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The importance of "extensive teaching" in the education of prospective teachers of mathematics

DE MARCHIS Martina (IT), MENGHINI Marta (IT), ROGORA Enrico (IT)

Abstract. In recent discussions on university curricula we fostered the idea that an *extensive teaching* of mathematics is needed in the education of prospective teachers. Most universities only provide courses on methodological didactical issues - if any - or *intensive teaching*, more suitable for prospective researchers. The goal of this paper is to clarify the meaning of extensive teaching, in the sense already considered in the work of Guido Castelnuovo, and to discuss its relevance today.

Keywords: math teacher training, extensive mathematics teaching, history of education, Guido Castelnuovo

Mathematics Subject Classification: Primary 00A35, 97B50; Secondary 01-XX

Introduction

At the beginning of his lectures for the course in higher geometry for the academic year 1910-11, Guido Castelnuovo[1] pondered about the training of prospective mathematics teachers, wondering if the programs of the higher courses for the degree in mathematics[2] were adequate to meet both the needs of prospective researchers and prospective teachers. He emphasized that the two needs are different hence the ways of teaching the mathematical contents of the courses should be, at least in part, diversified.
In particular, he criticized the fact that the need for mathematical content specifically designed for prospective teachers was not considered, both as regards the chosen topics and,

[1] Guido Castelnuovo (1867-1952) was one of the most representative Italian mathematicians of the "golden age" of Italian mathematics, that is, the period between the late nineteenth and the early twentieth centuries. Together with Corrado Segre and Federigo Enriques he is considered one of the fathers of Italian algebraic geometry. The passionate interest of Castelnuovo for the problems of teaching and of the preparation of high school teachers was an important part of his constant civil commitment [25], as evidenced by the notebooks of the lessons of Higher Geometry he held at the University of Rome and his numerous interventions in teachers' magazines [18].
[2] The structure of the degree course in mathematics at the time of Castelnuovo included a first preparatory biennium and a subsequent biennium of specialization. The first biennium is comparable to the current three-year bachelor's degree while the second biennium is comparable to a master's degree.

above all, the way of dealing with them. We believe that after more than a century the considerations made by Castelnuovo on how to teach Mathematics to prospective Mathematics teachers are still actual and interesting, at a time when Mathematics teaching and the Mathematics education of the citizen are considered, not only in Italy, a national priority.

> *Most people believe that mathematics is a strategic matter for today's economy and certain studies claim, for example, that mathematics contributes for around 15% of the gross national product of France. We live in a period of extraordinary accelerated changes: new knowledges; new technologies; new communication media - all directly connected with mathematics. Quantitative information, once reserved to specialists, is nowadays largely spread by media. The need to understand and use mathematics in every day's life and working has never been so urgent and it does not stop increasing. In a rapidly and continuously changing world, students need a solid background in mathematics. Mathematics provides essential instruments for an active citizenship. Mathematics is necessary for democracy since it favours autonomy and innovation*[3]. [28], p. 31.

1 Intensive and extensive teaching according to Castelnuovo

According to Castelnuovo, the different opinions on the most effective way to teach prospective teachers

> *can be grouped into the following three trends:*
> *a) intensive culture in the higher branches of mathematics; the teaching attitude will develop by itself;*
> *b) broad extensive culture in all mathematical addresses and in the sciences, which have the greatest affinities with mathematics;*
> *c) specific, methodological culture*[4]. [6], notebook 1910/11, part a), pp. 5-7.

Intensive teaching consists of exhausting all details of a well-defined and limited topic and proceeding from general to particular. Castelnuovo insisted on the inconveniences of teaching mathematics exclusively according to an intensive viewpoint to students who do not intend to become researchers. We also believe that following exclusively this method of teaching mathematics to prospective teachers, which seems to prevail today in the courses of the Master's Degree, is not effective. This does not mean that a future teacher should not know advanced topics, indeed he/she should know more than those that are taught to the future researcher but, in our opinion, these should be taught extensively, in accordance with the trend b) considered by Castelnuovo: that is, giving priorities to the connections (also with the other sciences), the general ideas and the examples rather than the deep knowledge of the most advanced techniques and to the search of the maximum generality. As Federigo Enriques wrote in [10], referring to the "science" that the teacher must possess

> *It is clear that the knowledge of the master must be much higher for this* [new role assigned to teachers by Gentile reform]. *It is not enough for him to know the results achieved by the heroes of thought, but it is required that he is able to relive in some way*

[3] Translated from French. All translations, except when explicitly noted, are provided by the authors.
[4] That is, the one transmitted with specific teachings on education, both general and disciplinary.

their work; that is, to understand the reasons for the problems, in the broader scientific context to which it belongs, rather than its solution. [10], p. 69.

Finally, regarding the specific educational knowledge, Castelnuovo suggests that it is not enough to cultivate it through the study of educational methodologies but that it must also be stimulated through the study of higher mathematical contents, treated in an extensive manner, i.e.

in the second biennium, [...] lessons will be held that aim to highlight the relationships between elementary and high school mathematics, to broaden the ideas of prospective teachers and to put the topics that are taught under the correct light. [6], notebook 1910/11, part a), p. 7.

Thus, pedagogical knowledge is not sufficient for future teachers and has to be completed by a specifically oriented mathematical knowledge.
Castelnuovo also believes that, in addition to dedicate suitable space to the extensive study of mathematics, space must be also given to enforce the relationships between mathematics and the sciences "which have the greatest affinities with mathematics". It is once again Enriques who reminds us in [9] of the importance of deepening these ties and some of the reasons for doing so.

In the concerns of our mathematical educators, "small logic" [takes] too much place in comparison to "big logic"! This depends, first of all, on the unsuitable separation that we usually make between mathematics and physics. And also, on the overly analytical dress of most of our teachers, which is also the result of a particularistic education. In my opinion what has to be required to mathematical teaching, conceived as formative of logical faculties, is first of all to carry out the "spirit of coordination", in that form which I have called macroscopic. This requires the teacher to pay constant attention to link the different parts of his teaching to each other: lessons isolated from each other, chapters that succeed one after the other without ever recalling the connections, even if carefully studied in their minutes particulars, will miss the purpose [9], p. 11.

The notebook [6] from which we extracted the reflections of Castelnuovo is that of the lessons of a course on non-Euclidean geometries (1910-11). According to Castelnuovo, this subject is particularly "suited to the purpose" of teacher training for the following reasons, which articulate its "extensive teaching" proposal in an operational manner:

i) Importance of the course in terms of educational goals; the topics are strictly related to secondary education, issues that no teacher should ignore[5].
ii) Importance of the course with respect to the relationship with (mathematical) culture; the various methods with which non-Euclidean geometry was studied (elementary; differential; projective; group-theoretic).
iii) Philosophical interest of the course: both from a logical point of view (questions about the independence and compatibility of postulates); and from the point of view of Physics (origin of the postulates; nature of the space) [6] notebook 1910/11, part a), p.7-9.

[5] Castelnuovo refers to axiomatic-deductive teaching of Euclidean geometry, fundamental in Italy in that period and still relevant today. The discussion of tentative proofs of Euclid's 5th postulate is the starting point for an elementary treatment of non-Euclidean geometries.

We believe that, in the century which separates us from these considerations, the teaching of the disciplinary contents of mathematics courses has increasingly gone in the direction of specialization leaving less and less space to cultural training needs of prospective teachers. We think that Castelnuovo's point of view still suggests an alternative worth of being taken seriously.

2 Guido Castelnuovo's lectures for prospective teachers

In this section we analyze the contents of the Notebooks of the courses taught by Guido Castelnuovo for prospective teacher training. The notebooks are available on line at [6] They suggest topics that can be treated according to the "extensive teaching" approach discussed in the first section. The first thing we notice is the variety of contents, taught by Castelnuovo in different years. This helps us to better define the concept of extensive teaching: not a course covering all topics judged useful for future teachers (as unfortunately we tend to do today), but a course limited to one or two topics at most, to which Castelnuovo devotes a broad study, presenting the historical development of ideas and the connections with other parts of mathematics and affine subjects. It is clear that contents are less important than the approach used to deal with them, very different from the one employed for prospective researchers' training.

Along the years, Castelnuovo considered many themes, covering topics in Geometry, Analysis and Algebra.

The topics taught by Castelnuovo in his courses for prospective teachers are still very relevant today in teacher training, since they trace the development of ideas during 2000 years of mathematics history. We believe however that in order to help teachers to play in full their crucial role in today's society, it is appropriate to add some other topics to those considered by Castelnuovo. We will come back to these in section 3.

Castelnuovo began to pursue extensive teaching within his course of Higher Geometry, alternating extensive teaching and intensive teachings of curves and algebraic surfaces. Already in the academic year 1903-04 he taught Geometric theories according to Klein's synthesis, and he chose the same topic in (1915-16.) In (1910-11) and in (1919-20) he taught Non-Euclidean geometry. In (1913-14) he considered the comparison between Mathematics of Precision and Mathematics of Approximation, again following Klein's ideas.

The course on "geometric theories" is inspired by the one held by Klein several times at the beginning of the 20th century and then merged into the second volume of the *Elementarmathematik vom höheren Standpunkte aus* series [14]. It focuses on the various types of groups of transformations and on the classification of geometries based on the invariants and covariants of such groups. Castelnuovo puts special attention into highlighting the historical evolution of concepts and compare possibly different demonstrative methods, without forgetting the relative importance of intuitive and heuristic ones. He believes that an extensive knowledge of all this is necessary to balance student's mathematical culture and allow prospective teachers to make conscious methodological choices.

Castelnuovo's lectures on "Mathematics of Precision and Mathematics of Approximation" took inspiration from those given by Klein in 1901 [13], republished in 1928 with the title *Präzisions- und Approximationsmathematik*, as volume III of the series on elementary mathematics from a higher view point (translated in English in 2016, [15]). Castelnuovo dedicates these lectures to prospective teachers and explains further why they are suitable for their training:

[To "extensive teaching" we can also] connect the opportunity to highlight the relationships between pure and applied mathematics. In particular: how mathematical concepts are formed starting from the observation of the external world; and how mathematical results can in turn occur in reality. These issues are very important from the educational point of view. The educational value of mathematics would be greatly enriched if, in teaching, alongside the logical procedures that are used to derive the theorems from the postulates, one [explains] by means of which [procedures] these are derived from observation, and, on the other hand, with what [precision] the theoretical results occur in reality [6], notebook 1913/14, p.2.

Therefore, Castelnuovo remarks, mathematical logical reasoning is not enough to ensure the result of an experience and on the other hand, instinctive and empirical procedures are not always sufficient to justify mathematical results. This is the reason why we distinguish between *mathematics of precision*, which concerns all propositions that are logically deduced from the postulates of geometry or analysis, and *mathematics of approximation*, which considers the results obtained with the degree of approximation that comes from experience.
The previous distinction is well exemplified by the relationship between the empirical curve and the ideal curve, described in depth in the course of Castelnuovo and entirely inspired by Klein's treatment. Castelnuovo then retraces the history of the concept of function which gives a vivid illustration of how history of mathematics could provide suggestions and valuable insights with regard to delicate teaching problems.
In 1923-24 Castelnuovo begins teaching the new course of *Matematiche Complementari*, explicitly addressed to prospective teachers.
Castelnuovo's courses for teachers were conceived in the wake of the in-depth reflections made by Felix Klein on mathematics contents for future teachers. Klein wanted future teachers to realize how important the study of higher mathematics was for their profession, but he also wanted to stress the importance - for prospective teachers - to get a different and more extensive approach to mathematics.

My task will always be to show the mutual connection between problems in the various disciplines, these connections use not to be sufficiently considered in the specialized lecture courses, and I want more specially to emphasize the relation of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you [...] vivid stimuli for your teaching. [16], p. 2.

3 Further mathematical content for contemporary Mathematics teachers.

At the time of Guido Castelnuovo society was very different from ours and, in particular, the relationship between science, society and technology was less complicated. The role of teachers consequently underwent radical changes and the attention that must be paid to teacher training, in particular that of mathematics teachers, is crucial. In our opinion, the renewal of teacher training must focus on the methods of acquisition of disciplinary contents. We believe that the proposal of Castelnuovo to guarantee ample space for an "extensive disciplinary teaching" is today more important than ever to ensure teachers are adequately prepared to carry out their work fully and assume the social role that belongs to them, which requires in particular a critical knowledge of the tools necessary to unravel the complex relationships between mathematics, science and society.

We believe that there are several topics that today have become indispensable in the education of the mathematics teacher. It is not simply a question of conveying new concepts, a task that is very difficult for their vastness, but that of developing a critical attitude and an informed awareness of their cultural value.
For example, an extensive treatment of probability seems important and urgent, highlighting its cultural relevance, its intertwining with not strictly mathematical problems, the ubiquity of its application in modern science and the need to develop a critical attitude towards its applications.
Many sciences, which at the time of Castelnuovo were considered "distant" from mathematics, such as economics or biology and new sciences such as robotics, psychometrics, computer science and artificial intelligence, have come much closer to mathematics, thanks to the use of probabilistic models. The relationship between these sciences and their mathematical models raises, in addition to the problem of understanding the mathematical principles on which they are based, also essentially new problems such as those relating to the adequacy of these models and the evaluation of the limits of their applicability. These themes have a cultural value that transcends science and deeply invests the relationships between science and society as these sciences have an increasing weight in everyday life. An "extensive" and critical knowledge of mathematics on which the models of new sciences and their applications are based seems necessary to us so that teachers can be put in a position to help young people in understanding the complex social phenomena related to their uses. Some knowledge of mathematical models oriented to the prediction and elaboration of decision-making strategies seems unavoidable to us to take conscious and informed part in the politically and socially relevant debates concerning these issues. This knowledge is also essential to develop, in future teachers and their pupils, a well-trained critical sense which is crucial to watch over the correct use of these models, for example in “objective evaluation”, or in economy just to recall two of the many fields in which mathematics enters heavily in the decision making process and for which we refer to [7].

4 A possible program for an “extensive course of Mathematics for prospective teachers

We have recently been involved in the design of a new course for prospective teachers for the students of the Master Degree in Mathematics that will be given at Sapienza University of Rome starting from the academic year 2020-21. This course aims at providing “extensive mathematical content" in the sense explained in this paper. In this course we will pay particular attention in exploring the connections between different areas of Mathematics and in following the historical development of ideas and its value for the teaching of Mathematics at high school level. We shall also consider the connections between Mathematics and other sciences and give space to critical discussions on the role of Mathematics in contemporary world.

Now, we outline the main objectives of this course, named "*Istituzioni di Matematiche Complementari*", and give a tentative program in order to exemplify concretely what we mean by "extensive teaching".

The course is divided into three parts.

Part 1 is dedicated to developing a topic with several connections to other parts of Mathematics, science and culture and directly linked to high school Mathematics. A possible choice is the topic of non-Euclidean geometry which, as already pointed out by Castelnuovo, may be studied from different points of view, employing at least four different methodological approaches: elementary, differential, projective and group theoretical. Therefore, it provides plenty of opportunities to study the links between these methodologies and to appreciate the specificity of each of them. Non-Euclidean geometry has also natural links with Physics, especially with the Special and General Theory of Relativity [11]. Many links of Non-Euclidean geometry with culture may be found in art, for example in connection with Escher’s paintings and philosophy, for its role in developing the concept of space. Part of the lectures will be devoted to read and discuss excerpts from [17], [4], [12] and [3].

Part 2 will be dedicated to developing a topic which highlights the social role of Mathematics in contemporary world and the necessity of a critical approach to its applications. A possible topic is science and pseudo-science. Its treatment may begin with Poincaré's memorial on the Dreyfus case, continue with a critical comparison between physical measures and psychometric measures and finish with studying Rasch model for test analysis and the consequences of its adoption for the politics of education [22,23]. Part of the lectures will be devoted to read and discuss excerpts from [19], and [20].

Part 3 will be dedicated to developing a topic which shows how an historical approach to some fundamental conceptual nodes can be successfully related to the teaching / learning of Mathematics. For example, the concept of function. Part of the lectures will be devoted to read and discuss excerpts from [21].

Another characteristic aspect of this course is the adoption of a laboratory approach in dealing with some topics, geared to develop prospective teachers’ familiarity with this kind of methodology [27]. The methodology is mainly used during the discussions of the above-mentioned extracts from original memories and during computer lab sessions. For Part 1 we use GeoGebra to implement and explore the Beltrami Poincaré model and *Mathematica* to make computations in differential geometry and to produce images of curves and surfaces in space. For part 2, we use the statistics software R to simulate hidden state probabilistic models and to perform test analysis according to Rasch model. For Part 3, we use *Mathematica* for familiarize with functional programming, computable functions and recursive functions. The links between theory and computer lab sessions and the discussion about using mathematical software in teaching add further layers to “extensive teaching” of mathematics.

5 Conclusions

Teacher training should anyway not be limited to the university period but should continue throughout the working period. “Extensive learning” of mathematics that teachers should have experienced in university courses should be followed by "extensive teaching" of mathematics to their students, promoting an extensive search of links and coordinated development with all topics covered at school (see [5]). In recent years we have experienced an "extended mathematics" for schools in the project “Liceo Matematico” [5]. The project and realization of activities for the classes which joined the project, designed and discussed in working groups made up of high school teachers and university researchers, becomes an ongoing training for teachers, which certainly does not replace university education but traces its spirit with new topics.

References

[1] BALL, D. L., BASS, H., 2000. “Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics”. In *Multiple prospectives on mathematics teaching and learning*, di J. Boaler, 83-104. Wetsport, CT: Ablex.

[2] BEGLE, E. G., 1979. *Critical variables in mathematics education: Findings from a survey of the empirical literature.* Washington: Mathematical Association of America and National Council of Teachers of Mathematics.

[3] BELTRAMI, E., 1868. “Saggio di interpretazione della Geometria non Euclidea”. *Giornale di Matematiche* 284-312.

[4] BOLYAI, J., 1832. “Appendix”, in BOLYAI, F., *Tentamen juventutem studiosam in elementa matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentiaque huic propria, introducendi : cum appendice triplici, Vol I*, 502-527. Maros-Vásárhely: Typis Collegii Reformatorum per Josephum, et Simeonem Kali de felsó Vist.

[5] CAPONE, R., ROGORA E., TORTORIELLO F.S., 2017. “La matematica come collante culturale nell'insegnamento”. *Matematica, Cultura e Società* 293-304.

[6] CASTELNUOVO, G.. *Quaderni delle Lezioni.* http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menuQ3.htm.

[7] DE MARCHIS, M., ROGORA E., 2017. “Attualità delle riflessioni di Guido Castelnuovo sulla formazione dell'insegnante di Matematica”. *Periodico di Matematiche* 71-79.

[8] EISENHART M., BORKO H., UNDERHILL R. G., BROWN C. A., JONES D., AGARD P. C., 1993. “Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding” *Journal for Research in Mathematical Education* 8-40.

[9] ENRIQUES F., 1921. “Insegnamento dinamico.” *Periodico di Matematiche* I (S. 4): 6-16.

[10] ENRIQUES, F., 1928. “La riforma Gentile e l'insegnamento della Matematica e della Fisica nella scuola media”. *Periodico di Matematiche* VIII (S. 4): 68-73.

[11] FANO G., 1936. *Geometria non euclidea (Introduzione geometrica alla Teoria della Relatività)*. Bologna: Zanichelli.

[12] GAUSS K. F., 1828. *Disquisitiones generales circa superficies curvas.* Gottinga: Typis Dieterichianis.

[13] KLEIN F. 1902. *Anwendung der Differential und Integralrechnung auf die Geometrie: eine Revision der Prinzipien: Vorlesung gehalten während des Sommersemesters 1901*. Leipzig: Teubner.

[14] KLEIN F., 1909. *Elementarmathematik vom höheren Standpunkte aus: Geometrie (Vol. 2)*. Leipzig: Teubner.

[15] KLEIN F., 2016. *Elementary Mathematics from a Higher Standpoint, Volume III: Precision Mathematics and Application Mathematics* (Menghini M. transl.). Heidelberg: Springer.
[16] KLEIN F., 2016. *Elementary Mathematics from a higher standpoint. Volume I: Arithmetic, Algebra, Analysis* (G. Schubring, trans.). Berlin: Springer.
[17] LOBACHEVSKY N. I., 1840. *Geometrische Untersuchungen zur Theorie der Parallellinien.* Berlin: in der Fincke'schen Buchhandlung.
[18] MENGHINI M., 2016. "Guido Castelnuovo e l'insegnamento della Matematica." *Rendiconti di Matematica e delle sue Applicazioni* 37: 185-197.
[19] POINCARÉ H., DARBOUX G., APPELL P. E.. 1908. "Rapport de MM. les experts". In *Affaire Dreyfus: La Revision du Procès de Rennes, Tome Troisième*.
[20] RASCH G., 1980. *Probabilistic models for Some Intelligence and Attainment Tests*. Chicago: University of Chicago Press.
[21] RIEMANN B., 1873. "Sur la possibilité de représenter une fonction par une sèrie trigonométrique". *Bullettin de Sciences Mathematiques et astronomique.*
[22] ROGORA E., 2014. "Il modello di Rasch". *ROARS review.* https://www.roars.it/online/il-modello-di-rasch/
[23] ROGORA, E., 2014. "Ancora su Invalsi, test di apprendimento e modello di Rasch". *ROARS Review.* https://www.roars.it/online/ancora-su-invalsi-test-di-apprendimento-e-modello-di-rasch/
[24] ROGORA, E., 2014. "Evaluating and choosing: the role of Mathematics". *Lettera Matematica* 161-164.
[25] ROGORA E. 2016. "Guido Castelnuovo e la matematica a Roma, tra Risorgimento e Belle Époque". *Rendiconti di Matematica e delle sue Applicazioni* 37: 219-233.
[26] SHULMAN L. S., 1987. "Knowledge and teaching: Foundations of the new reform". *Harvard Educational Review* 1-22.
[27] UMI-CIIM, "Riflessioni sul Laboratorio di Matematica". https://www.umi-ciim.it/materiali-umi-ciim/trasversali/riflessioni-sul-laboratorio-di-matematica/.
[28] VILLANI C., TOROSSIAN C., 2018. *21 mesures pour l'enseignement des mathématiques.* Paris: Ministère de l'éducation nationale.
[29] WHITELEY W., TAYLOR P., H. WHITE H "Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire." *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group, CMESG.* 81-84.
[30] WHITELEY W., TAYLOR P., H. WHITE H., 2003. "Types and Characteristics Desired of Courses in Mathematics Programs for Future (and In-Service) Teachers." *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group, CMESG.* 85-88.

Current address

De Marchis Martina, doc.
Institute of Education,
Edinburgh Napier University
17/2 Langton Road, EH93DA, Edinburgh
Scotland (UK)
E-mail: martinamdm23@gmail.com

Menghini Marta, Associate Professor

Faculty of Mathematics, Physics and Natural Sciences,
Sapienza University of Rome
p.le A. Moro 5, 00185 Rome, Italy
E-mail: marta.menghini@uniroma1.it

Rogora Enrico, Associate Professor
Faculty of Mathematics, Physics and Natural Sciences,
Sapienza University of Rome
p.le A. Moro 5, 00185 Rome, Italy
E-mail: enrico.rogora@uniroma1.it