

Probability of digits by dividing random numbers: a ψ and ζ functions approach

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Abstract

This paper begins with the statistics of the decimal digits of n/d with $(n, d) \in \mathbb{N}^2$ randomly chosen. Starting with a statement by E. Cesàro on probabilistic number theory, see [3] and [4], we evaluate, through the Euler ψ function, an integral appearing there. Furthermore the probabilistic statement itself is proved, using a different approach: in any case the probability of a given digit r to be the first decimal digit after dividing a couple of random integers is

$$p_r = \frac{1}{20} + \frac{1}{2} \left\{ \psi \left(\frac{r}{10} + \frac{11}{10} \right) - \psi \left(\frac{r}{10} + 1 \right) \right\}.$$

The theorem is then generalized to real numbers (Theorem 1, holding a proof of both $\frac{n}{d}$ results) and to the α th power of the ratio of integers (Theorem 2), via an elementary approach involving the ψ function and the Hurwitz ζ function. The article provides historic remarks, numerical examples, and original theoretical contributions: also it complements the recent renewed interest in Benford's law among number theorists.

Keywords: Elementary probability, Euler ψ function, Hurwitz ζ function

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1. Introduction: the $\frac{n}{d}$ theorems

At the end of the 18th century, Gauss was one of the first to use probabilistic arguments for investigating the number of products consisting of exactly k

distinct prime factors below a given bound. The case $k = 1$ led to the prime number theorem (proved in 1896), a milestone of analytic number theory. Instead of probabilistic number theory one should better speak about studying arithmetic functions with probabilistic methods. Anyway, the first approach in this direction dates back to Gauss, and to Ernesto Cesàro¹ who observed, 1881, that the probability that two randomly chosen integers are coprime is $6/\pi^2$. Substantial developments in probabilistic number theory started with the prime divisor counting function $\Omega(n)$ of a positive integer n , but only in 1917 Hardy and Ramanujan discovered the first deep results on such a function. The first of Cesàro's papers on the theory of numbers was followed, 1885, by a 117 pages book issued in Paris, [3], entitled: *Excursions arithmétiques à l'infini*, which reproduces nine articles from *Annali di matematica pura ed applicata* about some relevant problems in arithmetic. They looked at problems concerning the number of common divisors of two integers, determination of the values of the sum of their squares, the probability of the incommensurability of three arbitrary numbers, and so on; to these he attempted to apply results obtained in the theory of Fourier series. In this paper we focus on that collection's third article (pp. 35–56), entitled *Eventualités de la division arithmétique* originally appeared in [4]. In addition to the topics mentioned above, Cesàro contributed to founding probabilistic number theory studying the distribution of primes and trying to improve results obtained there by Chebyshev.

1.1. Cesàro's statements and the aim of the paper

In [3] p. 49, section 19, or [4], after a sequence of frequentistic arguments, he notes that

On trouve facilement que, si L et Λ sont les *fréquences* d'une certaine condition dans les séries

$$\begin{aligned} f\left(\frac{n}{1}\right), \quad f\left(\frac{n}{2}\right), \quad f\left(\frac{n}{3}\right), \dots \quad f\left(\frac{n}{n}\right) \\ f\left(\frac{1}{n}\right), \quad f\left(\frac{2}{n}\right), \quad f\left(\frac{3}{n}\right), \dots \quad f\left(\frac{n}{n}\right) \end{aligned} \tag{1}$$

¹E. Cesàro (1859-1906) entered the École des Mines where he studied mathematics with Catalan. In Paris he was particularly struck by lectures of Darboux on geometry which led him to his own studies of intrinsic geometry. Supported by Cremona, Battaglini, and Dini, he undertook research at the University of Rome from 1884 where the doctorate was awarded to him in 1887. He also developed the study of divergent series; furthermore his interest in mathematical physics is evident from two successful calculus texts he wrote. Cesàro was offered the chair of mathematics at Palermo where he remained until 1891, moving then to Naples where he held the chair of mathematical analysis until his death in 1906.

la probabilité que, en prenant u et v au hasard, $f\left(\frac{u}{v}\right)$ satisfasse à la même condition, est égale à la moyenne arithmétique des deux fréquences. Ainsi:

$$P = \frac{1}{2}(L + \Lambda) \quad (2)$$

At this point, taking $f(x) = x$ in the first of two series (1) he states that

D'autre part, dans la seconde des séries (1), chaque chiffre est également fréquent. Donc, d'après (2), on peut affirmer que: Dans une division quelconque, la probabilité que le premier chiffre décimal soit r , est:

$$\frac{1}{20} + 5 \int_0^1 \frac{1 - \varphi}{1 - \varphi^{10}} \varphi^{9+r} d\varphi. \quad (3)$$

Next he states that, performing an arbitrary division of two random integers, the i th decimal digit has a probability to be r given by

$$\frac{1}{20} + \frac{10^i}{2} \int_0^1 \frac{1 - \varphi}{1 - \varphi^{10}} \varphi^{10^i - 1 + r} d\varphi. \quad (4)$$

Our contribution starts by evaluating the integrals (3) and (4). Such integrals, concerning a ratio of rational functions,

$$\frac{1 - \varphi}{1 - \varphi^{10}} \varphi^{9+r} = \frac{\varphi^{9+r}}{(\varphi + 1)P(\varphi)},$$

where

$$P(\varphi) = \left(\varphi^2 - \frac{1}{2}(\sqrt{5} - 1)\varphi + 1 \right) \left(\varphi^2 + \frac{1}{2}(\sqrt{5} - 1)\varphi + 1 \right) \times \\ \times \left(\varphi^2 - \frac{1}{2}(1 + \sqrt{5})\varphi + 1 \right) \left(\varphi^2 + \frac{1}{2}(1 + \sqrt{5})\varphi + 1 \right),$$

are not easy. It is then better to compute them by standing on a higher viewpoint, i.e., introducing the Euler ψ function. Such a computation was not performed by Cesàro, who probably judged it too involved. But through an alternative representations of ψ , we realized that those integrals could be expressed by means of infinite series. This inspired an elementary proof of the same statement of Cesàro, followed by some generalizations to be found in our Sections 3 and 4.

A theorem of Gauss on ψ of rational arguments simplifies the process, avoiding decomposition into simple fractions.

1.2. Formal probability definition

In his arithmetic papers Cesàro freely mentions the probability $p_r^{(i)}$ that the i th decimal digit of the ratio of two integers to be a given digit, say $r \in \{0, \dots, 9\}$, but quite informally. For better clarity and chiefly to highlight the difference between the definition in our paper (two real positive numbers randomly chosen) and that implicitly followed by Cesàro (two integer numbers randomly chosen), we define this discrete version: if $d_i(v)$ is the i th decimal digit of the number v , we put:

$$p_r^{(i)} := \lim_{T \rightarrow \infty} \frac{\#\{(n_1, n_2) \in \mathbb{N}^2 \mid n_j \leq T \text{ for } j = 1, 2, d_i(n_1/n_2) = r\}}{\#\{(n_1, n_2) \in \mathbb{N}^2 \mid n_j \leq T \text{ for } j = 1, 2\}}$$

whenever such a limit exists. Here $\#A$ stands for the cardinality of A . When $i = 1$ we write p_r instead of $p_r^{(1)}$.

2. Cesàro's integrals evaluation through the ψ function

The main properties of Euler's ψ function will be recalled, other details being available in [1] and [6]. We have

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(x+n)}, \quad (5)$$

where γ is the Euler-Mascheroni constant,

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \right) = \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx.$$

where $[x]$ is the *floor* of x . From (5) follows the recursive relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \quad (6)$$

Furthermore these integral representations for $x > 0$ will be useful:

$$\psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt \quad (7)$$

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - u^{x-1}}{1 - u} du \quad (8)$$

Of course (8) is obtained from (7) by putting $e^{-t} = u$. From the second integral formula, taking $a, b > 0$, one obtains:

$$\int_0^1 \frac{y^a - y^b}{1-y} dy = \psi(b+1) - \psi(a+1)$$

of which the Cesàro integral in (3) is a special case:

$$\begin{aligned} \int_0^1 \frac{1-\varphi}{1-\varphi^{10}} \varphi^{9+r} d\varphi &= \frac{1}{10} \int_0^1 \frac{y^{\frac{r}{10}} - y^{\frac{r+1}{10}}}{1-y} dy \\ &= \frac{1}{10} \left\{ \psi\left(\frac{r+11}{10}\right) - \psi\left(\frac{r+10}{10}\right) \right\} \end{aligned} \quad (9)$$

In such a way the probability in (3) is given by

$$p_r = \frac{1}{20} + \frac{1}{2} \left\{ \psi\left(\frac{r}{10} + \frac{11}{10}\right) - \psi\left(\frac{r}{10} + 1\right) \right\}. \quad (10)$$

Gauss, see his *Gesammelte Werke*, Bd. 3, pp. 155–156, (Göttingen, 1866), found for ψ of rational arguments an expression involving elementary functions only. In fact, for any $p, q \in \mathbb{N}$ with $1 \leq p < q$,

$$\psi\left(\frac{p}{q}\right) = -\gamma - \frac{1}{2}\pi \cot\left(\frac{p\pi}{q}\right) - \ln q + \sum_{k=1}^{q-1} \cos\left(\frac{2kp\pi}{q}\right) \ln\left(2 \sin\left(\frac{k\pi}{q}\right)\right). \quad (11)$$

A relevant proof, more elementary than the original one, can be found in [6]. Joining (6), (10), and (11) we have a closed form expression of all the required probabilities:

$$\begin{aligned} p_0 &= \frac{101}{20} - \frac{\pi}{4} \sqrt{5+2\sqrt{5}} - \frac{1}{8} \left[\ln(800000) + \sqrt{5} \ln(9+4\sqrt{5}) \right] \\ p_1 &= \frac{1}{20} \left[-49 + \pi \sqrt{10(5+\sqrt{5})} + 5 \left(\ln 16 + \sqrt{5} \ln \frac{3+\sqrt{5}}{2} \right) \right] \\ p_2 &= \frac{1}{60} \left[-47 + 6\pi \sqrt{25-10\sqrt{5}} + 30 \left(\sqrt{5} \ln \frac{3+\sqrt{5}}{2} - \ln 4 \right) \right] \\ p_3 &= \frac{1}{60} \left[-22 + 3\pi(5-\sqrt{5}) \sqrt{5-2\sqrt{5}} - 15 \left(\sqrt{5} \ln \frac{3+\sqrt{5}}{2} - \ln 16 \right) \right] \\ p_4 &= \frac{1}{40} \left[-8 + 2\pi \sqrt{25-10\sqrt{5}} - (40+\sqrt{5}) \ln 2 + \right. \\ &\quad \left. + 25 \ln 5 + \sqrt{5} \ln(123-55\sqrt{5}) \right] \end{aligned}$$

$$\begin{aligned}
p_5 &= \frac{1}{120} \left[-14 + 6\pi \sqrt{25 - 10\sqrt{5}} - 15 \left(\ln \frac{3^{125}}{256} - 2\sqrt{5} \ln \frac{1+\sqrt{5}}{2} \right) \right] \\
p_6 &= -\frac{29}{420} + \frac{\pi}{2} \sqrt{\frac{25-11\sqrt{5}}{10}} + \ln \frac{\sqrt{1+\sqrt{5}}}{4} + \frac{1}{2} \left[\ln(\sqrt{5}-1) + \frac{\sqrt{5}}{2} \ln \frac{3+\sqrt{5}}{2} \right] \\
p_7 &= -\frac{11}{280} + \frac{\pi}{2} \sqrt{1 - \frac{2}{\sqrt{5}}} + \ln 2 + \frac{\sqrt{5}}{2} \ln \frac{3-\sqrt{5}}{2} \\
p_8 &= \frac{1}{360} \left(-7 + 18\pi \sqrt{10(5+\sqrt{5})} - 360 \ln 2 - 90\sqrt{5} \ln \frac{3+\sqrt{5}}{2} \right) \\
p_9 &= \frac{1}{360} \left\{ -2 - 90\pi \sqrt{5+2\sqrt{5}} + 45 \left[\ln(800000) + \sqrt{5} \ln(9+4\sqrt{5}) \right] \right\}.
\end{aligned}$$

With the help of Mathematica[®] it is immediate to verify that

$$\sum_{k=0}^9 p_k = 1.$$

The correspondent numerical values are

$$\begin{aligned}
p_0 &= 0.126730362245; & p_1 &= 0.117357521909; & p_2 &= 0.109924503863; \\
p_3 &= 0.103903172141; & p_4 &= 0.098937259282; & p_5 &= 0.094778739397; \\
p_6 &= 0.091250211050; & p_7 &= 0.088221779210; & p_8 &= 0.085596363935; \\
p_9 &= 0.083300086967.
\end{aligned}$$

The computed first digit probability is then decreasing with increasing digits, so that the highest is ‘zero’, and the lowest is ‘nine’.

Let us quote the recent paper of Qiu and Vuorinen, [11], where, formula (2.3) p. 727, it is proved that for any $x > 1$,

$$\ell(x) := \ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \ln x - \frac{1}{2x} - \frac{2\gamma-1}{2x^2} := u(x).$$

So, we see that

$$\sup_{x>1} (u(x) - \ell(x)) = \frac{7}{12} - \gamma \simeq 0.00611767$$

and then the use of either $u(x)$ or $\ell(x)$ instead of $\psi(x)$ produces an error less than 1/100.

2.1. An experimental check of the Cesàro's law

Before going ahead with the generalizations, the reader should know a possible experimental check on the above law's reliability. In order to do it, following the frequentist approach, after a very high number of divisions of two positive integers, chosen randomly, we will extract the first digit, recording the occurrence of each of ten digits. In such a sense we will follow the approach of the empirical checks, performed via Mathematica[®] which is the same as our tool, on the Weisstein web site [12] about Benford's law.

It will be recalled that the leading digit in tables and physical data is not evenly distributed among the digits 1 and 9. The first known written reference is a two-pages article by Newcomb, [10] 1881, who stated² that

$$\text{Prob}(\text{first significant digit} = r) = \log_{10} \left(1 + \frac{1}{r} \right), \quad r = 1, \dots, 9.$$

In such a way he conjectured that the digit '9' occurs about 4.6% of the time. In 1938, Benford published a paper [2] describing how well his 20,229 observations were fit by the logarithmic law, nowadays called the *Benford Law* stating that for a dimensionless data sequence, the digit '1' tends to occur with probability $\simeq 30\%$, almost three times greater than the naive value³, $\simeq 11.\bar{1}\%$. There is also a general significant digit Benford law including not only the first digit, but also the second, which may be zero, and all higher significant digits. For instance in [5], the probability that the first three significant digits are 3, 1, 4 in that order, is given by

$$\log_{10} \left(1 + \frac{1}{314} \right) \simeq 0.14\%$$

and similarly for other significant digit patterns.

Cesàro's law has a different object than Benford's: in fact Cesàro's argument is theoretical, founded upon number theory; on the contrary *in the beginning* the Benford law was nothing but an empirical statement about real-world datasets, applicable to some datasets but not all, and only approximate. The things are quite different *nowadays*, the law having been embodied as a theoretical mathematical

²Assuming that a universal probability distribution exists over such numbers, it will be invariant under a change of scale. Normalizing and differentiating such an invariance one will find a hyperbolic probability distribution, so that by integration the logarithmic phenomenon appears known as Benford's law.

³Namely the percentage one would get if the digits were equidistributed.

result (see for instance [9] and [7] and the references therein). Accordingly, a sequence of *positive* numbers x_n is defined *Benford (base b)* if the probability of observing the first digit of x_n in base b is $j \in \{j = 1, 2, \dots, b-1\}$ is given by:

$$\lim_{N \rightarrow \infty} \frac{\#\{n < N : \text{first digit of } x_n \text{ in base } b \text{ is } j\}}{N} = \log_b \left(1 + \frac{1}{j} \right)$$

From this definition, one can, for instance, see that the first digit of 2^n is Benford in base 10, but not Benford in base 2 because the first digit is always 1 in this second case. In [9] (Theorem 9.3.1 p. 220) some conditions are given in order that a recurrence relation of assigned length meets Benford; it is also shown why the first digit of a geometric Brownian motion is Benford.

But any sequence of digit obtained with the Cesàro procedure takes into account the ‘zero’ digit’s occurrences, so this phenomenon is not relevant for Benford’s, dealing with sequences of positive numbers.

Anyway: we did an experimental check by means of Mathematica[®], randomly taking couples of positive integers between 1 and 10^7 , taking the quotient, repeating 10^7 times, and recording the relative frequency of each digit. Comparing such values to the computed ones, we found for each digit a difference on the order of magnitude of $\pm 10^{-6}$, which has been successively confirmed by a similar test on the second place also.

2.2. Further digits

The integral in (4) can be again expressed through ψ :

$$p_r^{(i)} = \frac{1}{20} + \frac{10^i}{20} \left\{ \psi \left(\frac{r + 10^i + 1}{10} \right) - \psi \left(\frac{r + 10^i}{10} \right) \right\}. \quad (12)$$

Like in the previous analysis of the first digit, one can compute via elementary functions the probability that, dividing two randomly chosen integers, the i th digit will be $r = 0, 1, \dots, 9$. But, due to the recursive relationship on ψ ,

$$\psi(x+n) = \psi(x) + \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n-1}, \quad n \in \mathbb{N},$$

such formulae quickly become intractable, having to be evaluated with $n = 10^{i-1}$. For instance the probability the second decimal digit to be zero is

$$p_0^{(2)} = \frac{2991620812234909}{303502130011080} - \frac{\pi}{2} \sqrt{5 + 2\sqrt{5}} + \frac{\sqrt{5}}{4} \ln 5 - \frac{5 \ln 20}{4} - \frac{\sqrt{5} - 1}{2} \ln \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right) + \frac{1}{2} \left[\sqrt{5} \ln (7 - 3\sqrt{5}) + \ln (\sqrt{5} - 1) - \sqrt{5} \ln (3\sqrt{5} - 5) \right].$$

As $i \rightarrow \infty$, the events become equiprobable. Roughly speaking, if, instead of taking the first, we take for instance the millionth digit, then the probability distribution of the ten occurrences $(0, \dots, 9)$ is uniform.

In fact, see [6] p. 165, the following series expansion holds:

$$\psi(s) - \psi(s-r) = \sum_{n=1}^{\infty} \frac{(r)_n}{n(s)_n},$$

where $(s)_n = s(s+1) \cdots (s+n-1)$ is the Pochhammer symbol, so that for $k, \alpha, \beta > 0$,

$$k[\psi(\alpha+k) - \psi(\beta+k)] = k \left[\frac{\alpha - \beta}{\alpha + k} + \frac{(\alpha - \beta)(\alpha - \beta + 1)}{2(\alpha + k)(\alpha + k + 1)} + \cdots \right],$$

and then

$$\lim_{i \rightarrow \infty} p_r^{(i)} = \frac{1}{20} + \lim_{i \rightarrow \infty} \frac{10^i}{20} \left\{ \psi \left(\frac{r + 10^i + 1}{10} \right) - \psi \left(\frac{r + 10^i}{10} \right) \right\} = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}.$$

3. An elementary proof of both $\frac{n}{d}$ theorems

In this section Cesàro's statements will be proved by methods of elementary probability theory, and then they will be extended to the case of the quotient of positive real numbers with respect to an arbitrary base $b \geq 2$. Choose at random two real positive numbers n, d . We provide an alternative definition of the probability that the i th digit of the ratio n/d will be $r \in \{0, \dots, b-1\}$. There being a choice made in the whole continuum of possible outcomes, we cannot make use, as for the discrete case, of the ratio of two cardinal numbers. In such a way the definition can be applied to an arbitrary base b and not only to a simple ratio but

also to a power $\alpha > 0$ of such a ratio. Given two numbers $(n, d) \in \mathbb{R}_+^2$, we need to define the probability ${}^b_\alpha p_r^{(i)}$ that the ratio $(\frac{n}{d})^\alpha$ has, to base $b \geq 2$, the i th digit r . To do this first we point out formally how to extract the i th digit in base b of a given real number. Given an integer $b \geq 2$ called the base and using the b digits $c_i \in \{0, 1, \dots, b-1\}$, we have a representation of a positive real number v in base b as

$$v = \sum_{i=-\infty}^{\infty} c_i b^i,$$

Observe that there exists $N \in \mathbb{N}$ such that $i > N \implies c_i = 0$ thus

$$\sum_{i=0}^N c_i b^i \quad \text{and} \quad \sum_{i=-\infty}^{-1} c_i b^i$$

are respectively the integer part and the fractional part of v . The coefficient c_{-i} in front of b^{-i} , $i = 1, 2, \dots$ is called the i th digit in the base b representation and will be denoted $d_i^b(v)$. Thus

$$d_i^b(v) = r \in \{0, 1, \dots, b-1\}$$

if and only if there exists $k = 0, 1, \dots$ such that $b^{i-1}v \in [k + \frac{r}{b}, k + \frac{r+1}{b}[$.

Definition 1. Let $d_i^b(v)$ be the i th decimal digit for the real number v ; then, given two real positive numbers n, d and $\alpha > 0$ we define the probability ${}^b_\alpha p_r^{(i)}$ that the i th decimal digit to base b of the ratio $(n/d)^\alpha$ to be $r \in \{0, \dots, b-1\}$ as

$${}^b_\alpha p_r^{(i)} = \lim_{T \rightarrow \infty} \frac{\mu\{(n, d) \in \mathbb{R}_+^2 \mid n, d \leq T, d_i^b\left(\frac{n^\alpha}{d^\alpha}\right) = r\}}{\mu\{(n, d) \in \mathbb{R}_+^2 \mid n, d \leq T\}}$$

if the limit exists. Here μ is the Lebesgue measure on \mathbb{R}_+^2 .

We will keep the notation $p_r^{(i)}$ whenever $\alpha = 1$ and $b = 10$ and the notation p_r if $i = 1$, $\alpha = 1$, $b = 10$.

As a first step we will, by means of the series expansion (5) of ψ , rearrange (10). In fact from (5) we deduce the following relation, which holds for $b, x > 0$:

$$\psi\left(\frac{x+1}{b}\right) - \psi\left(\frac{x}{b}\right) = b \sum_{n=0}^{\infty} \frac{1}{(x+nb)(x+1+nb)} \quad (13)$$

In such a way, using (13) we see that (10) can be written using a numerical series:

$$p_r = \frac{1}{20} + 5 \sum_{k=0}^{\infty} \frac{1}{(10 + 10k + r)(11 + 10k + r)}. \quad (14)$$

This fact inspired us in looking for an alternative proof to the Cesàro statement, we are going to present in the following theorem. We take hereby the opportunity to thank the anonymous referee for his/her fruitful suggestions which simplify and clarify our original proof.

Theorem 1. *For a couple of real positive numbers n, d taken at random and represented in base b , the probability ${}_b p_r^{(i)}$ that the i th decimal digit of n/d will be $r \in \{0, 1, \dots, b-1\}$ can be computed by the formula*

$$\begin{aligned} {}_b p_r^{(i)} &= \frac{1}{2b} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{b^i}{(b^i + bk + r)(b^i + bk + r + 1)} \\ &= \frac{1}{2b} + \frac{b^{i-1}}{2} \left\{ \psi \left(\frac{b^i + 1 + r}{b} \right) - \psi \left(\frac{b^i + r}{b} \right) \right\}. \end{aligned} \quad (15)$$

Proof. We have

$$\left\{ (n, d) \in]0, T]^2 \mid d_i^b(n/d) = r \right\} = \bigcup_{k=0}^{\infty} \left\{ (n, d) \in]0, T]^2 \mid \frac{kb+r}{b^i} \leq \frac{n}{d} < \frac{kb+r+1}{b^i} \right\}.$$

This union is split as $U \cup L$, where

$$U = \bigcup_{k=0}^{b^{i-1}-1} \left\{ (n, d) \in]0, T]^2 \mid \frac{kb+r}{b^i} \leq \frac{n}{d} < \frac{kb+r+1}{b^i} \right\}$$

and

$$L = \bigcup_{k=b^{i-1}}^{\infty} \left\{ (n, d) \in]0, T]^2 \mid \frac{kb+r}{b^i} \leq \frac{n}{d} < \frac{kb+r+1}{b^i} \right\}$$

Let us explain the split. Reasoning in the plane (n, d) for $0 \leq k \leq b^{i-1} - 1$, each of the subsets forming U succeeds in being a triangle with the base on the horizontal straight line of equation $d = T$ bounded by the straight lines with slopes ≥ 1 . If, on the contrary, $k \geq b^{i-1}$, each of the subsets forming V is a triangle based on the vertical straight line $n = T$ and sided by the straight lines with slopes < 1 , as shown in the figure below.

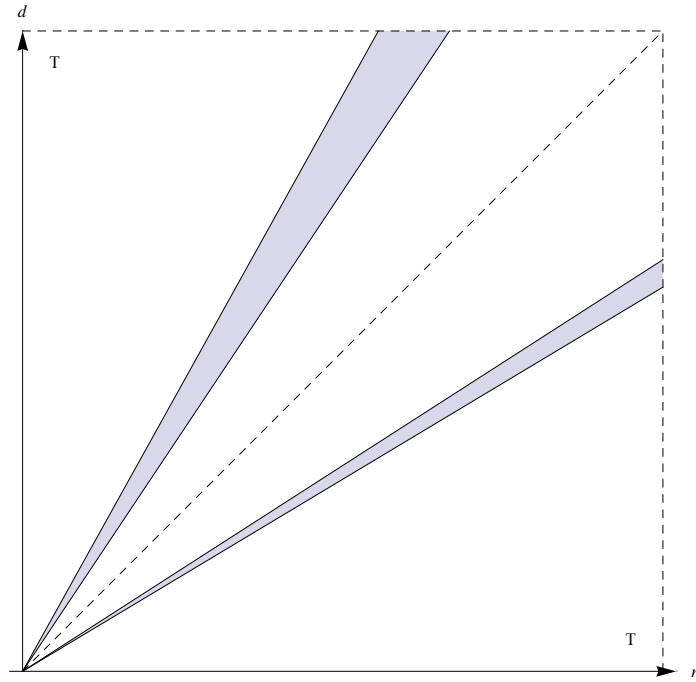


Figure 1: Reference (n, d) : the shaded areas sketch the standard subsets making up the set U (on the left) and the set L (on the right).

For $k = 0, \dots, b^{i-1} - 1$ the k th set in U is the triangle with vertices in the (n, d) -plane

$$(0, 0), \quad \left(\frac{kb+r}{b^i} T, T \right), \quad \left(\frac{kb+r+1}{b^i} T, T \right),$$

hence its Lebesgue measure is $T^2/(2b^i)$.

For $k = b^{i-1}, \dots$ the k th set in L is the triangle with vertices

$$(0, 0), \quad \left(T, \frac{b^i}{kb+r} T \right), \quad \left(T, \frac{b^i}{kb+r+1} T \right),$$

with Lebesgue measure

$$\frac{b^i T^2}{2(kb+r)(kb+r+1)}.$$

Therefore,

$$\begin{aligned} {}_b p_r^{(i)} &= \sum_{k=0}^{b^{i-1}-1} \frac{1}{2b^i} + \sum_{k=b^{i-1}}^{\infty} \frac{b^i}{2(kb+r)(kb+r+1)} \\ &= \frac{1}{2b} + \frac{b^i}{2} \sum_{n=0}^{\infty} \frac{1}{(b^i+bn+r)(b^i+bn+r+1)} \end{aligned}$$

This shows the first of (15), while the second relation follows from (13). \square

Remark 1.1. The theorem given above doesn't prove the statement of Cesàro, which concerns integer variables (discrete case) whilst our outcome concerns all the real ones (continuous case). Nevertheless it would be not difficult to retrace the original statement in combination with Gauss lattice point counting technique, see [8]. In fact, considering the lattice's "integer" points inside the triangles described by the sets U and L , following the approach given in [8] section 1.1, theorems 1.1 and 1.4, one can prove the number of integer points inside the triangle equates its area plus a term which is growing as the perimeter of the area itself: and then negligible with respect to the area when $T \rightarrow \infty$. In Theorem 1 the first part of the sum is found to be:

$$\sum_{k=0}^{b^{i-1}-1} \left(\frac{T^2}{2b^i} + O(T) \right)$$

dividing to T^2 , it goes to $1/2b$ for $T \rightarrow \infty$. On the contrary, the second part of the sum requires further attention, being an addition of infinitely many terms. There is no restriction in limiting the series from b^{i-1} to $\lfloor \sqrt{T} \rfloor$ and then to put T to infinite. In such a way, see e.g. the Theorem 1, the number of the points inside the lattice inside the triangles of the set L will be:

$$\lim_{T \rightarrow \infty} \sum_{k=b^{i-1}}^{\lfloor \sqrt{T} \rfloor} \left(\frac{b^i T^2}{(kb+r)(kb+r+1)} + O(T) \right)$$

When dividing to T^2 the expression

$$\sum_{k=b^{i-1}}^{\lfloor \sqrt{T} \rfloor} O(T) = O(T^{\frac{3}{2}})$$

passing to the limit, tends to 0. We take the opportunity to thank again the anonymous referee, for having pointed us this remark.

4. The $(\frac{n}{d})^\alpha$ theorems

In this section we can see how the same approach can easily lead to the any decimal digit of n^α/d^α for α real and positive.

Let us state in advance something about the *Hurwitz zeta* function. It will be recalled that the *Riemann zeta* function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re}(s) > 1$$

where s is a complex variable.

The Hurwitz zeta function $\zeta(s, x)$, generalizing Riemann's, is defined by

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad (16)$$

where $\operatorname{Re} s > 1$ and $\operatorname{Re} x > 0$. We have $\zeta(s) = \zeta(s, 1)$. Recall the Hurwitz ζ integral representation

$$\zeta(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-xt}}{1 - e^{-t}} t^{s-1} dt, \quad (17)$$

which holds for $\operatorname{Re} s > 1$ and $\operatorname{Re} x > 0$. It is known that the Hurwitz zeta function can be meromorphically continued to be defined for all complex numbers s with $s \neq 1$. At $s = 1$ it has a simple pole with residue 1. Its meromorphic continuation becomes explicit for $s \neq 1$ and $x > 0$ by means of Hermite's integral representation theorem:

$$\zeta(s, x) = \frac{1}{2}x^{-s} + \frac{1}{s-1}x^{1-s} + 2 \int_0^{\infty} \frac{\sin(s \arctan \frac{t}{x})}{(x^2 + t^2)^{s/2} (e^{2\pi t} - 1)} dt \quad (18)$$

More information about $\zeta(s, x)$ can be found in [13] chapter 13 or in [1].

The Hurwitz zeta function is linked to the polygamma function if its first variable is a positive integer, i.e., $n \in \mathbb{N}$:

$$\psi_n(z) := \frac{d^n}{dz^n} \psi(z) = (-1)^{n+1} n! \zeta(n+1, z). \quad (19)$$

In what follows, we will need the integration formula

Lemma 1. *Let $s \neq 1$ and $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ be such that $a + b \in \mathbb{R}^+$. Then*

$$\int_0^1 \zeta(s+1, a+bx) dx = \frac{1}{sb} \{ \zeta(s, a) - \zeta(s, a+b) \}. \quad (20)$$

Proof. Formula (20) follows by:

$$\int \zeta(s, x) dx = \frac{1}{1-s} \zeta(s-1, x)$$

which, in turn, follows from the relationship:

$$\frac{\partial \zeta}{\partial x}(s-1, x) = (1-s) \zeta(s, x),$$

which is a consequence of (16) □

Finally for $b > 0$, $\text{Re}(s) > 1$, $\text{Re}(x) > 0$ we will employ this relation which is immediate consequence of the definition of $\zeta(s, x)$

$$\sum_{n=0}^{\infty} \frac{1}{(bn+x)^s} = \frac{1}{b^s} \zeta\left(s, \frac{x}{b}\right) \quad (21)$$

We then can present our generalization of Cesàro's results.

Theorem 2. *For a couple of positive real numbers n, d taken at random and represented in base b , the probability ${}_b p_r^{(i)}$, with $\alpha > 0$, that the i th decimal digit of n^α/d^α will be $r \in \{0, 1, \dots, b-1\}$ is given by*

$$\begin{aligned} {}_b p_r^{(i)} &= \frac{1}{2^{\alpha/b^i}} \sum_{k=0}^{b^{i-1}-1} \left[(kb+r+1)^{1/\alpha} - (kb+r)^{1/\alpha} \right] \\ &+ \frac{1}{2} \frac{\alpha}{\sqrt[b^i]{b^i}} \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt[b^i]{b^i + bk + r}} - \frac{1}{\sqrt[b^i]{b^i + bk + r + 1}} \right) \end{aligned} \quad (22)$$

Moreover when $\alpha < 1$ in equation (22) the infinite series can be written in terms of Hurwitz zeta function:

$$\begin{aligned} {}_b p_r^{(i)} &= \frac{1}{2^{\alpha/b^i}} \sum_{k=0}^{b^{i-1}-1} \left[(kb+r+1)^{1/\alpha} - (kb+r)^{1/\alpha} \right] \\ &+ \frac{\alpha \sqrt[b^{i-1}]{b^{i-1}}}{2} \left\{ \zeta\left(\frac{1}{\alpha}, \frac{b^i+r}{b}\right) - \zeta\left(\frac{1}{\alpha}, \frac{b^i+r+1}{b}\right) \right\} \end{aligned} \quad (22b)$$

Proof. The proof is similar to Theorem 1. In this case we have

$$\begin{aligned} & \left\{ (n, d) \in]0, T]^2 \mid d_i^b (n^\alpha / d^\alpha) = r \right\} \\ &= \bigcup_{k=0}^{\infty} \left\{ (n, d) \in]0, T]^2 \mid \frac{kb+r}{b^i} \leq \frac{n^\alpha}{d^\alpha} < \frac{kb+r+1}{b^i} \right\} \\ &= \bigcup_{k=0}^{\infty} \left\{ (n, d) \in]0, T]^2 \mid \sqrt[\alpha]{\frac{kb+r}{b^i}} \leq \frac{n}{d} < \sqrt[\alpha]{\frac{kb+r+1}{b^i}} \right\}. \end{aligned}$$

As for the previous case in Theorem 1, this union can be split as $U \cup L$, where

$$U = \bigcup_{k=0}^{b^{i-1}-1} \left\{ (n, d) \in]0, T]^2 \mid \sqrt[\alpha]{\frac{kb+r}{b^i}} \leq \frac{n}{d} < \sqrt[\alpha]{\frac{kb+r+1}{b^i}} \right\}$$

and

$$L = \bigcup_{k=b^{i-1}}^{\infty} \left\{ (n, d) \in]0, T]^2 \mid \sqrt[\alpha]{\frac{kb+r}{b^i}} \leq \frac{n}{d} < \sqrt[\alpha]{\frac{kb+r+1}{b^i}} \right\}$$

For $k = 0, \dots, b^{i-1} - 1$ the k th set in U is the triangle with vertices in the (n, d) -plane

$$(0, 0), \quad \left(\sqrt[\alpha]{\frac{kb+r}{b^i}} T, T \right), \quad \left(\sqrt[\alpha]{\frac{kb+r+1}{b^i}} T, T \right),$$

hence its Lebesgue measure is

$$\frac{T^2 (\sqrt[\alpha]{kb+r+1} - \sqrt[\alpha]{kb+r})}{2 \sqrt[\alpha]{b^i}}.$$

For $k = b^{i-1}, \dots$ the k th set in L is the triangle with vertices

$$(0, 0), \quad \left(T, \sqrt[\alpha]{\frac{b^i}{kb+r}} T \right), \quad \left(T, \sqrt[\alpha]{\frac{b^i}{kb+r+1}} T \right),$$

with Lebesgue measure

$$\frac{1}{2} \left(\sqrt[\alpha]{\frac{b^i}{kb+r}} - \sqrt[\alpha]{\frac{b^i}{kb+r+1}} \right) T^2$$

Therefore summing the measures of the disjointed events in order to detect the required probability we find the first formula for ${}^b_\alpha P_r^{(i)}$ given in (22). To end the proof and get the formula (22b) for $\alpha < 1$, first notice that the series (22) converges also for this set of α 's, then use the following identity $a, k, \beta > 0$

$$\frac{1}{(a+k)^\beta} - \frac{1}{(1+a+k)^\beta} = \int_0^1 \frac{\beta}{(1+a+k-x)^{\beta+1}} dx$$

then summing on k and interchanging summation and integration, we see that

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt[\alpha]{b^i + bk + r}} - \frac{1}{\sqrt[\alpha]{b^i + bk + r + 1}} \right) \\ &= \frac{1}{\alpha} \int_0^1 \sum_{k=0}^{\infty} \frac{1}{(b^i + r + 1 + kb - x)^{\frac{1}{\alpha} + 1}} dx \\ &= \frac{1}{\alpha b^{\frac{1}{\alpha} + 1}} \int_0^1 \zeta \left(1 + \frac{1}{\alpha}, \frac{b^i + r - x + 1}{b} \right) dx \end{aligned}$$

Formula (22b) now follows using (21) and (20). \square

Remark 2.1. As for Theorem 1, we proved in such a way the theorem in the case of a continuous variable. For passing to the discrete one, we will refer once again to the theory of lattice points in plane domains, [8]. We get:

$$\sum_{k=0}^{b^i - 1} \left(T^2 \frac{\sqrt[\alpha]{kb + r + 1} - \sqrt[\alpha]{kb + r}}{2 \sqrt[\alpha]{b^i}} + O(T) \right)$$

and again dividing to T^2 , the last term will vanish for $T \rightarrow \infty$.

The same argument concerning the infinite series spent about Theorem 1 can be applied to the set L of Theorem 2.

Let us close with some points to note.

If in (22) we take the limit $\alpha \rightarrow 1$, (15) shall by necessity be obtained, and this is done easily using the expressions containing infinite series. If one instead wants to use the special functions representations, the pole at $s = 1$ of $\zeta(s, x)$ can be bypassed: in fact if x, y are two positive real numbers, then

$$A_{xy}(s) := \zeta(s, x) - \zeta(s, y)$$

is continuous in $s = 1$ attaining there the value

$$A_{xy}(1) = \psi(y) - \psi(x). \quad (23)$$

In fact, expressing A_{xy} by means of the integral representation (17) of $\zeta(s, \cdot)$, we have

$$A_{xy}(s) = \zeta(s, x) - \zeta(s, y) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt} - e^{-yt}}{1 - e^{-t}} t^{s-1} dt$$

the limit under the integral for $s \rightarrow 1$ is allowed due to the integrand's summability (including the origin). So that, by the first part of (7):

$$A_{xy}(1) = \int_0^\infty \frac{e^{-xt} - e^{-yt}}{1 - e^{-t}} dt = \int_0^\infty \frac{e^{-xt} - e^{-t} + e^{-t} - e^{-yt}}{1 - e^{-t}} dt = \psi(y) - \psi(x).$$

Then, by means of (23), we can put $\alpha = 1$ in (22), finding (15) again.

Whilst in Cesàro's formulae the digamma appears, in ours we use the Hurwitz ζ function: this depends on the non-integrality of the first argument of the generalized zeta function. Therefore if in (22b) one puts $\alpha = 1/n$ with $n \in \mathbb{N}$, then thanks to (19) it would be possible to express (22b) by a digamma.

Conclusions

In 1885 E. Cesàro considered in [4] the probability that a whichever digit r , *zero included*, could occur as the first when dividing two integers taken at random, say n and d . Such a probability was stated without proof as depending on the integral

$$\int_0^1 \frac{1 - \varphi}{1 - \varphi^{10}} \varphi^{9+r} d\varphi, \quad r = 0, 1, \dots, 9$$

which is a non-trivial function of r . In our article, an elementary proof is given of such a formula in the case of two arbitrary chosen positive real numbers; next, the integral is computed by reducing it to the Euler ψ function, namely the logarithmic derivative of the Γ function. Such a law has been tested by computer, taking random couples of positive integers between 1 and 10^7 , performing the division, repeating 10^7 times, recording the relative frequency of each digit. By comparison of such values to the computed ones, we found a difference for each digit of the order of $\pm 10^{-6}$, which has been successively confirmed when arranging a similar test of the second digit. In fact further theorems have been obtained generalizing such a result to the i th decimal digit of n/d and to the case of numbers in a whichever base b . Further generalizations, concerning the probabilistic occurrence to the second power and finally to the α th power of n/d , are proved by using the Hurwitz ζ function by means of theorems about that function whose role was long ago established through number theory.

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