COMBINATORIAL CHARACTERIZATION OF THE WEIGHT MONOIDS OF SMOOTH AFFINE SPHERICAL VARIETIES

GUIDO PEZZINI AND BART VAN STEIRTEGHEM

Abstract. Let G be a connected reductive group and X a smooth affine spherical G-variety, both defined over the complex numbers. A well known theorem of I. Losev's says that X is uniquely determined by its weight monoid, which is the set of irreducible representations of G that occur in the coordinate ring of X. In this paper, we use the combinatorial theory of spherical varieties and a smoothness criterion of R. Camus to characterize the weight monoids of smooth affine spherical varieties.

1. Introduction and main results

A natural invariant of a complex affine algebraic variety X equipped with an action of a complex connected reductive group G is its **weight monoid** $\Gamma(X)$. By definition, it is the set of (isomorphism classes of) irreducible representations of G that occur in the coordinate ring $\mathbb{C}[X]$ of X. In the 1990s, F. Knop conjectured that if X is a smooth affine spherical variety —i.e. if X is smooth and $\mathbb{C}[X]$ is multiplicity free as a representation of G— then $\Gamma(X)$ determines X up to equivariant automorphism. This conjecture was proved by I. Losev in [Los09a]. By work of Knop's [Kno11] it implies that multiplicity free (real) Hamiltonian manifolds (cf. [GS84]) are classified by their moment polytope and generic isotropy group. Recently, in [Kno16], Knop extended this result to multiplicity free quasi-Hamiltonian manifolds (cf. [AMM98]).

In this paper, we use the combinatorial theory of spherical varieties and a smoothness criterion of R. Camus [Cam01] to characterize the weight monoids of smooth affine spherical varieties. Our most general statement is Theorem 4.2. In this introduction we give a special case which is more elementary: in Theorem 1.12 we characterize the G-saturated weight monoids of smooth affine spherical varieties (see Definition 1.2).

As an application, we characterize in Theorem 1.16 when a semisimple and simply connected group G has a smooth affine model variety, i.e. a smooth affine G-variety in whose coordinate ring all irreducible representations of G occur with multiplicity one.

We point out that, for any given candidate weight monoid Γ, our criterion only requires finitely many elementary verifications. In fact, Theorem 4.2 can be implemented as an algorithm that given a set of generators of Γ decides whether Γ is the weight monoid of a smooth affine spherical variety. As part of his PhD thesis [Kim16] W. G. Kim has already implemented the case where $G = \mathrm{SL}(n)$ and Γ is G-saturated (as in Theorem 1.12) and free.

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Furthermore, thanks to [Kno11, Theorem 11.2] and [Kno16, Theorem 6.7], our main result gives a local combinatorial characterization of the moment polytopes of (real) multiplicity free Hamiltonian and quasi-Hamiltonian manifolds: by repeating the verifications of our criterion at every vertex of a candidate moment polytope $\mathcal{P}$, one can decide whether $\mathcal{P}$ is the momentum image of a multiplicity free (quasi-) Hamiltonian manifold.

We now describe our main result in the special case of G-saturated monoids. Fix a Borel subgroup B of G and a maximal torus T contained in B, denote by S the corresponding set of simple roots. Let U be the unipotent radical of B. When α is a root of (G, T), we will use $\alpha^\vee$ for the corresponding coroot. The weight lattice of G is denoted Λ. Recall that Λ is the character group of T, which we identify with the character group of B. The set of dominant weights of G with respect to B will be denoted Λ^+. Then Λ^+ is a finitely generated submonoid of Λ. We denote by $V(\lambda)$ the irreducible G-module of highest weight $\lambda \in \Lambda^+$. Given an affine G-variety X we identify its weight monoid $\Gamma(X)$ with a submonoid of Λ^+:

$$\Gamma(X) = \{\lambda \in \Lambda^+ \colon \operatorname{Hom}_G(V(\lambda), \mathbb{C}[X]) \neq 0\}.$$

For a subset $\mathcal{E}$ of Λ, we will write $\mathbb{Z}\mathcal{E}$ for the sublattice of Λ spanned by $\mathcal{E}$. We will use $\mathbb{N}\mathcal{E}$ for the submonoid (including 0) of Λ generated by $\mathcal{E}$. If $\mathcal{X}$ is a lattice, then we will write $\mathcal{X}^*$ for the dual lattice $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Z})$.

Definition 1.1. Let Γ be a submonoid of Λ^+. We say that Γ is a **smooth weight monoid** if there exists a smooth affine G-variety X such that

$$\mathbb{C}[X] \cong \oplus_{\lambda \in \Gamma} V(\lambda) \tag{1.1}$$

as G-modules.

Definition 1.2. Let Γ be a submonoid of Λ^+. We say that Γ is **G-saturated** if the following equality holds in Λ:

$$\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma. \tag{1.2}$$

For the remainder of this section, Γ is a G-saturated submonoid of Λ^+. Readers eager to check whether their favorite G-saturated weight monoid Γ is smooth can directly jump to Theorem 1.12 and work their way backwards from there following the provided cross-references.

As will be shown in Corollary 2.28, if Γ is G-saturated there is a unique affine G-variety for which the equality (1.1) holds and which *can be* smooth. We will denote it X_Γ. It is the "most generic" affine spherical G-variety with weight monoid Γ. We recall that an irreducible (not necessarily affine or smooth) G-variety is **spherical** if it is normal and has an open B-orbit.

From Γ we derive the following data:

- the set of N-spherical roots of X_Γ, denoted by $\Sigma^N(\Gamma)$ and defined in Definition 2.29 (see also Proposition 2.30),
- the valuation cone of X_Γ , i.e. the set

$$\mathcal{V}(\Gamma) = \{\nu \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q}) \colon \langle \nu, \overline{\sigma} \rangle \leq 0 \text{ for all } \overline{\sigma} \in \Sigma^N(\Gamma)\}, \tag{1.3}$$

- a set of simple roots S_Γ.

Proposition 1.6 below tells us how to compute the set $\Sigma^N(\Gamma)$ from Γ, and S_Γ is defined in Proposition 1.7. We first introduce the relevant notions.

Table 1. spherically closed spherical roots

Type of support	σ
A_1	α
A_1	2α
$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha + \alpha'$
A_n, $n \geq 2$	$\alpha_1 + \ldots + \alpha_n$
A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$
B_n, $n \geq 2$	$\alpha_1 + \ldots + \alpha_n$
	$2(\alpha_1 + \ldots + \alpha_n)$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
C_n, $n \geq 3$	$\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$
D_n, $n \geq 4$	$2(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$4\alpha_1 + 2\alpha_2$
	$\alpha_1 + \alpha_2$

Definition 1.3. Let σ be an element of the root lattice Λ_R of G and let $\sigma = \sum_{\alpha\in S} n_\alpha \alpha$ be its unique expression as a linear combination of the simple roots. The **support** of σ is $\operatorname{supp}(\sigma) = \{\alpha \in S \colon n_\alpha \neq 0\}$. The **type** of $\operatorname{supp}(\sigma)$ is the Dynkin type of the subsystem generated by $\operatorname{supp}(\sigma)$ in the root system of G. The **set $\Sigma^{sc}(G)$ of spherically closed spherical roots of G** is the subset of $\mathbb{N}S$ defined as follows: an element σ of $\mathbb{N}S$ belongs to $\Sigma^{sc}(G)$ if after numbering the simple roots in $\operatorname{supp}(\sigma)$ like Bourbaki (see [Bou68]) σ is listed in Table 1.

Remark 1.4. Note that $\Sigma^{sc}(G)$ is a finite set for every connected reductive group G. The notation $\Sigma^{sc}(G)$ is justified by Proposition 2.8 below.

Definition 1.5. Let Γ be a subset of Λ^+. Then we define

$$S^p(\Gamma) := \{\alpha \in S \colon \langle \alpha^\vee, \lambda\rangle = 0 \text{ for all } \lambda \in \Gamma\}.$$

Proposition 1.6. *Suppose Γ is a G-saturated submonoid of Λ^+. If $\overline{\sigma} \in \Sigma^{sc}(G)$, then $\overline{\sigma}$ is an element of $\Sigma^N(\Gamma)$ if and only if the following conditions are all satisfied:*

(i) $\overline{\sigma}$ is not a simple root;

(ii) $\overline{\sigma} \in \mathbb{Z}\Gamma$;

(iii) $\overline{\sigma}$ is compatible with $S^p(\Gamma)$, that is:

- *if $\overline{\sigma} = \alpha_1 + \ldots + \alpha_n$ with support of type B_n then $\{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\} \subseteq S^p(\Gamma)$ and $\alpha_n \notin S^p(\Gamma)$;*
- *if $\overline{\sigma} = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with support of type C_n then $\{\alpha_3, \alpha_4, \ldots, \alpha_n\} \subseteq S^p(\Gamma)$;*
- *if $\overline{\sigma}$ is any other element of $\Sigma^{sc}(G)$ then $\{\alpha \in \operatorname{supp}(\overline{\sigma}) \colon \langle \alpha^\vee, \overline{\sigma}\rangle = 0\} \subseteq S^p(\Gamma)$;*

(iv) if $\overline{\sigma} = 2\alpha \in 2S$ then $\langle \alpha^\vee, \gamma\rangle \in 2\mathbb{Z}$ for all $\gamma \in \mathbb{Z}\Gamma$;

(v) if $\overline{\sigma} = \alpha + \beta$ with $\alpha, \beta \in S$ and $\alpha \perp \beta$, then $\langle \alpha^\vee, \gamma\rangle = \langle \beta^\vee, \gamma\rangle$ for all $\gamma \in \mathbb{Z}\Gamma$.

Proposition 1.6 is a special case of [BVS16, Corollary 2.17], as we will show on page 19. The proof of the following proposition is in Section 4, on page 37.

Proposition 1.7. *Let Γ be a G-saturated submonoid of Λ^+. Among all the subsets F of S such that the relative interior of the cone spanned by $\{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in F\}$ in*

$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q})$ *intersects* $\mathcal{V}(\Gamma)$ *there is a unique one, denoted* S_Γ*, that contains all the others.*

Remark 1.8. (1) Note that $S^p(\Gamma) \subset S_\Gamma$ since $\alpha^\vee|_{\mathbb{Z}\Gamma} = 0$ for all $\alpha \in S^p(\Gamma)$.

(2) Determining S_Γ is a finite (algorithmic) process. Indeed, $S \smallsetminus S^p(\Gamma)$ is a finite set, and deciding for a given subset F of $S \smallsetminus S^p(\Gamma)$ whether the relative interior of the cone spanned by $\{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in F\}$ intersects $\mathcal{V}(\Gamma)$ is equivalent to deciding whether a certain system of linear inequalities with integer coefficients has a solution in the positive rational numbers. The Fourier-Motzkin elimination algorithm does the latter in finitely many steps.

Using $\Sigma^N(\Gamma)$ and S_Γ we apply a smoothness criterion due to Camus (see [Cam01, §6]) to decide whether X_Γ is smooth. We recall and explain the criterion in Section 3. To state the theorem, we need one more definition.

Definition 1.9. Let S be the set of simple roots of a root system. Let S^p be a subset of S. Let Σ^N be a subset of $\mathbb{N}S$. We say that the triple (S, S^p, Σ^N) is **admissible** if there exist a finite set I, for every $i \in I$ a triple (S_i, S_i^p, Σ_i) from List 1.10 below, and a bijection φ from S to the disjoint union $\bigcup_i S_i$, such that:

(1) φ induces an isomorphism of the corresponding Dynkin diagrams,
(2) $\varphi(S^p) = \bigcup_i S_i^p$,
(3) $\varphi(\Sigma^N) = \bigcup_i \Sigma_i$, where we have extended φ linearly to the $\mathbb{Z}$-span of S.

List 1.10 (Primitive admissible triples)**.** 1. $(S, S, \emptyset)$ where S is the set of simple roots of an irreducible root system;
2. $(\mathsf{A}_n, \{\alpha_2, \alpha_3, \ldots, \alpha_n\}, \emptyset)$ for $n \geq 1$;
3. $(\mathsf{A}_n, \{\alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{n-1}\}, \{\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3 + 2\alpha_4 + \alpha_5, \ldots, \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1}\})$ for $n \geq 4$, n even;
4. $(\mathsf{A}_n \times \mathsf{A}_k, \{\alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_n\}, \{\alpha_1 + \alpha_1', \alpha_2 + \alpha_2', \ldots, \alpha_k + \alpha_k'\})$ for $n > k \geq 2$;
5. $(\mathsf{C}_n, \{\alpha_2, \alpha_3, \ldots, \alpha_n\}, \emptyset)$ for $n \geq 2$;
6. $(\mathsf{D}_5, \{\alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5\})$.

Remark 1.11. Note that we allow $I = \emptyset$ in Definition 1.9: the triple $(\emptyset, \emptyset, \emptyset)$ is admissible.

Here is the main result of this paper (Theorem 4.2), specialized to the case of G-saturated weight monoids. The proof is given on page 37 in Section 4.

Theorem 1.12. *Let* Γ *be a* G*-saturated submonoid of* Λ^+*. Let* $S^p(\Gamma)$ *be the set of simple roots that are orthogonal to* Γ *as in Definition 1.5,* S_Γ *the set of simple roots as in Proposition 1.7 and* $\Sigma^N(\Gamma)$ *the set given by Proposition 1.6. Then* Γ *is the weight monoid of a smooth affine spherical* G*-variety if and only if*

(a) $\{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in S_\Gamma \smallsetminus S^p(\Gamma)\}$ *is a subset of a basis of* $(\mathbb{Z}\Gamma)^*$*; and*
(b) for all $\alpha, \beta \in S_\Gamma \smallsetminus S^p(\Gamma)$ *such that* $\alpha \neq \beta$ *and* $\alpha^\vee|_{\mathbb{Z}\Gamma} = \beta^\vee|_{\mathbb{Z}\Gamma}$ *we have* $\alpha + \beta \in \mathbb{Z}\Gamma$*; and*
(c) the triple $(S_\Gamma, S^p(\Gamma), \Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma)$ *is admissible (see Definition 1.9).*

Example 1.13. To illustrate Theorem 1.12, we apply it to the monoid

$$\Gamma = \mathbb{N}(3\omega_1) + \mathbb{N}(\omega_1 + \omega_2) + \mathbb{N}(3\omega_2) \quad \text{for } G = \mathrm{SL}(3).$$

Here ω_1 and ω_2 are the fundamental weights. We illustrate our more general Theorem 4.2 in Example 4.4.

Recall that for $G = \mathrm{SL}(3)$ the set of simple roots is $S = \{\alpha_1, \alpha_2\}$, the weight lattice is $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Lambda^+ = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$. Moreover $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -\omega_1 + 2\omega_2$.

It is clear that

$$\mathbb{Z}\Gamma = \mathbb{Z}(3\omega_1) \oplus \mathbb{Z}(\omega_1 + \omega_2) = \mathbb{Z}(\omega_1 + \omega_2) \oplus \mathbb{Z}(3\omega_2).$$

A straightforward calculation shows that Γ satisfies condition (1.2), that is, Γ is G-saturated and Theorem 1.12 applies. Using that for $i, j \in \{1, 2\}$ we have $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$, it is immediate that

$$S^p(\Gamma) = \emptyset.$$

It follows from Definition 1.3 that

$$\Sigma^{sc}(G) = \{\alpha_1, \alpha_2, 2\alpha_1, 2\alpha_2, \alpha_1 + \alpha_2\}.$$

By applying Proposition 1.6 to each element of $\Sigma^{sc}(G)$ one finds that

$$\Sigma^N(\Gamma) = \{\alpha_1 + \alpha_2\}.$$

Indeed α_1 and α_2 do not satisfy condition (i) of the Proposition, $2\alpha_1$ and $2\alpha_2$ do not satisfy condition (iv), while $\alpha_1 + \alpha_2$ satisfies all the conditions. We now determine S_Γ using Proposition 1.7. Observe that here equation (1.3) becomes

$$\mathcal{V}(\Gamma) = \{\nu \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q}) \colon \langle \nu, \alpha_1 + \alpha_2 \rangle \leq 0\}.$$

Consider the inequality

$$\langle q_1(\alpha_1^\vee|_{\mathbb{Z}\Gamma}) + q_2(\alpha_1^\vee|_{\mathbb{Z}\Gamma}), \alpha_1 + \alpha_2 \rangle \leq 0. \tag{1.4}$$

Since $\alpha_1 + \alpha_2 = \omega_1 + \omega_2$, this inequality is equivalent to

$$q_1 + q_2 \leq 0.$$

It follows that the only solution to inequality (1.4) with $q_1, q_2 \in \mathbb{Q}_{\geq 0}$ is $q_1 = q_2 = 0$, which implies that

$$S_\Gamma = \emptyset.$$

We can now verify the three conditions in Theorem 1.12. Conditions (a) and (b) are vacuously met. Condition (c) is also satisfied since the triple

$$(S_\Gamma, S^p(\Gamma), \Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma) = (\emptyset, \emptyset, \emptyset)$$

is admissible. We have shown that there exists a smooth affine spherical SL(3)-variety with weight monoid Γ.

Remark 1.14. Our proof of Theorem 1.12 relies on the classification of wonderful varieties by spherical systems, which was conjectured in [Lun01] and was known as the Luna Conjecture. It is proved in [BP16]. Specifically, we use the classification in Proposition 2.24; see also Remark 4.3(a). Another proof of the Luna Conjecture has been proposed in [Cup14].

Remark 1.15. (a) Several of the statements in Sections 2.3 – 2.5 appeared in [ACF18b], cf. Remark 2.5(b). However, the present paper is independent of *loc.cit.* The content of Sections 2.4 and 2.5 was inspired by an unpublished working document of Luna's from 2005.

(b) In Section 3 we present Camus's smoothness criterion, with a complete exposition of its original proof following [Cam01]. Gagliardi has published another proof of the criterion in the paper [Gag15], which also includes the so-called "Luna diagrams" of the spherical modules.

(c) Other smoothness criteria for spherical varieties, quite different from this one, have appeared in the literature: one by Brion in [Bri91], and two conjectural criteria, resp. in [BM13] and [GH16].

As an application we study affine model varieties of simply connected semisimple groups. Recall that a quasi-affine G-variety Y is called a **model** G-variety if the G-module $\mathbb{C}[Y]$ contains every irreducible representation of G with multiplicity one. This is equivalent to being a quasi-affine spherical G-variety with weight monoid equal to Λ^+. Homogeneous model G-varieties were first introduced in [BGG81] and then further studied in [GZ84, GZ85, AHV98, Lun07].

When G is simply connected and semisimple, Λ^+ is free and G-saturated. Applying Theorem 1.12 to the monoid Λ^+, we obtain the following theorem. Its proof is given in Section 5.

Theorem 1.16. *Let G be a simply connected semisimple linear algebraic group. There exists a smooth affine model G-variety if and only if the simple factors of G are of type* A *or of type* C.

The "if" part of Theorem 1.16 is not new, see Example 1.17 below.

Example 1.17. For all $n, k \in \mathbb{Z}_{>0}$ the groups $\mathrm{SL}(n)$, $\mathrm{Sp}(2n)$ and $\mathrm{SL}(k) \times \mathrm{Sp}(2n)$ have smooth affine model varieties, while e.g. $\mathrm{Spin}(k)$ does not. The existence of such varieties for $\mathrm{SL}(n)$ and $\mathrm{Sp}(2n)$ was already known: the variety for the former group with $n > 1$ odd is $\mathrm{SL}(n)/\mathrm{Sp}(n-1)$, and for n even it is the homogeneous vector bundle $\mathrm{SL}(n) \times^{\mathrm{Sp}(n)} \mathbb{C}^n$, while the variety for $\mathrm{Sp}(2n)$ is the homogeneous vector bundle $\mathrm{Sp}(2n) \times^{\mathrm{Sp}(2a)\times\mathrm{Sp}(2b)} \mathbb{C}^{2b}$ where $a = b = n/2$ if n is even, and $a = b-1 = (n-1)/2$ if n is odd. In order to decide the existence of a smooth affine model variety in general, one can apply Theorem 1.12 for $\Gamma = \Lambda^+$ when G is any connected reductive group. For example, this way one can recover that the group $\mathrm{SO}(2n+1)$, with $n \geq 1$, has a smooth affine model variety. In fact, it is $\mathrm{SO}(2n+1)/\mathrm{GL}(n)$ (cf. [Lun07]).

Remark 1.18. As exhibited in [PPVS18], it is possible to apply our combinatorial criterion, in a way similar to the proof of Theorem 1.16, to classify smooth affine spherical varieties whose weight monoids satisfy additional conditions, e.g. G-saturatedness, restrictions on the rank, etc. In his thesis [Kim16, Theorem 4.1], Kim used our criterion to classify the smooth weight monoids for $G = \mathrm{SL}(n)$ generated by subsets of the set of fundamental weights.

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2. Spherical combinatorics and Alexeev and Brion's moduli scheme

2.1. **Alexeev and Brion's moduli scheme.** We recall that the weight monoid Γ of an affine spherical variety X is finitely generated, because $\mathbb{C}[X]$ is a finitely generated ring. Since X is normal, its weight monoid satisfies the following equality in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\mathbb{Z}\Gamma \cap \mathbb{Q}_{\geq 0}\Gamma = \Gamma. \tag{2.1}$$

We will call a finitely generated submonoid Γ of Λ^+ satisfying condition (2.1) **normal**. Notice that a G-saturated submonoid of Λ^+ is normal.

Let now Γ be a normal submonoid of Λ^+. In [AB05], Alexeev and Brion introduced a moduli scheme M_Γ, which parametrizes affine spherical G-varieties with weight monoid Γ. To describe M_Γ more precisely, put $V(\Gamma) := \oplus_{\lambda\in\Gamma} V(\lambda)$ and equip $V(\Gamma)^U$ with a T-multiplication law by choosing an isomorphism $V(\Gamma)^U \cong \mathbb{C}[\Gamma]$. The (closed) points of M_Γ are in one-to-one correspondence with the G-multiplication laws on $V(\Gamma)$ that extend the multiplication law on $V(\Gamma)^U$. Alexeev and Brion showed that M_Γ *is an affine scheme of finite type over* $\mathbb{C}$ *which represents the functor* Schemes $\to$ Sets *that associates with any scheme Z, the set of families of algebra structures of type Γ over Z*. For an introduction to this moduli scheme, we refer to [Bri13, Section 4.3].

Thanks to M_Γ, we can make precise the notion of "generic" affine spherical G-variety with weight monoid Γ. Alexeev and Brion equipped M_Γ with an action of the maximal torus T of G and showed that *there is a natural bijection between the T-orbits on M_Γ and the isomorphism classes of affine spherical G-varieties with weight monoid* Γ, see [AB05, Theorem 1.12]. When X is an affine spherical variety with weight monoid Γ, we will write $T\cdot[X]$ for the T-orbit on M_Γ corresponding to the G-isomorphism class of X.

Definition 2.1. Let X be an affine spherical G-variety with weight monoid Γ. We will say that X is **generic** if $T\cdot[X]$ is an open subset of M_Γ.

Proposition 2.2 ([AB05, Corollary 2.9])**.** *If X is a smooth affine spherical G-variety with weight monoid Γ, then X is generic.*

Thanks to [AB05, Corollary 3.4] we know that that there are, up to isomorphism, only finitely many affine spherical varieties with a given weight monoid Γ. Equivalently, M_Γ *contains only finitely many T-orbits.* This implies the following proposition.

Proposition 2.3. *Let Γ be a normal submonoid of Λ^+. Every irreducible component of M_Γ contains a (unique) open (and dense) T-orbit. Equivalently, for every irreducible component Z of M_Γ there exists a unique T-orbit $T\cdot[X]$ on M_Γ such that $Z = \overline{T\cdot[X]}$, where Z is equipped with its reduced induced scheme structure and $\overline{T\cdot[X]}$ is the closure of $T\cdot[X]$.*

It follows from Proposition 2.3 that the map $X \mapsto \overline{T\cdot[X]}$ yields a bijection between isomorphism classes of generic varieties with weight monoid Γ and irreducible components of M_Γ.

Here is another result from [AB05] we will need. It establishes a crucial link between the geometry of M_Γ and a combinatorial invariant of the varieties M_Γ parametrizes: their root monoids. We recall that the **root monoid** $\mathscr{M}_X$ of a quasi-affine G-variety X is the submonoid of Λ generated by

$$\{\lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Lambda^+ \text{ such that } \mathbb{C}[X]_{(\nu)} \cap (\mathbb{C}[X]_{(\lambda)}\mathbb{C}[X]_{(\mu)}) \neq 0\},$$

where for $\gamma \in \Lambda^+$ we used $\mathbb{C}[X]_{(\gamma)}$ for the isotypic component of type γ in $\mathbb{C}[X]$ and $\mathbb{C}[X]_{(\lambda)}\mathbb{C}[X]_{(\mu)}$ is the subspace of $\mathbb{C}[X]$ spanned by the set $\{fg \colon f \in \mathbb{C}[X]_{(\lambda)}, g \in \mathbb{C}[X]_{(\mu)}\}$. By [Kno96, Theorem 1.3], the saturation of $\mathscr{M}_X$, that is, the intersection of the cone spanned by $\mathscr{M}_X$ and the group generated by $\mathscr{M}_X$, is a free submonoid of Λ_R (cf. Remark 2.5).

Proposition 2.4 ([AB05, Proposition 2.13])**.** *If X is an affine spherical G-variety with weight monoid Γ, then the closure $\overline{T \cdot [X]}$ of the corresponding T-orbit on M_Γ is T-isomorphic to* $\operatorname{Spec} \mathbb{C}[\mathscr{M}_X]$*, where $\mathscr{M}_X$ is the root monoid of X.*

Remark 2.5. (a) In their recent paper [ACF18a] Avdeev and Cupit-Foutou have given a proof of Knop's conjecture that the root monoid $\mathscr{M}_X$ of an affine spherical variety X is free, and of Brion's conjecture in [Bri13] that the irreducible components of M_Γ, equipped with their reduced induced scheme structure, are affine spaces, under the assumption that Γ is normal.

(b) Proposition 2.13(a), Proposition 2.13(b), Corollary 2.14, Corollary 2.15, Proposition 2.24 and Corollary 2.28(a) below are similar to Proposition 4.7 ((1) $\Leftrightarrow$ (2)), Proposition 3.7(b), Theorem 7.1 (first assertion), Corollary 7.2, Theorem 6.9 and Theorem 7.27(c), respectively, in [ACF18b].

2.2. **Luna invariants and spherical closure.** In this section, we first recall some basic notions in the theory of spherical varieties. For more details we refer to [Kno91, Lun01]. We then state Proposition 2.7, which is essentially due to Losev and which describes the relationship between the three standard normalizations of the so-called 'spherical roots' of a spherical variety.

Let X be a spherical G-variety with open orbit G/H. The basic invariants the theory of spherical varieties associates to X are defined as follows.

1. The **lattice** of X, denoted $\Lambda(X)$, is the subgroup of Λ consisting of the B-weights of B-eigenvectors in the field of rational functions $\mathbb{C}(X)$.
2. Let $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Q}$ be a discrete valuation. Then ν induces an element of $\operatorname{Hom}_\mathbb{Z}(\Lambda(X), \mathbb{Q})$, denoted $\rho_X(\nu)$, by
$$\langle \rho_X(\nu), \gamma \rangle = \nu(f_\gamma)$$
where $f_\gamma \in \mathbb{C}(X)$ is a B-eigenvector of B-weight $\gamma \in \Lambda(X)$. If $D \subset X$ is a prime divisor, we denote by ν_D the associated discrete valuation, and for simplicity by $\rho_X(D)$ the element $\rho_X(\nu_D)$ of $\operatorname{Hom}_\mathbb{Z}(\Lambda(X), \mathbb{Q})$.
3. A **color** of X is a B-stable but not G-stable prime divisor of X. The set of colors of X is denoted $\Delta(X)$.
4. The **Cartan pairing** of X is the bilinear map
$$c_X \colon \mathbb{Z}\Delta(X) \times \Lambda(X) \to \mathbb{Z}$$
given by extending by linearity the elements $\rho_X(D) \in \operatorname{Hom}_\mathbb{Z}(\Lambda(X), \mathbb{Q})$ with $D \in \Delta(X)$. In particular, for $D \in \Delta(X)$ and $\gamma \in \Lambda(X)$,
$$c_X(D, \gamma) = \langle \rho_X(D), \gamma \rangle.$$

5. Let P_X be the stabilizer of the open B-orbit of X and denote by $S^p(X)$ the subset of simple roots corresponding to P_X, which is a parabolic subgroup of G containing B.
6. We use $\mathcal{V}(X)$ for the set of G-invariant $\mathbb{Q}$-valued discrete valuations of $\mathbb{C}(X)$ and identify $\mathcal{V}(X)$ with its image in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$ via the map ρ_X. We call $\mathcal{V}(X)$ the **valuation cone** of X.
7. By [Bri90], $\mathcal{V}(X)$ is a co-simplicial cone. The set of **spherical roots** $\Sigma(X)$ of X is the minimal set of primitive elements of $\Lambda(X)$ such that

$$\mathcal{V}(X) = \{\eta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}) \mid \langle \eta, \sigma \rangle \leq 0 \ \forall \sigma \in \Sigma(X)\}\,. \tag{2.2}$$

8. A color D of X is **moved** by a simple root $\alpha \in S$ if D is not stable under the minimal parabolic subgroup of G containing B and associated with α. If $\alpha \in S \cap \Sigma(X)$ then we denote by $\mathbf{A}(X, \alpha)$ the set of colors of X moved by α, and we set

$$\mathbf{A}(X) := \bigcup_{\alpha \in S \cap \Sigma(X)} \mathbf{A}(X, \alpha).$$

Note that all of these invariants are equivariantly birational: they only depend on G/H. We will also need two other sets of spherical roots associated to X, namely $\Sigma^{sc}(X)$ and $\Sigma^N(X)$. Like $\Sigma(X)$, they consist of normal vectors to $\mathcal{V}(X)$, but possibly of other lengths. Before defining them, we recall Luna's notion of spherical closure [Lun01, §6.1]. Recall that subgroup H of G is called a **spherical subgroup** if G/H is a spherical G-variety. Then the quotient $N_G(H)/H$ naturally acts on G/H by G-equivariant automorphisms, inducing an action of $N_G(H)$ on the set of colors of G/H. The kernel of this last action is the **spherical closure** of H, denoted by $\overline{H}$. If $H = \overline{H}$, then H is said to be **spherically closed**. In general, the normalizer of $\overline{H}$ may be bigger that the normalizer of H, but $\overline{H}$ is always spherically closed, that is

$$\overline{\overline{H}} = \overline{H}. \tag{2.3}$$

This follows from [BL11, Lemma 2.4.2]; see [Pez15, Proposition 3.1] for a direct proof.

Definition 2.6. Let X be a spherical G-variety with open orbit G/H. We define

$$\Sigma^{sc}(X) := \Sigma^{sc}(G/H) := \Sigma(G/\overline{H}) \tag{2.4}$$

$$\Sigma^N(X) := \Sigma^N(G/H) := \Sigma(G/N_G(H)). \tag{2.5}$$

The latter is called the set of **N-spherical roots** of X.

It follows from [Kno96, Theorem 1.3] that when X is quasi-affine, the set $\Sigma^N(X)$ defined in equation (2.5) is the basis of the saturation of $\mathscr{M}_X$. Thanks to [Los09b], the relation between $\Sigma(X)$, $\Sigma^{sc}(X)$ and $\Sigma^N(X)$ is well understood: the three sets have the same cardinality, and for every $\sigma \in \Sigma(X)$, either σ or its double belongs to $\Sigma^{sc}(X)$; and similarly for $\Sigma^N(X)$. In the next proposition we precisely say which elements of $\Sigma(X)$ have to be doubled.

Proposition 2.7 (Losev)**.** *Let X be a spherical variety. Then $\Sigma^N(X)$ is obtained from $\Sigma(X)$ by replacing σ with 2σ for all σ satisfying any one of the following conditions:*

(1) $\sigma \in \Sigma \cap S$ with $\rho_X(D_\sigma^+) = \rho_X(D_\sigma^-)$ where $\{D_\sigma^+, D_\sigma^-\} = \mathbf{A}(X, \sigma)$,

(2) *$\sigma = \alpha_1 + \ldots + \alpha_n$, where $\{\alpha_1, \ldots, \alpha_n\} \subseteq S$ has type B_n and $\alpha_i \in S^p$ for all $i \in \{2, \ldots, n\}$,*
(3) *$\sigma = 2\alpha_1 + \alpha_2$, where $\{\alpha_1, \alpha_2\} \subseteq S$ has type G_2,*
(4) *σ is not in the root lattice of G.*

The set $\Sigma^{sc}(X)$ is obtained from $\Sigma(X)$ by replacing σ with 2σ for all σ satisfying conditions (2), (3), or (4).

Proof. The statement about $\Sigma^N(X)$ is exactly [Los09b, Theorem 2], and the statement about $\Sigma^{sc}(X)$ is exactly [Lun01, Lemme 7.1].

For the second statement, we point out that the proof of *loc.cit.* uses the general classification of spherical homogeneous spaces. A similar but more self-contained argument, essentially relying only on [Los09b, Theorem 2], goes as follows.

Let G/H be the open orbit of X and let Σ' be equal to $\Sigma(X)$ where we replace σ with 2σ for all σ satisfying conditions (2), (3), and (4). By [Los09b, Theorem 2] we have $\mathbb{Z}\Sigma^N(X) \subseteq \mathbb{Z}\Sigma'$, and it holds

$$\frac{\mathbb{Z}\Sigma'}{\mathbb{Z}\Sigma^N(X)} \subseteq \frac{\Lambda(X)}{\mathbb{Z}\Sigma^N(X)} = \frac{\Lambda(X)}{\Lambda(G/N_G H)} \cong \mathcal{X}\left(\frac{N_G H}{H}\right),$$

where the equality is [Kno96, Corollary 6.5], the isomorphism is [Gan11, Lemma 2.4], and $\mathcal{X}(N_G(H)/H)$ denotes the group of characters of $N_G(H)/H$. Then $\mathbb{Z}\Sigma'$ corresponds to a subgroup K of $N_G H$ containing H, such that $\Lambda(G/K) = \mathbb{Z}\Sigma'$, again by [Gan11, Lemma 2.4]. From [Kno91, Theorem 4.4 and proof of Theorem 6.1] we deduce that $\Sigma(G/K) = \Sigma'$.

According to [Lun01, Section 2.3] the number of colors of G/H and of G/K are equal. In other words the natural map $G/H \to G/K$ induces a bijection between the sets of colors of G/H and G/K, whence $K \subseteq \overline{H}$. Applying once again [Kno91, Theorem 4.4 and proof of Theorem 6.1] we have that Σ' and $\Sigma^{sc}(X)$ are equal up to replacing some elements of the first set with positive rational multiples, and thanks to the classification of spherical roots we have that the coefficients can only be 1 or 2, i.e. in particular $\Sigma^{sc}(X) \subseteq \Sigma' \cup 2\Sigma'$.

Assume that there exists an element in $\Sigma^{sc}(X)$ not in Σ'. It has the form 2σ with $\sigma \in \Sigma'$. Then $\sigma \in \Sigma(X)$ and $2\sigma \in \Sigma^N(X)$, and by definition of Σ' the only possibility is that σ satisfies condition (1), i.e. $\sigma \in S \cap \Sigma(G/H)$. But in this case G/H and $G/\overline{H}$ would have a different number of colors moved by σ (resp. 2 and 1), which is impossible.

Therefore $\Sigma^{sc}(X) \subseteq \Sigma'$, and since the two sets have the same finite cardinality, they are equal. □

The notation $\Sigma^{sc}(G)$ of Definition 1.3 is justified by the following.

Proposition 2.8 (see [Lun01, §1.2])**.** *An element σ of $\mathbb{N}S$ belongs to $\Sigma^{sc}(G)$ if and only if there exists a spherically closed spherical subgroup K of G such that $\Sigma(G/K) = \{\sigma\}$.*

Given a spherical G-variety X, the triple

$$\mathscr{S}(X) = (S^p(X), \Sigma^{sc}(X), \mathbf{A}(X)),$$

equipped with the restriction of the Cartan pairing c_X to $\mathbb{Z}\mathbf{A}(X) \times \mathbb{Z}\Sigma^{sc}(X)$, is a spherically closed spherical system in the following sense.

Definition 2.9. Let $(S^p, \Sigma, \mathbf{A})$ be a triple where S^p is a subset of S, Σ is a subset of $\Sigma^{sc}(G)$ and $\mathbf{A}$ is a finite set endowed with a $\mathbb{Z}$-bilinear pairing $c\colon \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma \to \mathbb{Z}$. For every $\alpha \in \Sigma \cap S$, let $\mathbf{A}(\alpha)$ denote the set $\{D \in \mathbf{A} : c(D, \alpha) = 1\}$. Such a triple is called a **spherically closed spherical G-system** if all the following axioms hold:

(A1) for every $D \in \mathbf{A}$ and every $\sigma \in \Sigma$, we have that $c(D, \sigma) \leq 1$ and that if $c(D, \sigma) = 1$ then $\sigma \in S$;
(A2) for every $\alpha \in \Sigma \cap S$, $\mathbf{A}(\alpha)$ contains exactly two elements, which we denote by D_α^+ and D_α^-, and for all $\sigma \in \Sigma$ we have $c(D_\alpha^+, \sigma) + c(D_\alpha^-, \sigma) = \langle \alpha^\vee, \sigma \rangle$;
(A3) the set $\mathbf{A}$ is the union of $\mathbf{A}(\alpha)$ for all $\alpha \in \Sigma \cap S$;
(Σ1) if $2\alpha \in \Sigma \cap 2S$ then $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ is a non-positive integer for all $\sigma \in \Sigma \smallsetminus \{2\alpha\}$;
(Σ2) if $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to Σ then $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$;
(S) every $\sigma \in \Sigma$ is **compatible** with S^p, that is, for every $\sigma \in \Sigma$ there exists a spherically closed spherical subgroup K of G with $S^p(G/K) = S^p$ and $\Sigma(G/K) = \{\sigma\}$.

Remark 2.10. 1. Condition (S) of Definition 2.9 can be stated in purely combinatorial terms as follows (see [BL11, §1.1.6]). A spherically closed spherical root σ is compatible with S^p if and only if:

- in case $\sigma = \alpha_1 + \ldots + \alpha_n$ with support of type B_n

$$\{\alpha \in \mathrm{supp}(\sigma)\colon \langle \alpha^\vee, \sigma \rangle = 0\} \smallsetminus \{\alpha_n\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\} \smallsetminus \{\alpha_n\},$$

- in case $\sigma = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with support of type C_n

$$\{\alpha \in \mathrm{supp}(\sigma)\colon \langle \alpha^\vee, \sigma \rangle = 0\} \smallsetminus \{\alpha_1\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\},$$

- in the other cases

$$\{\alpha \in \mathrm{supp}(\sigma)\colon \langle \alpha^\vee, \sigma \rangle = 0\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\}.$$

2. Definition 2.9 is the same as [BVS16, Definition 2.5]. It combines the standard definition of spherical system, see [Lun01, §2], with the requirement that it be spherically closed, see [Lun01, §7.1] and [BL11, §2.4].
3. Axiom (Σ1) of Definition 2.9 can be replaced by the weaker condition that $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ is an integer for all $\sigma \in \Sigma \smallsetminus \{2\alpha\}$; we thank Friedrich Knop for communicating this fact to us. Indeed, the case-by-case verification of the proof of Lemma 2.25 below shows that the inequality $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle \leq 0$ follows from this weaker version together with the other axioms.

2.3. **Spherical roots of a generic affine spherical variety with weight monoid Γ.** In this section, we recall from [BVS16] the definition of spherical roots that are 'adapted' to a given normal submonoid Γ of Λ^+. We deduce that the generic affine spherical varieties X with weight monoid Γ are those for which $\Sigma^{sc}(X)$ is a maximal set of spherical roots adapted to Γ; see Corollary 2.14.

Definition 2.11. Let Γ be a normal submonoid of Λ^+. We say that a subset Σ of $\Sigma^{sc}(G)$ is **adapted to** Γ if there exists an affine spherical variety X such that $\Gamma(X) = \Gamma$ and $\Sigma^{sc}(X) = \Sigma$. We say that an element σ of $\Sigma^{sc}(G)$ is adapted to Γ if $\{\sigma\}$ is adapted to Γ. We use $\Sigma^{sc}(\Gamma)$ for the set of all $\sigma \in \Sigma^{sc}(G)$ that are adapted to Γ.

Remark 2.12. In general, $\Sigma^{sc}(\Gamma)$ is not adapted to Γ. The following example is due to Luna: when $G = \mathrm{SL}(2)\times\mathrm{SL}(2)$ and $\Gamma = \mathbb{N}\{2\omega, 2\omega+4\omega'\}$, then $\Sigma^{sc}(\Gamma) = \{\alpha,\alpha'\}$ and one checks that this set is not adapted to Γ.

Proposition 2.13. *Let X be an affine spherical G-variety with weight monoid Γ.*

(a) If Y is also an affine spherical G-variety with weight monoid Γ, then $T\cdot[Y] \subseteq \overline{T\cdot[X]}$ as subsets of M_Γ if and only if $\Sigma^{sc}(Y) \subseteq \Sigma^{sc}(X)$.

(b) For every subset Σ' of $\Sigma^{sc}(X)$ there exists an affine G-spherical variety Y' with weight monoid Γ and $\Sigma^{sc}(Y') = \Sigma'$.

(c) $\Sigma^{sc}(X) \subseteq \Sigma^{sc}(\Gamma)$.

Proof. We begin with assertion (a). As recalled in Proposition 2.4, Alexeev and Brion showed that $\overline{T\cdot[X]} = \mathrm{Spec}\,\mathbb{C}[\mathscr{M}_X]$. A basic fact in the theory of (not necessarily normal) affine toric varieties is that we have the following inclusion preserving one-to-one correspondence [CLS11, Theorem 3.A.3]:

$$\{\text{faces of } \mathbb{Q}_{\geq 0}\mathscr{M}_X\} \to \{\text{orbit closures in } \overline{T\cdot[X]}\} \tag{2.6}$$

$$\mathcal{F} \mapsto \mathrm{Spec}\,\mathbb{C}[\mathscr{M}_X \cap \mathcal{F}] \tag{2.7}$$

Consequently, $T\cdot[Y] \subseteq \overline{T\cdot[X]}$ if and only if $\overline{T\cdot[Y]} \subseteq \overline{T\cdot[X]}$ if and only if

$$\mathscr{M}_Y = \mathscr{M}_X \cap \mathcal{F} \text{ for some face } \mathcal{F} \text{ of } \mathbb{Q}_{\geq 0}\mathscr{M}_X. \tag{2.8}$$

By [Kno96, Theorem 1.3], the equality (2.8) holds if and only if $\Sigma^N(Y) \subseteq \Sigma^N(X)$ up to multiples. Recall that the elements of $\Sigma^N(X)$ are integer multiples of the elements of $\Sigma(X)$, and that the elements of $\Sigma(X)$ are primitive in the lattice $\Lambda(X)$. Since $\Lambda(X) = \Lambda(Y) = \mathbb{Z}\Gamma$, it follows that (2.8) is equivalent to $\Sigma(Y) \subseteq \Sigma(X)$. By Proposition 2.7, the latter inclusion holds if and only if $\Sigma^{sc}(Y) \subseteq \Sigma^{sc}(X)$. This proves assertion (a).

Assertion (b) is a formal consequence of the correspondence (2.6) and [Kno96, Theorem 1.3]. Indeed, $\mathbb{Q}_{\geq 0}\Sigma'$ is a face of $\mathbb{Q}_{\geq 0}\Sigma^{sc}(X) = \mathbb{Q}_{\geq 0}\mathscr{M}_X$ and so corresponds to a T-orbit closure $\overline{T\cdot[Y']}$ in $\overline{T\cdot[X]}$. Applying [Kno96, Theorem 1.3] as above, it follows that $\Sigma^{sc}(Y') = \Sigma'$.

Finally, assertion (c) follows by applying (b) to the singletons in $\Sigma^{sc}(X)$. ☐

Corollary 2.14. *If X is an affine spherical G-variety with weight monoid Γ, then X is generic if and only if there is no subset of $\Sigma^{sc}(G)$ that strictly contains $\Sigma^{sc}(X)$ and is adapted to Γ.*

Proof. This is a formal consequence of Propositions 2.13 and 2.3 above and [AB05, Theorem 1.12], as we now explain. We first assume that X is generic, and show that $\Sigma^{sc}(X)$ is a maximal subset of $\Sigma^{sc}(G)$ that is adapted to Γ. Let Σ' be a subset of $\Sigma^{sc}(G)$ that is adapted to Γ and such that $\Sigma^{sc}(X) \subseteq \Sigma'$. By Definition 2.11, there exits an affine variety Y with weight monoid Γ such that $\Sigma^{sc}(Y) = \Sigma'$. Proposition 2.13 now implies that $T\cdot[X] \subseteq \overline{T\cdot[Y]}$. Since X is generic, which means that $\overline{T\cdot[X]}$ is an irreducible component of M_Γ, this implies that $\overline{T\cdot[X]} = \overline{T\cdot[Y]}$. Consequently $T\cdot[X] = T\cdot[Y]$, and so X is G-equivariantly isomorphic to Y by [AB05, Theorem 1.12]. In particular, $\Sigma^{sc}(X) = \Sigma^{sc}(Y) = \Sigma'$.

Conversely, suppose that $\Sigma^{sc}(X)$ is a maximal subset of $\Sigma^{sc}(G)$ that is adapted to Γ. If X were not generic, then Proposition 2.3 would imply the existence of an affine spherical G-variety Y with weight monoid Γ such that $\overline{T\cdot[X]} \subsetneq \overline{T\cdot[Y]}$. But then $\Sigma^{sc}(X) \subset \Sigma^{sc}(Y)$ by Proposition 2.13, and $\Sigma^{sc}(X) \neq \Sigma^{sc}(Y)$, since otherwise we would have $T\cdot[Y]\subseteq\overline{T\cdot[X]}$ again by Proposition 2.13. This contradicts the maximality of $\Sigma^{sc}(X)$, and finishes the proof. ☐

Corollary 2.15. M_Γ *is irreducible if and only if* $\Sigma^{sc}(\Gamma)$ *is adapted to* Γ.

Proof. This is a formal consequence of Proposition 2.13. We first assume that $\Sigma^{sc}(\Gamma)$ is adapted to Γ, and denote by X an affine spherical G-variety such that $\Gamma(X)=\Gamma$ and $\Sigma^{sc}(X)=\Sigma^{sc}(\Gamma)$. We claim that $\overline{T\cdot[X]}=\mathrm{M}_\Gamma$. To prove the claim, it suffices to show that if Y is an affine spherical G-variety with $\Gamma(Y)=\Gamma$, then $T\cdot[Y]\subseteq\overline{T\cdot[X]}$. By Proposition 2.13(c) we have that $\Sigma^{sc}(Y)\subseteq\Sigma^{sc}(\Gamma)=\Sigma^{sc}(X)$. By Proposition 2.13(a), it follows that $T\cdot[Y]\subseteq\overline{T\cdot[X]}$.

We turn to the the reverse implication. Since M_Γ is irreducible, it has a unique dense T-orbit $T\cdot[X]$. In particular

$$\overline{T\cdot[X]}=\mathrm{M}_\Gamma. \tag{2.9}$$

We claim that $\Sigma^{sc}(X)=\Sigma^{sc}(\Gamma)$. By Proposition 2.13(c) we only have to show that $\Sigma^{sc}(\Gamma)\subseteq\Sigma^{sc}(X)$. Let $\sigma\in\Sigma^{sc}(\Gamma)$. By the definition of $\Sigma^{sc}(\Gamma)$ there exists an affine spherical G-variety Y with $\Sigma^{sc}(Y)=\{\sigma\}$ and $\Gamma(Y)=\Gamma$. By the equality (2.9), it follows that $T\cdot[Y]\subseteq\overline{T\cdot[X]}$. Proposition 2.13(a) finishes the proof. ☐

Remark 2.16. Note that $\Sigma^{sc}(\Gamma)$ is adapted to Γ if Γ is G-saturated, and when $\Sigma^{sc}(\Gamma)$ does not contain any simple roots; see Corollary 2.27 below.

2.4. **Spherical roots adapted to a weight monoid.** In this subsection, we begin by recalling some results from [BVS16], including the combinatorial characterization of $\sigma\in\Sigma^{sc}(G)$ that are adapted to Γ; see Proposition 2.23. We proceed with a proof of a criterion formulated by Luna which characterizes the subsets of $\Sigma^{sc}(G)$ that are adapted to Γ; see Proposition 2.24.

The first three results are taken from [BVS16]. Before we state them, we recall from [Lun01] the definition of the colors, and of an augmentation, of a spherical system. We use the formulation of [BVS16].

Definition 2.17. Let $\mathscr{S}=(S^p,\Sigma,\mathbf{A})$ be a (spherically closed) spherical G-system. The **set of colors of** $\mathscr{S}$ is the finite set Δ obtained as the disjoint union $\Delta=\Delta^a\cup\Delta^{2a}\cup\Delta^b$ where:

- $\Delta^a=\mathbf{A}$,
- $\Delta^{2a}=\{D_\alpha:\alpha\in S\cap\frac{1}{2}\Sigma\}$,
- $\Delta^b=\{D_\alpha:\alpha\in S\smallsetminus(S^p\cup\Sigma\cup\frac{1}{2}\Sigma)\}/\sim$, where $D_\alpha\sim D_\beta$ if α and β are orthogonal and $\alpha+\beta\in\Sigma$.

Let $\alpha\in S$ and $D\in\Delta$. We say that D is **moved** by α if $D=D_\alpha\in\Delta^{2a}\cup\Delta^b$ or if $D\in\Delta^a$ and $c(D,\alpha)=1$.

Definition 2.18. Let $\mathscr{S}=(S^p,\Sigma,\mathbf{A})$ be a spherically closed spherical G-system with Cartan pairing $c:\mathbb{Z}\mathbf{A}\times\mathbb{Z}\Sigma\to\mathbb{Z}$. An **augmentation** of $\mathscr{S}$ is a lattice $\Lambda'\subset\Lambda$ endowed with a pairing $c'\colon\mathbb{Z}\mathbf{A}\times\Lambda'\to\mathbb{Z}$ such that $\Lambda'\supset\Sigma$ and

(a1) c' extends c;
(a2) if $\alpha\in S\cap\Sigma$ then $c'(D_\alpha^+,\xi)+c'(D_\alpha^-,\xi)=\langle\alpha^\vee,\xi\rangle$ for all $\xi\in\Lambda'$;
(σ1) if $2\alpha\in 2S\cap\Sigma$ then $\alpha\notin\Lambda'$ and $\langle\alpha^\vee,\xi\rangle\in 2\mathbb{Z}$ for all $\xi\in\Lambda'$;
(σ2) if α and β are orthogonal elements of S with $\alpha+\beta\in\Sigma$ then $\langle\alpha^\vee,\xi\rangle=\langle\beta^\vee,\xi\rangle$ for all $\xi\in\Lambda'$; and
(s) if $\alpha\in S^p$ then $\langle\alpha^\vee,\xi\rangle=0$ for all $\xi\in\Lambda'$.

Let Δ be the set of colors of $\mathscr{S}$. The **full Cartan pairing** of the augmentation is the $\mathbb{Z}$-bilinear map $c' : \mathbb{Z}\Delta \times \Lambda' \to \mathbb{Z}$ given by

$$c'(D,\gamma) = \begin{cases} c'(D,\gamma) & \text{if } D \in \Delta^a; \\ \frac{1}{2}\langle \alpha^\vee, \gamma\rangle & \text{if } D = D_\alpha \in \Delta^{2a}; \\ \langle \alpha^\vee, \gamma\rangle & \text{if } D = D_\alpha \in \Delta^b. \end{cases} \tag{2.10}$$

Remark 2.19. Let X be a spherical G-variety. The set of colors of X is naturally identified with the set of colors of $\mathscr{S}(X)$, thanks to [Lun01, Proposition 3.2]. The lattice $\Lambda(X)$ together with the Cartan pairing c_X is an augmentation of $\mathscr{S}(X)$, thanks to [Lun01, Proposition 6.4].

We also need some additional notation. Let Γ be a normal submonoid of Λ^+. We will denote the dual cone to Γ by $\Gamma^\vee$, that is,

$$\Gamma^\vee := \{v \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q}) : \langle v, \gamma\rangle \geq 0 \text{ for all } \gamma \in \Gamma\}.$$

It is a strictly convex polyhedral cone, and we denote the set of primitive vectors on its extremal rays by $E(\Gamma)$:

$$E(\Gamma) := \{\delta \in (\mathbb{Z}\Gamma)^* \colon \delta \text{ spans an extremal ray of } \Gamma^\vee \text{ and } \delta \text{ is primitive}\}. \tag{2.11}$$

Observe that

$$E(\Gamma) = \{\delta \in (\mathbb{Z}\Gamma)^* \colon \delta \text{ is primitive}, \delta(\Gamma) \subseteq \mathbb{Z}_{\geq 0}, \\ \delta \text{ is the equation of a face of codim 1 of } \mathbb{Q}_{\geq 0}\Gamma\}. \tag{2.12}$$

For $\alpha \in S \cap \mathbb{Z}\Gamma$, we define

$$a(\alpha) := \{\delta \in (\mathbb{Z}\Gamma)^* \colon \langle \delta, \alpha\rangle = 1 \text{ and } \big(\delta \in E(\Gamma) \text{ or } \alpha^\vee|_{\mathbb{Z}\Gamma} - \delta \in E(\Gamma)\big)\}.$$

Proposition 2.20 ([BVS16, Proposition 2.13 and Remark 2.14])**.** *Let Γ be a normal submonoid of Λ^+. Suppose that a subset Σ of $\Sigma^{sc}(G)$ is adapted to Γ, let X be as in Definition 2.11, and set $S^p = S^p(X)$, $\mathbf{A} = \mathbf{A}(X)$. Then $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ satisfies*

1. *$S^p = S^p(\Gamma)$; and*
2. *$\mathbb{Z}\Gamma$ is the lattice of an augmentation of $\mathscr{S}$, such that*
3. *if $\delta \in E(\Gamma)$, then $\langle \delta, \sigma\rangle \leq 0$ for all $\sigma \in \Sigma$ or there exists $D \in \Delta$ such that $c(D, \cdot)$ is a positive multiple of δ; where Δ is the set of colors of $\mathscr{S}$ and $c : \mathbb{Z}\Delta \times \mathbb{Z}\Gamma \to \mathbb{Z}$ is the full Cartan pairing of the augmentation; and*
4. *$c(D, \cdot) \in \Gamma^\vee$ for all $D \in \mathbf{A}$.*

Viceversa, let Σ be a subset of $\Sigma^{sc}(G)$ and suppose that there exists a spherically closed spherical system $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ with the properties (1)–(4). Then Σ is adapted to Γ, and $\mathscr{S}$ and the augmentation are uniquely determined by these properties.

Remark 2.21. (a) The proof of the fact that Σ is adapted to Γ in the viceversa statement of Proposition 2.20 relies on the (now proven) Luna Conjecture, that is on [BP16, Theorem 1.2.3].

(b) In condition (4) of Proposition 2.20, we could replace $\mathbf{A}$ by the set Δ of all colors of $\mathscr{S}$. Indeed, if $D \in \Delta \setminus \mathbf{A}$, then $c(D, \cdot)$ takes the same values on $\mathbb{Z}\Gamma$ as a simple coroot or its half, and therefore takes nonnegative values on $\Gamma \subseteq \Lambda^+$.

The following lemma, extracted from the proof of [BVS16, Corollary 2.15], explains the "meaning" of the set $a(\alpha)$.

Lemma 2.22. *Let $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ and $(\mathbb{Z}\Gamma, c)$ be the spherically closed spherical system and the augmentation as in Proposition 2.20. If $\alpha \in \Sigma \cap S$ and $\mathbf{A}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$, then*

$$a(\alpha) = \{c(D_\alpha^+, \cdot), c(D_\alpha^-, \cdot)\}. \tag{2.13}$$

While Proposition 2.20 depends on the Luna Conjecture, the following combinatorial characterization of $\sigma \in \Sigma^{sc}(G)$ that are adapted to Γ only uses the classification of spherical varieties of rank 1 [Ahi83, Bri89].

Proposition 2.23 ([BVS16, Corollary 2.16])**.** *Let Γ be a normal submonoid of Λ^+. An element σ of $\Sigma^{sc}(G)$ is adapted to Γ if and only if all of the following conditions hold:*

(1) $\sigma \in \mathbb{Z}\Gamma$;
(2) σ is compatible with $S^p(\Gamma)$;
(3) if $\sigma \notin S$ and $\delta \in E(\Gamma)$ such that $\langle \delta, \sigma \rangle > 0$ then there exists $\beta \in S \setminus S^p(\Gamma)$ such that $\beta^\vee|_{\mathbb{Z}\Gamma}$ is a positive multiple of δ;
(4) if $\sigma \in S$ then
  (a) $a(\sigma)$ has one or two elements; and
  (b) $\langle \delta, \gamma \rangle \geq 0$ for all $\delta \in a(\sigma)$ and all $\gamma \in \Gamma$; and
  (c) $\langle \delta, \sigma \rangle \leq 1$ for all $\delta \in E(\Gamma)$;
(5) if $\sigma = 2\alpha \in 2S$, then $\alpha \notin \mathbb{Z}\Gamma$ and $\langle \alpha^\vee, \gamma \rangle \in 2\mathbb{Z}$ for all $\gamma \in \Gamma$;
(6) if $\sigma = \alpha + \beta$ with $\alpha, \beta \in S$ and $\alpha \perp \beta$, then $\alpha^\vee = \beta^\vee$ on Γ.

The following criterion was formulated by Luna in 2005 in an unpublished note.

Proposition 2.24. *Let Γ be a normal submonoid of Λ^+. A subset Σ of $\Sigma^{sc}(G)$ is adapted to Γ if and only if the following two conditions hold:*

(a) Σ is a subset of $\Sigma^{sc}(\Gamma)$;
(b) If $\alpha \in S \cap \Sigma$, $\delta \in a(\alpha)$ and $\gamma \in \Sigma$ satisfy $\langle \delta, \gamma \rangle > 0$, then $\gamma \in S$ and $\delta \in a(\gamma)$.

Proof. We first prove the necessity of the two conditions. Assume that Σ is adapted to Γ. Condition (a) is Proposition 2.13(c). Thanks to Lemma 2.22, condition (b) follows from Luna's axiom (A1).

To show that the two conditions are sufficient, we will use Proposition 2.20. We first construct a triple $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ and a pairing $c\colon \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Gamma \to \mathbb{Z}$, and then show that (a) and (b) imply that $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ and c satisfy all the conditions in Proposition 2.20. Note that (a) means exactly that every $\sigma \in \Sigma$ satisfies conditions (1) – (6) in Proposition 2.23.

We put $S^p := S^p(\Gamma)$. For every $\alpha \in \Sigma \cap S$ we formally put $\mathbf{A}(\alpha) := \{D_\alpha^+, D_\alpha^-\}$. If $a(\alpha)$ has one element, then we put $c(D_\alpha^+, \cdot) := c(D_\alpha^-, \cdot) := \frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}$. If $a(\alpha)$ has two elements, say $a(\alpha) = \{\delta_\alpha^+, \delta_\alpha^-\}$, then we set $c(D_\alpha^+, \cdot) := \delta_\alpha^+$ and $c(D_\alpha^-, \cdot) := \delta_\alpha^-$. Finally, we put

$$\mathbf{A} := \frac{\coprod_{\alpha \in \Sigma^{sc} \cap S} \mathbf{A}(\alpha)}{\sim}$$

where $D_1 \sim D_2$ if there exist $\alpha, \beta \in \Sigma \cap S$ such that $\alpha \neq \beta$, $D_1 \in \mathbf{A}(\alpha)$, $D_2 \in \mathbf{A}(\beta)$ and $c(D_1, \cdot) = c(D_2, \cdot)$.

Step 1: We check that $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ is a spherically closed spherical system. Strictly speaking, this triple is equipped with the restriction of $c : \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Gamma \to \mathbb{Z}$ to $\mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma$, but we will also denote this restriction by c, since no confusion will arise. We begin by verifying axiom (A1). Let $D \in \mathbf{A}(\alpha)$ for some $\alpha \in \Sigma \cap S$ and let $\gamma \in \Sigma$. Then $c(D, \cdot) \in a(\alpha)$. If $c(D, \gamma) > 0$, then $\gamma \in S$ and $c(D, \cdot) \in a(\gamma)$ by (b). By the definition of $a(\gamma)$, it follows that $c(D, \gamma) = 1$ and so (A1) holds.

Axioms (A2) and (A3) hold by the construction of $\mathbf{A}$, where we identify $\mathbf{A}(\alpha)$ with its image in $\mathbf{A}$. Axiom (Σ2) follows from (6) in Proposition 2.23. Axiom (S) follows from (2) in Proposition 2.23.

Next, we turn to axiom (Σ1). Let $2\alpha \in \Sigma \cap 2S$ and let $\sigma \in \Sigma \smallsetminus \{2\alpha\}$. The fact that $\langle \frac{1}{2}\alpha^\vee, \sigma\rangle \in \mathbb{Z}$ follows from (5) in Proposition 2.23. We need to show that

$$\langle \alpha^\vee, \sigma \rangle \leq 0, \tag{2.14}$$

but this follows from Lemma 2.25 below.

Step 2: We now check that $(\mathbb{Z}\Gamma, c)$ is an augmentation of $\mathscr{S}$; that is, we check all the conditions of Definition 2.18. Since $\Sigma \subsetneq \Sigma^{sc}(\Gamma)$, it follows from Proposition 2.23(1) that $\mathbb{Z}\Sigma \subseteq \mathbb{Z}\Gamma$. Axiom (a1) holds because $\mathscr{S}$ was equipped with the restriction of $c : \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Gamma \to \mathbb{Z}$ to $\mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma$. Axiom (a2) holds by the construction of c. Axiom (σ1) is an immediate consequence of Proposition 2.23(5). Similarly, axiom (σ2) follows from Proposition 2.23(6). Axiom (s), finally, is true by the definition of $S^p = S^p(\Gamma)$, cf. Definition 1.5.

Step 3: We verify condition (3) of Proposition 2.20. Let $\delta \in E(\Gamma)$ and $\sigma \in \Sigma$ such that $\langle \delta, \sigma \rangle > 0$. Let Δ be the set of colors of $\mathscr{S}$. We have to prove that there exists $D \in \Delta$ such that $c(D, \cdot)$ is a positive multiple of δ. We will consider two cases:

(i) $\sigma \notin S$;
(ii) $\sigma \in S$.

Suppose we are in case (i). Then Proposition 2.23(3) tells us there exists $\beta \in S \smallsetminus S^p$ such that $\beta^\vee|_{\mathbb{Z}\Gamma} \in \mathbb{Q}_{>0}\delta$. If $\beta \notin \Sigma$, then the construction of the full Cartan pairing of $\mathscr{S}$ implies that there exists $D \in \Delta$ such that $c(D, \cdot)$ is equal to $\beta^\vee|_{\mathbb{Z}\Gamma}$ or to $\frac{1}{2}\beta^\vee|_{\mathbb{Z}\Gamma}$. It follows that $c(D, \cdot)$ is a positive rational multiple of δ. On the other hand we claim that $\beta \in \Sigma$ is impossible. Indeed, if β were an element of Σ, then $\langle \delta, \beta \rangle = 1$ by (4c) of Proposition 2.23 and consequently $\delta \in a(\beta)$, which would imply, by (b) of the present proposition, that $\sigma \in S$. But this contradicts our assumption (i).

We now consider case (ii). Then $\delta \in a(\sigma)$ by Proposition 2.23 (4c). By the construction of c above in this proof, it follows that $\delta = c(D, \cdot)$ for at least one of the colors $D \in \mathbf{A}(\alpha)$.

Step 4: Finally, we verify condition (4) of Proposition 2.20. Suppose $D \in \mathbf{A}(\alpha)$ for some $\alpha \in S \cap \Sigma$. If $|a(\alpha)| = 1$, then $c(D, \cdot)$ is a positive rational multiple of a simple coroot and therefore takes nonnegative values on $\Gamma \subseteq \Lambda^+$. If $|a(\alpha)| = 2$, then we conclude, through (a), by condition (4b) of Proposition 2.23. ☐

Lemma 2.25. *Let Γ be a normal submonoid of Λ^+ and let $\sigma, \beta \in \Sigma^{sc}(\Gamma)$ with $\sigma \neq \beta$. If $\beta = 2\alpha \in 2S$, or $\beta = \alpha \in S$ with $|a(\alpha)| = 1$, then*

$$\langle \alpha^\vee, \sigma \rangle \leq 0. \tag{2.15}$$

Proof. We prove the lemma case-by-case for the different types of spherically closed spherical roots σ. For all but two of the types (the spherical root $\alpha_1 + \alpha_2$ with support of type B_2 and $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with $\operatorname{supp}(\sigma)$ of type C_n), we do this by showing that

$$\alpha \notin \operatorname{supp}(\sigma). \tag{2.16}$$

This implies (2.15) because σ is a linear combination with positive coefficients of the simple roots that make up $\mathrm{supp}(\sigma)$.

We will use a few times that the hypotheses on β imply that

$$\langle \alpha^\vee, \delta\rangle \in 2\mathbb{Z} \text{ for all } \delta \in \mathbb{Z}\Gamma. \tag{2.17}$$

If $\beta = 2\alpha$, then (2.17) is part of Proposition 2.23(5). On the other hand, if $\beta = \alpha$ with $|a(\alpha)| = 1$, then it follows form the definition of $a(\alpha)$ that $a(\alpha) = \{\frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}\}$ and that $\frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}$ takes integer values on $\mathbb{Z}\Gamma$. This means that (2.17) holds in this case as well.

For some of the types of spherical roots, we will take advantage of the following consequence of the fact that $\beta \in \mathbb{Z}\Gamma$:

$$\langle \alpha^\vee, \gamma\rangle = 0 \text{ for all } \gamma \in S^p(\Gamma). \tag{2.18}$$

We now proceed with the case-by-case verification.

- $\sigma \in S$: it is enough to show that $\sigma \neq \alpha$, since then (2.16) is trivial. If $\beta = \alpha$ then $\sigma \neq \alpha$ holds by assumption. If $\beta = 2\alpha$, then $2\alpha \in \Sigma^{sc}(\Gamma)$ and therefore, by Proposition 2.23(5), $\alpha \notin \mathbb{Z}\Gamma$, and in particular $\alpha \notin \Sigma^{sc}(\Gamma) \ni \sigma$.
- $\sigma \in 2S$: it is enough to show that $\sigma \neq 2\alpha$. If $\beta = 2\alpha$ then this is true by assumption. If $\beta = \alpha$ then it follows from Proposition 2.23(1) that $\alpha \in \mathbb{Z}\Gamma$ and then from (5) in the same Proposition that $2\alpha \notin \Sigma^{sc}(\Gamma)$.
- $\sigma = \alpha' + \beta'$ with $\mathrm{supp}(\sigma)$ of type $\mathsf{A}_1 \times \mathsf{A}_1$: since $\beta \in \Sigma^{sc}(\Gamma)$, Proposition 2.23(1) implies that $\beta \in \mathbb{Z}\Gamma$. Because $\sigma \in \Sigma^{sc}(\Gamma)$ it then follows from Proposition 2.23(6) that $\alpha \notin \mathrm{supp}(\sigma) = \{\alpha', \beta'\}$.
- $\sigma = \alpha_1 + \ldots + \alpha_n$ with $\mathrm{supp}(\sigma)$ of type A_n, $n \geq 2$: since σ is compatible with $S^p(\Gamma)$, the subset $\{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$ of $\mathrm{supp}(\sigma)$ belongs to $S^p(\Gamma)$. If $n \geq 3$, this implies, using (2.18), that $\alpha \notin \mathrm{supp}(\sigma)$. We now consider the case $n = 2$. Then $\mathrm{supp}(\sigma) = \{\alpha_1, \alpha_2\}$ and $\langle \alpha_1^\vee, \sigma\rangle = 1 = \langle \alpha_2^\vee, \sigma\rangle$. Since $\sigma \in \mathbb{Z}\Gamma$, it follows from (2.17) that $\alpha \notin \mathrm{supp}(\sigma)$.
- $\sigma = \alpha_1 + \ldots + \alpha_n$ with $\mathrm{supp}(\sigma)$ of type B_n, $n \geq 2$: for $n \geq 3$, the argument that $\alpha \notin \mathrm{supp}(\sigma)$ is the same as for the previous spherical root. When $n = 2$, then $\alpha \neq \alpha_1 \in \mathrm{supp}(\sigma)$ by (2.17), since $\langle \alpha_1^\vee, \sigma\rangle = 1$. If $\alpha = \alpha_2 \in \mathrm{supp}(\sigma)$, then $\langle \alpha^\vee, \sigma\rangle = 0$ and so (2.15) holds.
- $\sigma = \alpha_1 + \alpha_2$ with $\mathrm{supp}(\sigma)$ of type G_2: in this case $\alpha \notin \mathrm{supp}(\sigma) = \{\alpha_1, \alpha_2\}$ by (2.17) since $\langle \alpha_1^\vee, \sigma\rangle = -1 \notin 2\mathbb{Z}$ and $\langle \alpha_2^\vee, \sigma\rangle = 1 \notin 2\mathbb{Z}$.
- $\sigma = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with $\mathrm{supp}(\sigma)$ of type C_n, $n \geq 3$: it follows from the compatibility of σ with $S^p(\Gamma)$ that $\{\alpha_3, \alpha_4, \ldots, \alpha_n\} \subseteq S^p(\Gamma) \cap \mathrm{supp}(\sigma)$. This implies, using (2.18), that either $\alpha \notin \mathrm{supp}(\sigma)$ or $\alpha = \alpha_1 \in \mathrm{supp}(\sigma)$. Since $\langle \alpha_1^\vee, \sigma\rangle = 0$, equation (2.15) holds either way.
- The remaining six types of spherically closed spherical roots are all handled in the same way: if σ is of one of these types, then it follows from the compatibility of σ with $S^p(\Gamma)$, for each type, that $S^p(\Gamma)$ contains all but one of the simple roots in $\mathrm{supp}(\sigma)$. It then easily follows that (2.18) cannot hold for any $\alpha \in \mathrm{supp}(\sigma)$. To be more precise:
  - if $\sigma = \alpha_1 + 2\alpha_2 + \alpha_3$ with $\mathrm{supp}(\sigma)$ of type A_3, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains α_1 and α_3;
  - if $\sigma = 2(\alpha_1 + \ldots + \alpha_n)$ with $\mathrm{supp}(\sigma)$ of type B_n where $n \geq 2$, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$;
  - if $\sigma = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with $\mathrm{supp}(\sigma)$ of type B_3, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains α_1 and α_2;

- if $\sigma = 2(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ with $\mathrm{supp}(\sigma)$ of type D_n where $n \geq 4$, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$;
- if $\sigma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ with $\mathrm{supp}(\sigma)$ of type F_4, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains $\{\alpha_1, \alpha_2, \alpha_3\}$;
- if $\sigma = 4\alpha_1 + 2\alpha_2$ with $\mathrm{supp}(\sigma)$ of type G_2, then $S^p(\Gamma) \cap \mathrm{supp}(\sigma)$ contains α_2.

□

2.5. **G-saturated weight monoids.** In this section, we will look at the combinatorics of affine spherical varieties with G-saturated weight monoids (cf. Definition 1.2).

Proposition 2.26. *If Γ is a G-saturated submonoid of Λ^+, then $|a(\sigma)| = 1$ for every $\sigma \in \Sigma^{sc}(\Gamma) \cap S$.*

Proof. We will prove the contrapositive. Recall from Proposition 2.23 that $a(\sigma)$ has one or two elements for every $\sigma \in \Sigma^{sc}(\Gamma) \cap S$. Let $\sigma \in \Sigma^{sc}(\Gamma) \cap S$ with $|a(\sigma)| = 2$. We will show that there then exists a dominant weight γ' in $\mathbb{Z}\Gamma$ with $\gamma' \notin \Gamma$.

Let $\delta \in E(\Gamma)$ such that $a(\sigma) = \{\delta, \sigma^\vee|_{\mathbb{Z}\Gamma} - \delta\}$. It follows from $|a(\sigma)| = 2$ that δ and $\sigma^\vee|_{\mathbb{Z}\Gamma}$ are not proportional: if they were, then $\langle \delta, \sigma \rangle = 1 = \langle \sigma^\vee|_{\mathbb{Z}\Gamma} - \delta, \sigma \rangle$ would imply $|a(\sigma)| = 1$.

Since $\delta \in E(\Gamma)$, the set $\{\gamma \in \Gamma \colon \langle \delta, \gamma \rangle = 0\}$ spans $\ker \delta$, which is a sublattice of corank 1 of $\mathbb{Z}\Gamma$. As δ and $\sigma^\vee|_{\mathbb{Z}\Gamma}$ are not proportional and therefore have different kernels, this implies that there exists $\gamma \in \Gamma$ such that $\langle \delta, \gamma \rangle = 0$ and $\langle \sigma^\vee, \gamma \rangle \neq 0$. Then $\langle \sigma^\vee, \gamma \rangle \geq 0$ since γ is a dominant weight.

Let $\gamma' := 2\gamma - \sigma$. Clearly $\gamma' \in \mathbb{Z}\Gamma$. Moreover, $\langle \alpha^\vee, \gamma' \rangle \geq 0$ for all $\alpha \in S \smallsetminus \{\sigma\}$ since γ is dominant and $\langle \alpha^\vee, \sigma \rangle \leq 0$. Furthermore $\langle \sigma^\vee, \gamma' \rangle = \langle \sigma^\vee, 2\gamma \rangle - \langle \sigma^\vee, \sigma \rangle \geq 2 - 2 = 0$. Consequently, γ' is a dominant weight.

On the other hand $\langle \delta, \gamma' \rangle = 2\langle \delta, \gamma \rangle - \langle \delta, \sigma \rangle = -1$ which implies that $\gamma' \notin \Gamma$. This proves that Γ is not G-saturated. □

Corollary 2.27. *If $|a(\alpha)| = 1$ for all $\alpha \in \Sigma^{sc}(\Gamma) \cap S$, then $\Sigma^{sc}(\Gamma)$ is adapted to Γ. In particular, $\Sigma^{sc}(\Gamma)$ is adapted to Γ if Γ is G-saturated.*

Proof. The first assertion follows from Proposition 2.24 and Lemma 2.25; indeed, the Lemma implies that condition (b) of the Proposition is trivially met. The second assertion follows from the first, by Proposition 2.26. □

Corollary 2.28. *If Γ is a G-saturated submonoid of Λ^+, then*

(a) M_Γ is irreducible;
(b) up to G-equivariant isomorphism, there is exactly one generic *affine spherical G-variety X_Γ with weight monoid Γ;*
(c) $\Sigma^{sc}(X_\Gamma) = \Sigma^{sc}(\Gamma)$.

Proof. Assertion (a) follows from Corollaries 2.15 and 2.27. Assertion (b) follows from (a) and Proposition 2.3. Assertion (c), finally, is a consequence of Proposition 2.13. □

We now give the proof of Proposition 1.6 on page 3. Before doing so, we recall the following Definition from [BVS16].

Definition 2.29. Let Γ be a normal submonoid of Λ^+. We say that an element σ of $\Sigma^{sc}(G)$ is **N-adapted to** Γ if there exists an affine spherical variety X such that $\Gamma(X) = \Gamma$ and $\Sigma^N(X) = \{\sigma\}$. We use $\Sigma^N(\Gamma)$ for the set of all $\sigma \in \Sigma^{sc}(G)$ that are N-adapted to Γ.

Proof of Proposition 1.6. This is a special case of [BVS16, Corollary 2.17]. Since Γ is G-saturated, condition (4) of that Corollary is equivalent to (i) of Proposition 1.6, by Proposition 2.26. Condition (3) of the Corollary is redundant, by the Definition 1.2 of a G-saturated weight monoid. Conditions (1), (2), (5) and (6) of the Corollary are identical to conditions (ii), (iii), (iv) and (v), respectively, of Proposition 1.6. Finally, in statement (iii) of Proposition 1.6 we reformulate the compatibility of $\overline{\sigma}$ with $S^p(\Gamma)$, disregarding the condition $S^p(\Gamma) \subseteq \{\alpha \in S : \langle\alpha^\vee, \overline{\sigma}\rangle = 0\}$. This condition is redundant here, thanks to the definition of $S^p(\Gamma)$ and the fact that $\overline{\sigma} \in \mathbb{Z}\Gamma$. ☐

Proposition 2.30. *If Γ is a G-saturated submonoid of Λ^+ and X_Γ is as in Corollary 2.28(b), then $\Sigma^N(X_\Gamma) = \Sigma^N(\Gamma)$.*

Proof. The proposition follows from Corollary 2.28(c) and Proposition 2.7 by comparing Propositions 2.23 and 1.6. ☐

We end this section with a proposition formulated by Luna in his aforementioned 2005 working document. Together with equation (2.13) it gives a geometric characterization of the affine spherical G-varieties with G-saturated weight monoid.

Proposition 2.31 (Luna)**.** *Let X be an affine spherical G-variety with open G-orbit X_o. The weight monoid $\Gamma(X)$ of X is G-saturated if and only if the following two conditions hold:*

(1) $\operatorname{codim}_X(X \smallsetminus X_o) \geq 2$*;*
(2) $|a(\sigma)| = 1$ *for every* $\sigma \in \Sigma^{sc}(X) \cap S$*.*

Proof. We begin with the "only if" part, so assume that $\Gamma(X)$ is G-saturated. Part (2) follows from Proposition 2.26, because $\Sigma^{sc}(X) \subseteq \Sigma^{sc}(\Gamma(X))$ by Proposition 2.13. Let us show part (1). The weight monoid $\Gamma(X_o)$ of the quasi-affine spherical G-variety X_o is defined just like for an affine spherical variety. Since X_o is a dense G-stable subset of X, we have the inclusions

$$\Gamma(X) \subseteq \Gamma(X_o) \subseteq \mathbb{Z}\Gamma(X) \cap \Lambda^+,$$

which together with the fact that $\Gamma(X)$ is G-saturated yield $\Gamma(X) = \Gamma(X_o)$. Since the coordinate ring $\mathbb{C}[X_o]$ is a multiplicity free G-module, this implies that $\mathbb{C}[X] = \mathbb{C}[X_o]$. If now D is an irreducible subvariety of $X \smallsetminus X_o$ of codimension 1, then there exist two regular functions $f, g \in \mathbb{C}[X]$ such that f vanishes on D, and g vanishes on all irreducible components of $\{f = 0\}$ except D. It follows that we can find a positive integer n such that the quotient g^n/f has no poles on X except for D. Since X_o is normal, because it's open in the normal variety X, this implies that $g^n/f \in \mathbb{C}[X_o] \smallsetminus \mathbb{C}[X]$: contradiction. This proves part (1).

We prove the "if" part. It follows from the normality of X that the monoid $\Gamma(X)$ is the subset of $\mathbb{Z}\Gamma(X)$ where the valuations of all colors and all G-stable prime divisors of X take non-negative values. By condition (1) we have that X has no G-stable prime divisors. By condition (2) we know that the valuations of all colors of X moved by simple roots in $\Sigma^{sc}(X)$ are actually multiples of some simple coroot of G. This implies that all colors of X are multiples of simple coroots. Consequently

$\Gamma(X)$ is the set of all elements of $\mathbb{Z}\Gamma(X)$ on which some simple coroots take non-negative values. This shows that $\mathbb{Z}\Gamma(X) \cap \Lambda^+ \subseteq \Gamma(X)$, which implies that $\Gamma(X)$ is G-saturated. □

Remark 2.32. By Proposition 2.7, condition (2) in Proposition 2.31 is equivalent to

$$\Sigma^N(X) \cap S = \emptyset$$

3. Camus's smoothness criterion

We report in this section a smoothness criterion for spherical varieties due to R. Camus [Cam01], with a complete exposition of its original proof. We underline that this criterion does not require the varieties to be affine. Table 2 below includes groups of any type, whereas in *loc.cit.* only the cases with groups of type A were listed.

In this section, if A is any algebraic group, we denote by $\mathcal{X}(A)$ the group of its characters and we will use A^r for its radical. Recall that when G is a connected reductive group, G^r is the connected component of the center of G containing the identity.

Definition 3.1 ([Cam01])**.** Let X be a spherical G-variety.

(1) If X has a unique closed G-orbit then we say that X is **simple**. A simple spherical variety is **quasi-vectorial** if all its colors contain the closed G-orbit.
(2) If X is simple, we denote by $\mathcal{V}_X$ the set of G-stable prime divisors of X, by Δ_X the set of the colors containing the closed G-orbit, and we set $\mathcal{D}_X = \Delta_X \cup \mathcal{V}_X$.
(3) If X is a simple spherical variety, then its **socle** is

$$\mathrm{soc}(X) := (S, S^p(X), \Sigma^{sc}(X), \mathbf{A}(X), \Delta_X, \mathcal{V}_X, \rho'_X \colon \mathcal{D}_X \to (\mathbb{Z}\Sigma^{sc}(X))^*)$$

where $\rho'_X(D) := \rho_X(D)|_{\mathbb{Z}\Sigma^{sc}(X)}$ for all $D \in \mathcal{D}_X$.
(4) Equality of socles of two different varieties is defined as equality on the components S^p and Σ^{sc}, and bijections on the components $\mathbf{A}$, Δ, $\mathcal{V}$ compatibly with the maps ρ'.

Remark 3.2. In [Cam01] the socle $\mathrm{soc}(X)$ did not have the set of simple roots of G as a component. We include it to avoid confusion regarding the group we are considering, because the smoothness criterion (Theorem 3.16) and consequently our main result (Theorem 4.2) involve the socle of a certain spherical L-variety, where L is a Levi subgroup of G corresponding to a certain subset of S.

Notice that an affine spherical variety X is simple, since $\mathbb{C}[X]^G = \mathbb{C}$ and therefore $X/\!\!/G = \operatorname{Spec} \mathbb{C}[X]^G$ is a single point.

Let G_i be a connected reductive group and X_i a simple spherical G_i-variety for all $i \in \{1, 2\}$. Then the socle of the $G_1 \times G_2$-variety $X_1 \times X_2$ is $\mathrm{soc}(X_1) \times \mathrm{soc}(X_2)$, where the **product of two socles** is defined as the union of the two factors on the first six components, and defined accordingly on the component ρ'. The socle of a point under the action of a group G is not considered "trivial" when G is not abelian; in particular, $S^p = S$.

Spherical modules, i.e. finite dimensional G-modules that are spherical as G-varieties, will play a crucial role in what follows. The following well known proposition justifies the terminology "quasi-vectorial" (of Definition 3.1).

Proposition 3.3. *Let E be a spherical G-module. Then every color of E contains $\{0\}$, which is the unique closed orbit of E.*

Proof. Clearly $\{0\}$ is a closed orbit, and since E is affine and spherical, it is the only one. Let D be a color of E. Since the G-action on $\mathbb{C}[E]$ respects the grading and $\mathbb{C}[E]$ is multiplicity free, every irreducible submodule of $\mathbb{C}[E]$ is homogeneous. Since $\mathbb{C}[E]$ is a UFD, the prime divisor D has an irreducible global equation f_D, unique up to multiplication by a nonzero scalar. Since D is B-stable, f_D is a highest weight vector. It follows that f_D is homogeneous, and consequently $f_D(0) = 0$. □

Definition 3.4. If E is a spherical G-module, and

$$E = \bigoplus_{i \in I} E_i$$

a decomposition into irreducibles, then γ_i^E (for any $i \in I$) denotes the B-eigenvalue of a B-eigenvector $f_i \in \mathbb{C}[E_i]$ of degree 1. It is also the highest weight of E_i^*. The set

$$D_i := \{v \in E_i : f_i(v) = 0\} \times \bigoplus_{j \in I \smallsetminus \{i\}} E_j$$

is a B-stable hyperplane of E. Write

$$\mathcal{D}_E^1 = \{D_i \mid i \in I\},$$

and for each $D = D_i$ as above denote γ_i^E also by γ_D^E.

Lemma 3.5. *In the notations of Definition 3.4, the group $\Lambda(E)$ is generated by $\Sigma^N(E) \cup \{\gamma_i^E \mid i \in I\}$.*

Proof. Since $\Sigma^N(E) \cup \{\gamma_i^E \mid i \in I\}$ is a subset of $\Lambda(E)$ and since $\Lambda(E) = \mathbb{Z}\Gamma(E)$ it is enough to show that the subgroup Ξ of $\Lambda(E)$ generated by $\Sigma^N(E)$ and $\{\gamma_i^E \mid i \in I\}$ contains all the highest weights of $\mathbb{C}[E]$. Let M be a simple submodule of $\mathbb{C}[E]$; then M is contained in a product

$$M \subseteq \prod_{i \in I} (E_i^*)^{a_i}$$

where each $a_i \in \mathbb{Z}_{\geq 0}$. We proceed by induction on $s = \sum_i a_i$. If $s = 1$ then $M = E_i$ for some i, whence the highest weight of M is γ_i^E.

We now show the induction step. Write

$$M \subseteq N E_i^* \tag{3.1}$$

for some $i \in I$, where

$$N = \left(\prod_{j \in I \smallsetminus \{i\}} (E_j^*)^{a_j} \right) \cdot (E_i^*)^{a_i - 1}.$$

By the induction hypothesis, the highest weights of all simple submodules of N are in the group Ξ. We want to replace N by an irreducible submodule in (3.1), so we write a decomposition

$$N = \bigoplus_k N_k$$

of N into simple submodules. The product NE_i^* is the sum of the products $N_k E_i^*$, hence M (which is an isotypic component of $\mathbb{C}[E]$, by the sphericity of E) is contained in $N_k E_i^*$ for some k. By the definition of $\Sigma^N(E)$ and because both N_k and

E_i^* are irreducible, the highest weight of M is then of the form $\mu = \eta + \gamma_i^E - \sigma$, where η is the highest weight of N_k and σ is in the monoid generated by $\Sigma^N(E)$. It follows that μ is in Ξ. □

Remark 3.6. If E is a spherical G-module, then $\Sigma^N(E) = \Sigma^{sc}(E)$ by Proposition 2.7. Indeed, since $\mathbb{C}[E]$ is a UFD, no two colors can have the same valuation.

Lemma 3.7. *For every $i \in \{1,2\}$ let G_i be a connected reductive group, suppose that $(G_1,G_1) \cong (G_2,G_2)$ and identify these two groups via a fixed isomorphism; choose moreover Borel subgroups $B_i \subseteq G_i$ in such a way that $B_1 \cap (G_1,G_1) = B_2 \cap (G_2,G_2)$. Let X be a simple spherical G_i-variety for every $i \in \{1,2\}$, suppose that the actions of (G_1,G_1) and of (G_2,G_2) coincide, and that the actions of G_1^r and G_2^r commute. Then the socles of X with respect to the actions of G_1 and G_2 (resp. computed with respect to the Borel subgroups B_1 and B_2) coincide.*

Proof. We may suppose that $G_i = (G_i,G_i) \times G_i^r$, whence $B_i = (B_i \cap (G_i,G_i)) \times G_i^r$, and we denote X by X_i when the G_i-action is considered. Notice that the images in $\operatorname{Aut}(X)$ of G_1 and of G_2 normalize each other, so G_i permutes the orbits of G_{3-i} for all $i \in \{1,2\}$. Since G_{3-i} has finitely many orbits, and G_i is connected, we deduce that G_i stabilizes each G_{3-i}-orbit. It follows that G_1 and G_2 have the same orbits on X, and a similar argument yields that the B_1-orbits and the B_2-orbits also coincide. As a consequence, X has the same colors and the same G_i-invariant divisors with respect to both actions: the equalities $\Delta_{X_1} = \Delta_{X_2}$, and $\mathcal{V}_{X_1} = \mathcal{V}_{X_2}$ follow. The same argument as above, applied to the parabolic subgroups of G_i that contain B_i, shows that a simple root moves a color of X_1 if and only if it moves the same color of X_2. This implies the equality $\mathbf{A}(X_1) = \mathbf{A}(X_2)$.

Notice that for all $i \in \{1,2\}$ the parabolic subgroup P_{X_i} is equal to $(P_{X_i} \cap (G_i,G_i)) \times G_i^r$, and $P_{X_i} \cap (G_i,G_i)$ is the stabilizer of the open B_i-orbit of X_i in (G_i,G_i). Since the open B_1-orbit coincides with the open B_2-orbit, we have $P_{X_1} \cap (G_1,G_1) = P_{X_2} \cap (G_2,G_2)$, which yields $S^p(X_1) = S^p(X_2)$.

Moreover, the group B_i stabilizes the set $\mathbb{C}(X_{3-i})^{(B_{3-i})}_\lambda$ for all $\lambda \in \Lambda(X_{3-i})$ and all $i \in \{1,2\}$, where $\mathbb{C}(X_{3-i})^{(B_{3-i})}_\lambda$ denotes the set of B_{3-i}-eigenvectors of B_{3-i}-weight λ. Therefore a rational function on X is a B_1-eigenvector if and only if it is a B_2-eigenvector, and if λ_i is its B_i-eigenvalue, then $\lambda_1|_{B_1\cap(G_1,G_1)} = \lambda_2|_{B_2\cap(G_2,G_2)}$. Let $\mathcal{X}'$ denote the subgroup of $\mathcal{X}(B_i)$ of those elements whose restriction to G_i^r is zero (notice that $\mathcal{X}'$ is naturally a sublattice of both $\mathcal{X}(B_1)$ and $\mathcal{X}(B_2)$). The considerations above imply that $\Lambda(X_1) \cap \mathcal{X}' = \Lambda(X_2) \cap \mathcal{X}'$.

Suppose now that X is quasi-affine, and consider the root monoids $\mathscr{M}_{X_i}$. The primitive elements in $\Lambda(X_i)$ on the extremal rays of $\mathbb{Q}_{\geq 0}\mathscr{M}_{X_i}$ are the spherical roots of X_i. Now, the irreducible submodules of $\mathbb{C}[X_1]$ and $\mathbb{C}[X_2]$ are the same subspaces, and the two highest weights of the same irreducible submodule coincide on $B_i \cap (G_i,G_i)$. This implies that $\mathscr{M}_{X_1} = \mathscr{M}_{X_2}$, so $\Sigma(X_1) = \Sigma(X_2)$ since $\Sigma(X_i) \subseteq \Lambda(X_i) \cap \mathcal{X}'$. This implies that $\Sigma^{sc}(X_1) = \Sigma^{sc}(X_2)$ by Proposition 2.7. Even more is true: for all $\sigma \in \Sigma^{sc}(X_i)$ we have $\mathbb{C}(X)^{(B_1)}_\sigma = \mathbb{C}(X)^{(B_2)}_\sigma$, since $\sigma \in \Lambda(X_i) \cap \mathcal{X}'$.

It follows that $\rho'_{X_1} = \rho'_{X_2}$, since these two maps are computed considering the vanishing of the same rational functions on X (precisely, those in $\mathbb{C}(X)^{(B_i)}_\sigma$ for $\sigma \in \Sigma^{sc}(X_i)$) along the same prime divisors of X. We have shown that $\operatorname{soc}(X_1) = \operatorname{soc}(X_2)$ when X is quasi-affine.

In general X may not be quasi-affine, but being simple it is quasi-projective by a theorem of H. Sumihiro [Sum74, Theorem 1]. Therefore we can consider X under the action of $G_1 \times G_2^r$ (which naturally contains both groups G_1 and G_2 thanks to our assumptions), and we can choose an embedding of X as a locally closed subset of $\mathbb{P}(V)$, where V is a $G_1 \times G_2^r$-module, in such a way that the embedding is both G_1- and G_2-equivariant. Let $X' \subset V$ be the cone over $X \subseteq \mathbb{P}(V)$, and define $Y = X' \smallsetminus \{0\}$.

Then Y is a quasi-affine spherical $G_i \times \mathbb{G}_m$-variety, and it satisfies the hypotheses of the lemma with respect to the $G_i \times \mathbb{G}_m$-actions (where all invariants are computed with the Borel subgroups $B_i \times \mathbb{G}_\mathrm{m}$). Thanks to the first part of the proof we have $\mathrm{soc}(Y_1) = \mathrm{soc}(Y_2)$, where Y_i is Y under the action of $G_i \times \mathbb{G}_\mathrm{m}$.

Let X_i^m be X_i equipped with the action of $G_i \times \mathbb{G}_\mathrm{m}$, where $\mathbb{G}_\mathrm{m}$ acts trivially, and observe that the natural projection $\pi_i \colon Y_i \to X_i^m$ is $G_i \times \mathbb{G}_\mathrm{m}$-equivariant. This map induces a bijection of the sets of $G_i \times \mathbb{G}_\mathrm{m}$-orbits, of the sets of $B_i \times \mathbb{G}_\mathrm{m}$-orbits, and by pull-back an inclusion of the groups of $B_i \times \mathbb{G}_\mathrm{m}$-semiinvariant rational functions compatible with the maps ρ. It follows $S^p(Y_i) = S^p(X_i^m)$, $\mathbf{A}(Y_i) = \mathbf{A}(X_i^m)$, $\Delta_{Y_i} = \Delta_{X_i^m}$, and $\mathcal{V}_{Y_i} = \mathcal{V}_{X_i^m}$ for each $i \in \{1, 2\}$.

Moreover, a generic stabilizer $H_{X_i^m}$ of X_i^m is equal to $H_{Y_i} \cdot (\{e\} \times \mathbb{G}_\mathrm{m})$, where H_{Y_i} is a generic stabilizer of Y_i. It follows that $H_{X_i^m}/H_{Y_i}$ is connected. We claim that this implies $\Sigma^{sc}(Y_i) = \Sigma^{sc}(X_i^m)$ for all $i \in \{1, 2\}$.

To prove the claim, we consider the inclusion $\pi_i^* \colon \Lambda(X_i^m) \to \Lambda(Y_i)$ (which is induced by pulling back rational functions along π_i), and the corresponding dual map $\pi_{i,*} \colon \mathrm{Hom}_\mathbb{Z}(\Lambda(Y_i), \mathbb{Z}) \to \mathrm{Hom}_\mathbb{Z}(\Lambda(X_i^m), \mathbb{Z})$. Since $H_{X_i^m}$ normalizes H_{Y_i}, from [Kno91, First part of the proof of Theorem 6.1] it follows that the kernel of π_* is contained in the linear part of the valuation cone of Y_i. Then [Kno91, Theorem 4.4] yields the equality $\Sigma^{sc}(Y_i) = \Sigma^{sc}(X_i^m)$ up to replacing some elements with positive rational multiples.

Since $H_{X_i^m}/H_{Y_i}$ is connected, [Gan11, Lemma 2.4] implies that $\Lambda(Y_i)/\Lambda(X_i^m)$ has no torsion, whence an element in $\Lambda(Y_i)$ is primitive if and only if it is primitive in $\Lambda(X_i^m)$. The equality $\Sigma(Y_i) = \Sigma(X_i^m)$ follows, and we deduce the claimed equality $\Sigma^{sc}(Y_i) = \Sigma^{sc}(X_i^m)$ from Proposition 2.7, thanks to the equality $S^p(Y_i) = S^p(X_i^m)$.

This concludes the proof that $\mathrm{soc}(Y_i) = \mathrm{soc}(X_i^m)$ for each $i \in \{1, 2\}$. Since obviously $\mathrm{soc}(X_i^m) = \mathrm{soc}(X_i)$, the proof is complete. □

Lemma 3.8. *Let X be a simple spherical variety. If its socle is equal to the socle of a quasi-vectorial variety Y, then X is quasi-vectorial.*

Proof. Since Y is quasi-vectorial then $|\Delta_Y| = |\Delta(Y)|$. By Remark 2.19 the spherical system $\mathscr{S}(Z)$ of a spherical variety Z determines the number of colors of Z. We deduce that $|\Delta_X| = |\Delta(X)|$, i.e. X is quasi-vectorial. □

The following proposition is a key step in the proof of the smoothness criterion. We remark that it uses the uniqueness statement in the Luna Conjecture, which was proved by Losev in [Los09b].

Proposition 3.9. *A simple spherical variety X is G-isomorphic to a spherical module if and only if*

1. *its socle is the socle of a spherical module; and*
2. *the $|\mathcal{D}_X|$-tuple $(\rho_X(D))_{D \in \mathcal{D}_X}$ is a basis of $\Lambda(X)^*$.*

Proof. Suppose that X is a spherical G-module. Then any element $D \in \mathcal{D}_X$ has a global equation $f_D \in \mathbb{C}[X]^{(B)}$, where $\mathbb{C}[X]^{(B)}$ denotes the set of B-eigenvectors of $\mathbb{C}[X]$. Its B-eigenvalue γ_D belongs to $\Lambda(X)$, takes value 1 on $\rho_X(D)$ and value 0 on $\rho_X(D')$ where D' is any element of $\mathcal{D}_X$ different from D. This shows that both $(\rho_X(D))_{D\in\mathcal{D}_X}$ and $(\gamma_D)_{D\in\mathcal{D}_X}$ are linearly independent in $\Lambda(X)^*$ and $\Lambda(X)$, respectively.

Now part (2) follows if we show that $(\gamma_D)_{D\in\mathcal{D}_X}$ generates $\Lambda(X)$. Pick $\lambda \in \Lambda(X)$, choose $f \in \mathbb{C}(X)^{(B)}_\lambda$, where $\mathbb{C}(X)^{(B)}_\lambda$ denotes the set of B-eigenvectors of B-weight λ, and consider

$$F = \prod_{D\in\mathcal{D}_X} f_D^{\langle \rho_X(D),\lambda\rangle}.$$

Then F is a B-semiinvariant rational function on X, with $\operatorname{div}(F) = \operatorname{div}(f)$ and B-eigenvalue belonging to the group generated by $(\gamma_D)_{D\in\mathcal{D}_X}$. The quotient F/f is then an invertible rational function on X, therefore constant. We deduce that F and f have the same B-eigenvalue, hence λ is in the group generated by $(\gamma_D)_{D\in\mathcal{D}_X}$.

Now we show that the conditions (1) and (2) are sufficient for X to be G-isomorphic to a spherical module. Without loss of generality, we suppose that $G = (G, G) \times G^r$.

Let E be a spherical G-module with the same socle as X, and let $E = \bigoplus_{i\in I} E_i$ be its decomposition into irreducibles as in Definition 3.4. The G-action on E is not uniquely determined by the socle; in particular the socle gives no information on the action of G^r (see e.g. Lemma 3.7). To prevent any difficulty arising from this fact, we let the bigger group $\widetilde{G} = G \times \mathrm{GL}(E)^G$ act on E in the obvious way, where we denote by $\mathrm{GL}(E)^G$ the centralizer in $\mathrm{GL}(E)$ of the image of G. We also let $\widetilde{G}$ act on X, by letting the $|I|$-dimensional torus $\mathrm{GL}(E)^G$ act trivially. Denote by $\widetilde{C}$ the radical $G^r \times \mathrm{GL}(E)^G$ of $\widetilde{G}$, by $\widetilde{T}$ the maximal torus $T \times \mathrm{GL}(E)^G$, and by $\widetilde{B}$ the Borel subgroup $B \times \mathrm{GL}(E)^G$ of $\widetilde{G}$. Notice that the assumptions (1) and (2) of the proposition also hold for the $\widetilde{G}$-action, and that proving that X is $\widetilde{G}$-isomorphic to E implies that X is G-isomorphic to E. In what follows, all invariants are now relative to the $\widetilde{G}$-action. We will write Σ^{sc} for the two equal sets $\Sigma^{sc}(X) = \Sigma^{sc}(E)$, and since $\mathcal{D}_X$ is identified with $\mathcal{D}_E$ via a fixed bijection, we may sometimes write just $\mathcal{D}$ for both sets, and the same for Δ_X and $\mathcal{V}_X$.

For any $D \in \mathcal{D}_E$ (resp. $\mathcal{D}_X$), let γ^E_D (resp. γ^X_D) be the element corresponding to D in the basis of $\Lambda(E)$ (resp. $\Lambda(X)$) dual to $\rho_E(\mathcal{D}_E)$ (resp. $\rho_X(\mathcal{D}_X)$). Notice that this notation is compatible with Definition 3.4. A rational function $f \in \mathbb{C}(X)^{(\widetilde{B})}$ with $\widetilde{B}$-eigenvalue γ^X_D vanishes on D and has no zero nor pole on any other $\widetilde{B}$-stable prime divisor, since $\langle \rho_X(D'), \gamma^X_D\rangle = \delta_{D',D}$ for all $D' \in \mathcal{D}_X$ and $\mathcal{D}_X$ is the set of all B-stable prime divisors on X because X is quasi-vectorial by Proposition 3.3 and Lemma 3.8. Consequently f is a global equation of D.

Write $\gamma^E_D = \eta^E_D + \epsilon^E_D$ and $\gamma^X_D = \eta^X_D + \epsilon^X_D$, where $\eta^E_D, \eta^X_D \in \mathcal{X}(\widetilde{T} \cap (\widetilde{G}, \widetilde{G}))$ and $\epsilon^E_D, \epsilon^X_D \in \mathcal{X}(\widetilde{C})$.

Consider now the set $\mathcal{D}^1_E$ of Definition 3.4. The restrictions $(\epsilon^E_D)|_{\mathrm{GL}(E)^G}$ for D varying in $\mathcal{D}^1_E$ are by construction a basis of $\mathcal{X}(\mathrm{GL}(E)^G)$. It follows that the weights ϵ^E_D are a basis of the lattice Ξ they generate and that the latter is saturated inside $\mathcal{X}(\widetilde{C})$. Hence Ξ is a direct summand of $\mathcal{X}(\widetilde{C})$, and there exists a homomorphism $\mathcal{X}(\widetilde{C}) \to \mathcal{X}(\widetilde{C})$ sending ϵ^E_D to ϵ^X_D for all $D \in \mathcal{D}^1_E$.

We extend the corresponding homomorphism $\widetilde{C} \to \widetilde{C}$ to $\widetilde{G}$ via the identity on $(\widetilde{G}, \widetilde{G})$, and we denote the extension by $\phi\colon \widetilde{G} \to \widetilde{G}$. It also induces a homomorphism $\phi^*\colon \mathcal{X}(\widetilde{T}) \to \mathcal{X}(\widetilde{T})$.

Now consider generic stabilizers $H_E \subseteq \widetilde{G}$ of E and H_X of X. The homogeneous spaces $\widetilde{G}/\overline{H}_X$ and $\widetilde{G}/\overline{H}_E$ have the same spherical system. By [Los09b, Theorem 1] we may assume that $\overline{H}_E = \overline{H}_X$.

It follows that the pull-back on $\widetilde{G}$ of any $D \in \Delta_X$ along $\widetilde{G} \to \widetilde{G}/H_X$ coincides with the pull-back of the corresponding color of E along $\widetilde{G} \to \widetilde{G}/H_E$, because D is the pull-back of a color of $\widetilde{G}/\overline{H}_X = \widetilde{G}/\overline{H}_E$ along $\widetilde{G}/H_X \to \widetilde{G}/\overline{H}_X$. Since γ_D^X is the eigenvalue of a global equation of D in X and γ_D^E is the eigenvalue of a global equation of the corresponding color in E, the weights γ_D^E and γ_D^X are the $\widetilde{B}$-eigenvalues of two global equations in $\mathbb{C}[\widetilde{G}]$ of this pull-back, hence they differ only by a character of $\widetilde{C}$. This shows that $\eta_D^X = \eta_D^E$ for all $D \in \Delta$.

On the other hand if $D \in \mathcal{V}_X$ (resp. $\mathcal{V}_E$), then D is $\widetilde{G}$-stable and γ_D^X (resp. γ_D^E) is the $\widetilde{G}$-eigenvalue of a global equation of D in X (resp. E). It follows that γ_D^X and γ_D^E are $\widetilde{G}$-characters, and therefore $\eta_D^X = \eta_D^E = 0$ for all $D \in \mathcal{V}$.

At this point we have shown that $\phi^*(\gamma_D^E) = \gamma_D^X$ for all $D \in \mathcal{D}_E^1$. Since we know that ϕ^* is the identity on Σ^{sc}, it follows from Lemma 3.5 and Remark 3.6 that $\phi^*(\Lambda(E)) \subseteq \Lambda(X)$, hence ϕ^* also induces a dual map $\phi^{**}\colon \Lambda(X)^* \to \Lambda(E)^*$.

We compare now $\rho_E(D)$ and $\phi^{**}(\rho_X(D))$ for all $D \in \mathcal{D}$. They are equal on Σ^{sc} by hypothesis, and

$$\langle \phi^{**}(\rho_X(D)), \gamma_{D'}^E \rangle = \langle \rho_X(D), \phi^*(\gamma_{D'}^E) \rangle = \langle \rho_X(D), \gamma_{D'}^X \rangle = \delta_{D,D'} = \langle \rho_E(D), \gamma_{D'}^E \rangle$$

for all $D' \in \mathcal{D}_E^1$. This means that $\rho_E(D)$ and $\phi^{**}(\rho_X(D))$ coincide on a set of generators of $\Lambda(E)$, therefore they are equal for all $D \in \mathcal{D}$. Since $(\rho_E(D))_{D \in \mathcal{D}}$ is a basis of $\Lambda(E)^*$ and $(\rho_X(D))_{D \in \mathcal{D}}$ is a basis of $\Lambda(X)^*$, we also have that ϕ^{**} is an isomorphism. Consequently, so is $\phi^*|_{\Lambda(E)}\colon \Lambda(E) \to \Lambda(X)$.

We define a new action of $\widetilde{G}$ on E, denoting the obtained module by E'. Let $\psi\colon \widetilde{G} \to \mathrm{GL}(E)$ be the homomorphism induced by our original action, set $E' = E$ as vector spaces, and define $\psi'\colon \widetilde{G} \to \mathrm{GL}(E')$ to be $\psi' = \psi \circ \phi$. We claim that E' is spherical, that $\Lambda(E') = \Lambda(X)$, $\mathcal{V}(E') = \mathcal{V}(X)$, and that $\mathcal{D}_{E'}$ can be identified with $\mathcal{D}_X$ compatibly with the maps $\rho_{E'}$ and ρ_X.

For the first claim, decompose

$$\mathbb{C}[E] = \bigoplus_{\lambda \in \Gamma(E)} V(\lambda)$$

into a sum of irreducibles. Each $V(\lambda)$ is also an irreducible submodule of $\mathbb{C}[E']$, of highest weight $\phi^*(\lambda)$. Since $\mathbb{C}[E]$ is multiplicity free and ϕ^* is injective on $\Lambda(E)$, we deduce that $\mathbb{C}[E']$ is also multiplicity free. This proves that E' is spherical.

For the second claim, notice that E' and E have the same socle thanks to Lemma 3.7; then, with similar considerations as for the first claim, we have equality of the lattices $\Lambda(E') = \Lambda(X)$. The equality $\rho_{E'}(D) = \rho_X(D)$ for all $D \in \mathcal{D}$ follows from the definition of E'.

By [Los09b, Theorem 1], the open $\widetilde{G}$-orbits of X and E' are isomorphic. Finally, we apply [Kno91, Theorem 2.3] to the simple varieties X and E' and we deduce that they are equivariantly isomorphic. □

Corollary 3.10. *For an affine spherical G-variety X the following are equivalent:*

(1) X is smooth and quasi-vectorial;
(2) X is isomorphic to a product $G/K \times V$, where $K \subseteq G$ is a subgroup containing (G,G) and V is a spherical G-module;
(3) the following two conditions hold:
(a) its socle is the socle of a spherical module,
(b) the $|\mathcal{D}_X|$-tuple $(\rho_X(\mathcal{D}_X))_{D\in\mathcal{D}_X}$ can be completed to a basis of $\Lambda(X)^*$.

Proof. It is harmless to assume in the whole proof that $G = (G,G) \times G^r$.

(1) ⇒ (2). Since X is smooth and affine, it is isomorphic to a vector bundle

$$X \cong G \times^K V$$

on its closed G-orbit G/K, with fiber a K-module V, where K is a reductive group and V is spherical under the action of the connected component of K containing the identity; see [KVS06, Corollary 2.2]. The closed orbit G/K has no colors, otherwise the inverse image in X of one of its colors would be a color of X not containing G/K. It follows from [Kno94, Proposition 2.4] that $K \supset (G,G)$, thus $K = (G,G) \times R$ where $R \subseteq G^r$. Notice that then $\mathcal{X}(R)$ is a quotient of the free abelian group $\mathcal{X}(G^r)$. The homomorphism $R \to \mathrm{GL}(V)$ given by the action has image contained in a maximal torus H of $\mathrm{GL}(V)$, and the corresponding homomorphism $\mathcal{X}(H) \to \mathcal{X}(R)$ lifts to a homomorphism $\mathcal{X}(H) \to \mathcal{X}(G^r)$. This implies that there exists an action of G^r, and thus of G, on V extending that of K. We conclude that $X \cong G/K \times V$.

(2) ⇒ (1). We only have to show that all colors of V contain 0, its unique closed G-orbit. This is Proposition 3.3.

(2) ⇒ (3). We know that $K = (G,G)\times R$ where $R \subseteq G^r$, and $X \cong (G^r/R)\times V$. Let us consider the action of $C\times G$ on X where $C \cong G^r$ and acts only on the factor G^r/R via its isomorphism with G^r, and G acts only on the factor V. Denote the newly obtained $C \times G$-variety by X'.

Then $\mathrm{soc}(X) = \mathrm{soc}(X')$ by Lemma 3.7. On the other hand X' is of the form $X_1 \times X_2$ where one factor of $C \times G$ acts only on X_1, and the other only on X_2. Therefore $\mathrm{soc}(X') = \mathrm{soc}(G^r/R) \times \mathrm{soc}(V)$. Since C is abelian, the socle $\mathrm{soc}(G^r/R)$ is trivial, and $\mathrm{soc}(X') = \mathrm{soc}(V)$, which is condition (3a). Then, notice that

$$\mathbb{C}[X] = \mathbb{C}[G/K] \otimes_{\mathbb{C}} \mathbb{C}[V],$$

which implies that $\Lambda(X) = \Lambda(G/K) \oplus \Lambda(V)$.

Consider the projection $\pi\colon X \to V$. The open B-orbit X_0 of X is mapped onto the open B-orbit V_0 of V. For an element $v \in V_0$, the stabilizer B_v of v in B acts on $\pi^{-1}(v) \cap X_0$ with an open orbit. The fiber $\pi^{-1}(v)$ is isomorphic to G/K, and we conclude that B_v has an open orbit $B_v \cdot gK$ on G/K for some $g \in G$. In other words the subgroup B_vK/K of G/K has an open right coset, so $B_vK = G$ and hence $\pi^{-1}(v) \cap X_0 = G/K \times \{v\}$. This shows that $X_0 = G/K \times V_0$.

It follows that the map

$$\begin{array}{ccc} \mathcal{D}_V & \to & \mathcal{D}_X \\ D & \mapsto & G/K \times D \end{array}$$

is a bijection, and the element $\rho_X(G/K\times D)$ is equal to $\rho_V(D)$ on $\Lambda(V)$ and is zero on $\Lambda(G/K)$. Thanks to Proposition 3.9 the $|\mathcal{D}_V|$-tuple $(\rho_V(D))_{D\in\mathcal{D}_V}$ is a basis of $\Lambda(V)^*$, and it follows that $(\rho_X(D))_{D\in\mathcal{D}_X}$ can be completed to a basis of $\Lambda(X)^*$. This shows (3b).

(3) ⇒ (2). Since the socle of X is that of a spherical module, the set Δ_X is the whole set of colors of X by Lemma 3.8 and Proposition 3.3.

Consider the vector subspace N_1 spanned by $\rho_X(\mathcal{D}_X)$ in the vector space $N = \Lambda(X)^* \otimes_{\mathbb{Z}} \mathbb{Q}$. We claim that (N_1, Δ_X) is a colored subspace of N in the sense of [Kno91, Section 4]. To show the claim, it is enough to prove that N_1 is generated as a convex cone by $\rho_X(\Delta_X)$ together with $N_1 \cap \mathcal{V}(X)$ (observe that $N_1 \cap \mathcal{V}(X)$ is a polyhedral cone and therefore generated by finitely many elements of $\mathcal{V}(X)$). To show this, we first observe that N_1 contains the cone generated by $\rho_X(\Delta_X)$ and $N_1 \cap \mathcal{V}(X)$. To prove the other inclusion, notice that $\mathcal{V}(X)$ together with $\rho_X(\Delta_X)$ generates N as a convex cone, thanks to [Kno91, Theorem 4.4] applied to the surjective G-equivariant map $X \to \{\mathrm{pt}\}$. Let $n \in N_1$. Since $n \in N$ there exists $u \in \mathbb{Q}_{\geq 0}\rho_X(\Delta_X)$ and $v \in \mathbb{Q}_{\geq 0}\mathcal{V}(X)$ such that $n = u + v$. Since $\mathcal{V}(X)$ is a convex cone and $n - u \in N_1$, it follows that $v \in N_1 \cap \mathcal{V}(X)$. This finishes the proof of the claim.

Let X_0 be the open G-orbit of X. By [Kno91, Theorem 4.4], there exists a spherical G-homogeneous space $Y := G/K$ and a surjective G-equivariant map $\varphi : X_0 \to Y$ with connected fibers such that the sequence

$$0 \to N_1 \to N \stackrel{\varphi_*}{\to} \Lambda^*(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

is exact and Δ_X is exactly the set of colors of X_0 mapped dominantly onto Y (up to identifying the colors of X and of X_0). By the same theorem, Y has no colors and so $(G, G) \subset K$ by [Kno94, Proposition 2.4]. By [Kno91, Lemma 4.3] the map $\varphi^* \colon \Lambda(Y) \to \Lambda(X_0)$, i.e. the map induced by the pull-back of functions from Y to X_0, identifies $\Lambda(Y)$ with $\{\chi \in \Lambda(X_0) \colon \langle \chi, v\rangle = 0 \text{ for all } v \in N_1\}$, which is a direct summand of $\Lambda(X_0)$. Moreover, since $\rho_X(\mathcal{D}_X) \subseteq N_1$, by [Kno91, Theorem 4.1] the map $X_0 \to Y$ extends to a G-equivariant map $X \to Y$.

Now, by [Tim11, page 4] we can write X as a G-equivariant bundle over $Y = G/K$:

$$X \cong G \times^K V$$

where V is the fiber over $eK \in G/K$ of the map $X \to Y$. As in the proof of "(1) $\Rightarrow$ (2)" the variety V is also a G-variety and the bundle is trivial:

$$X \cong G/K \times V.$$

Hence $\Lambda(X) = \Lambda(G/K) \oplus \Lambda(V)$, and $N_1 = \Lambda(G/K)^{\perp}$. With the same argument as in the proof of (2) $\Rightarrow$ (3) we have that $\mathrm{soc}(X) = \mathrm{soc}(V)$ and that $(\rho_V(D))_{D \in \mathcal{D}_V}$ is identified with $(\rho_X(D))_{D \in \mathcal{D}_X}$ under the identification of $\Lambda(V)^*$ with $N_1 \cap \Lambda(X)^*$. By assumption $(\rho_X(D))_{D \in \mathcal{D}_X}$ can be extended to a basis of $\Lambda(X)^*$, and so it is a basis of $N_1 \cap \Lambda(X)^*$, since $(\rho_X(D))_{D \in \mathcal{D}_X}$ spans N_1 as a vector space. In other words, $(\rho_V(D))_{D \in \mathcal{D}_V}$ is a basis of $\Lambda(V)^*$.

Now V satisfies the hypotheses of Proposition 3.9, so it is a spherical G-module and the proof is complete. □

In order to apply Corollary 3.10 to a general spherical variety, we need to introduce a certain form of *localization* of simple spherical varieties (and of socles). It is related to the notion of localization on a subset of simple roots introduced in [Lun01, Section 3]. Recall that we defined what it means for color to be moved by a simple root on page 9.

Definition 3.11. Let

$$\mathrm{soc}(X) = (S, S^p(X), \Sigma^{sc}(X), \mathbf{A}(X), \Delta_X, \mathcal{V}_X, \rho'_X \colon \mathcal{D}_X \to (\mathbb{Z}\Sigma^{sc}(X))^*)$$

be the socle of a simple spherical G-variety X. Let S' be the set of simple roots of G that only move colors of X which contain the closed orbit of X, that is,

$$S' = \{\alpha \in S \colon \text{if } D \in \Delta(X) \text{ and } \alpha \text{ moves } D, \text{ then } D \in \Delta_X\}.$$

The **localization** $\overline{\mathrm{soc}}(X)$ **of** $\mathrm{soc}(X)$ is defined as follows:

$$\overline{\mathrm{soc}}(X) = (S', S^p(X)_{S'}, \Sigma^{sc}(X)_{S'}, \mathbf{A}(X)_{S'}, \Delta_{X,S'}, \mathcal{V}_{X,S'}, \rho'_{X,S'})$$

where

(1) $S^p(X)_{S'} = S^p(X) \cap S'$,
(2) $\Sigma^{sc}(X)_{S'} = \Sigma^{sc}(X) \cap \mathbb{Z}S'$,
(3) $\mathbf{A}(X)_{S'} = \bigcup_{\alpha \in S' \cap \Sigma(X)} \mathbf{A}(X, \alpha)$,
(4) $\Delta_{X,S'}$ is the set of colors of the spherically closed spherical system

$$(S^p(X)_{S'}, \Sigma^{sc}(X)_{S'}, \mathbf{A}(X)_{S'}),$$

as defined in Definition 2.17 (notice that $\Delta_{X,S'}$ is naturally identified with the subset $\{D \in \Delta_X \colon D \text{ is moved by some } \alpha \text{ in } S'\}$ of Δ_X),
(5) $\mathcal{V}_{X,S'} = \mathcal{V}_X \cup (\Delta_X \smallsetminus \Delta_{X,S'})$,
(6) $\mathcal{D}_{X,S'} = \Delta_{X,S'} \cup \mathcal{V}_{X,S'}$ and $\rho'_{X,S'}$ is the restriction of ρ'_X to $\mathbb{Z}\Sigma^{sc}(X)_{S'}$ (notice that $\mathcal{D}_{X,S'}$ is equal to $\mathcal{D}_X$ under the identification in (4) of $\Delta_{X,S'}$ with a subset of Δ_X).

Remark 3.12. (a) The construction of $\overline{\mathrm{soc}}(X)$ in Definition 3.11 only depends on $\mathrm{soc}(X)$ as a combinatorial object, and not on X.
(b) Observe that the fifth component $\Delta_{X,S'}$ of $\overline{\mathrm{soc}}(X)$ is determined by the first four components.

Definition 3.13. Let X be a spherical G-variety and $Y \subseteq X$ be a G-orbit. We define

$$X_{Y,G} = \{x \in X \mid \overline{Gx} \supseteq Y\}.$$

Notice that Y is the unique closed G-orbit of $X_{Y,G}$, which is open in X and G-stable.

We recall the local structure theorem for spherical varieties, see e.g. [Kno94, Theorem 2.3].

Theorem 3.14. *Let X be a spherical variety and $Y \subseteq X$ a G-orbit. Let P be the stabilizer of all colors of X not containing Y, and let L be a Levi subgroup of P. Then there exists an affine, L-stable and L-spherical, locally closed subvariety Z of X such that*

$$\begin{array}{ccc} P^u \times Z & \to & X \smallsetminus D \\ (p,z) & \mapsto & pz \end{array}$$

is a P-equivariant isomorphism, where D is the union of all B-stable prime divisors of X not containing Y and $P = P^u L$ acts on $P^u \times Z$ by $ul \cdot (p,z) = (ulpl^{-1}, lz)$.

Proof. Let us explain how [Kno94, Theorem 2.3] implies this version of the local structure theorem. Denote $D' = D \cap X_{G,Y}$. Then D' doesn't contain any G-orbit, so each of its irreducible components is a Cartier divisor of $X_{G,Y}$ by [Kno94, Lemma 2.2]. Moreover $X \smallsetminus D = X_{G,Y} \smallsetminus D'$. At this point [Kno94, Theorem 2.3] applied to $X_{G,Y}$ and D' yields our theorem. □

Proposition 3.15. *Let X be a spherical G-variety and $Y \subseteq X$ a G-orbit. Let L and Z be as in the local structure theorem (Theorem 3.14) with the additional assumption that L contains T. Then the socle of Z as a spherical L-variety is equal to the localization $\overline{\mathrm{soc}}(X_{Y,G})$. In particular, Z is quasi-vectorial.*

Proof. We fix the Borel subgroup $B_L = B \cap L$ of L. With this choice the set S' of Definition 3.11 is the set of simple roots of L.

First we claim that the spherical system of Z is given by the second, third and fourth components of $\overline{\mathrm{soc}}(X_{Y,G})$. Notice that X (resp. Z) has the same spherical system and the same B-stable (resp. B_L-stable) prime divisors as the smooth locus X^{reg} (resp. Z^{reg}). Here we are identifying, as we will do also in the rest of the proof, such prime divisors with their (non-empty) intersections with the smooth locus of the respective ambient variety.

The local structure theorem gives an isomorphism

$$\begin{array}{ccc} P^u \times Z^{\mathrm{reg}} & \to & X^{\mathrm{reg}} \setminus D \\ (p,z) & \mapsto & pz \end{array}$$

and our first claim now follows from [Los09b, Lemma 3.5.5], where we set $\mathscr{D}'$ of *loc.cit.* equal to the set of colors of X not containing Y.

To complete the proof of the proposition, given the definition of $\overline{\mathrm{soc}}(X_{Y,G})$, we show that

1. all B_L-stable prime divisors of Z contain its closed L-orbit, and that
2. the set of the L-stable prime divisors of Z can be identified with the set $\mathcal{V}_{X,S'} = \mathcal{V}_X \cup (\Delta_X \setminus \Delta_{X,S'})$ compatibly with the two maps ρ', the one of $\mathrm{soc}(Z)$ and the one of $\overline{\mathrm{soc}}(X_{Y,G})$.

The local structure theorem assures that intersecting a prime divisor with Z induces a bijection between the set of B-stable prime divisors of X containing Y and the set of B_L-stable prime divisors of Z. The same theorem assures that $\Lambda(X) = \Lambda(Z)$, that the above bijection between prime divisors preserves the induced valuations, and that a simple root $\alpha \in S'$ moves a B-stable prime divisor E of X if and only it moves $E \cap Z$.

On the other hand Y intersects $X \setminus D$, hence it intersects Z, and we deduce that Y contains the closed L-orbit of Z. This implies statement (1).

We turn to the L-stable prime divisors of Z: any such divisor is the intersection $E \cap Z$, where E is either a G-stable prime divisor of X containing Y, or E is a color of X, such that E contains Y and is L-stable. Notice that, if E is a color of X containing Y, then $E \cap Z$ is L-stable if and only if it is not a color of Z. This implies statement (2), and the proof is complete. □

We come to the smoothness criterion. We remark that spherical modules were classified in [Kac80, Bri85, BR96, Lea98]; see also [Kno98]. Their socles can be deduced from the list in Table 2, thanks to Theorem 3.22 below. The same list is found in [Gag15], where *Luna diagrams* are used to denote spherical systems.

Theorem 3.16. *Let X be a spherical G-variety and $Y \subseteq X$ be a G-orbit. Then X is smooth in all points of Y if and only if*

(1) the localization $\overline{\mathrm{soc}}(X_{Y,G})$ of $\mathrm{soc}(X_{Y,G})$ is the socle of a spherical module.
(2) the $|\mathcal{D}_{X_{Y,G}}|$-tuple $(\rho_X(D))_{D \in \mathcal{D}_{X_{Y,G}}}$ can be completed to a basis of $\Lambda(X)^*$.

Proof. Let Z be as in Proposition 3.15. The smoothness of X along Y is equivalent to the smoothness of Z, which is an affine simple spherical variety whose socle is the localization of $\mathrm{soc}(X_{Y,G})$ by Proposition 3.15. The theorem now follows from Corollary 3.10. ☐

Remark 3.17. Example 4.5 in Section 4 below shows that it is not possible to replace, in Theorem 3.16, the $|\mathcal{D}_{X_{Y,G}}|$-tuple $(\rho_X(D))_{D\in\mathcal{D}_{X_{Y,G}}}$ with the *set* $\rho_X(\mathcal{D}_{X_{Y,G}})$.

Finally, we relate the socle of any spherical module to those in Table 2. First we recall some definitions.

Definition 3.18 ([Cam01])**.** Consider the socles

$$\mathrm{soc}_i = (S_i, S_i^p, \Sigma_i^{sc}, \mathbf{A}_i, \Delta_i, \mathcal{V}_i, \rho_i' \colon \mathcal{D}_i \to (\mathbb{Z}\Sigma_i^{sc})^*) \text{ for } i \in \{1,2\}$$

of two simple spherical G_i-varieties. They are **isomorphic** if they are equal up to an isomorphism φ of the Dynkin diagrams of G_1 and G_2, i.e. if $S_2 = \varphi(S_1)$, $S_2^p = \varphi(S_1^p)$ and $\varphi(\Sigma_1^{sc}) = \Sigma_2^{sc}$ (where we have extended φ to a map between the two root lattices), and if $\mathbf{A}_1$, Δ_1, $\mathcal{V}_1$ can be identified with $\mathbf{A}_2$, Δ_2, $\mathcal{V}_2$, respectively, in such a way that $\langle \rho_1'(D), \sigma\rangle = \langle \rho_2'(D), \varphi(\sigma)\rangle$ for all $D \in \mathbf{A}_1 \cup \Delta_1 \cup V_1$ and all $\sigma \in \Sigma_1^{sc}$.

Notice that the above definition includes the case where $G_1 = G_2$ and φ is an automorphism of its Dynkin diagram.

Definition 3.19 ([Kno98, Section 5])**.** Two representations $\eta_i \colon G_i \to \mathrm{GL}(V_i)$ for $i \in \{1,2\}$ are called **geometrically equivalent** if there is an isomorphism $\Psi \colon V_1 \to V_2$ such that for the induced isomorphism $\mathrm{GL}(\Psi) \colon \mathrm{GL}(V_1) \to \mathrm{GL}(V_2)$ we have $\mathrm{GL}(\Psi)(\eta_1(G_1)) = \eta_2(G_2)$.

Lemma 3.20. *Let $\eta_i \colon G_i \to \mathrm{GL}(V_i)$ for $i \in \{1,2\}$ be two spherical modules that are geometrically equivalent and such that no simple normal subgroup of G_i acts trivially on V_i. Then the socles of V_1 and of V_2 are isomorphic.*

Proof. Fix a map Ψ as in Definition 3.19. Using Ψ we may identify V_1 and V_2 as vector spaces, denoting them both V. This yields $\eta_1(G_1) = \eta_2(G_2)$ as subgroups of $\mathrm{GL}(V)$; denote them both by G. Notice that the kernel of η_i is contained in the center of G_i by our assumptions, so it is contained in all Borel subgroups and all maximal tori. As a consequence, taking the image under η_i induces a bijection between the sets of the Borel subgroups of G_i and those of G, and between the sets of the maximal tori of G_i and those of G, with inverses induced by taking the inverse images under η_i.

Choose a Borel subgroup B_1 and a maximal torus $T_1 \subseteq B_1$ of G_1. We fix the Borel subgroup $\eta_1(B_1)$ and the maximal torus $\eta_1(T_1)$ of G, and we fix the Borel subgroup $\eta_2^{-1}(\eta_1(B_1))$ and the maximal torus $\eta_2^{-1}(\eta_1(T_1))$ of G_2.

Since no simple normal subgroup of G_i acts trivially on V, we obtain an identification of the Dynkin diagrams of G_1, G, and G_2. This, and the fact that G_i acts on V via the image $\eta_i(G_i)$, implies that for all $i \in \{1,2\}$ the socle $\mathrm{soc}(V)$ defined with respect to the action of G_i is equal to the socle defined with respect to the action of $\eta_i(G_i)$.

At this point, considered under the action of G, the modules V_1 and V_2 are the same spherical module under the action of the same group G, thus they have the same socle. Considered under the action of G_i, their socles are equal up to the above identification of the Dynkin diagrams, which finishes the proof. ☐

Remark 3.21. Observe that the converse to Lemma 3.20 does not hold. For example, the standard actions of $\mathrm{SL}(n)$ and $\mathrm{GL}(n)$ on $\mathbb{C}^n$ have isomorphic socles but are not geometrically equivalent.

Theorem 3.22. *Let V be a spherical G-module. Then its socle is, up to isomorphism, a product of socles of Table 2.*

Proof. If G has some simple normal subgroup G' (with set of simple roots S') acting trivially on V, then $\mathrm{soc}(V)$ is the product of the socle $(S', S', \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ and the socle of V under the action of G/G', hence we may assume that no such G' exists.

Suppose that $G = G_1 \times \ldots \times G_n$ and $V = V_1 \oplus \ldots \oplus V_n$ such that for all $i \in \{1, \ldots, n\}$ the factor G_i acts non-trivially on the i-th summand V_i and trivially on the other summands. Then the socle of V under the action of G is the product of the socles of the modules $V_1, \ldots, V_n$ under the action of resp. $G_1, \ldots, G_n$.

Thanks to Lemma 3.20, we may prove the theorem assuming that there is no such decomposition, not even up to geometric equivalence. We recall that this is the definition of an **indecomposable** module in the sense of [Kno98, Section 5].

The module V might be a reducible G-module, so we denote by $V = W_1 \oplus \ldots \oplus W_m$ the decomposition into irreducible summands, unique by sphericity of V.

Extend the G-action on V to the group $G_0 = G \times (\mathbb{C}^*)^m$ by letting the i-th $\mathbb{C}^*$-factor act on W_i by multiplication. The socle of V with respect to the action of G is equal to the socle with respect to the action of G_0 by Lemma 3.7. Denote by $\psi \colon G_0 \to \mathrm{GL}(V)$ this representation: then the center of $\psi(G_0)$ has dimension equal to m. We recall that this is the definition of a **saturated** module in the sense of [Kno98, Section 5].

By [Kno98, Theorem 5.1], the module V appears up to geometric equivalence in the list of modules in [Kno98, Section 5]. Their socles are given in Table 2; see Remark 3.23. □

We explain the notations of Table 2. It is straightforward to compute the data given in the Table from the information contained in [Kno98, Section 5].

For each module V under the action of the group G we give the root system S of G and the sets $S^p(V)$ and $\Sigma^{sc}(V)$. Here the simple roots are denoted by $\alpha_1, \alpha_2, \ldots$ and numbered as in [Bou68]; we use the notation α_1', α_1'', ... if the Dynkin diagram of G has more than one connected component.

Then, whenever $\mathbf{A}(V) \neq \emptyset$, instead of specifying the whole set $\mathbf{A}(V)$ we give the values of $\rho'(D)$ on all spherical roots for only one color $D \in \mathbf{A}(V)$. The whole set $\mathbf{A}(V)$ is the unique one containing D and such that $(S^p(V), \Sigma^{sc}(V), \mathbf{A}(V))$ is a spherical system. Indeed, using Luna's axioms (A1) and (A2), cf. Definition 2.9, one can deduce in each case of the table the values of the other colors from the one we give. The set Δ_V is equal to the whole set of colors $\Delta(V)$ of the spherical system $(S^p(V), \Sigma^{sc}(V), \mathbf{A}(V))$, and is described in Definition 2.17.

Finally, we describe the elements of $\mathcal{V}_V$ and their values on the spherical roots as follows. In each case, either $\mathcal{V}_V$ is empty, or it contains exactly one element ν, or it contains exactly two elements. If $\mathcal{V}_V = \{\nu\}$, then either $\Sigma^{sc}(V)$ is empty, or there exists a spherical root $\gamma \in \Sigma^{sc}(V)$ such that $\rho'(\nu)(\gamma) = -1$, and $\rho'(\nu)(\sigma) = 0$ for all $\sigma \in \Sigma^{sc}(V) \smallsetminus \{\gamma\}$. If $\Sigma^{sc}(V)$ is empty then $\rho'(\nu)$ is the zero map, and we report $\mathcal{V}_V$ only as $\{\nu\}$. Otherwise we denote ν by $-\gamma^*$ and we report $\mathcal{V}_V$ as $\{-\gamma^*\}$. If $\Sigma^{sc}(V)$ contains more than one element we indicate $\gamma \in \Sigma^{sc}(V)$ explicitly, otherwise γ is obviously the unique element of $\Sigma^{sc}(V)$.

If $\mathcal{V}_V$ contains two elements ν_1, ν_2, then $\Sigma^{sc}(V)$ contains two elements γ_1, γ_2 such that for all $i \in \{1, 2\}$ we have $\langle \rho'(\nu_i), \gamma_i \rangle = -1$ and $\langle \rho'(\nu_i), \sigma \rangle = 0$ for all spherical roots σ different from γ_i. In this case we report $\mathcal{V}_V$ as $\{-\gamma_1^*, -\gamma_2^*\}$, and indicate $\gamma_1, \gamma_2 \in \Sigma^{sc}(V)$ explicitly. These facts about $\mathcal{V}_V$ follow by inspection of the data in [Kno98, Section 5].

Table 2: Socles of spherical modules.

	S	$S^p(V)$	$\Sigma^{sc}(V)$	$\mathbf{A}(V)$	$\mathcal{V}_V$
1	Any[1]	S	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	ν

	S	$S^p(V)$	$\Sigma^{sc}(V)$	$\mathbf{A}(V)$	$\mathcal{V}_V$
3	A_n, $n \geq 1$	$\emptyset$	$2\alpha_1, \ldots, 2\alpha_{n-1}, \gamma = 2\alpha_n$	$\emptyset$	$-\gamma^*$
4	A_n, $n \geq 3$	$\emptyset$	$\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\ldots$,$\gamma = \alpha_{n-1} + \alpha_n$	$\emptyset$	$-\gamma^*$
5	A_n, $n \geq 5$ odd	$\emptyset$	$\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\ldots$, $\gamma = \alpha_{n-2} + \alpha_{n-1}$, $\alpha_{n-1} + \alpha_n$	$\emptyset$	$-\gamma^*$
6	$\mathsf{A}_n \times \mathsf{A}_n$ $n \geq 1$	$\emptyset$	$\alpha_1 + \alpha'_1, \ldots, \alpha_{n-1} + \alpha'_{n-1}$ $\gamma = \alpha_n + \alpha'_n$	$\emptyset$	$-\gamma^*$

	S	$S^p(V)$	$\Sigma^{sc}(V)$	$\mathbf{A}(V)$	$\mathcal{V}_V$
7	A_1	$\emptyset$	α_1	1	$-\gamma^*$
8	A_n, $n \geq 4$ even	$\emptyset$	$\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\ldots$, $\alpha_{n-2} + \alpha_{n-1}$, α_n	$0, \ldots, 0, 1$	$\emptyset$
9	$\mathsf{A}_n \times \mathsf{A}_n$ $n \geq 1$	$\emptyset$	$\alpha_1, \ldots, \gamma = \alpha_n$, $\alpha'_1, \ldots, \alpha'_n$	$1, -1, 0, \ldots, 0,$ $-1, 1, 0, \ldots, 0$	$-\gamma^*$
10	$\mathsf{A}_n \times \mathsf{A}_n$ $n \geq 1$	$\emptyset$	$\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \gamma = \alpha'_n$	$1, 0, \ldots, 0,$ $1, -1, 0, \ldots, 0$	$-\gamma^*$
11	$\mathsf{A}_n \times \mathsf{A}_{n+1}$ $n \geq 1$	$\emptyset$	$\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \gamma = \alpha'_{n+1}$	$1, -1, 0, \ldots, 0,$ $-1, 1, 0, \ldots, 0$	$-\gamma^*$
12	$\mathsf{A}_2 \times \mathsf{C}_2$	$\emptyset$	S	$1, 0, 1, -1$	$\emptyset$
13	$\mathsf{A}_3 \times \mathsf{C}_2$	$\emptyset$	$S, \gamma = \alpha_3$	$1, 0, -1, 1, -1$	$-\gamma^*$
14	$\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$	$\emptyset$	$\gamma_1 = \alpha_1, \alpha'_1, \gamma_2 = \alpha''_1$	$1, 1, -1$	$-\gamma_1^*$, $-\gamma_2^*$

Notation: $\alpha_{a,b} = \alpha_a + \alpha_{a+1} + \ldots + \alpha_{b-1} + \alpha_b$					
	S	$S^p(V)$	$\Sigma^{sc}(V)$	$\mathbf{A}(V)$	$\mathcal{V}_V$
15	A_n, $n \geq 1$	$\alpha_2, \ldots, \alpha_n$	$\emptyset$	$\emptyset$	$\emptyset$
16	A_n, $n \geq 2$	$\alpha_2, \ldots, \alpha_{n-1}$	$\alpha_{1,n}$	$\emptyset$	$-\gamma^*$
17	A_n $n \geq 4$ even	$\alpha_1, \alpha_3, \ldots,$ $\alpha_{n-3}, \alpha_{n-1}$	$\alpha_1 + 2\alpha_2 + \alpha_3,$ $\alpha_3 + 2\alpha_4 + \alpha_5,$ $\ldots,$ $\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1}$	$\emptyset$	$\emptyset$
18	A_n $n \geq 3$ odd	$\alpha_1, \alpha_3,$ $\alpha_{n-2} \ldots, \alpha_n$	$\alpha_1 + 2\alpha_2 + \alpha_3,$ $\alpha_3 + 2\alpha_4 + \alpha_5,$ $\ldots,$ $\gamma = \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$	$\emptyset$	$-\gamma^*$
19	B_3	α_1, α_2	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$\emptyset$	$-\gamma^*$
20	B_4	α_2, α_3	$\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\alpha_2 + 2\alpha_3 + 3\alpha_4$	$\emptyset$	$-\gamma^*$
21	B_n, $n \geq 2$	$\alpha_2, \ldots, \alpha_n$	$2\alpha_{1,n}$	$\emptyset$	$-\gamma^*$
22	C_n, $n \geq 2$	$\alpha_2, \ldots, \alpha_n$	$\emptyset$	$\emptyset$	$\emptyset$
23	D_4	α_2	$\gamma_1 = \alpha_1 + \alpha_2 + \alpha_3,$ $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_4,$ $\alpha_2 + \alpha_3 + \alpha_4$	$\emptyset$	$-\gamma_1^*$, $-\gamma_2^*$
24	D_5	$\alpha_2, \alpha_3, \alpha_4$	$\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$	$\emptyset$	$\emptyset$
25	D_n, $n \geq 4$	$\alpha_2, \ldots, \alpha_n$	$2\alpha_{1,n-2} + \alpha_{n-1} + \alpha_n$	$\emptyset$	$-\gamma^*$
26	G_2	α_2	$4\alpha_1 + 2\alpha_2$	$\emptyset$	$-\gamma^*$

[1]This is the socle of the module $\{0\}$, and the next socle is the one of a one-dimensional module under the action of a torus.

27	E_6	$\alpha_2, \alpha_3, \alpha_4, \alpha_5$	$2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$, $\gamma = \alpha_2 + \alpha_3 + 2\alpha_{4,6}$	$\emptyset$	$-\gamma^*$
28	$\mathsf{A}_n \times \mathsf{A}_m$ $m > n \geq 1$	$\alpha'_{n+2}, \dots, \alpha'_m$	$\alpha_1 + \alpha'_1, \dots, \alpha_n + \alpha'_n$	$\emptyset$	$\emptyset$
29	$\mathsf{A}_1 \times \mathsf{C}_m$ $m \geq 2$	$\alpha'_3, \dots, \alpha'_m$	$\alpha_1 + \alpha'_1$, $\gamma = \alpha'_1 + 2\alpha'_{2,m-1} + \alpha'_m$	$\emptyset$	$-\gamma^*$

Notation: $\alpha_{a,b} = \alpha_a + \alpha_{a+1} + \ldots + \alpha_{b-1} + \alpha_b$					
	S	$S^p(V)$	$\Sigma^{sc}(V)$	$\mathbf{A}(V)$	$\mathcal{V}_V$
30	A_n, $n \geq 2$	$\alpha_3, \dots, \alpha_n$	α_1	1	$\emptyset$
31	C_n, $n \geq 2$	$\alpha_3, \dots, \alpha_n$	α_1, $\gamma = \alpha_1 + 2\alpha_{2,n-1} + \alpha_n$	$1, 0$	$-\gamma^*$
32	$\mathsf{A}_2 \times \mathsf{C}_m$ $m \geq 3$	$\alpha'_4, \dots, \alpha'_m$	$\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$, $\alpha'_2 + 2\alpha'_{3,m-1} + \alpha'_m$	$1, 0, 1, -1$, 0	$\emptyset$
33	$\mathsf{A}_n \times \mathsf{C}_2$ $n \geq 4$	$\alpha_5, \dots, \alpha_n$	$\alpha_1, \alpha_2, \alpha_3$, α'_1, α'_2	$1, 0, -1$, $1, -1$	$\emptyset$
34	$\mathsf{C}_n \times \mathsf{A}_1$ $n \geq 2$	$\alpha_3, \dots, \alpha_n$	α_1, α'_1, $\gamma = \alpha_1 + 2\alpha_{2,n-1} + \alpha_n$	$1, 1$, 0	$-\gamma^*$
35	$\mathsf{A}_n \times \mathsf{A}_m$ $m - 2 \geq n \geq 1$	$\alpha'_{n+3}, \dots$, α'_m	$\alpha_1, \dots, \alpha_n$, $\alpha'_1, \dots, \alpha'_{n+1}$	$1, -1, 0, \dots, 0$, $-1, 1, 0, \dots, 0$	$\emptyset$
36	$\mathsf{A}_n \times \mathsf{A}_m$ $n > m \geq 1$	$\alpha_{m+2}, \dots$, α_n	$\alpha_1, \dots, \alpha_m$, $\alpha'_1, \dots, \alpha'_m$	$1, -1, 0, \dots, 0$, $-1, 1, 0, \dots, 0$	$\emptyset$
37	$\mathsf{A}_n \times \mathsf{A}_m$ $n > m \geq 1$	$\alpha_{m+2}, \dots$, α_{n-1}	$\alpha_1, \dots, \alpha_m$, $\alpha'_1, \dots, \alpha'_m$, $\alpha_{m+1,n}$	$1, 0, \dots, 0$, $1, -1, 0, \dots, 0$, 0	$\emptyset$
38	$\mathsf{A}_n \times \mathsf{A}_m$ $n > m \geq 1$	$\alpha_{m+2}, \dots$, α_n	$\alpha_1, \dots, \alpha_m$, $\alpha'_1, \dots, \alpha'_m$	$1, 0, 0, \dots, 0$, $-1, 0, 0, \dots, 0$	$\emptyset$
39	$\mathsf{A}_n \times \mathsf{A}_1 \times \mathsf{A}_1$ $n \geq 2$	$\alpha_3, \dots, \alpha_n$	$\alpha_1, \alpha'_1, \gamma = \alpha''_1$	$1, 1, -1$	$-\gamma^*$
40	$\mathsf{A}_n \times \mathsf{A}_1 \times \mathsf{A}_m$ $n, m \geq 2$	$\alpha_3, \dots, \alpha_n$, $\alpha''_3, \dots, \alpha''_m$	$\alpha_1, \alpha'_1, \alpha''_1$	$1, 1, -1$	$\emptyset$
41	$\mathsf{C}_n \times \mathsf{A}_1 \times \mathsf{A}_1$ $n \geq 2$	$\alpha_3, \dots, \alpha_n$	$\alpha_1, \alpha'_1, \gamma_1 = \alpha''_1$, $\gamma_2 = \alpha_1 + 2\alpha_{2,n-1} + \alpha_n$	$1, 1, -1$, 0	$-\gamma_1^*$, $-\gamma_2^*$
42	$\mathsf{C}_n \times \mathsf{A}_1 \times \mathsf{A}_m$ $n, m \geq 2$	$\alpha_3, \dots, \alpha_n$, $\alpha''_3, \dots, \alpha''_m$	$\alpha_1, \alpha'_1, \alpha''_1$, $\gamma = \alpha_1 + 2\alpha_{2,n-1} + \alpha_n$	$1, 1, -1$, 0	$-\gamma^*$
43	$\mathsf{C}_n \times \mathsf{A}_1 \times \mathsf{C}_m$ $n, m \geq 2$	$\alpha_3, \dots, \alpha_n$, $\alpha''_3, \dots, \alpha''_m$	$\alpha_1, \alpha'_1, \alpha''_1$, $\gamma_1 = \alpha_1 + 2\alpha_{2,n-1} + \alpha_n$, $\gamma_2 = \alpha''_1 + 2\alpha''_{2,m-1} + \alpha''_m$	$1, 1, -1$, 0, 0	$-\gamma_1^*$, $-\gamma_2^*$

Remark 3.23. To demonstrate the completeness of Table 2 let us compare the cases in the table with the list of [Kno98, Section 5] (using the revised version available on Knop's website).

1. $\mathrm{GL}(m) \times \mathrm{GL}(n)$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $1 \leq m \leq n$: case (2) for $1 = m = n$, case (15) for $1 = m < n$, case (6) for $1 < m = n$, case (28) for $1 < m < n$.
2. $\mathrm{GL}(n)$ on $S^2(\mathbb{C}^n)$ with $1 \leq n$: case (3).
3. $\mathrm{GL}(m)$ on $\bigwedge^2(\mathbb{C}^m)$ with $2 \leq m$: case (2) for $m = 2$, case (18) (with $n = 3$) for $m = 4$, case (17) for $m \geq 5$ odd, case (18) (with $n \geq 5$) for $m \geq 6$ even. For $m = 3$ this module is geometrically equivalent to $\mathrm{GL}(3)$ on $\mathbb{C}^3$, so it has already appeared previously in this list.
4. $\mathrm{Sp}(2n) \times \mathbb{C}^*$ on $\mathbb{C}^{2n}$ with $1 \leq n$: case (22) for $n \geq 2$. For $n = 1$ it is geometrically equivalent to $\mathrm{GL}(2)$ on $\mathbb{C}^2$.
5. $\mathrm{Sp}(2n) \times \mathrm{GL}(2)$ on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ with $2 \leq n$: case (29).
6. $\mathrm{Sp}(2n) \times \mathrm{GL}(3)$ on $\mathbb{C}^{2n} \otimes \mathbb{C}^3$ with $3 \leq n$: case (32).
7. $\mathrm{Sp}(4) \times \mathrm{GL}(3)$ on $\mathbb{C}^4 \otimes \mathbb{C}^3$: case (12).
8. $\mathrm{Sp}(4) \times \mathrm{GL}(n)$ on $\mathbb{C}^4 \otimes \mathbb{C}^n$ with $4 \leq n$: case (13) for $n = 4$, case (33) for $n \geq 5$.
9. $\mathrm{SO}(n) \times \mathbb{C}^*$ on $\mathbb{C}^n$ with $3 \leq n$: case (21) for $n \geq 5$ odd, case (25) for $n \geq 8$ even. For $n = 3$ it is geometrically equivalent to $\mathrm{GL}(2)$ on $S^2(\mathbb{C}^2)$, for $n = 4$

it is geometrically equivalent to $\mathrm{GL}(2)\times\mathrm{GL}(2)$ on $\mathbb{C}^2\otimes\mathbb{C}^2$, for $n=6$ it is geometrically equivalent to $\mathrm{GL}(4)$ on $\bigwedge^2(\mathbb{C}^4)$.

10. $\mathrm{Spin}(10)\times\mathbb{C}^*$ on $\mathbb{C}^{16}$: case (24).
11. $\mathrm{Spin}(7)\times\mathbb{C}^*$ on $\mathbb{C}^{8}$: case (19).
12. $\mathrm{Spin}(9)\times\mathbb{C}^*$ on $\mathbb{C}^{16}$: case (20).
13. $\mathsf{G}_2\times\mathbb{C}^*$ on $\mathbb{C}^{7}$: case (26).
14. $\mathsf{E}_6\times\mathbb{C}^*$ on $\mathbb{C}^{27}$: case (27).
15. $\mathrm{GL}(n)\times\mathbb{C}^*$ on $\bigwedge^2(\mathbb{C}^n)\oplus\mathbb{C}^n$ with $4\leq n$: case (4).
16. $\mathrm{GL}(n)\times\mathbb{C}^*$ on $\bigwedge^2(\mathbb{C}^n)\oplus(\mathbb{C}^n)^*$ with $4\leq n$: case (5) for $n\geq 6$ even, case (8) for n odd. For $n=4$ it is geometrically equivalent to the previous module in this list.
17. $\mathrm{GL}(m)\times\mathrm{GL}(n)$ on $(\mathbb{C}^m\otimes\mathbb{C}^n)\oplus\mathbb{C}^n$ with $1\leq m$, $2\leq n$: case (7) for $m=1$ and $n=2$, case (30) for $m=1$ and $n\geq 3$, case (35) for $2\leq m<n-1$, case (11) for $2\leq m=n-1$, case (9) for $2\leq m=n$, case (36) for $m>n$.
18. $\mathrm{GL}(m)\times\mathrm{GL}(n)$ on $(\mathbb{C}^m\otimes\mathbb{C}^n)\oplus(\mathbb{C}^n)^*$ with $1\leq m$, $2\leq n$: case (16) for $m=1$ and $n\geq 3$, case (37) for $2\leq m\leq n-1$, case (10) for $2\leq m=n$, case (38) for $m>n$. For $m=1$ and $n=2$ it is geometrically equivalent to the previous module in this list.
19. $\mathrm{Sp}(2n)\times\mathbb{C}^*\times\mathbb{C}^*$ on $\mathbb{C}^{2n}\oplus\mathbb{C}^{2n}$ with $2\leq n$: case (31).
20. $(\mathrm{Sp}(2n)\times\mathbb{C}^*)\times\mathrm{GL}(2)$ on $(\mathbb{C}^{2n}\otimes\mathbb{C}^2)\oplus\mathbb{C}^2$ with $2\leq n$: case (34).
21. $\mathrm{GL}(m)\times\mathrm{SL}(2)\times\mathrm{GL}(n)$ on $(\mathbb{C}^m\otimes\mathbb{C}^2)\oplus(\mathbb{C}^2\otimes\mathbb{C}^n)$ with $2\leq m\leq n$: case (14) for $n=m=2$, case (39) for $m=2$ and $n\geq 3$, case (40) for $m,n\geq 3$.
22. $(\mathrm{Sp}(2m)\times\mathbb{C}^*)\times\mathrm{SL}(2)\times\mathrm{GL}(n)$ on $(\mathbb{C}^{2m}\otimes\mathbb{C}^2)\oplus(\mathbb{C}^2\otimes\mathbb{C}^n)$ with $2\leq m,n$: case (41) for $n=2$, case (42) for $n\geq 3$.
23. $(\mathrm{Sp}(2m)\times\mathbb{C}^*)\times\mathrm{SL}(2)\times(\mathrm{Sp}(2n)\times\mathbb{C}^*)$ on $(\mathbb{C}^{2m}\otimes\mathbb{C}^2)\oplus(\mathbb{C}^2\otimes\mathbb{C}^{2n})$ with $2\leq m,n$: case (43).
24. $\mathrm{Spin}(8)\times\mathbb{C}^*\times\mathbb{C}^*$ on $\mathbb{C}^8_+\oplus\mathbb{C}^8_-$: case (23).

4. Combinatorial characterization of smooth weight monoids

In this section we state and prove the main result of this paper. In order to do so we introduce a few more notions.

Definition 4.1. Let Γ be a normal submonoid of Λ^+, and let Σ be a subset of $\Sigma^{sc}(G)$ that is adapted to Γ. We define the following:

1. $\mathscr{S}(\Gamma,\Sigma)=(S^p(\Gamma),\Sigma,\mathbf{A}(\Gamma,\Sigma))$ is the spherical system constructed in the proof of Proposition 2.24, $\Delta(\Gamma,\Sigma)$ is the set of colors of this spherical system, and $(\mathbb{Z}\Gamma,c)$ is the augmentation constructed in the same proposition.
2. $\mathcal{V}(\Gamma,\Sigma):=\{v\in\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma,\mathbb{Q}):\langle v,\sigma\rangle\leq 0 \text{ for all } \sigma\in\Sigma\}$.
3. $\mathcal{C}(\Gamma,\Sigma)$ is the maximal face of $\Gamma^\vee$ whose relative interior meets $\mathcal{V}(\Gamma,\Sigma)$.
4. $\mathcal{F}(\Gamma,\Sigma):=\{D\in\Delta(\Gamma,\Sigma)\colon c(D,\cdot)\in\mathcal{C}(\Gamma,\Sigma)\}$
5. $\mathcal{B}(\Gamma,\Sigma)$ is the set the primitive elements in $(\mathbb{Z}\Gamma)^*$ that lie on extremal rays of $\mathcal{C}(\Gamma,\Sigma)$ which do not contain any element of $\{c(D,\cdot):D\in\mathcal{F}(\Gamma,\Sigma)\}$.
6. $\mathcal{D}(\Gamma,\Sigma):=\mathcal{F}(\Gamma,\Sigma)\cup\mathcal{B}(\Gamma,\Sigma)$.
7. $\rho\colon\mathcal{D}(\Gamma,\Sigma)\to(\mathbb{Z}\Gamma)^*$ is defined by $\rho(D)=c(D,\cdot)$ if $D\in\mathcal{F}(\Gamma,\Sigma)$ and $\rho(D)=D$ for $D\in\mathcal{B}(\Gamma,\Sigma)$.
8. $\mathrm{soc}(\Gamma,\Sigma):=(S,S^p(\Gamma),\Sigma,\mathbf{A}(\Gamma,\Sigma),\mathcal{F}(\Gamma,\Sigma),\mathcal{B}(\Gamma,\Sigma),\rho'\colon\mathcal{D}(\Gamma,\Sigma)\to(\mathbb{Z}\Sigma)^*)$, with $\rho'(D)=\rho(D)|_{\mathbb{Z}\Sigma}$.
9. $S(\Gamma,\Sigma):=\{\alpha\in S\colon \text{if } D\in\Delta(\Gamma,\Sigma) \text{ and } \alpha \text{ moves } D \text{ then } D\in\mathcal{F}(\Gamma,\Sigma)\}$.

10. $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is the localization of $\mathrm{soc}(\Gamma,\Sigma)$, see Remark 3.12(a) (notice that for the socle $\mathrm{soc}(\Gamma,\Sigma)$ the set S' in Definition 3.11 is equal to $S(\Gamma,\Sigma)$).

Theorem 4.2. *Let Γ be a normal submonoid of Λ^+. Then Γ is the weight monoid of a smooth affine spherical G-variety if and only if there exists a subset Σ of $\Sigma^{sc}(\Gamma)$ such that*

(1) Σ is adapted to Γ, and there is no subset of $\Sigma^{sc}(\Gamma)$ that strictly contains Σ and is adapted to Γ;
(2) $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is isomorphic to the socle of a spherical module; and
(3) the $|\mathcal{D}(\Gamma,\Sigma)|$-tuple $(\rho(D))_{D\in\mathcal{D}(\Gamma,\Sigma)}$ can be completed to a basis of $\mathbb{Z}\Gamma^*$.

The proof of Theorem 4.2 is given after Example 4.4.

Remark 4.3. (a) Recall that by Proposition 2.23, determining the set $\Sigma^{sc}(\Gamma)$ is a finite, combinatorial problem. Similarly, Proposition 2.24, which relies on the (proved) Luna Conjecture, reduces checking condition (1) in Theorem 4.2 to a finite, combinatorial problem.
(b) Verifying conditions (2) and (3) of Theorem 4.2 is also a finite problem. Indeed, (2) reduces to checking that $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is the product of socles from Table 2, up to isomorphism (see Theorem 3.22). By the Elementary Divisors Theorem, condition (3) comes down to checking that the maximal minors of an integer matrix have greatest common divisor equal to 1.
(c) The second part of statement (1) of Theorem 4.2, i.e. the maximality of Σ, could be omitted from the theorem. It is included because it provides, in applications, a useful necessary condition for Γ to be the weight monoid of a smooth affine spherical G-variety.
(d) If $\Sigma^{sc}(\Gamma)$ is adapted to Γ, then $\Sigma=\Sigma^{sc}(\Gamma)$ is the only set that satisfies condition (1) in Theorem 4.2. By Remark 2.16 this is the case when Γ is G-saturated or when $\Sigma^{sc}(\Gamma)$ does not contain any simple roots.

Example 4.4. We illustrate Theorem 4.2 by applying it to the monoid

$$\Gamma=\mathbb{N}(\omega_1)+\mathbb{N}(\omega_1+\omega_2)\quad\text{for } G=\mathrm{SL}(3).$$

We will use the same notations for the simple roots and fundamental weights of SL(3) as in Example 1.13. Recall from that example that

$$\Sigma^{sc}(G)=\{\alpha_1,\alpha_2,2\alpha_1,2\alpha_2,\alpha_1+\alpha_2\}.$$

It is clear that

$$\mathbb{Z}\Gamma=\mathbb{Z}(\omega_1)\oplus\mathbb{Z}(\omega_1+\omega_2)=\Lambda$$

and that

$$S^p(\Gamma)=\emptyset.$$

We will apply Proposition 2.23 to each element of $\Sigma^{sc}(G)$ to show that

$$\Sigma^{sc}(\Gamma)=\{\alpha_1+\alpha_2\}.$$

Indeed $2\alpha_1$ and $2\alpha_2$ do not satisfy condition (5) of the Proposition, since $\alpha_1,\alpha_2\in\mathbb{Z}\Gamma$. To show that $\alpha_1,\alpha_2\notin\Sigma^{sc}(\Gamma)$, let $\{\delta_1,\delta_2\}$ be the basis of $(\mathbb{Z}\Gamma)^*$ that is dual to the basis $\{\omega_1,\omega_1+\omega_2\}$ of $\mathbb{Z}\Gamma$. Since Γ is a free monoid (i.e. its generators $\omega_1,\omega_1+\omega_2$ are linearly independent elements of Λ), we have that $E(\Gamma)=\{\delta_1,\delta_2\}$. Now, $\langle\delta_1,\alpha_1\rangle=3$, because $\alpha_1=3(\omega_1)-(\omega_1+\omega_2)$, and therefore α_1 does not satisfy condition (4c) of Proposition 2.23. Similarly, $\alpha_2\notin\Sigma^{sc}(\Gamma)$ because $\langle\delta_2,\alpha_2\rangle=2$. Finally, we check that $\alpha_1+\alpha_2\in\Sigma^{sc}(\Gamma)$, by verifying the necessary and sufficient conditions of the Proposition: condition (1) holds because $\alpha_1 + \alpha_2 = \omega_1 + \omega_2 \in \mathbb{Z}\Gamma$; condition (2) is easy to check using Remark 2.10; condition (3) holds because $\langle \delta_1, \alpha_1 + \alpha_2 \rangle = 0$ and $\delta_2 = \alpha_2^\vee$.

Next, we set

$$\Sigma = \Sigma^{sc}(\Gamma) = \{\alpha_1 + \alpha_2\}$$

and we verify the three conditions in Theorem 4.2. Condition (1) is clear. To compute $\overline{\mathrm{soc}}(\Gamma, \Sigma)$, we determine the combinatorial invariants of Definition 4.1:

1. $\mathscr{S}(\Gamma, \Sigma) = (\emptyset, \{\alpha_1 + \alpha_2\}, \emptyset)$; $\Delta(\Gamma, \Sigma) = \{D_{\alpha_1}, D_{\alpha_2}\}$; for $i \in \{1, 2\}$ we have $c(D_{\alpha_i}, \cdot) = \alpha_i^\vee$.
2. $\mathcal{V}(\Gamma, \Sigma) = \{q_1\delta_1 + q_2\delta_2 \in (\mathbb{Z}\Gamma)^* \otimes_{\mathbb{Z}} \mathbb{Q} \colon q_1, q_2 \in \mathbb{Q}$ and $q_2 \leq 0\}$ since $\langle q_1\delta_1 + q_2\delta_2, \alpha_1 + \alpha_2 \rangle = q_2$.
3. $\mathcal{C}(\Gamma, \Sigma) = \mathbb{Q}_{\geq 0}\delta_1$ since $\Gamma^\vee = \{q_1\delta_1 + q_2\delta_2 \in (\mathbb{Z}\Gamma)^* \otimes_{\mathbb{Z}} \mathbb{Q} \colon q_1, q_2 \in \mathbb{Q}_{\geq 0}\}$.
4. $\mathcal{F}(\Gamma, \Sigma) = \emptyset$ since for neither of the two elements D of $\Delta(\Gamma, \Sigma)$ we have that $c(D, \cdot)$ is a nonnegative rational multiple of δ_1.
5. $\mathcal{B}(\Gamma, \Sigma) = \{\delta_1\}$.
6. $\mathcal{D}(\Gamma, \Sigma) = \{\delta_1\}$.
7. $\rho \colon \mathcal{D}(\Gamma, \Sigma) \to (\mathbb{Z}\Gamma)^*$ is defined by $\rho(\delta_1) = \delta_1$.
8. $\mathrm{soc}(\Gamma, \Sigma) = (\{\alpha_1, \alpha_2\}, \emptyset, \{\alpha_1 + \alpha_2\}, \emptyset, \emptyset, \{\delta_1\}, \rho' \colon \mathcal{D}(\Gamma, \Sigma) \to (\mathbb{Z}\Sigma)^*)$, with $\rho'(\delta_1) = 0 \in (\mathbb{Z}\Sigma)^*$.
9. $S(\Gamma, \Sigma) = \emptyset$ because for each $i \in \{1, 2\}$ we have that α_i moves D_{α_i} but $D_{\alpha_i} \notin \mathcal{F}(\Gamma, \Sigma)$.
10. $\overline{\mathrm{soc}}(\Gamma, \Sigma) = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \{\delta_1\}, \{\delta_1\} \to \{0\})$.

We see that $\overline{\mathrm{soc}}(\Gamma, \Sigma)$ is isomorphic to socle #2 in Table 2, and therefore Condition (2) of Theorem 3.16 holds. Finally, Condition (3) of the Theorem holds because δ_1 is part of a basis of $(\mathbb{Z}\Gamma)^*$. It follows that there exists a smooth affine spherical SL(3)-variety with weight monoid Γ.

Proof of Theorem 4.2. We first show that the conditions on Γ in the Theorem are necessary for Γ to be smooth. Let X be a smooth affine spherical G-variety X such that $\Gamma(X) = \Gamma$. We put $\Sigma = \Sigma^{sc}(X)$. Then condition (1) holds by the definition of 'adapted' and by Proposition 2.2 and Corollary 2.14. Let Y be the unique closed G-orbit of X and observe that $X_{Y,G} = X$. Conditions (2) and (3) follow from Theorem 3.16 because

(i) $S(\Gamma, \Sigma)$ is equal to the set S' of Definition 3.11;
(ii) $\overline{\mathrm{soc}}(\Gamma, \Sigma)$ is equal to $\overline{\mathrm{soc}}(X)$; and
(iii) the $|\mathcal{D}_X|$-tuple $(\rho_X(D))_{D \in \mathcal{D}_X}$ is equal to the $|\mathcal{D}(\Gamma, \Sigma)|$-tuple $(\rho(D))_{D \in \mathcal{D}(\Gamma, \Sigma)}$.

The three claims (i), (ii) and (iii) are consequences of standard facts in the combinatorial theory of spherical varieties, which can be found in [Kno91], [Lun01] and [Tim11], together with our analysis in Section 2. More specifically, the invariants in Definition 4.1 are the combinatorial descriptions of certain geometric invariants of X:

1. $\mathscr{S}(\Gamma, \Sigma) = \mathscr{S}(X)$, by the uniqueness statement in Proposition 2.20: indeed $\mathscr{S}(\Gamma, \Sigma)$ satisfies properties (1)-(4) of Proposition 2.20 by construction (see the proof of Proposition 2.24), and $\mathscr{S}(X)$ satisfies them by the first part of Proposition 2.20. Thanks to Remark 2.19 we may and will identify $\Delta(\Gamma, \Sigma)$ with $\Delta(X)$. Since $\Gamma(X) = \Gamma$, we have $\Lambda(X) = \mathbb{Z}\Gamma$ and, again by Proposition 2.20, the Cartan pairing c in the augmentation $(\mathbb{Z}\Gamma, c)$ agrees with c_X.

2. $\mathcal{V}(\Gamma, \Sigma) = \mathcal{V}(X)$ by equation (2.2).
3. $\mathcal{C}(\Gamma, \Sigma)$ is equal to $\mathcal{C}(X)$ of [Kno91, Section 2]: this can be deduced from [Kno91, Theorem 6.7].
4. $\mathcal{F}(\Gamma, \Sigma)$ is equal to $\mathcal{F}(X)$ of [Kno91, Section 2] by Theorem 6.7 of *loc.cit.*
5. $\mathcal{B}(\Gamma, \Sigma)$ is equal to $\mathcal{B}(X)$ of [Kno91, Section 2] thanks to Lemma 2.4 of *loc.cit.* and the sentence preceding it.
6. $\mathcal{D}(\Gamma, \Sigma)$ is identified with $\mathcal{D}_X$ because $\mathcal{F}(\Gamma, \Sigma) = \mathcal{F}(X)$ and because $\mathcal{B}(X)$ is identified with $\mathcal{V}_X$ by $(D \in \mathcal{V}_X) \mapsto (\nu_D \in \mathcal{B}(X))$.
7. $\rho\colon \mathcal{D}(\Gamma, \Sigma) \to \mathbb{Z}\Gamma^*$ is equal to with $\rho_X \colon \mathcal{D}_X \to \Lambda(X)^*$.
8. $\mathrm{soc}(\Gamma, \Sigma)$ is equal to $\mathrm{soc}(X)$ using to the identifications above.
9. $S(\Gamma, \Sigma)$ is equal to the set S' of Definition 3.11 for the socle $\mathrm{soc}(\Gamma, \Sigma)$ because the geometric and combinatorial notion of color *moved* by a simple root agree (cf. Remark 2.19).
10. $\overline{\mathrm{soc}}(\Gamma, \Sigma)$ is equal to $\overline{\mathrm{soc}}(X)$ by points 8. and 9.

We now prove the sufficiency of the conditions in the Theorem for Γ to be smooth. Condition (1) says, by Definition 2.11, that there exists an affine spherical G-variety X with $\Gamma(X) = \Gamma$ and $\Sigma^{sc}(X) = \Sigma$. Using, once again, that $\overline{\mathrm{soc}}(X) = \overline{\mathrm{soc}}(\Gamma, \Sigma)$ and that $\rho : \mathcal{D}(\Gamma, \Sigma) \to (\mathbb{Z}\Gamma)^*$ is equal to $\rho_X : \mathcal{D}_X \to \Lambda(X)^*$ (up to the standard identifications), the Camus smoothness criterion (i.e. Theorem 3.16) and conditions (2) and (3) of Theorem 4.2 imply that X is smooth. ☐

We can now prove Proposition 1.7 and Theorem 1.12 from Section 1.

Proof of Proposition 1.7. Consider the convex cone C in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q})$ generated by $\{\alpha^\vee|_{\mathbb{Z}\Gamma} : \alpha \in S\}$. It intersects $\mathcal{V}(\Gamma)$ at least in 0, so there are faces of C whose relative interior intersect $\mathcal{V}(\Gamma)$ (at the very least, the linear part of C). By convexity of $\mathcal{V}(\Gamma)$, for any two of such faces C_1, C_2 of C there is a point in the relative interior of their convex envelope lying in $\mathcal{V}(\Gamma)$, so C has a face C_3 containing C_1 and C_2 and such that the relative interior of C_3 intersects $\mathcal{V}(\Gamma)$. It follows that C has a maximal such face C_m. The set S_Γ with the required properties is the maximal subset of S such that C_m is the convex cone generated by $\{\alpha^\vee|_{\mathbb{Z}\Gamma} : \alpha \in S_\Gamma\}$. ☐

Proof of Theorem 1.12. Let Γ be a G-saturated submonoid of Λ^+. Thanks to Corollary 2.27, the set $\Sigma^{sc}(\Gamma)$ is adapted to Γ, and so $\Sigma := \Sigma^{sc}(\Gamma)$ is the unique subset of $\Sigma^{sc}(\Gamma)$ that satisfies (1) of Theorem 4.2. We will show that

$$\begin{aligned} &\Gamma \text{ satisfies conditions (a), (b) and (c) of Theorem 1.12} \\ &\iff \text{conditions (2) and (3) of Theorem 4.2 hold for } (\Gamma, \Sigma). \end{aligned} \tag{4.1}$$

We begin by deducing some consequences of the G-saturatedness of Γ for the invariants in Definition 4.1. We start with $\mathbf{A}(\Gamma, \Sigma)$. By Proposition 2.26, $|a(\alpha)| = 1$ for all $\alpha \in S \cap \Sigma$. Recall, for example from the proof of Proposition 2.24, that this implies that if D_α^+ and D_α^- are the two colors moved by α, then

$$\rho(D_\alpha^+) = \rho(D_\alpha^-) = \tfrac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}. \tag{4.2}$$

Next we consider $\mathcal{C}(\Gamma, \Sigma)$. Because Γ is G-saturated, the cone $\Gamma^\vee$ is generated by $\{\alpha^\vee|_{\mathbb{Z}\Gamma}\colon \alpha \in S\}$. More precisely,

$$\Gamma^\vee = \mathbb{Q}_{\geq 0}\{\alpha^\vee|_{\mathbb{Z}\Gamma}\colon \alpha \in S \smallsetminus S^p(\Gamma)\}. \tag{4.3}$$

Consequently, $\mathcal{C}(\Gamma,\Sigma)$ is equal to the cone C in the proof of Proposition 1.7, that is:

$$\begin{aligned}\mathcal{C}(\Gamma,\Sigma) &= \mathbb{Q}_{\geq 0}\{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in S_\Gamma\} \\ &= \mathbb{Q}_{\geq 0}\{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in S_\Gamma \smallsetminus S^p(\Gamma)\}.\end{aligned} \tag{4.4}$$

The maximality of S_Γ implies that

$$\text{if } \alpha \in S \smallsetminus S_\Gamma \text{ then } \alpha^\vee|_{\mathbb{Z}\Gamma} \notin \mathcal{C}(\Gamma,\Sigma). \tag{4.5}$$

Using Definition 2.17 and equations (2.10) and (4.2) we see that

(i) for each color $D \in \Delta(\Gamma,\Sigma)$, there exists a simple root $\alpha \in S \smallsetminus S^p(\Gamma)$ such that $\rho(D) \in \{\alpha^\vee|_{\mathbb{Z}\Gamma}, \frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}\}$;

(ii) for each $\alpha \in S \smallsetminus S^p(\Gamma)$ there exists a color $D \in \Delta(\Gamma,\Sigma)$ such that $\rho(D) \in \{\alpha^\vee|_{\mathbb{Z}\Gamma}, \frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}\}$; and

(iii) if $\alpha \in S$ and $D \in \Delta(\Gamma,\Sigma)$, then D is moved by α if and only if $\rho(D) \in \{\alpha^\vee|_{\mathbb{Z}\Gamma}, \frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}\}$.

Together with (4.4) and (4.5) these three facts imply that

$$\mathcal{F}(\Gamma,\Sigma) = \Big\{D \in \Delta(\Gamma,\Sigma) \colon \text{ there exists } \alpha \in S_\Gamma \smallsetminus S^p(\Gamma) \\ \text{such that } \rho(D) \in \{\alpha^\vee|_{\mathbb{Z}\Gamma}, \tfrac{1}{2}\alpha^\vee|_{\mathbb{Z}\Gamma}\}\Big\}; \tag{4.6}$$

and that

$$\mathcal{B}(\Gamma,\Sigma) = \emptyset; \text{ and} \tag{4.7}$$

$$S(\Gamma,\Sigma) = S_\Gamma. \tag{4.8}$$

Observe that (4.6) is equivalent to

$$\mathcal{F}(\Gamma,\Sigma) = \{D \in \Delta(\Gamma,\Sigma) \colon D \text{ is moved by some } \alpha \in S_\Gamma\}. \tag{4.9}$$

Using the identification in Definition 3.11(4), this implies that the fifth entry of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is equal to the fifth entry of $\mathrm{soc}(\Gamma,\Sigma)$, which is $\mathcal{F}(\Gamma,\Sigma)$. With (4.7), this implies that the sixth entry of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is the empty set.

We now prove $\boxed{\Leftarrow}$ in (4.1). Condition (3) of Theorem 4.2 and equations (4.2) and (4.6) imply that $\mathbf{A}(\Gamma,\Sigma) \cap \mathcal{D}(\Gamma,\Sigma) = \emptyset$. Therefore

$$S_\Gamma \cap \Sigma = \emptyset \tag{4.10}$$

and the fourth entry of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is the empty set. By Proposition 2.7, it also follows from (4.10) that

$$\Sigma^{sc}(\Gamma) \cap \mathbb{Z}S_\Gamma = \Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma \tag{4.11}$$

and therefore that the three first components of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ are $(S_\Gamma, S^p(\Gamma), \Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma)$. The only socles in Table 2 for which the fourth and sixth components are empty are socles $\#1, \#15, \#17, \#22, \#24$ and $\#28$. The first three components of these socles are triples number 1, 2, 3, 5, 6 and 4, respectively, in List 1.10. Consequently, condition (2) of Theorem 4.2 implies condition (c) of Theorem 1.12.

Condition (2) of Theorem 4.2 also implies that

$$2\alpha \notin \Sigma \text{ for all } \alpha \in S_\Gamma \smallsetminus S^p(\Gamma) \tag{4.12}$$

because none of the six socles $\#1, \#15, \#17, \#22, \#24$ and $\#28$ in Table 2 have the double of a simple root in their third component (the set of spherical roots). Combined with (4.10) and (2.10), this implies that for every color $D \in \mathcal{F}(\Gamma,\Sigma)$ the functional $\rho(D)$ is equal to the restriction of some simple coroot to $\mathbb{Z}\Gamma$. With (ii), (4.6) and (4.7) we deduce that

$$\rho(\mathcal{D}(\Gamma,\Sigma)) = \rho(\mathcal{F}(\Gamma,\Sigma)) = \{\alpha^\vee|_{\mathbb{Z}\Gamma} \colon \alpha \in S_\Gamma \smallsetminus S^p(\Gamma)\}. \tag{4.13}$$

Condition (a) in Theorem 1.12 now follows from condition (3) of Theorem 4.2.

To show (b) in Theorem 1.12, assume that $\alpha, \beta \in S_\Gamma \smallsetminus S^p(\Gamma)$ with $\alpha \neq \beta$ and $\alpha^\vee|_{\mathbb{Z}\Gamma} = \beta^\vee|_{\mathbb{Z}_\Gamma}$. By (4.13), (4.9) and (iii) there exist colors $D_\alpha, D_\beta \in \mathcal{F}(\Gamma,\Sigma)$ moved by α and β, respectively, and such that $\rho(D_\alpha) = \rho(D_\beta)$. Condition (3) of Theorem 4.2 implies that $\rho : \mathcal{D}(\Gamma,\Sigma) \to (\mathbb{Z}\Gamma)^*$ is injective, and consequently $D_\alpha = D_\beta$, and so there is a color moved by both α and β. Since $\alpha, \beta \notin \Sigma$, the combinatorial description of the colors of $\mathscr{S}(\Gamma,\Sigma)$, cf. Definition 2.17, implies that α and β are orthogonal and $\alpha + \beta \in \Sigma^{sc}(\Gamma)$. In particular $\alpha + \beta \in \mathbb{Z}\Gamma$ by Proposition 2.23, which proves (b) in Theorem 1.12.

Finally, we prove $\boxed{\Rightarrow}$ in (4.1). Since none of the triples in List 1.10 contain a simple root or its double in their third component, it follows from condition (c) of Theorem 1.12 that

$$\Sigma^N(\Gamma) \cap (S_\Gamma \cup 2S_\Gamma) = \emptyset. \tag{4.14}$$

By Proposition 2.7, this implies that (4.10) and (4.12) hold again, and therefore, so does (4.13).

In view of deducing condition (3) of Theorem 4.2 we now show that

$$\rho \text{ is injective on } \mathcal{D}(\Gamma,\Sigma). \tag{4.15}$$

To do so, we first observe that, if α and β are two different elements of $S_\Gamma \smallsetminus S^p(\Gamma)$, then

$$\alpha^\vee|_{\mathbb{Z}\Gamma} = \beta^\vee|_{\mathbb{Z}\Gamma} \Rightarrow \alpha \perp \beta. \tag{4.16}$$

Indeed, if α and β are not orthogonal, then they belong to the same connected component of the Dynkin diagram of S_Γ, so they appear in the same primitive admissible triple. But, among the primitive admissible triples given in List 1.10, only in case 4. does the set $S_\Gamma \smallsetminus S^p(\Gamma)$ contain two non-orthogonal simple roots, and for any such couple of simple roots the two corresponding simple coroots take different values on some element of $\Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma$.

To show (4.15), let $D, E \in \mathcal{D}(\Gamma,\Sigma) = \mathcal{F}(\Gamma,\Sigma)$ be such that $\rho(D) = \rho(E)$, and let α and β be simple roots in $S_\Gamma \smallsetminus S^p(\Gamma)$ moving resp. D and E. If $\alpha = \beta$, then we know that $D = E$ thanks to (4.10) because if a simple root moves more than one color of the spherical system $\mathscr{S}(\Gamma,\Sigma)$, then it must belong to Σ. So, we assume that $\alpha \neq \beta$. From (4.13) we know that $\alpha^\vee|_{\mathbb{Z}\Gamma} = \rho(D) = \rho(E) = \beta^\vee|_{\mathbb{Z}\Gamma}$. It follows from (b) of Theorem 1.12 that $\alpha + \beta \in \mathbb{Z}\Gamma$. Since $\alpha \perp \beta$ by (4.16), it follows from Proposition 2.23 that $\alpha + \beta \in \Sigma^{sc}(\Gamma)$, and from Definition 2.17 of the colors of $\mathscr{S}(\Gamma,\Sigma)$ that $D = E$. This proves (4.15). Together with (4.13) and condition (a) of Theorem 1.12, statement (3) of Theorem 4.2 follows.

What remains is to deduce condition (2) of Theorem 4.2. Recall that by (4.14) we have that (4.10) holds, and therefore that (4.11) holds as well. It follows, as in the proof of "$\Leftarrow$", that the first three components of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ are $(S_\Gamma, S^p(\Gamma), \Sigma^N(\Gamma) \cap \mathbb{Z}S_\Gamma)$. Moreover, by (4.10) the fourth component of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is the empty set. We already know that the sixth component of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is empty whenever Γ is G-saturated. Since the fourth and the seventh component of $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ are determined by the other components (for the seventh this is because the sixth component is empty), it follows from condition (c) that $\overline{\mathrm{soc}}(\Gamma,\Sigma)$ is isomorphic to a product of socles #1, #15, #17, #22, #24 and #28 in Table 2. By Theorem 3.22 this implies statement (2) of Theorem 4.2. This finishes the proof. □

Example 4.5. This example shows that condition (b) in Theorem 1.12 cannot be removed — and therefore also that it is not possible to replace, in Theorem 4.2, the $|\mathcal{D}(\Gamma,\Sigma)|$-tuple $(\rho(D))_{D\in\mathcal{D}(\Gamma,\Sigma)}$ with the *set* $\rho(\mathcal{D}(\Gamma,\Sigma))$. Let $G = \mathrm{SL}(3)\times\mathrm{SL}(3)$ and $\Gamma = \mathbb{N}\{\omega_1+\omega_1'\}$. Then $S^p(\Gamma) = \{\alpha_2,\alpha_2'\}$. One checks that $\Sigma^N(\Gamma) = \emptyset$ and that $S_\Gamma = S$. It follows that the triple $(S_\Gamma, S^p(\Gamma), \Sigma^N(\Gamma))$ is admissible. Moreover, condition (a) of Theorem 1.12 is met because $\langle\alpha_1^\vee, \omega_1+\omega_1'\rangle = \langle(\alpha_1')^\vee, \omega_1+\omega_1'\rangle = 1$. On the other hand, condition (b) of the theorem is not met because $\alpha_1+\alpha_1' \notin \mathbb{Z}\{\omega_1+\omega_1'\}$. Since $\Sigma^N(\Gamma)=\emptyset$ there is, up to isomorphism, only one affine spherical G-variety with weight monoid Γ. It is $X_0 := \overline{G\cdot x_0} \subseteq V(\omega_2+\omega_2')$, where x_0 is a highest weight vector in $V(\omega_2+\omega_2')$. As is well-known, X_0 is not smooth. This shows that Theorem 1.12 would be false without condition (b).

5. Smooth affine model varieties

In this section we apply our smoothness criterion to show Theorem 1.16. Here we focus on the monoid $\Gamma = \Lambda^+$, which is G-saturated. For this reason we apply our criterion in the version of Theorem 1.12.

Proposition 5.1. *Suppose that G is semisimple and simply connected. Then:*

(1) $S^p(\Lambda^+) = \emptyset$,
(2) $\Sigma^N(\Lambda^+) = \{\alpha+\alpha' \mid \alpha,\alpha'\in S,\ \alpha\neq\alpha',\ \alpha\not\perp\alpha'\}$.

Proof. Part (1) is obvious. Part (2) follows from [Lun07], let us give a direct proof here. We observe that the only elements of $\Sigma^{sc}(G)$ compatible with $S^p(\Lambda^+)=\emptyset$ are either sums $\alpha+\alpha'$ of two different simple roots, or simple roots, or doubles of simple roots. Simple roots are excluded from $\Sigma^N(\Lambda^+)$ by Proposition 1.6. The double of any simple root α is excluded by the same proposition, because if ω_α is the fundamental dominant weight corresponding to α then $\omega_\alpha\in\Lambda^+$ (since G is simply connected) and $\langle\alpha^\vee,\omega_\alpha\rangle = 1\notin 2\mathbb{Z}$.

Finally, let $\sigma = \alpha+\beta$ where $\alpha,\beta\in S$ with $\alpha\neq\beta$. If α is orthogonal to β, then $\sigma\notin\Sigma^N(\Lambda^+)$ by Proposition 1.6 because $\alpha^\vee\neq\beta^\vee$. On the other hand, if α is not orthogonal to β, then $\sigma\in\Sigma^N(\Lambda^+)$, again by Proposition 1.6. □

Thanks to part (1) of the proposition above, we discuss now the admissibility of $(S_{\Lambda^+}, \emptyset, \Sigma^N(\Lambda^+)\cap\mathbb{Z}S_{\Lambda^+})$ in view of applying Theorem 1.12.

Part (2) of the proposition implies that no spherical root in $\Sigma^N(\Lambda^+)$ is the sum of simple roots in different connected components of the Dynkin diagram of G.

It follows that the triple $(S_{\Lambda^+}, \emptyset, \Sigma^N(\Lambda^+)\cap\mathbb{Z}S_{\Lambda^+})$ is admissible if and only if $(S'\cap S_{\Lambda^+}, \emptyset, \Sigma^N(\Lambda^+)\cap\mathbb{Z}(S'\cap S_{\Lambda^+}))$ is admissible for all $S'\subseteq S$ corresponding to a connected component of the Dynkin diagram of G.

Lemma 5.2. *Under the assumptions of Proposition 5.1, let $S'\subseteq S$ correspond to a connected component of the Dynkin diagram of G.*

(1) If S' is of type A_n with $n\geq 1$ odd, then $S'\cap S_{\Lambda^+} = \{\alpha_i \mid i \text{ odd}\}$.
(2) If S' is of type A_n with $n\geq 2$ even, then $S'\cap S_{\Lambda^+} = \emptyset$.
(3) If S' is of type B_n with $n\geq 3$ odd, then $S'\cap S_{\Lambda^+} = S'$.
(4) If S' is of type B_n with $n\geq 4$ even, then $S'\cap S_{\Lambda^+} = S'\smallsetminus\{\alpha_1\}$.
(5) If S' is of type C_n with $n\geq 2$, then $S'\cap S_{\Lambda^+} = \{\alpha_i \mid i \text{ odd}\}$.

(6) If S' is of type D_n with $n \geq 5$ odd, then $S' \cap S_{\Lambda^+} = S' \smallsetminus \{\alpha_1\}$.
(7) If S' is of type D_n with $n \geq 4$ even, then $S' \cap S_{\Lambda^+} = S'$.
(8) If S' is of type E_6, then $S' \cap S_{\Lambda^+} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.
(9) If S' is of type E_7, then $S' \cap S_{\Lambda^+} = S'$.
(10) If S' is of type E_8, then $S' \cap S_{\Lambda^+} = S'$.
(11) If S' is of type F_4, then $S' \cap S_{\Lambda^+} = \{\alpha_1, \alpha_2, \alpha_3\}$.
(12) If S' is of type G_2, then $S' \cap S_{\Lambda^+} = S'$.

Proof. Recall that $S' \cap S_{\Lambda^+}$ is the maximal subset of S' such that there exists a linear combination of the corresponding simple coroots, with strictly positive coefficients, that is non-positive on $\Sigma^N(\Lambda^+)$. We prove case by case that the sets indicated in the statement of the lemma satisfy this requirement.

We prove it by exhibiting some linear combinations ζ of this kind, showing that some simple roots are in $S' \cap S_{\Lambda^+}$. We also use the following elementary fact: if some $\tau \in \mathbb{Q}_{\geq 0}\Sigma^N(\Lambda^+)$ is positive on a subset of coroots C, and non-negative on the simple coroots not in C, then the simple roots corresponding to the elements of C are not in $S' \cap S_{\Lambda^+}$. It is convenient to use the notation $\sigma_i = \alpha_i + \alpha_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$ in types A_n, B_n, C_n, for $i \in \{1, 2, 3\}$ in type F_4, and also in type D_n for $i \in \{1, \ldots, n-2\}$.

Case (1). Taking $\zeta = \sum_{i \text{ odd}} \alpha_i^\vee$ shows that $S' \cap S_{\Lambda^+} \supset \{\alpha_i \mid i \text{ odd}\}$. For $i \in \{1, 2, \ldots, (n-3)/2\}$, taking $\tau_i = \sum_{j=1}^{i}(\sigma_{2j-1} + \sigma_{2j} + \ldots + \sigma_{n-1})$ shows that α_{2i} and α_{n-1} are not in $S' \cap S_{\Lambda^+}$.

Case (2). Taking $\tau = \sum_{j \text{ odd}} 2\sigma_j + \sum_{j \text{ even}} \sigma_j$ shows that $\alpha_1, \alpha_n \notin S' \cap S_{\Lambda^+}$. For all $i \in \{1, 2, \ldots, n/2 - 1\}$, taking $\tau_i = \sum_{j=1}^{i}(\sigma_{2j-1} + \sigma_{2j} + \ldots + \sigma_{n-1})$ shows that $\alpha_{2i} \notin S' \cap S_{\Lambda^+}$, and $\tau_i' = \sum_{j=1}^{i}(\sigma_1 + \sigma_2 + \ldots + \sigma_{n-2j+1})$ shows that $\alpha_{n-2i+1} \notin S' \cap S_{\Lambda^+}$. Hence $S' \cap S_{\Lambda^+} = \emptyset$.

Case (3). One can take $\zeta = \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \alpha_4^\vee) + \ldots + \frac{n-3}{2}(\alpha_{n-4}^\vee + \alpha_{n-3}^\vee) + \frac{n-1}{2}(\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee)$.

Case (4). Taking $\zeta = \alpha_2^\vee + \alpha_3^\vee + 2(\alpha_4^\vee + \alpha_5^\vee) + \ldots + \frac{n-4}{2}(\alpha_{n-4}^\vee + \alpha_{n-3}^\vee) + \frac{n-2}{2}(\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee)$ shows that $S' \cap S_{\Lambda^+} \supset S' \smallsetminus \{\alpha_1\}$. To prove that $\alpha_1 \notin S' \cap S_{\Lambda^+}$, we use the element $\tau = \sigma_1 + \sigma_3 + \ldots + \sigma_{n-1}$.

Case (5). Taking $\zeta = \sum_{i \text{ odd}} \alpha_i^\vee$ shows that $S' \cap S_{\Lambda^+} \supset \{\alpha_i \mid i \text{ odd}\}$. For all $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we use the element $\tau_i = \sum_{j=1}^{i}(\sigma_{2j-1} + \sigma_{2j} + \ldots + \sigma_{n-1})$ to conclude that $\alpha_{2i} \notin S' \cap S_{\Lambda^+}$.

Case (6). Taking $\zeta = \alpha_2^\vee + \alpha_3^\vee + 2(\alpha_4^\vee + \alpha_5^\vee) + \ldots + \frac{n-1}{2}(\alpha_{n-1}^\vee + \alpha_n^\vee)$ shows that $S' \cap S_{\Lambda^+} \supset S' \smallsetminus \{\alpha_1\}$. To prove that $\alpha_1 \notin S' \cap S_{\Lambda^+}$, we use the element $\tau = 2(\sigma_1 + \sigma_3 + \ldots + \sigma_{n-4}) + \sigma_{n-2} + (\alpha_{n-2} + \alpha_n)$.

Case (7). One can take $\zeta = \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \alpha_4^\vee) + \ldots + \frac{n}{2}(\alpha_{n-1}^\vee + \alpha_n^\vee)$.

Case (8). Taking $\zeta = \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee + \alpha_5^\vee$ shows that $S' \cap S_{\Lambda^+} \supset \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Taking $\tau_1 = 2(\alpha_1 + \alpha_3) + 3(\alpha_2 + \alpha_4) + 2(\alpha_3 + \alpha_4) + (\alpha_4 + \alpha_5) + 4(\alpha_5 + \alpha_6)$ shows that $\alpha_6 \notin S' \cap S_{\Lambda^+}$, and taking $\tau_2 = 4(\alpha_1 + \alpha_3) + 3(\alpha_2 + \alpha_4) + (\alpha_3 + \alpha_4) + 2(\alpha_4 + \alpha_5) + 2(\alpha_5 + \alpha_6)$ shows that $\alpha_1 \notin S' \cap S_{\Lambda^+}$.

Case (9). One can take $\zeta = \alpha_1^\vee + 3\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee + 2\alpha_7^\vee$.

Case (10). One can take $\zeta = 2\alpha_1^\vee + 4\alpha_2^\vee + 5\alpha_3^\vee + 7\alpha_4^\vee + 6\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee + \alpha_8^\vee$.

Case (11). One can take $\zeta = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$ to conclude $S' \cap S_{\Lambda^+} \supset \{\alpha_1, \alpha_2, \alpha_3\}$, and the element $\tau = \sigma_1 + \sigma_2 + 2\sigma_3$ to prove that $\alpha_4 \notin S' \cap S_{\Lambda^+}$.

Case (12). One can take $\zeta = \alpha_1^\vee + \alpha_2^\vee$. □

Corollary 5.3. *The triple* $(S' \cap S_{\Lambda^+}, \emptyset, \Sigma^N(\Lambda^+) \cap \mathbb{Z}(S' \cap S_{\Lambda^+}))$ *is admissible if and only if* S' *has type* A_n *or* C_n*.*

Proof. For S' of type A_n with n even, the triple is $(\emptyset, \emptyset, \emptyset)$, which is admissible (cf. Remark 1.11).

Assume S' has type A_n with n odd or C_n with arbitrary n. Then the triple above is obtained as a "union", in the sense of Definition 1.9, of triples of the form $(\mathsf{A}_1, \emptyset, \emptyset)$, which is the second admissible triple in List 1.10 for $n = 1$.

In all other cases $S' \cap S_{\Lambda^+}$ is connected, so expressing the above triple as a union only results in a trivial decomposition. The triple itself does not appear in List 1.10, so these cases are excluded and the proof is complete. □

Since again $S^p(\Lambda^+) = \emptyset$, it remains to show that $\{\alpha^\vee|_{\mathbb{Z}\Gamma} : \alpha \in S_{\Lambda^+}\}$ is a subset of a basis of $\mathbb{Z}(\Lambda^+)^*$ if G has simple factors of type A_n or C_n. This is obvious, since $\{\alpha^\vee \mid \alpha \in S\}$ is the dual basis of the basis of Λ consisting of the fundamental dominant weights of G. This completes the proof of Theorem 1.16.

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Dipartimento di Matematica "Guido Castelnuovo", "Sapienza" Università di Roma
E-mail address: pezzini@mat.uniroma1.it

Department Mathematik, Emmy–Noether–Zentrum, FAU Erlangen-Nürnberg & Department of Mathematics, Medgar Evers College - City University of New York
E-mail address: bartvs@mec.cuny.edu