# Asymptotic low-temperature critical behavior of two-dimensional multiflavor lattice SO($N_c$) gauge theories

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We address the interplay between global and local gauge non-Abelian symmetries in lattice gauge theories with multicomponent scalar fields. We consider two-dimensional lattice scalar non-Abelian gauge theories with a local SO($N_c$) ($N_c \geq 3$) and a global O($N_f$) invariance, obtained by partially gauging a maximally O($N_f N_c$)-symmetric multicomponent scalar model. Correspondingly, the scalar fields belong to the coset $S^{N_f N_c - 1}/\mathrm{SO}(N_c)$, where $S^N$ is the $N$-dimensional sphere. In agreement with the Mermin-Wagner theorem, these lattice SO($N_c$) gauge models with $N_f \geq 3$ do not have finite-temperature transitions related to the breaking of the global non-Abelian O($N_f$) symmetry. However, in the zero-temperature limit they show a critical behavior characterized by a correlation length that increases exponentially with the inverse temperature, similarly to nonlinear O($N$) $\sigma$ models. Their universal features are investigated by numerical finite-size scaling methods. The results show that the asymptotic low-temperature behavior belongs to the universality class of the two-dimensional RP$^{N_f - 1}$ model.

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## I. INTRODUCTION

Lattice gauge models provide effective theories in various physical contexts, ranging from fundamental interactions [1,2] to emerging phenomena in condensed matter physics [3,4]. They provide mechanisms for fundamental phenomena, such as confinement and the Higgs mechanism, which explain the spectrum of subnuclear systems interacting via strong and electroweak forces, superconductivity, etc. The interplay between global and local gauge symmetries is crucial to determining the main features of the theory, such as the nature of the spectrum, the degeneracy of the energy levels, the phase diagram, and the nature and universality classes of their thermal and quantum transitions.

In the case of two-dimensional (2D) lattice gauge models, the interplay of non-Abelian global symmetries and local gauge symmetries determines the large-scale properties of the system in the zero-temperature limit, and therefore, the statistical field theory realized in the corresponding continuum limit [5]. These issues have been addressed in the multicomponent Abelian-Higgs model [6], characterized by a global U($N_f$) symmetry ($N_f \geq 2$) and a local U(1) gauge symmetry, and in the multiflavor scalar quantum chromodynamics [7], characterized by a global U($N_f$) symmetry and a local SU($N_c$) gauge symmetry. The results of Refs. [6,7] provide numerical evidence that the asymptotic low-temperature behavior of these 2D lattice gauge models always belongs to the universality class of the 2D CP$^{N_f - 1}$ field theory [5]. Therefore, the universality class of the low-temperature behavior is only determined by the global U($N_f$) symmetry of the model. The local gauge symmetry apparently does not play any role: models with different gauge symmetry but with the same global invariance have the same large-scale low-temperature behavior. These results may be interpreted as numerical evidence of a more general conjecture [7]: the renormalization-group flow determining the asymptotic low-temperature behavior is generally controlled by the 2D statistical field theories associated with the symmetric spaces [5,8] that have the same global symmetry. This is indeed the case of the Abelian-Higgs model and of scalar chromodynamics, whose low-temperature behavior is always controlled by the 2D CP$^{N_f - 1}$ field theory.

To gain additional evidence of the above conjecture, we extend the analysis to other 2D lattice models, characterized by different global and local gauge symmetries. For this purpose, we consider 2D lattice models with real scalar fields, which are invariant under global and local gauge transformations that belong to orthogonal groups. In particular, we consider lattice gauge models that are invariant under SO($N_c$) local transformations and under O($N_f$) global transformations ($N_c$ will be referred to as the number of colors and $N_f$ as the number of flavors), focusing on the case $N_f \geq 3$, so that the global symmetry group is non-Abelian. According to the Mermin-Wagner theorem [9], these lattice gauge models do not present a finite-temperature transition associated with the breaking of the global $O(N_f)$ symmetry. However, they are expected to develop a critical behavior in the zero-temperature limit [for $N_f = 2$, the global symmetry group is the Abelian group O(2), so that a finite-temperature Berezinskii-Kosterlitz-Thouless transition is possible]. We study the universal features of the asymptotic zero-temperature behavior for $N_f = 3$, 4 and $N_c = 3$, 4 by means of finite-size scaling (FSS) analyses of Monte Carlo (MC) results. According to the above-mentioned conjecture, the asymptotic behavior should be that of a statistical theory defined on a symmetric space with the same global symmetry. We provide a theoretical argument that shows that the appropriate model is the 2D $\mathrm{RP}^{N_f-1}$ model, in which the fields effectively belong to the real projective space in $N_f$ dimension, a symmetric space which is invariant under global $O(N_f)$ transformations. Note that the associated symmetric space with the same global $O(N_f)$ symmetry group is not the $N_f$-dimensional sphere. This is due to the fact that the low-energy behavior is essentially characterized by a bilinear operator (a projector) that is invariant under local $\mathbb{Z}_2$ transformations. We anticipate that the numerical results will confirm the conjecture.

The paper is organized as follows: In Sec. II, we introduce the lattice non-Abelian gauge models that we consider. In Sec. III, we discuss the general strategy we use to investigate the nature of the low-temperature critical behavior. Then, in Sec. IV, we report the numerical results for lattice models with $N_f = 3, 4$ and $N_c = 3, 4$. Finally, in Sec. V, we summarize and draw our conclusions. In the Appendix, we report some results on the minimum-energy configurations of the models considered.

## II. THE MULTIFLAVOR LATTICE SO($N_c$) GAUGE MODEL

We define a 2D lattice scalar gauge theory by partially gauging a maximally symmetric model of real matrix variables $\phi_x^{af}$, with $a = 1, \ldots, N_c$ and $f = 1, \ldots, N_f$ (we will refer to these two indices as *color* and *flavor* indices, respectively). We start from the action

$$S_{\mathrm{sym}} = -t \sum_{x,\mu} \mathrm{Tr} \phi_x^t \phi_{x+\hat{\mu}}, \qquad \mathrm{Tr} \phi_x^t \phi_x = 1, \tag{1}$$

where the sum is over all links of a square lattice and $\hat{\mu} = \hat{1}, \hat{2}$ denotes the unit vectors along the lattice directions.[1]

[1]Model (1) with the unit-length constraint for the $\phi_x$ variables is a particular limit of a model with a quartic potential $\sum_x V(\mathrm{Tr}\phi_x^\dagger \phi_x)$ of the form $V(X) = rX + \frac{1}{2}uX^2$. Formally, it can be obtained by setting $r + u = 0$ and taking the limit $u \to \infty$.

Without loss of generality, we can set $t = 1$. The action $S_{\mathrm{sym}}$ is invariant under global $O(M)$ transformations with $M = N_f N_c$. Indeed, it can be written in terms of $M$-component unit-length real vectors $s_x$, as $S_{\mathrm{sym}} = -\sum_{x,\mu} s_x \cdot s_{x+\hat{\mu}}$, which is the standard nearest-neighbor $M$-vector lattice model.

We proceed by gauging some of the degrees of freedom using the Wilson approach [1]. We associate an SO($N_c$) matrix $V_{x,\mu}$ with each lattice link [$(x, \mu)$ denotes the link that starts at site $x$ in the $\hat{\mu}$ direction] and add a Wilson kinetic term [1] for the gauge fields. We thus obtain the model with action

$$S_g = -N_f \sum_{x,\mu} \mathrm{Tr} \phi_x^t V_{x,\mu} \phi_{x+\hat{\mu}} - \frac{\gamma}{N_c} \sum_x \mathrm{Tr} \Pi_x, \tag{2}$$

where $\Pi_x$ is the plaquette operator

$$\Pi_x = V_{x,1} V_{x+\hat{1},2} V_{x+\hat{2},1}^t V_{x,2}^t. \tag{3}$$

The plaquette parameter $\gamma$ plays the role of inverse gauge coupling. The partition function reads

$$Z = \sum_{\{\phi,V\}} e^{-\beta S_g}, \qquad \beta \equiv 1/T. \tag{4}$$

One can easily check that the lattice model (2) is invariant under SO($N_c$) gauge transformations:

$$\phi_x \to W_x \phi_x, \qquad V_{x,\mu} \to W_x V_{x,\mu} W_{x+\hat{\mu}}^t, \tag{5}$$

with $W_x \in \mathrm{SO}(N_c)$. For $\gamma \to \infty$, the link variables $V_x$ become equal to the identity (modulo gauge transformations); thus, one recovers the ungauged model (1), or equivalently the nearest-neighbor $M$-vector model.

For $N_c = 2$, the global symmetry group of model (2) is actually larger than $O(N_f)$. Indeed, one can show that [10] the model can be exactly mapped onto the lattice Abelian-Higgs model

$$S_{\mathrm{AH}} = -N_f \sum_{x,\mu} \mathrm{Re}[\bar{z}_x \cdot \lambda_{x,\mu} z_{x+\hat{\mu}}] - \gamma \sum_{x,\mu>\nu} \mathrm{Re}[\lambda_{x,\mu} \lambda_{x+\hat{\mu},\nu} \bar{\lambda}_{x+\hat{\nu},\mu} \bar{\lambda}_{x,\nu}], \tag{6}$$

where $z_x$ is a unit-length $N_f$-component complex vector, and $\lambda_{x,\nu}$ a U(1) link variable. The Abelian-Higgs model is invariant under local U(1) and global $U(N_f)$ transformations. There is therefore an enlargement of the global symmetry of the model: the global symmetry group is $U(N_f)$ instead of $O(N_f)$. The asymptotic zero-temperature behavior of these models has been studied in Ref. [6]. Therefore, in the following, we focus on the asymptotic low-temperature behavior for $N_c \geq 3$.

We mention that the phase diagram and critical behavior of model (2) in three dimensions was already discussed in Refs. [10,11], and similar results were presented in Refs. [12,13] for $SU(N_c)$ gauge theories. In this work, we focus on the 2D case. According to the Mermin-Wagner theorem [9], lattice $SO(N_c)$ gauge theories are not expected to show finite-temperature transitions with a low-temperature phase in which the global $O(N_f)$ symmetry is broken. Therefore, there are only two possibilities: either the system is always disordered for any $\beta$, or a finite-temperature transition occurs with a low-temperature phase in which there is no long-range order, but correlations decay algebraically with the distance. We expect the first behavior whenever the global symmetry group is non-Abelian, and the second one whenever the symmetry group is isomorphic to U(1).

For $N_f \geq 3$, the global $O(N_f)$ symmetry group is non-Abelian. Therefore, we expect a nontrivial critical behavior only in the zero-temperature limit, analogous to that occurring in the nonlinear $O(N)$ $\sigma$ model or in the $CP^{N-1}$ model—see, e.g., Ref. [5]. Infinite-volume correlation functions are characterized by a length scale $\xi$ that diverges as

$$\xi \sim \beta^p e^{c\beta}. \tag{7}$$

For $N_f = 2$ and $N_c \geq 3$, the model has an Abelian O(2) global symmetry. It is therefore possible that it undergoes a finite-temperature Berezinskii-Kosterlitz-Thouless transition [14–18], with a spin-wave low-temperature phase characterized by correlation functions decaying algebraically. For $N_f = 2$ and $N_c = 2$, due to the mapping to the Abelian-Higgs model (6), the global symmetry group, the U(2) group, is non-Abelian. Therefore, the model is only critical for $\beta \to \infty$. The low-temperature behavior belongs to the universality class of the 2D $CP^1$ model [6], which is equivalent to that of the nonlinear O(3) $\sigma$ model.

The global symmetry group of the model is $O(N_f)$, which is not a simple group. Therefore, in principle, one may have both the breaking of the $\mathbb{Z}_2$ subgroup and of the $SO(N_f)$ subgroup. However, on the basis of the results for the same model in three dimensions [10], we do not expect the $\mathbb{Z}_2$ subgroup to play any role (a similar decoupling occurs in the unitary case [7]). The critical low-temperature behavior is therefore associated with the order parameter for the breaking of the $SO(N_f)$ subgroup, which is the bilinear operator

$$Q_{\boldsymbol{x}}^{fg} = \sum_a \phi_{\boldsymbol{x}}^{af} \phi_{\boldsymbol{x}}^{ag} - \frac{1}{N_f}\delta^{fg}, \tag{8}$$

which is a symmetric and traceless $N_f \times N_f$ matrix.

In the following sections, we provide numerical evidence that, for $N_c \geq 3$ and $N_f \geq 3$, the asymptotic zero-temperature limit of the $SO(N_c)$ gauge model (2) is the same as that of the 2D $RP^{N_f-1}$ models, which are also invariant under $O(N_f)$ transformations. The $RP^{N-1}$ models can be defined by associating a real $N$-component unit-length vector $\varphi_{\boldsymbol{x}}$ with each lattice site and considering actions that are invariant under global $O(N)$ rotations of the fields and local $\mathbb{Z}_2$ transformations $\varphi_{\boldsymbol{x}} \to s_x \varphi_{\boldsymbol{x}}$ ($s_x = \pm 1$). The standard nearest-neighbor $RP^{N-1}$ model is defined by the lattice action

$$S_{\rm RP} = -t \sum_{\boldsymbol{x},\mu} (\varphi_{\boldsymbol{x}} \cdot \varphi_{\boldsymbol{x}+\hat{\mu}})^2. \tag{9}$$

Alternatively, one may introduce an explicit link variable $\sigma_{\boldsymbol{x},\mu} = \pm 1$, and consider the lattice action

$$S_{\rm RP\sigma} = -t \sum_{\boldsymbol{x},\mu} \varphi_{\boldsymbol{x}} \cdot \sigma_{\boldsymbol{x},\mu} \varphi_{\boldsymbol{x}+\hat{\mu}}. \tag{10}$$

The nature of their low-temperature behavior for $N \geq 3$ has been the object of a long debate—see, e.g., Refs. [19–23]. The main question has been whether the 2D $RP^{N-1}$ model belongs to the same universality class as the $O(N)$ vector model. We refer to Ref. [23] for a thorough discussion of this point. There, we report extensive numerical results that indicate that the universal low-temperature long-distance behavior of the 2D $RP^{N-1}$ models differs from that of the 2D $O(N)$ vector models: they appear as distinct universality classes.

We will show that renormalization-group-invariant quantities defined in terms of $Q^{fg}$ in the non-Abelian gauge theory have the same universal behavior as the corresponding $RP^{N_f-1}$ quantities defined in terms of the local gauge-invariant operator

$$P_{\boldsymbol{x}}^{fg} = \varphi_{\boldsymbol{x}}^f \varphi_{\boldsymbol{x}}^g - \frac{1}{N_f}\delta^{fg}. \tag{11}$$

Such correspondence can be established using the same arguments we used for unitary models in Ref. [7]. As discussed in the Appendix, for $\beta \to \infty$ the $\phi$ configurations can be parametrized by a single $N_f$-dimensional unit vector $\varphi^f$. Modulo gauge transformations, we have

$$\begin{aligned} \phi^{af} &= 0, \qquad a < N_c, \\ \phi^{af} &= \varphi^f, \qquad a = N_c, \end{aligned} \tag{12}$$

which implies that the bilinear $Q_{\boldsymbol{x}}$ becomes equivalent in this limit to the $RP^{N_f-1}$ operator $P_{\boldsymbol{x}}$. Since the $\mathbb{Z}_2$ global symmetry does not play any role, in the zero-temperature limit the gauge model can be described by an effective theory only in terms of the $SO(N_f)$ order parameter $P_{\boldsymbol{x}}$. The natural candidate for the action is

$$H_{\rm eff} = -\kappa \sum_{\boldsymbol{x},\mu} {\rm Tr} P_{\boldsymbol{x}} P_{\boldsymbol{x}+\hat{\mu}}, \tag{13}$$

which gives Eq. (9) apart from an irrelevant constant. We have thus obtained the $RP^{N_f-1}$ model.

## III. UNIVERSAL FINITE-SIZE SCALING

We exploit FSS techniques [24–27] to study the nature of the asymptotic critical behavior of the model for $T \to 0$. For this purpose we consider models defined on square lattices of linear size $L$ with periodic boundary conditions. We focus on the correlations of the gauge-invariant variable $Q_{\boldsymbol{x}}$ defined in Eq. (8). The corresponding two-point correlation function is defined as

$$G(\boldsymbol{x}-\boldsymbol{y}) = \langle \mathrm{Tr} Q_{\boldsymbol{x}} Q_{\boldsymbol{y}} \rangle, \tag{14}$$

where the translation invariance of the system has been taken into account. We define the susceptibility $\chi = \sum_{\boldsymbol{x}} G(\boldsymbol{x})$ and the correlation length

$$\xi^2 = \frac{1}{\sin^2(\pi/L)} \frac{\tilde{G}(\boldsymbol{0}) - \tilde{G}(\boldsymbol{p}_m)}{\tilde{G}(\boldsymbol{p}_m)}, \tag{15}$$

where $\tilde{G}(\boldsymbol{p}) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} G(\boldsymbol{x})$ is the Fourier transform of $G(\boldsymbol{x})$, and $\boldsymbol{p}_m = (2\pi/L, 0)$. We also consider the quartic cumulant (Binder) parameter defined as

$$U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \qquad \mu_2 = \frac{1}{V^2} \sum_{\boldsymbol{x},\boldsymbol{y}} \mathrm{Tr} Q_{\boldsymbol{x}} Q_{\boldsymbol{y}}, \tag{16}$$

where $V = L^2$.

To identify the universality class of the asymptotic zero-temperature behavior, we consider the Binder parameter $U$ as a function of the ratio

$$R_\xi \equiv \xi/L. \tag{17}$$

Indeed, in the FSS limit we have (see, e.g., Ref. [6])

$$U(\beta, L) \approx F(R_\xi), \tag{18}$$

where $F(x)$ is a universal scaling function that completely characterizes the universality class of the transition. The asymptotic values of $F(R_\xi)$ for $R_\xi \to 0$ and $R_\xi \to \infty$ correspond to the values that $U$ takes in the small-$\beta$ and large-$\beta$ limits. For $R_\xi \to 0$, we have

$$\lim_{R_\xi \to 0} U = 1 + \frac{4}{(N_f - 1)(N_f + 2)}, \tag{19}$$

independently of the value of $N_c$. In the large-$\beta$ limit, we have $U \to 1$, as discussed in the Appendix.

Equation (18) allows us to check the universality of the asymptotic zero-temperature behavior without the need of tuning any parameter. Corrections to Eq. (18) are expected to decay as a power of $L$. In the case of asymptotically free models, such as the 2D $CP^{N-1}$ and $O(N)$ vector models, corrections decrease as $L^{-2}$, multiplied by powers of $\ln L$ [28]. However, we note that sometimes, when the available data are not sufficiently asymptotic, the approach to the asymptotic behavior may appear slower, and corrections apparently decay as $L^{-p}$ with $p < 2$ [29].

Because of the universality of relation (18), we can use the plots of $U$ versus $R_\xi$ to identify the models that belong to the same universality class. If the data of $U$ for two different models follow the same curve when plotted versus $R_\xi$, their critical behavior is described by the same continuum quantum field theory. This implies that any other dimensionless RG-invariant quantity has the same critical behavior in the two models, both in the thermodynamic and in the FSS limit. An analogous strategy for the study of the asymptotic zero-temperature behavior of 2D models was employed in Refs. [6,7].

## IV. NUMERICAL RESULTS

In this section, we study the large-$\beta$ critical behavior of the lattice scalar gauge model (2) for some values of $N_f \geq 3$

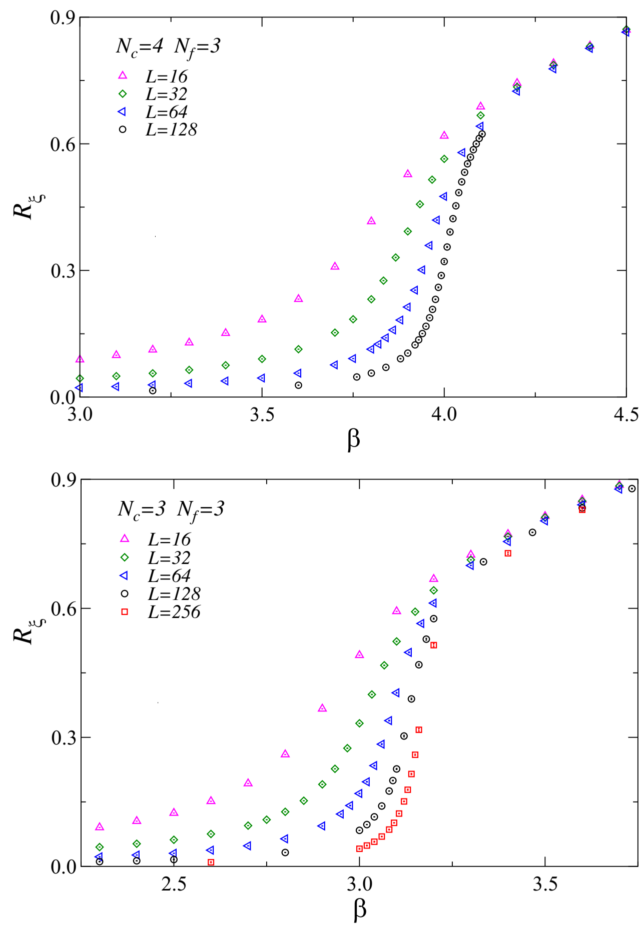


FIG. 1. $R_\xi \equiv \xi/L$ for the three-flavor SO(3) and SO(4) gauge theories [Eq. (2)] with $\gamma = 0$. We show data up to $L = 256$ for $N_c = 3$ (bottom) and up to $L = 128$ for $N_c = 4$ (top). Data for different sizes do not show evidence of crossing points. Statistical errors are hardly visible on the scale of the figure.

and $N_c \geq 3$. We perform MC simulations, using the same upgrading algorithm employed in Ref. [10] for three-dimensional lattice $SO(N_c)$ gauge models. We show that the FSS curves (18) of the Binder parameter $U$ versus $R_\xi$ computed in the model (2) agree with those computed in $RP^{N-1}$ models (we use the results reported in Ref. [23]). These results provide numerical evidence that, for $N_c \geq 3$, the critical behavior belongs to the universality class of the 2D $RP^{N_f-1}$ field theory, in agreement with the arguments of the previous section.

We first mention that the data of $R_\xi \equiv \xi/L$ corresponding to different lattice sizes—see Fig. 1—do not intersect, confirming the absence of a phase transition at finite $\beta$, as expected on the basis of the Mermin-Wagner theorem [9]. In Fig. 2, we show the estimates of the correlation length for the three-flavor SO(3) and SO(4) gauge theories [Eq. (2)] with $\gamma = 0$, up to lattice sizes $L = 256$ and $L = 128$, respectively. When data for different lattice sizes match, they can be considered as a good approximation of the correlation length in the thermodynamic limit at the given inverse temperature $\beta$. The data in this regime are substantially consistent with an exponential dependence of $\xi$ on $\beta$—see Eq. (7)—as expected for asymptotically free models.

In Fig. 3, we plot $U$ versus $R_\xi$ for the three-flavor SO(3) and SO(4) gauge theories with $\gamma = 0$, up to $L = 256$ and $L = 128$, respectively. We observe that the data of $U$ appear to approach a FSS curve in the large-$L$ limit, in agreement with the FSS prediction [Eq. (18)]. In the same figure, we also report data for the standard $RP^2$ lattice model with action (9), and for the $RP^2$ gauge model with action (10) (as shown in Ref. [23], the data for $L = 320$ provide a good approximation of the asymptotic curve). The $RP^2$ results are consistent with the asymptotic FSS curve for the $SO(N_c)$ gauge model, confirming our claim

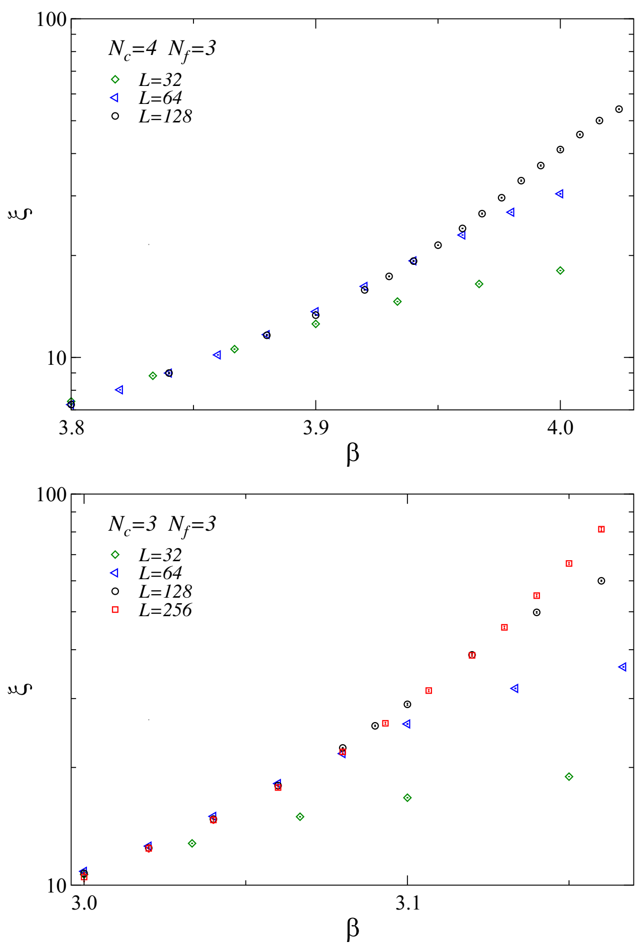


FIG. 2. The correlation length $\xi$ versus $\beta$ for $N_f = 3$, $N_c = 3$ (bottom) and $N_f = 3$ $N_c = 4$ (top). We set $\gamma = 0$. When data for different values of $L$ match, they can be considered as good approximations of the infinite-volume correlation length, within their errors. The behavior of the infinite-volume data is consistent with an exponential dependence on $\beta$ (we use a logarithmic scale on the vertical axis).

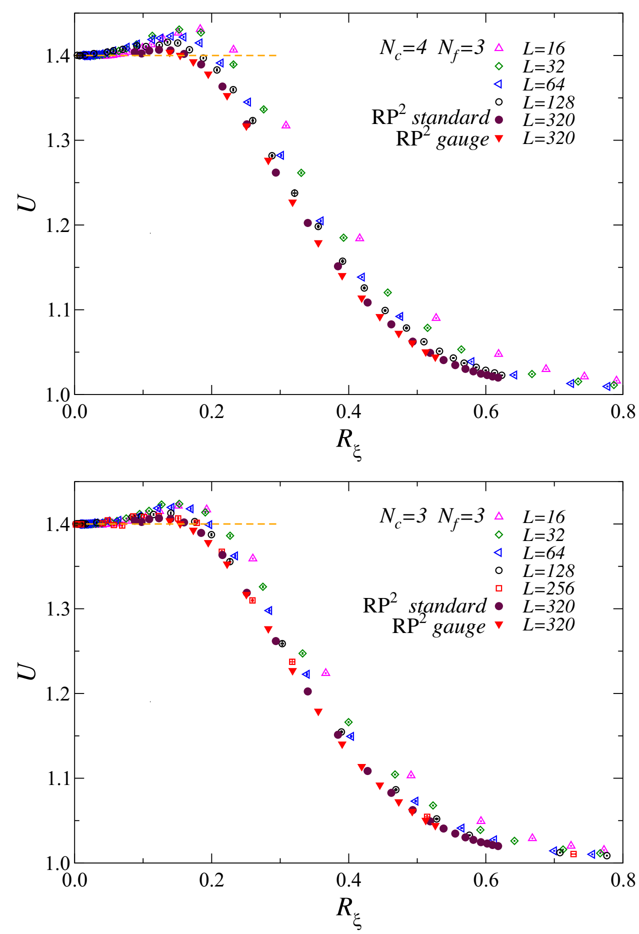


FIG. 3. Plot of $U$ versus $R_\xi$ for the three-flavor SO(3) (bottom) and SO(4) (top) gauge theory at $\gamma = 0$. The horizontal dashed line shows the $R_\xi \to 0$ limit $U = 7/5$. The data approach the asymptotic curve of the 2D $RP^2$ models (9) and (10) (labeled as *standard* and *gauge*, respectively; the corresponding data for $L = 320$ are taken from Ref. [23]). Statistical errors are so small as to be hardly visible.

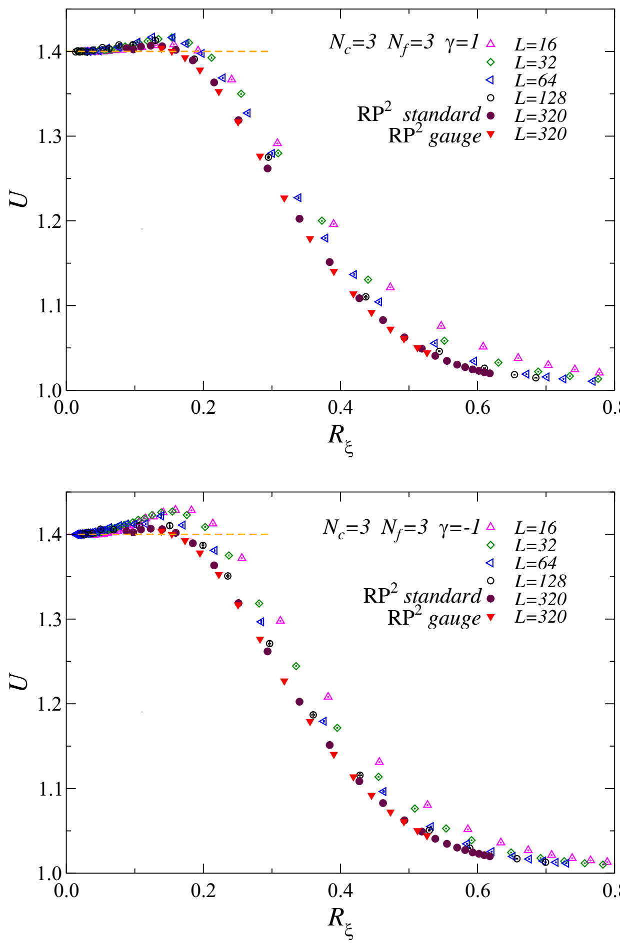


FIG. 4. Plot of $U$ versus $R_\xi$ for $N_f = 3$, $N_c = 3$, $\gamma = -1$ (lower panel) and $\gamma = 1$ (upper panel). The horizontal dashed line shows the $R_\xi \to 0$ limit $U = 7/5$. Data approach the same universal FSS curve obtained for the $\gamma = 0$ $SO(N_c)$ gauge model and the $RP^2$ models (9) and (10) (see Fig. 3).

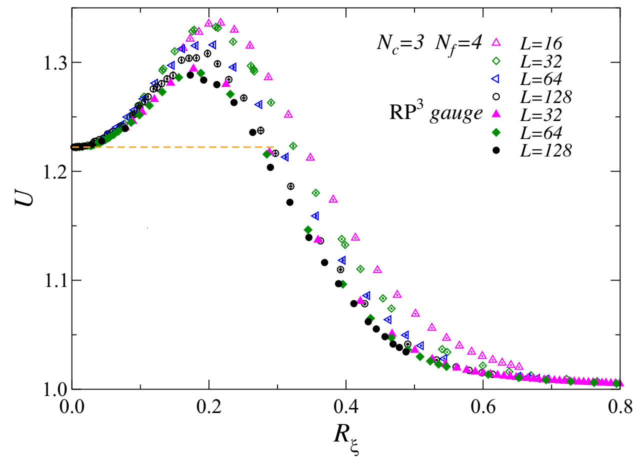


FIG. 5. Plot of $U$ versus $R_\xi$ for $N_f = 4$, $N_c = 3$, and $\gamma = 0$, up to $L = 128$. The horizontal dashed line indicates the $R_\xi \to 0$ limit $U = 11/9$. Data approach the same universal FSS curve obtained for the $RP^3$ models, for which we show MC data for the model (10) taken from Ref. [23].

that the $RP^2$ model and the $SO(N_c)$ gauge model with $N_f = 3$ and any $N_c \geq 3$ have the same large-distance universal behavior in the critical limit $\beta \to \infty$.

We have also performed MC simulations for nonvanishing values of $\gamma$. Figure 4 reports data for the three-flavor SO(3) gauge theory [Eq. (2)] with $\gamma = \pm 1$, up to $L = 128$. They appear to approach the asymptotic FSS curve of the $RP^2$ universality class, demonstrating that the universal features of the asymptotic low-temperature behavior are independent of the inverse gauge coupling $\gamma$, at least in a wide interval around $\gamma = 0$. Data up to $L = 64$ for $\gamma = \pm 2$ (not shown) also approach the $RP^2$ curve as $L$ increases. As discussed in Sec. II, the asymptotic FSS curves must change if we take the limit $\gamma \to \infty$ and then the limit $\beta \to \infty$, In this case the SO(3) and SO(4) gauge theories turn into the O(9) and O(12) models, respectively.

These results should be considered as a robust evidence that the asymptotic low-temperature behavior of the three-flavor lattice gauge theory with SO(3) and SO(4) gauge symmetry belongs to the universality class of the 2D $RP^2$ universality class, in a large interval of values of $\gamma$ around $\gamma = 0$.

As an additional check of the arguments presented in Sec. II, we have performed simulations of the model (2) for $N_f = 4$, $N_c = 3$ and $\gamma = 0$. The results for the Binder parameter $U$ are plotted versus $R_\xi$ in Fig. 5. For comparison, we also report results for the $RP^3$ gauge model. The SO(3) gauge data show a significant size dependence, but with a clear trend towards the $RP^3$ data. In particular, the SO(3) gauge data corresponding to $L = 128$ are essentially consistent with the $RP^3$ data, confirming again the asymptotic equivalence of the universal large-distance behavior of the SO(3) gauge model and of the $RP^3$ model.

## V. CONCLUSIONS

We have studied a class of 2D lattice non-Abelian $SO(N_c)$ gauge models with multicomponent scalar fields, focusing on the role that global and local non-Abelian gauge symmetries play in determining the universal features of the asymptotic low-temperature behavior. Such lattice gauge models are obtained by partially gauging a maximally $O(M)$-symmetric multicomponent scalar model, $M = N_f N_c$, using the Wilson lattice approach. For $N_c \geq 3$, the resulting theory is locally invariant under v gauge transformations and globally invariant under $O(N_f)$ transformations. For $N_c = 2$, these lattice gauge models are instead equivalent to the 2D Abelian-Higgs model and therefore have a larger $U(N_f)$ global invariance group. The fields belong to the coset $S^{M-1}/SO(N_c)$, where $M = N_f N_c$ and $S^{M-1}$ is the $(M-1)$-sphere in an $M$-dimensional space.

Since for $N_c = 2$ these lattice gauge models are equivalent to the 2D Abelian-Higgs models, already studied in

Ref. [6], we only consider $N_c \geq 3$. Moreover, we will only consider models with $N_f \geq 3$. In this case the global symmetry group is non-Abelian, and thus one expects the system to develop a critical behavior only in the zero-temperature limit. For $N_f = 2$ the behavior is expected to be different, since the global Abelian O(2) symmetry may allow finite-temperature Berezinskii-Kosterlitz-Thouless transitions.

The universal features of the zero-temperature behavior are determined by means of MC simulations. We consider the lattice $\mathrm{SO}(N_c)$ gauge models (2) for $N_c = 3$, 4 and for $N_f = 3$, 4. The FSS analyses of the MC results provide numerical evidence that the asymptotic low-temperature behavior is the same as that of the 2D $\mathrm{RP}^{N_f-1}$ models, characterized by the same global $\mathrm{O}(N_f)$ symmetry and by a local $\mathbb{Z}_2$ gauge symmetry. The numerical results are supported by theoretical arguments that show that $\mathrm{RP}^{N_f-1}$ models and $\mathrm{SO}(N_c)$ gauge theories with $N_f$ flavors have the same ground-state (zero-temperature) properties. Moreover, the gauge degrees of freedom decouple as $\beta \to \infty$.

These results provide further support to the conjecture put forward in Ref. [7], that the renormalization-group flow determining the asymptotic low-temperature behavior is generally controlled by the 2D statistical theories associated with the symmetric spaces that have the same global symmetry. For models with complex fields and $\mathrm{U}(N_f)$ global invariance—for instance, the multicomponent lattice Abelian-Higgs model and the multiflavor lattice scalar chromodynamics considered in Ref. [7]—the universal behavior is described by the 2D $\mathrm{CP}^{N_f-1}$ field theory. For the lattice $\mathrm{SO}(N_c)$ gauge models with $N_c \geq 3$ and $N_f \geq 3$, instead, the $\mathrm{RP}^{N_f-1}$ field theory is the relevant one.

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## APPENDIX: MINIMUM-ENERGY CONFIGURATIONS

In this appendix, we identify the minimum-energy configurations for the action (2). The analysis is very similar to that presented for unitary models in Ref. [7]. We refer the reader to this work for additional details.

We start by considering the simplest case, $\gamma = 0$. The minimum-energy configurations are those that satisfy the condition

$$\mathrm{Tr}[\phi_{\boldsymbol{x}}^t V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\hat{\mu}}] = 1 \tag{A1}$$

for each link. This condition is satisfied if $\phi_{\boldsymbol{x}+\hat{\mu}} = V_{\boldsymbol{x},\mu}^t \phi_{\boldsymbol{x}}$, and therefore $Q_{\boldsymbol{x}} = Q_{\boldsymbol{x}+\hat{\mu}}$, thus entailing the breaking of the global symmetry for $\beta \to \infty$. To understand which type of configurations dominate the large-$\beta$ limit, we have again resorted to numerical simulations for large values of $\beta$. Their $\beta \to \infty$ extrapolations provide information on the relevant configurations minimizing the energy. The results are reported in Table I.

The results for the trace of the square of the operator

$$B_{\boldsymbol{x}}^{fg} = \sum_a \phi_{\boldsymbol{x}}^{af} \phi_{\boldsymbol{x}}^{ag} = Q_{\boldsymbol{x}}^{fg} + \frac{1}{N_f} \delta^{fg} \tag{A2}$$

[such that $\mathrm{Tr} B_{\boldsymbol{x}} = 1$ due to the unit length of the matrix variables $\phi_{\boldsymbol{x}}$, cf. Eq. (1)], indicate that

$$\lim_{\beta\to\infty} \langle \mathrm{Tr} B_{\boldsymbol{x}}^2 \rangle = 1, \tag{A3}$$

with great accuracy. This shows that the bilinear operator $B_{\boldsymbol{x}}$ effectively behaves as a projector in the large-$\beta$ limit, thus implying that the operator $Q_{\boldsymbol{x}}$ defined in Eq. (8) becomes equivalent to the operator $P_{\boldsymbol{x}}$ defined in the $\mathrm{RP}^{N_f-1}$ theory. This is also supported by the results of the Binder parameter $U$. Indeed, for consistency, $U$ should converge to 1 in the large-$\beta$ limit, as also shown by the results reported in Table I.

To deepen our understanding of the above nontrivial results, we note that the minimum-energy conditions imply the consistency condition $\phi_{\boldsymbol{x}} = \Pi_{\boldsymbol{x}} \phi_{\boldsymbol{x}}$, where $\Pi_{\boldsymbol{x}}$ is the plaquette operator [Eq. (3)]. For $N_c \geq 3$, such a consistency condition has several classes of different solutions. The plaquette $\Pi_{\boldsymbol{x}}$ must satisfy

$$\Pi_{\boldsymbol{x}} = A \oplus 1 = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \tag{A4}$$

where $A$ is an $\mathrm{SO}(N_c - 1)$ matrix, modulo a gauge transformation. The corresponding configurations of the fields $\phi_x$ depend on the structure of the matrix $A$. If $A$ is a generic unitary matrix which does not have unit eigenvalues, the field $\phi$ is necessarily given by

$$\begin{aligned} &\phi^{af} = 0, \qquad a < N_c, \\ &\phi^{af} = v^f, \qquad a = N_c, \end{aligned} \tag{A5}$$

where $v^f$ is a unit $N_f$-dimensional vector. Different $\phi$ configurations are only possible if $A$ has some unit eigenvalues. For instance, if $A = A_1 \oplus 1$, with $A_1$ belonging to the $\mathrm{SO}(N_c - 2)$ subgroup, then the $\phi$ field configurations of the form

$$\begin{aligned} &\phi^{af} = 0, \qquad a < N_c - 1, \\ &\phi^{af} = w^f, \qquad a = N_c - 1, \\ &\phi^{af} = v^f, \qquad a = N_c \end{aligned} \tag{A6}$$

TABLE I. Estimates of several observables on the minimum-energy configurations for $\gamma = 0$, for two lattice sizes $L = 4$, 8. They are obtained by fitting large-$\beta$ numerical data (we use the same procedure discussed in the appendix of Ref. [7]).

| $(N_c, N_f)$ | $L$ | $\langle \mathrm{Tr}\Pi_{\boldsymbol{x}} \rangle / N_c$ | $S_g/(2VN_f)$ | $U$ | $\langle \mathrm{Tr}B_{\boldsymbol{x}}^2 \rangle$ |
|---|---|---|---|---|---|
| (3,3) | 4 | 0.3504(2) | −1.00000(1) | 1.000000(1) | 1.0000(1) |
| | 8 | 0.3501(1) | −0.99999(1) | 1.000001(1) | 0.99998(1) |
| (3,4) | 4 | 0.3600(3) | −1.00000(1) | 1.000000(1) | 1.00000(1) |
| | 8 | 0.3587(1) | −1.00000(1) | 0.999999(1) | 0.99999(1) |
| (4,3) | 4 | 0.2564(1) | −0.99999(1) | 0.999999(1) | 1.00000(1) |
| | 8 | 0.2563(1) | −0.99999(1) | 0.999999(1) | 1.00000(2) |
| (4,4) | 4 | 0.2599(1) | −1.00000(1) | 1.00000(1) | 1.00001(2) |
| | 8 | 0.2595(1) | −1.00000(1) | 1.00000(1) | 1.00000(1) |

($v^f$ and $w^f$ are generic $N_f$-dimensional vectors) satisfy the condition $\phi_{\boldsymbol{x}} = \Pi_{\boldsymbol{x}} \phi_{\boldsymbol{x}}$.

For the plaquette operator $\Pi_{\boldsymbol{x}}$, see Eq. (3), the results reported in Table I show that $\langle \mathrm{Tr}\Pi_{\boldsymbol{x}} \rangle \approx 1$. This indicates that $\beta \to \infty$ configurations are mostly obtained by randomly choosing $\mathrm{SO}(N_c - 1)$ matrices $A$ in Eq. (A4). For instance, if $A = A_1 \oplus 1$ with a generic $A_1 \in \mathrm{SO}(N_c - 2)$, one would instead predict $\langle \mathrm{Tr}\Pi_{\boldsymbol{x}} \rangle = 2$. The results for the plaquette are substantially consistent with the form (A5) for the field $\phi_{\boldsymbol{x}}$ in the large-$\beta$ limit. If this is the case, the operator $Q_{\boldsymbol{x}}$, defined in Eq. (8), takes the form $Q_{\boldsymbol{x}}^{fg} = v^f v^g - \delta^{fg}/N_f$ in the large-$\beta$ regime. Therefore, $Q_{\boldsymbol{x}}$ becomes equivalent to the operator $P_{\boldsymbol{x}}$ defined in the $\mathrm{RP}^{N_f - 1}$ theory.

When $\gamma \neq 0$, the analysis of the minimum-energy configurations becomes more complicated, as is also the case for lattice $\mathrm{SU}(N_c)$ gauge theories (see the appendix of Ref. [7]). We do not repeat here the arguments of Ref. [7]. They apply to $\mathrm{SO}(N_c)$ theories as well, as we have explicitly verified numerically for $\gamma = -1$ and $\gamma = 1$. We only mention that, as in the case of $\mathrm{SU}(N_c)$ gauge theories, the gauge parameter $\gamma$ is relevant for gauge properties, but not for the behavior of the $\phi$ correlations, which dominate the large-$\beta$ limit.


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