 9.1 and 10.2), the set $\Omega \setminus (S \cup U_x)$ is Sobolev-open. By Sobolev-connectedness of D and the inclusion

$$D \subset U_x \cup \big(\Omega \setminus (S \cup U_x)\big),$$

it follows that D is contained in one of the Sobolev-open sets in the right-hand side. For $x \in D$, we then must have $D \subset U_x$. $\qquad\square$

A deeper property concerns the Sobolev-connectedness of all sets U_x:

Proposition 11.2. *For every $x \in \Omega \setminus S$, the set U_x is Sobolev-connected.*

To prove Proposition 11.2 we need a counterpart of Poincaré's balayage method for the Schrödinger operator on a non-empty Sobolev-open set $O \subset \Omega$. We use the following notation

$$\mathcal{W}(O, \Omega) = \big\{ v \in W_0^{1,2}(\Omega) : v = 0 \text{ almost everywhere in } \Omega \setminus O \big\}.$$

Observe that $\mathcal{W}(O, \Omega)$ contains any function ξ that verifies the Sobolev-openness of O. As a vector space, $\mathcal{W}(O, \Omega)$ is then nontrivial and also complete with respect to the $W^{1,2}$ norm.

Lemma 11.3. *Given a non-empty Sobolev-open set $O \subset \Omega \setminus S$ and a nonnegative function $h \in L^2(\Omega)$ such that $h = 0$ almost everywhere in $\Omega \setminus O$, let u be the minimizer of*

$$E(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + V v^2) - \int_\Omega hv \quad \text{in } \mathcal{W}(O, \Omega) \cap L^2(\Omega; V \, \mathrm{d}x).$$

Then, there exists a nonnegative locally finite diffuse Borel measure $\tau \in L^1(\Omega) + (W_0^{1,2}(\Omega))'$ such that

$$\int_\Omega uf = \int_\Omega \zeta_f h - \int_\Omega \widehat{\zeta}_f \, \mathrm{d}\tau \quad \text{for every } f \in L^\infty(\Omega),$$

with

$$\tau(O) = 0$$

and

$$\tau(T) = 0 \quad \text{for every Sobolev-open set } T \subset \Omega \setminus O.$$

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The measure τ can be interpreted as the density of charges in Ω needed to obtain a zero potential outside O, starting from a given potential u that satisfies the equation $-\Delta u + Vu = h$ in O and vanishes on the Sobolev-boundary of O. The properties $\tau(O) = \tau(T) = 0$ encode the concentration of τ on the Sobolev-boundary of O.

Proof of Lemma 11.3. The minimizer u exists and satisfies the Euler-Lagrange equation

$$\int_\Omega (\nabla u \cdot \nabla v + Vuv) = \int_\Omega hv \quad \text{for every } v \in \mathcal{W}(O,\Omega) \cap L^2(\Omega; V\,\mathrm{d}x). \tag{11.1}$$

Using various choices of test functions in (11.1), one shows that u is nonnegative, $u \in L^1(\Omega; V\,\mathrm{d}x)$ and

$$-\Delta u + Vu \leq h \quad \text{in the sense of distributions in } \Omega. \tag{11.2}$$

Indeed, taking $v = \min\{u, 0\}$ in (11.1), one sees that $u \geq 0$ almost everywhere in Ω. Taking $v = T_1(ku)$ with $k \in \mathbb{N}$ and letting $k \to \infty$, one deduces that

$$\|Vu\|_{L^1(\Omega)} \leq \|h\|_{L^1(\Omega)}.$$

Finally, to show (11.2), one chooses $v = T_1(ku)\varphi$ for any nonnegative $\varphi \in C_c^\infty(\Omega)$. Dropping the nonnegative term $\nabla u \cdot \nabla T_1(ku)\,\varphi$, as $k \to \infty$ one gets

$$\int_\Omega (\nabla u \cdot \nabla \varphi\, \chi_{\{u>0\}} + Vu\varphi) \leq \int_\Omega h\varphi,$$

from which (11.2) follows since $\nabla u = 0$ almost everywhere on $\{u = 0\}$.

By (11.2) and a classical property of positive distributions, there exists a nonnegative locally finite Borel measure τ in Ω such that

$$-\Delta u + Vu = h - \tau \quad \text{in the sense of distributions in } \Omega. \tag{11.3}$$

Since $u \in W_0^{1,2}(\Omega)$, $Vu \in L^1(\Omega)$ and $h \in L^2(\Omega)$, we have that τ is diffuse with respect to the $W^{1,2}$ capacity [18] and belongs to $L^1(\Omega) + (W_0^{1,2}(\Omega))'$. The latter property is a general fact satisfied by diffuse measures that has been established in [5]. We now prove that

$$\int_\Omega (\nabla u \cdot \nabla z + Vuz) = \int_\Omega zh - \int_\Omega \widehat{z}\,\mathrm{d}\tau \quad \text{for every } z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{11.4}$$

To this end, we write in functional form the action on any $\varphi \in C_c^\infty(\Omega)$ in (11.3) as

$$\int_\Omega (\nabla u \cdot \nabla \varphi + Vu\varphi) = \int_\Omega \varphi h - \int_\Omega \varphi\,\mathrm{d}\tau = \int_\Omega \varphi h - \tau[\varphi].$$

Since $\tau \in L^1(\Omega) + (W_0^{1,2}(\Omega))'$, by an approximation of $z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with functions in $C_c^\infty(\Omega)$ we get

$$\int_\Omega (\nabla u \cdot \nabla z + Vuz) = \int_\Omega zh - \tau[z].$$

The identification of $\tau[z]$ as integration with respect to τ then gives (11.4). In particular, for every $f \in L^\infty(\Omega)$ we can apply (11.4) with $z = \zeta_f$. Using u as test function in the Euler-Lagrange equation (3.1) satisfied by ζ_f we deduce that

$$\int_\Omega uf = \int_\Omega (\nabla u \cdot \nabla \zeta_f + Vu\zeta_f) = \int_\Omega \zeta_f h - \int_\Omega \widehat{\zeta_f}\,\mathrm{d}\tau.$$

We now prove that $\tau(T) = 0$ holds for every Sobolev-open set $T \subset \Omega \setminus O$. Replacing the function ξ coming from the definition of Sobolev-openness of T by the truncated function $T_1(\xi)$, we may assume from the beginning that $\xi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We are thus entitled to take in (11.4) the test function $z = \xi$. Since $\xi = 0$ almost everywhere in O and $u = h = 0$ almost everywhere in $\Omega \setminus O$, we get

$$\int_\Omega \widehat{\xi} \, \mathrm{d}\tau = \int_\Omega \xi h - \int_\Omega (\nabla u \cdot \nabla \xi + V u \xi) = 0.$$

As $\widehat{\xi} > 0$ in T, we conclude that $\tau(T) = 0$.

We are left to prove that $\tau(O) = 0$. For this purpose, we now take $v \in \mathcal{W}(O, \Omega) \cap L^2(\Omega; V \, \mathrm{d}x) \cap L^\infty(\Omega)$, which is an admissible test function for both (11.1) and (11.4). Comparison between both identities gives

$$\int_\Omega \widehat{v} \, \mathrm{d}\tau = 0. \tag{11.5}$$

As the function ξ coming from the definition of the Sobolev-openness of O belongs to $\mathcal{W}(O, \Omega)$ and the torsion function ζ_1 belongs to $W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, \mathrm{d}x) \cap L^\infty(\Omega)$, we may apply (11.5) with $v := \min\{\xi, \zeta_1\}$. Observe that every point in Ω is a Lebesgue point of v and

$$\widehat{v} = \min\{\widehat{\xi}, \widehat{\zeta_1}\}, \tag{11.6}$$

which is a consequence of the facts that $\min\{a, b\} = a - (a - b)^+$ for every $a, b \in \mathbb{R}$ and composition with Lipschitz functions preserves Lebesgue points. Moreover, the assumption $O \subset \Omega \setminus S$ implies that $\widehat{\zeta_1}(x) > 0$ for every $x \in O$ and then by (11.6) and the choice of ξ we have $\widehat{v} > 0$ in O. As τ is nonnegative, we deduce from (11.5) that $\tau(O) = 0$. $\qquad\square$

Proof of Proposition 11.2. Assume that $U_x \subset A \cup B$, where $A, B \subset \Omega$ are disjoint Sobolev-open sets, and $A \cap U_x \neq \emptyset$. Since $A \cap U_x$ is Sobolev-open, it has positive Lebesgue measure. We then let $h := \chi_{A \cap U_x}$. As ζ_h is a duality solution of (1.8) with datum $\mu = h \, \mathrm{d}x$, by the representation formula (3.8) we have

$$\widehat{\zeta_h}(y) = \int_\Omega G_y h = \int_{A \cap U_x} G_y \quad \text{for every } y \in \Omega. \tag{11.7}$$

We then observe that

$$\widehat{\zeta_h} > 0 \quad \text{in } U_x. \tag{11.8}$$

Indeed, since $U_y = U_x$ for $y \in U_x$ (by Proposition 9.1) and $A \cap U_x$ has positive Lebesgue measure, from (11.7) we get

$$\widehat{\zeta_h}(y) = \int_{A \cap U_y} G_y > 0 \quad \text{for every } y \in U_x.$$

In view of (11.8), the proof of $U_x \subset A$ will be complete once we show that

$$\widehat{\zeta_h} = 0 \quad \text{in } B. \tag{11.9}$$

The heart of the matter lies in the following

Claim. $\zeta_h = 0$ almost everywhere in B.

Proof of the Claim. It suffices to prove that $\zeta_h = u$, where u is the function given by Poincaré's balayage method with $h = \chi_{A \cap U_x}$ as above and $O = A \cap U_x$. Indeed, we recall that $u = 0$ almost everywhere in $\Omega \setminus O$ and, by the choice of O, we have

$$B \subset \Omega \setminus A \subset \Omega \setminus O.$$

By Lemma 11.3, the function u satisfies

$$\int_\Omega uf = \int_\Omega \zeta_f h - \int_\Omega \widehat{\zeta}_f \, \mathrm{d}\tau \quad \text{for every } f \in L^\infty(\Omega), \tag{11.10}$$

where τ is a nonnegative measure in Ω. Let us first show that τ is carried by the Sobolev-closed set S, that is,

$$\tau = 0 \quad \text{in } \Omega \setminus S. \tag{11.11}$$

Since τ is nonnegative and, as a consequence of Proposition 9.1, $\Omega \setminus S$ can be covered by at most countably many sets U_y, it suffices to prove that $\tau(U_y) = 0$ for every $y \in \Omega \setminus S$.

When $y \notin U_x$, an application of Proposition 9.1 gives

$$U_y \subset \Omega \setminus U_x \subset \Omega \setminus O$$

and one applies Lemma 11.3 with $T = U_y$. We are left to prove that $\tau(U_x) = 0$. To this end, we observe that $U_x \subset O \cup B$. Thus, by monotonicity and additivity of τ,

$$0 \le \tau(U_x) \le \tau(O \cup B) = \tau(O) + \tau(B).$$

By Lemma 11.3 we have $\tau(O) = 0$. Since $B \subset \Omega \setminus O$ is Sobolev-open, once again by Lemma 11.3 we have $\tau(B) = 0$. Thus, $\tau(U_x) = 0$ and (11.11) is satisfied.

Since $\widehat{\zeta}_f = 0$ in S, we thus have

$$\int_\Omega \widehat{\zeta}_f \, \mathrm{d}\tau = \int_{\Omega \setminus S} \widehat{\zeta}_f \, \mathrm{d}\tau = 0 \quad \text{for every } f \in L^\infty(\Omega).$$

Inserting this identity in (11.10), we conclude that u is the duality solution of (1.8) with datum $\mu = h \, \mathrm{d}x$ and then, by uniqueness, $u = \zeta_h$. $\qquad \square$

We now proceed with the proof of (11.9). By the Claim, for every ball $B_r(x) \subset \Omega$ we have

$$0 \le \fint_{B_r(x)} \zeta_h = \frac{1}{|B_r(x)|} \int_{B_r(x) \setminus B} \zeta_h \le \frac{|B_r(x) \setminus B|}{|B_r(x)|} \|\zeta_h\|_{L^\infty(\Omega)}.$$

Since B is Sobolev-open, every $x \in B$ is a density point of B by Proposition 10.1. In this case, the right-hand side converges to zero as $r \to 0$ and we conclude that $\widehat{\zeta}_h(x) = 0$, which is (11.9). $\qquad \square$

12. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Each superlevel set U_x from Definition 9.1 is Sobolev-open (Proposition 10.2), Sobolev-connected (Proposition 11.2) and $U_x \subset \Omega \setminus Z$ whenever $x \in \Omega \setminus Z$ (Proposition 10.3). Since U_x is non-empty and Sobolev-open, it has positive Lebesgue measure. Thus, by Proposition 9.1,

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the set $\Omega \setminus Z$ is a finite or countably infinite disjoint union of components $(D_j)_{j \in J}$ of the form $D_j = U_{x_j}$ for some $x_j \in \Omega \setminus Z$.

Uniqueness of the decomposition is based on a standard topological argument. Indeed, let $(\widetilde{D}_i)_{i \in I}$ be another finite or infinite countable decomposition of $\Omega \setminus Z$ in terms of disjoint Sobolev-connected-open sets. If $D_k \cap \widetilde{D}_l \neq \emptyset$, then as

$$D_k \subset \widetilde{D}_l \cup \bigcup_{i \in I \setminus \{l\}} \widetilde{D}_i$$

and the sets in the right-hand side are disjoint and Sobolev-open, it follows from the definition of Sobolev-connectedness that $D_k \subset \widetilde{D}_l$. Interchanging the roles of D_k and $\widetilde{D}_l$, the reverse inclusion also holds. Hence, both families $(D_j)_{j \in J}$ and $(\widetilde{D}_i)_{i \in I}$ coincide up to a bijection between indices.

It remains to prove that a function $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfying (1.1) with nonnegative $f \in L^\infty(\Omega)$ is either positive or zero in each component D_j. To this end, take $x \in D_j$. By Proposition 9.1, we have $U_x = D_j$ and then $G_x = 0$ almost everywhere in $\Omega \setminus U_x = \Omega \setminus D_j$. By Green's representation formula in Theorem 1.3 we thus have

$$\widehat{w}(x) = \int_\Omega G_x f = \int_{D_j} G_x f \quad \text{for every } x \in D_j. \tag{12.1}$$

If $\widehat{w}(x) = 0$ for some $x \in D_j$, then by positivity of G_x in $U_x = D_j$ and nonnegativity of f, we must have

$$f = 0 \quad \text{almost everywhere in } D_j. \tag{12.2}$$

By (12.1) applied at a point $y \in D_j$ and (12.2) we conclude that

$$\widehat{w}(y) = \int_{D_j} G_y f = 0 \quad \text{for every } y \in D_j. \qquad \square$$

The representation formula (12.1) in terms of each Sobolev-connected component D_j makes more transparent the fact that the strong-maximum-principle alternative in D_j is independent of the behavior of the solution in the other components. Observe that for any given subset of indices $L \subset J$, by Theorem 1.4, see also Proposition 5.6, there exists a solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (1.1) with $f = \chi_B$ and $B = \bigcup_{l \in L} D_l$. We then deduce from (12.1) in this case that $\widehat{w} > 0$ in D_j if and only if $j \in L$.

The proof of Theorem 1.1 adapts automatically to duality solutions after replacing the universal zero-set Z by the zero-set of the torsion function,

$$S = \{\widehat{\zeta_1} = 0\}.$$

As the Sobolev-connected components of $\Omega \setminus S$ are obtained from all distinct superlevel sets U_x with $x \in \Omega \setminus S$ (and not only $x \in \Omega \setminus Z$ as in Theorem 1.1), from Proposition 10.3 they are formed by the collection $(D_j)_{j \in J}$ of Sobolev-connected components of $\Omega \setminus Z$ and the Sobolev-connected components of the Sobolev-open set $Z \setminus S$. In this respect, there can be more components when $Z \neq S$, but they can never get larger by replacing $\Omega \setminus Z$ with $\Omega \setminus S$. We may then summarize the counterpart of Theorem 1.1 for duality solutions as follows:

Theorem 12.1. *The Sobolev-open set $\Omega \setminus S$ can be uniquely decomposed as a finite or countably infinite family $(D_j)_{j \in \widetilde{J}}$ of Sobolev-connected-open sets that contains $(D_j)_{j \in J}$. In addition, every duality solution ζ_f of (1.8) with $\mu = f \, dx$ and nonnegative $f \in L^\infty(\Omega)$ satisfies, in each component D_j with $j \in \widetilde{J}$,*

$$\text{either} \quad \widehat{\zeta}_f > 0 \ \text{ in } D_j \quad \text{or} \quad \widehat{\zeta}_f \equiv 0 \ \text{ in } D_j.$$

We now present a stronger version of Corollary 1.2, where the Hausdorff-measure assumption is made upon S and gives a sufficient condition for equality with the universal zero-set Z:

Proposition 12.2. *If $\mathcal{H}^{N-1}(S) = 0$ and $Z \neq \Omega$, then $S = Z$ and the Sobolev-open set $\Omega \setminus Z$ is Sobolev-connected. Hence, every solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (1.1) for some nonnegative $f \in L^\infty(\Omega)$ with $\int_\Omega f > 0$ satisfies*

$$\widehat{w}(x) = 0 \quad \text{if and only if} \quad x \in Z.$$

The assumption $\mathcal{H}^{N-1}(S) = 0$ can be verified with the help of Proposition 4.4. To check that $Z \neq \Omega$, it is enough to know there is a distributional solution of the Dirichlet problem (1.8) for some finite nonnegative measure $\mu \neq 0$, since by Theorem 1.4 one must have $\mu(Z) = 0$.

Example 12.1. If there exists $v \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V \, dx)$ such that

$$\widehat{v} > 0 \quad \text{quasi-everywhere in } \Omega,$$

then $\mathrm{cap}_{W^{1,2}}(S) = 0$. As the measure λ in Proposition 4.1 is diffuse and carried by S, we then have $\lambda = 0$. Since S is negligible for the Lebesgue measure, the torsion function ζ_1 satisfies (1.1) with $f \equiv 1$, whence $Z \neq \Omega$. Thus, by Proposition 12.2, we have $S = Z$ and $\Omega \setminus Z$ is Sobolev-connected.

Proof of Proposition 12.2. Since Ω is connected and $\mathcal{H}^{N-1}(S) = 0$, the set $\Omega \setminus S$ is Sobolev-connected. Indeed, let $A, B \subset \Omega$ be disjoint Sobolev-open sets such that

$$\Omega \setminus S \subset A \cup B$$

and assume by contradiction that the sets $\widetilde{A} := A \setminus S$ and $\widetilde{B} := B \setminus S$ are both non-empty. Observe that $\widetilde{A}$ and $\widetilde{B}$ are also Sobolev-open and

$$\Omega \setminus S = \widetilde{A} \cup \widetilde{B}.$$

Let $\xi_1, \xi_2 \in W_0^{1,2}(\Omega)$ be nonnegative functions such that their Lebesgue sets coincide with Ω and $\widetilde{A} = \{\widehat{\xi_1} > 0\}$ and $\widetilde{B} = \{\widehat{\xi_2} > 0\}$. The function $\widehat{\xi_1} - \widehat{\xi_2}$ is positive on $\widetilde{A}$, negative on $\widetilde{B}$, and vanishes on S. By the Intermediate value theorem for Sobolev functions [32, Proposition 2.11], we have $\mathcal{H}^{N-1}(S) > 0$, which is a contradiction. We conclude that $\Omega \setminus S$ is Sobolev-connected.

It thus follows from Proposition 11.1 that $\Omega \setminus S \subset U_x$ for any $x \in \Omega \setminus S$. Since $U_x \subset \Omega \setminus S$, equality holds and we can write

$$\Omega = S \cup U_x.$$

When $Z \neq \Omega$, we can take $x \in \Omega \setminus Z$ and deduce from Proposition 10.3 and the decomposition (10.3) that

$$U_x = \Omega \setminus Z \quad \text{and} \quad Z \setminus S = \emptyset.$$

Therefore, $S = Z$. Since $\Omega \setminus Z = \Omega \setminus S$ contains only one Sobolev-connected component, the conclusion follows from Theorem 1.1. $\qquad\qquad\square$

13. Strong maximum principle for distributional solutions involving measures

We prove in this last section a counterpart of Theorem 1.1 for distributional solutions of the Dirichlet problem (1.8):

Theorem 13.1. *Let $(D_j)_{j \in \mathbb{N}}$ be the Sobolev-connected components of $\Omega \setminus Z$. If u is a distributional solution of (1.8) for some nonnegative measure $\mu \in \mathcal{M}(\Omega)$ and if there exists a Lebesgue point $x \in D_j$ such that $\widehat{u}(x) = 0$, then*

$$\widehat{u} = 0 \quad \text{in } D_j \quad \text{and} \quad \mu(D_j) = 0.$$

Proof. We first observe that

$$u = 0 \quad \text{almost everywhere in } D_j. \tag{13.1}$$

To this end, we apply the comparison principle to deduce that $u \geq \zeta_{H(u)}$ almost everywhere in Ω, where $\zeta_{H(u)}$ satisfies (1.1) with $f = H(u)$. Since $\zeta_{H(u)}$ is nonnegative and $\widehat{u}(x) = 0$, we get $\widehat{\zeta_{H(u)}}(x) = 0$. From (12.2), we thus have $H(u) = 0$ almost everywhere in D_j and then (13.1) is satisfied by positivity of H on $(0, +\infty)$. Now, at any Lebesgue point $y \in D_j$, by (13.1) and the fact that y is a density point of D_j, we then have $\widehat{u}(y) = 0$. $\qquad\square$

To prove that $\mu(D_j) = 0$, we first need a weak form of Green's representation formula for general duality solutions of (1.8):

Lemma 13.2. *If u is a duality solution of (1.8) with nonnegative datum $\mu \in \mathcal{M}(\Omega)$, then for almost every $x \in \Omega$ we have $\mu(\{x\}) = 0$ and*

$$\widehat{u}(x) = \int_\Omega \widehat{G_x} \, \mathrm{d}\mu.$$

Proof of Lemma 13.2. Let $(\rho_k)_{k \in \mathbb{N}}$ be a sequence of mollifiers and let u_k be the duality solution associated to $\rho_k * \mu$. Passing to a subsequence if necessary, by Proposition 3.5 we have that $(\widehat{u_k})_{k \in \mathbb{N}}$ converges to $\widehat{u}$ in $L^1(\Omega)$ and everywhere in $\Omega \setminus E_1$ for some negligible set E_1. By the representation formula (3.8) for bounded data,

$$\widehat{u_k}(x) = \int_\Omega G_x \, \rho_k * \mu \quad \text{for every } x \in \Omega.$$

When $x \in S$, we have $\widehat{u_k}(x) = 0$ and $G_x = 0$ almost everywhere in Ω. Thus, for every $x \in S \setminus E_1$, it follows that $\widehat{u}(x) = 0$ and the representation formula is satisfied almost everywhere in S.

When $x \notin S$, we first apply Fubini's theorem,

$$\widehat{u_k}(x) = \int_\Omega \breve{\rho}_k * G_x \, \mathrm{d}\mu. \tag{13.2}$$

Taking ρ_k of the form $\rho_k(z) = \frac{1}{r_k^N} \rho(\frac{z}{r_k})$, where $\rho \in C_c^\infty(\Omega)$ and $(r_k)_{k \in \mathbb{N}}$ converges to zero, we have

$$(\breve{\rho}_k * G_x)(y) \to \widehat{G_x}(y) \quad \text{for every } y \in \Omega \setminus \{x\},$$

since every point in $\Omega \setminus \{x\}$ is a Lebesgue point of G_x. To apply the Dominated convergence theorem in (13.2), we first observe that, for every $x \in \Omega$,

$$0 \leq G_x(a) \leq F(x - a) \quad \text{for almost every } a \in \Omega, \tag{13.3}$$

where F is the fundamental solution of the Laplacian:

$$F(z) = \begin{cases} \dfrac{1}{\gamma_N} \dfrac{1}{|z|^{N-2}} & \text{if } N \geq 3, \\ \dfrac{1}{2\pi} \log \dfrac{d}{|z|} & \text{if } N = 2, \end{cases}$$

and we take $d > \operatorname{diam}(\Omega)$ in dimension two to make sure that $F(x - y) > 0$ for every $x, y \in \overline{\Omega}$. The second inequality in (13.3) follows from the weak maximum principle since $G_x \in W_0^{1,1}(\Omega)$ satisfies (9.1) for some nonnegative measure λ and $F(\cdot - a)$ is a positive function on $\overline{\Omega}$ that satisfies the Poisson equation with δ_a; see [28, Example 6.2].

Assuming that ρ is radial, by (13.3) and superharmonicity of $F(x - \cdot)$, we then have

$$0 \leq (\breve{\rho}_k * G_x)(y) \leq F(x - y) \quad \text{for every } y \in \Omega. \tag{13.4}$$

Let $E_2 \subset \Omega$ be a negligible set such that $(F * \mu)(x) < \infty$ for every $x \in \Omega \setminus E_2$. For such a point x, $\mu(\{x\}) = 0$, the function $F(x - \cdot)$ is summable with respect to μ and, by (13.4), we can apply the Dominated convergence theorem to get

$$\lim_{k \to \infty} \int_\Omega \breve{\rho}_k * G_x \, \mathrm{d}\mu = \int_\Omega \widehat{G_x} \, \mathrm{d}\mu \quad \text{for every } x \in \Omega \setminus (S \cup E_2).$$

We deduce the representation formula for every $x \in \Omega \setminus (S \cup E_1 \cup E_2)$ as $k \to \infty$ in (13.2). $\qquad\square$

Proof of Theorem 13.1 completed. Take a Lebesgue point $y \in D_j$ such that $\widehat{u}(y) = 0$, $\mu(\{y\}) = 0$ and the representation formula in Lemma 13.2 holds. Then,

$$\int_\Omega \widehat{G_y} \, \mathrm{d}\mu = \widehat{u}(y) = 0.$$

This implies that $\mu(\{\widehat{G_y} > 0\}) = 0$. Since by Proposition 9.1 we have

$$D_j = U_y = \{y\} \cup \{\widehat{G_y} > 0\},$$

we conclude using the additivity of μ that

$$\mu(D_j) = \mu(\{y\}) + \mu(\{\widehat{G_y} > 0\}) = 0. \qquad\square$$

Although all distributional solutions with bounded datum vanish on Z, the same need not be true in the case of measures or even L^1 functions:

Example 13.1. For $N \geq 3$, take $V(x) = 1/|x - a|^2$ for some fixed $a \in \Omega$. Any nontrivial superharmonic function $\psi \in C_0^\infty(\overline{\Omega})$ satisfies the Schrödinger equation

$$-\Delta \psi + V\psi =: f \quad \text{in the sense of distributions in } \Omega,$$

where the function f is nonnegative and belongs to $L^p(\Omega)$ for every $1 \leq p < N/2$. However,

$$Z = \{a\} \quad \text{and} \quad \psi(a) > 0.$$

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From our proof of Theorem 1.4, see (8.6), we know that all distributional solutions vanish almost everywhere in Z. We now show the stronger property that this actually holds except for a set of $W^{1,2}$ capacity zero:

Proposition 13.3. *If u is a distributional solution of* (1.8) *for some nonnegative measure $\mu \in \mathcal{M}(\Omega)$, then*

$$\widehat{u} = 0 \quad \text{quasi-everywhere in } Z.$$

Proof. We first assume that the Newtonian potential $F * \mu$ is bounded. In particular, μ is diffuse with respect to the $W^{1,2}$ capacity and also belongs to $(W_0^{1,2}(\Omega))'$. In this case, $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and every element in this space is an admissible test function.

Given a sequence of mollifiers $(\rho_k)_{k \in \mathbb{N}}$, let u_k be the distributional solution of (1.8) with datum $\chi_{\Omega \setminus Z}(\rho_k * \mu)$, which exists by Theorem 1.4; see also Proposition 5.6. We claim that $(u_k)_{k \in \mathbb{N}}$ converges to u in $W_0^{1,2}(\Omega)$. To this end, we apply $u_k - u$ as test function in the equation satisfied by $u_k - u$ to get

$$\int_\Omega |\nabla(u_k - u)|^2 + \int_\Omega V(u_k - u)^2 = \int_{\Omega \setminus Z}(u_k - u)\rho_k * \mu - \int_\Omega (\widehat{u_k} - \widehat{u}) \, \mathrm{d}\mu. \qquad (13.5)$$

We write the action of μ as an element of the dual $(W_0^{1,2}(\Omega))'$ in the form

$$\int_\Omega (\widehat{u_k} - \widehat{u}) \, \mathrm{d}\mu = \mu[u_k - u].$$

Since the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$ and converges to u in $L^1(\Omega)$, we have weak convergence in $W_0^{1,2}(\Omega)$ and then

$$\lim_{k \to \infty} \int_\Omega (\widehat{u_k} - \widehat{u}) \, \mathrm{d}\mu = 0.$$

We next recall that $u_k = u = 0$ almost everywhere in Z. Thus, using Fubini's theorem,

$$\int_{\Omega \setminus Z}(u_k - u)\rho_k * \mu = \int_\Omega (u_k - u)\rho_k * \mu = \int_\Omega \breve{\rho_k} * (u_k - u) \, \mathrm{d}\mu.$$

The sequence $(\breve{\rho_k} * (u_k - u))_{k \in \mathbb{N}}$ converges weakly to zero in $W^{1,2}(\mathbb{R}^N)$, and then one has as before,

$$\lim_{k \to \infty} \int_{\Omega \setminus Z}(u_k - u)\rho_k * \mu = 0.$$

As $k \to \infty$ in (13.5), we get

$$\lim_{k \to \infty} \int_\Omega |\nabla(u_k - u)|^2 = 0,$$

which implies the claim. Now passing to a subsequence $(u_{k_j})_{j \in \mathbb{N}}$, one deduces that

$$\widehat{u_{k_j}} \to \widehat{u} \quad \text{quasi-everywhere in } \Omega.$$

Since every u_{k_j} satisfies an equation in the sense of distributions with bounded datum, we have

$$\widehat{u_{k_j}} = 0 \quad \text{in } Z.$$

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The conclusion thus follows when $F * \mu$ is bounded.

In the case of a general nonnegative measure μ, it suffices to prove that the truncated function $T_1(u)$ satisfies the conclusion. Observe that $T_1(u) \in W_0^{1,1}(\Omega) \cap L^1(\Omega; V \, dx)$ and

$$- \Delta T_1(u) + V T_1(u) = \widetilde{\mu} \quad \text{in the sense of distributions in } \Omega,$$

for some nonnegative diffuse measure $\widetilde{\mu} \in \mathcal{M}(\Omega)$; see [12, 8, 9]. By a classical property in Potential theory [19, Theorem 3.6.3], this measure can be strongly approximated in $\mathcal{M}(\Omega)$ by a nondecreasing sequence of measures $(\nu_k)_{k \in \mathbb{N}}$ with bounded Newtonian potential, for which the conclusion holds from the first part of the proof. This implies the theorem as the distributional solutions v_k of (1.8) with data ν_k converge strongly to $T_1(u)$ in $W_0^{1,2}(\Omega)$ and, for each $k \in \mathbb{N}$, they satisfy $\widehat{v_k} = 0$ quasi-everywhere in Ω. $\qquad \square$

Acknowledgements

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