The influence of a line with fast diffusion on Fisher-KPP propagation

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Dedicated in friendship to Odo Diekmann

Abstract

We propose here a new model to describe biological invasions in the plane when a strong diffusion takes place on a line. We establish the main properties of the system, and also derive the asymptotic speed of spreading in the direction of the line. For low diffusion, the line has no effect, whereas, past a threshold, the line enhances global diffusion in the plane and the propagation is directed by diffusion on the line. It is shown here that the global asymptotic speed of spreading in the plane, in the direction of the line, grows as the square root of the diffusion on the line. The model is much relevant to account for the effects of fast diffusion lines such as roads on spreading of invasive species.

Keywords: KPP equations, reaction-diffusion system, fast diffusion on a line, asymptotic speed of propagation.

MSC: 35K57, 92D25, 35B40, 35K40, 35B53.

1   The model

It has long been known that fast diffusion on roads can have a driving effect on the spread of epidemics. A classical example is the spread of the "Black death" plague in the middle of the 14th century, considered to be one of the most devastating in human history. This pandemics is known to have spread first along the silk road. After reaching the port of Marseilles, carried by merchant boats from Crimea, it spread northwards in Europe at a fast pace along the commercial roads connecting

the cities that had trade fairs. It then also spread more slowly away from the roads, inland, bringing about a dramatic invasion. See, for instance, the account by [15].

More recently, it has been observed that invasive species such as the Processionary caterpillar of the pine tree in Europe, have been moving faster than anticipated. One plausible explanation is that enough individuals might have been carried on further distances than usual by vehicles travelling on roads going through infested areas. In the same vein, the invasion of the Aedes albopictus mosquito (also known as "Asian tiger mosquito") is a concern of public health in Europe, see [3]. The invasion by this insect is driven by roads. Rivers may accelerate the spread of plant pathologies: see for instance [11].

Another example of the effect of lines on propagation in open space comes from the observation of the population of wolves in the Western Canadian Forest. GPS observations reported by McKenzie et al. [14], 2012, suggest that wolves move and concentrate on seismic lines. These are straight lines (with a width of about 5m) used by oil exploration companies for testing of oil reservoirs. A kinetic model leading to an anisotropic diffusion equation was recently proposed by Hillen and Painter [9]; see also [8]. Performing a Chapman-Enskog analysis (with a hyperbolic scaling) they derive a drift-diffusion equation where the drift term is directed along the lines, and an anisotropic diffusion tensor. They carry out numerical simulations that show qualitative agreement with observations: the lines clearly enhance motion.

From a mathematical point of view, this raises the question of whether the inclusion of a line with fast diffusion may affect the overall invasion speed for species that usually thrive and reproduce in regions with moderate diffusion capacities. In addition, a natural aim is to quantify this effect. The present paper addresses these questions and derives precise answers.

We propose here a model where the two-dimensional environment includes a line on which fast diffusion takes place while reproduction and usual diffusion only occur outside this line. Thus we consider the line $\{(x,0) : x \in \mathbb{R}\}$ in the plane $\mathbb{R}^2$. For the sake of simplicity, we will refer to the plane as "the field" and the line as "the road". But clearly, the model we propose here is a very general one that is relevant not only to ecology and population dynamics but also for a wide array of situations in biology. For a single species, we consider a system that combines the density of this population in the field $v(x,y,t)$ and the density on the line $u(x,t)$. Exchanges of populations take place between the road and the field. Namely a fraction ν of individuals from the field at the road, that is, $v(x,0,t)$, join the road while a fraction μ of the population on the road, $u(x,t)$, goes in the fields. It is assumed that the population in the field is subject to a logistic type of growth resulting in a Fisher-KPP type of reaction term $f(v)$ in the field. We assume that no such reaction is relevant on the road. The diffusion coefficient in the field is represented by d and in the road by D. By reasons of symmetry, it will be enough to consider the half plane $\Omega := \{(x,y) : x \in \mathbb{R}, \ y > 0\}$. Translating into equations all that we have

just described, we are led to the following system:

$$\begin{cases} \partial_t u - D\partial_{xx} u = \nu v(x, 0, t) - \mu u & x \in \mathbb{R}, \ t > 0 \\ \partial_t v - d\Delta v = f(v) & (x, y) \in \Omega, \ t > 0 \\ -d\partial_y v(x, 0, t) = \mu u(x, t) - \nu v(x, 0, t) & x \in \mathbb{R}, \ t > 0. \end{cases} \tag{1.1}$$

The various parameters are supposed constant $D \geq 0$, d, μ, $\nu > 0$ and are given.

System (1.1) incorporates the case where the population also lives in the lower half-plane $\mathbb{R} \times (-\infty, 0)$ and is symmetric with respect to the x axis. Indeed, since u takes the contribution by v from both sides of the road and split its contribution to the upper and lower half-plane, the problem reduces to (1.1) with ν replaced by 2ν in the first equation and μ by $\frac{\mu}{2}$ in the last one. This change does not affect the linearised system and then, as we will see in the sequel, the spreading properties of solutions. Indeed, one come back to the original system by simply multiplying v by 2.

Our assumptions on the reaction term f are that $f \in C^1([0, 1])$ and satisfies

$$f(0) = f(1) = 0, \qquad \forall s \in (0, 1), \quad 0 < f(s) \leq f'(0)s.$$

We extend it to a negative function outside $[0, 1]$.

Finally, rescaling the time by a factor $1/\nu$ in (1.1), we get a multiplicative term $1/\nu$ in front of ν, μ, D, d, f. Thus, we assume in the rest of the paper that $\nu = 1$; this does not entail any loss of generality.

The quantity

$$c_{KPP} := 2\sqrt{df'(0)}$$

is the spreading velocity in the usual KPP equation [1],

$$u_t - du_{xx} = f(u).$$

As we will see, the problem has the structure of a monotone system [10]. See [2], [7] for results of front propagation in monotone systems, with which our problem shares many features. The originality here lies in the 1D-2D coupling. The main result of this paper is the

Theorem 1.1 *(i). Spreading. There is an asymptotic speed of spreading $c_* = c_*(\mu, d, D) > 0$ such that the following is true. Let (u, v) be a solution of (1.1) with a nonnegative, compactly supported initial datum $(u_0, v_0) \not\equiv (0, 0)$. Then:*
- *for all $c > c_*$, we have $\lim\limits_{t \to +\infty} \sup\limits_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$.*
- *For all $c < c_*$, we have $\lim\limits_{t \to +\infty} \inf\limits_{|x| \leq ct} (u(x, t), v(x, y, t)) = (1/\mu, 1)$.*

(ii). The spreading velocity. If d and μ are fixed, and D varies in $(0, +\infty)$, the following holds true.
- *If $D \leq 2d$, then $c_*(\mu, d, D) = c_{KPP}$.*
- *If $D > 2d$, then $c_*(\mu, d, D) > c_{KPP}$ and $\lim\limits_{D \to +\infty} c_*(\mu, d, D)/\sqrt{D}$ exists and is a positive real number.*

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Note that, in the statement of Theorem 1.1, the convergence holds pointwise in y. A more precise study of the domains where convergence to $(1/\mu, 1)$ or $(0, 0)$ holds is outside the scope of this paper, and will be done elsewhere.

In the recent years, there have been a considerable number of works on propagation in heterogeneous media, and how heterogeneities may enhance, or block, propagation. See for instance [16], [4]. See also [5] for many other references.

2 Conservation of total population

If (u, v) is a solution of (1.1) with $f \equiv 0$, the quantity $\|u(\cdot, t)\|_{L^1(\mathbb{R})} + \|v(\cdot, t)\|_{L^1(\Omega)}$ does not depend on t. To see this, suppose that u and v decay faster than some exponential functions at time $t = 0$. Anticipating on the next sections, we assert that this property still holds for $t > 0$, also for the derivatives of u and v, owing to parabolic estimates. We can therefore integrate by parts the first two equations in (1.1) and we find:

$$\|u(\cdot, T)\|_{L^1(\mathbb{R})} - \|u(\cdot, 0)\|_{L^1(\mathbb{R})} = \int_0^T \int_{-\infty}^{+\infty} (v(x, 0, t) - \mu u(x, t)) dx\, dt,$$

$$\|v(\cdot, T)\|_{L^1(\Omega)} - \|v(\cdot, 0)\|_{L^1(\Omega)} = -d \int_0^T \int_{-\infty}^{+\infty} \partial_y v(x, 0, t) dx\, dt$$

$$= \int_0^T \int_{-\infty}^{+\infty} (\mu u(x, t) - v(x, 0, t)) dx\, dt.$$

Whence,

$$\|u(\cdot, T)\|_{L^1(\mathbb{R})} + \|v(\cdot, T)\|_{L^1(\Omega)} = \|u(\cdot, 0)\|_{L^1(\mathbb{R})} + \|v(\cdot, 0)\|_{L^1(\Omega)}.$$

Biological interpretation. Our model is consistent with the conservation of the total population in the case of zero natality/mortality rate. The exchange between the line and the open plane exactly compensate each other as is natural. The only modification in the total size of the population comes from the "effective birth rate" represented by the term $f(v)$. $\qquad\square$

3 The Cauchy problem

In this section, we derive the existence and uniqueness result for the Cauchy problem associated with (1.1). We prove it when Ω is the $N + 1$-dimensional half space and, for later use, with the addition of a drift term $q \in \mathbb{R}^N$, $r \in \mathbb{R}^{N+1}$. Namely, denoting the generic point in Ω by $(x, y) \in \mathbb{R}^N \times (0, +\infty)$, we study the problem

$$\begin{cases} \partial_t u - D\Delta_x u - q \cdot \nabla_x u = v(x, 0, t) - \mu u & x \in \mathbb{R}^N, \ t > 0 \\ \partial_t v - d\Delta v - r \cdot \nabla v = f(v) & (x, y) \in \Omega, \ t > 0 \\ -d\partial_y v(x, 0, t) = \mu u(x, t) - v(x, 0, t) & x \in \mathbb{R}^N, \ t > 0, \end{cases} \qquad (3.1)$$

combined with the initial condition

$$\begin{cases} u|_{t=0} = u_0 & \text{in } \mathbb{R}^N \\ v|_{t=0} = v_0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

We always assume that u_0 and v_0 are nonnegative, bounded and continuous. We look for solutions satisfying (3.1)-(3.2) in the classical sense.

The system (3.1) is not very standard because the coupling appears in the Robin boundary condition for v. Therefore, well-posedness has to be proved.

Proposition 3.1 *The Cauchy problem* (3.1)-(3.2) *admits a unique nonnegative, bounded solution.*

The proof of the existence part is given in Appendix A. We derive the uniqueness part by showing, more in general, that comparison between sub and supersolutions[1] is preserved during the evolution in Problem (3.1).

Proposition 3.2 *Let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be respectively a subsolution bounded from above and a supersolution bounded from below of* (3.1) *satisfying* $\underline{u} \le \overline{u}$ *and* $\underline{v} \le \overline{v}$ *at* $t = 0$. *Then, either* $\underline{u} < \overline{u}$ *and* $\underline{v} < \overline{v}$ *for all* t, *or there exists* $T > 0$ *such that* $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ *for* $t \le T$.

PROOF. Let l be the Lipschitz constant of f. Define the following functions:

$$(\tilde{u}, \tilde{v}) := (\underline{u}, \underline{v})e^{-lt}, \qquad (\check{u}, \check{v}) := (\overline{u}, \overline{v})e^{-lt}.$$

They are sub and supersolution of the problem

$$\begin{cases} \partial_t u - D\Delta_x u - q \cdot \nabla_x u + (\mu + l)u = v(x, 0, t) & x \in \mathbb{R}^N, \ t > 0 \\ \partial_t v - d\Delta v - r \cdot \nabla v = h(t, v) & (x, y) \in \Omega, \ t > 0 \\ v(x, 0, t) - d\partial_y v(x, 0, t) = \mu u(x, t) & x \in \mathbb{R}^N, \ t > 0, \end{cases} \tag{3.3}$$

where $h(t, v) := e^{-lt}f(ve^{lt}) - lv$ is nonincreasing with respect to v. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function satisfying

$$\chi(0) = 0, \qquad \chi' = 0 \ \text{in } [0, 1], \qquad \lim_{\rho \to +\infty} \chi(\rho) = +\infty,$$

$$((N - 1)(d + D) + |q| + 2|r|)|\chi'| + (2d + D)|\chi''| \le 1 \ \text{in } \mathbb{R}.$$

Then, for $\varepsilon > 0$, set

$$\hat{u}(x, t) := \check{u}(x, t) + \varepsilon(\chi(|x|) + t + 1), \qquad \hat{v}(x, y, t) := \check{v}(x, y, t) + \mu\varepsilon(\chi(|x|) + \chi(y) + t + 1).$$

Using the fact that

$$|\nabla_x(x \mapsto \chi(|x|))| \le |\chi'(|x|)|, \qquad |\Delta_x(x \mapsto \chi(|x|))| \le |\chi''(|x|)| + (N - 1)|\chi'(|x|)|,$$

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[1]A subsolution (resp. supersolution) is a couple satisfying the system (in the classical sense) with the $=$ signs replaced by $\le$ (resp. $\ge$) signs, which is also continuous up to time 0.

one readily checks that $(\hat{u}, \hat{v})$ is still a supersolution of (3.3). Moreover, $(\hat{u}, \hat{v})$ is strictly above $(\tilde{u}, \tilde{v})$ at $t = 0$. Assume by contradiction that this property does not hold for all $t > 0$. Then,

$$T := \sup\{\tau \geq 0 \; : \; \tilde{u} \leq \hat{u} \text{ in } \mathbb{R}^N \times [0, \tau], \; \tilde{v} \leq \hat{v} \text{ in } \overline{\Omega} \times [0, \tau]\} \in [0, +\infty).$$

It follows that $\tilde{u} \leq \hat{u}$ in $\mathbb{R}^N \times [0, T]$, $\tilde{v} \leq \hat{v}$ in $\overline{\Omega} \times [0, T]$. Moreover, the continuity of the functions and the fact that $\hat{u}$ and $\hat{v}$ tend to $+\infty$ as the space variable goes to infinity, uniformly in time, implies that $T > 0$ and either $\hat{u} - \tilde{u}$ or $\hat{v} - \tilde{v}$ vanish somewhere at time T. If $\min_{\mathbb{R}^N}(\hat{u} - \tilde{u})(\cdot, T) = 0$ then, since for $x \in \mathbb{R}^N$, $0 < t \leq T$,

$$\partial_t \hat{u} - D\Delta_x \hat{u} - q \cdot \nabla_x \hat{u} + (\mu + l)\hat{u} \geq \hat{v}(x, 0, t) \geq \tilde{v}(x, 0, t),$$

the parabolic strong maximum principle yields $\hat{u} = \tilde{u}$ in $\mathbb{R}^N \times [0, T]$, which is impossible. Thus, $\min_{\mathbb{R}^N}(\hat{u} - \tilde{u})(\cdot, T) > 0$ and $\min_{\overline{\Omega}}(\hat{v} - \tilde{v})(\cdot, T) = 0$. The strong maximum principle implies that the latter cannot be attained inside Ω. Thus, necessarily, $(\hat{v} - \tilde{v})(\xi, 0, T) = 0$, for some $\xi \in \mathbb{R}^N$. This case is ruled out too, because it implies

$$0 \geq (\hat{v} - \tilde{v})(\xi, 0, T) - d\partial_y(\hat{v} - \tilde{v})(\xi, 0, T) \geq \mu(\hat{u} - \tilde{u})(\xi, T) > 0.$$

We have shown that $(\hat{u}, \hat{v})$ is above $(\tilde{u}, \tilde{v})$ for all $t > 0$, and then $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ due to the arbitrariness of ε. Suppose now that there exists a contact point between $\underline{u}$ and $\overline{u}$ at some time T. Applying the strong maximum principle to the first equation in (3.1) we derive $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ for $t \leq T$. If $\underline{u} < \overline{u}$ for all t, then the same arguments as before show that $\underline{v} < \overline{v}$. □

Biological interpretation. If two pairs of population densities ("road" and "field") are initially ordered, the exchanges between them always occur in such a way that the order will continue to hold separately for each population at the microscopic level. This effect is somewhat more surprising than the mass conservation we discussed earlier. Actually, such a monotonicity property is not a priori included in the model but rather comes out as an interesting property. □

The above comparison principle immediately extends to *generalised* sub and supersolutions, given by the supremum of subsolutions and the infimum of supersolutions respectively. In Section 6, we will need a comparison principle for a more general class of subsolutions, obtained as the supremum of two subsolutions in a given set and extended outside to the one which is larger on the boundary. In the case of a single parabolic equation, the comparison principle always holds for such kind of generalised subsolutions, but this is no longer true for our system (3.1). We indeed need an extra assumption.

Proposition 3.3 *Let $E \subset \mathbb{R}^N \times (0, +\infty)$ and $F \subset \Omega \times (0, +\infty)$ be two open sets and let (u_1, v_1), (u_2, v_2) be two subsolutions of (3.1) bounded from above and satisfying*

$$u_1 \leq u_2 \quad \text{on } (\partial E) \cap (\mathbb{R}^N \times (0, +\infty)), \qquad v_1 \leq v_2 \text{ on } (\partial F) \cap (\Omega \times (0, +\infty)).$$

If the functions $\underline{u}$, $\underline{v}$ defined by

$$\underline{u}(x,t) := \begin{cases} \max(u_1(x,t), u_2(x,t)) & \textit{if } (x,t) \in \overline{E} \\ u_2(x,t) & \textit{otherwise,} \end{cases}$$

$$\underline{v}(x,y,t) := \begin{cases} \max(v_1(x,y,t), v_2(x,y,t)) & \textit{if } (x,y,t) \in \overline{F} \\ v_2(x,y,t) & \textit{otherwise,} \end{cases}$$

satisfy

$$\underline{u}(x,t) > u_2(x,t) \implies \underline{v}(x,0,t) \geq v_1(x,0,t),$$
$$\underline{v}(x,0,t) > v_2(x,0,t) \implies \underline{u}(x,t) \geq u_1(x,t),$$

then, any supersolution $(\overline{u},\overline{v})$ of (3.1) bounded from below and such that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ at $t=0$, satisfies $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ for all $t > 0$.

PROOF. Since $\overline{u} \geq \underline{u} \geq u_2$ and $\overline{v} \geq \underline{v} \geq v_2$ at $t = 0$, Proposition 3.2 yields $\overline{u} \geq u_2$ and $\overline{v} \geq v_2$ for all t. Assume by contradiction that $(\overline{u},\overline{v})$ is below $(\underline{u},\underline{v})$ somewhere. Using the same transformations as in the proof of Proposition 3.2, with ε small enough, one reduces to the case where (u_1,v_1), (u_2,v_2) are subsolutions of (3.3) and $(\overline{u},\overline{v})$ is a supersolution of (3.3) which is strictly above (u_2,v_2) for all t, is strictly above $(\underline{u},\underline{v})$ for t less than some $T > 0$ and touches it somewhere at $t = T$. We argue differently depending on the fact that $\overline{u}$ touches $\underline{u}$ or not at $t = T$.

Case 1. $\overline{u}(\xi,T) = \underline{u}(\xi,T)$ for some $\xi \in \mathbb{R}^N$.
Since $u_2(\xi,T) < \overline{u}(\xi,T) = \underline{u}(\xi,T) = u_1(\xi,T)$, it follows that $(\xi,T) \in E$ and there exists $\delta > 0$ such that $u_2 < u_1 = \underline{u}$ in $B_\delta(\xi) \times (T-\delta, T+\delta) \subset E$. Let us call $Q := B_\delta(\xi) \times (T-\delta, T)$. For $(x,t) \in Q$, the hypothesis on $(\underline{u},\underline{v})$ yields $v_1(x,0,t) \leq \underline{v}(x,0,t) < \overline{v}(x,0,t)$. Whence,

$$\partial_t u_1 - D\Delta_x u_1 - q \cdot \nabla_x u_1 + (\mu + l)u_1 \leq v_1(x,0,t) < \overline{v}(x,0,t) \quad \text{in } Q.$$

But $u_1 \leq \overline{u}$ on the parabolic boundary of Q and then the strong maximum principle yields $u_1 = \overline{u}$ in Q, which is impossible.

Case 2. $\underline{u} < \overline{u}$ for $t = T$ and $\underline{v}(\xi,\eta,T) = \overline{v}(\xi,\eta,T)$ for some $(\xi,\eta) \in \overline{\Omega}$.
Since $v_2(\xi,\eta,T) < \overline{v}(\xi,\eta,T) = \underline{v}(\xi,\eta,T) = v_1(\xi,\eta,T)$, it follows that $(\xi,\eta,T) \in \overline{F}$ and there exists $\delta > 0$ such that $v_2 < v_1$ in $(B_\delta(\xi,\eta) \cap \Omega) \times (T-\delta, T+\delta)$. By the hypothesis on v_1 and v_2, this set cannot intersect ∂F and then it is contained in F, because $(\xi,\eta,T) \in \overline{F}$. Whence, we find that $v_2 < v_1 = \underline{v} < \overline{v}$ in $Q := (B_\delta(\xi,\eta) \cap \Omega) \times (T-\delta, T)$. If $\eta > 0$, the parabolic strong maximum principle implies that $v_1 = \overline{v}$ in Q, which is impossible. Therefore, $\eta = 0$ and then the hypothesis on $\underline{v}$ yields $u_1(\xi,T) \leq \underline{u}(\xi,T)$. Consequently,

$$0 \geq (\overline{v} - v_1)(\xi,0,T) - d\partial_y(\overline{v} - v_1)(\xi,0,T) = \mu(\overline{u} - u_1)(\xi,T) \geq \mu(\overline{u} - \underline{u})(\xi,T).$$

We have reached a contradiction, because we are in the case $\underline{u} < \overline{u}$ for $t = T$. $\square$

Remark 3.4 *It is clear that in the statement of Lemma 6.2 one can require that (u_1,v_1) is a subsolution of the three equations of (3.1) only in E, F and for $(x,0,t) \in \partial F$ respectively.*

4 Long time behaviour

We derive a Liouville-type result for stationary solutions of system (3.1) in the absence of drift, that is

$$\begin{cases} -D\Delta_x U = V(x,0) - \mu U & x \in \mathbb{R}^N \\ -d\Delta V = f(V) & (x,y) \in \Omega \\ -d\partial_y V(x,0) = \mu U(x) - V(x,0) & x \in \mathbb{R}^N. \end{cases} \tag{4.1}$$

Proposition 4.1 *The unique nonnegative, bounded solutions of* (4.1) *are* $(U,V) \equiv (0,0)$ *and* $(U,V) \equiv (1/\mu, 1)$.

PROOF. Let (U,V) be a nonnegative, bounded solution of (4.1). By the last equation of (4.1), the statement is proved if we show that $V \equiv 0$ or 1. Hence, by the strong maximum principle, it is sufficient to show that if $V > 0$ in Ω then $V \equiv 1$. We do it in three steps.

Step 1. *The function* V *satisfies*

$$\forall r > 0, \quad \inf_{\mathbb{R}^N \times [r,+\infty)} V > 0. \tag{4.2}$$

This step is the only part of the proof in which the absence of drift in the system is required. Take $R > 0$ large enough in such a way that the principal eigenvalue of $-\Delta$ in $B_R \subset \mathbb{R}^{N+1}$, under Dirichlet boundary condition, is less than $f'(0)/d$. Let φ be the associated principal eigenfunction. It follows that there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, $-d\Delta(\varepsilon\varphi) < f(\varepsilon\varphi)$ in B_R. As a consequence, for given $\hat{x} \in \mathbb{R}^N$, $\hat{y} > R$, the elliptic maximum principle yields

$$\forall (x,y) \in B_R(\hat{x}, \hat{y}), \quad V(x,y) \geq \varepsilon_0 \varphi(x - \hat{x}, y - \hat{y}).$$

This proves (4.2).

Step 2. $V \geq 1$.

Set $m := \inf V$ and consider $((x_n, y_n))_n$ such that $V(x_n, y_n) \to m$. We assume by way of contradiction that $m < 1$ and we distinguish two cases.

   Case 1: $(y_n)_{n \in \mathbb{N}}$ tends to 0.

We set

$$U_n(x) := U(x + x_n), \qquad V_n(x,y) := V(x + x_n, y).$$

By standard elliptic estimates, we have that $(U_n)_n$ and $(V_n)_n$ converge (up to subsequences) locally uniformly to some functions $\check{U}, \check{V}$ satisfying the same system (4.1). Furthermore, $\check{V}(0,0) = m = \min_{\overline{\Omega}} \check{V}$. Since $\Delta\check{V} = -f(\check{V})/d \leq 0$ in the intersection of Ω with a neighbourhood of $(0,0)$, the strong maximum principle and Hopf's lemma imply that either $\check{V} \equiv m$ or $\check{V} > m$ in Ω and $-\partial_y \check{V}(0,0) < 0$. The first situation cannot occur because $f(m) > 0$ by (4.2). In the second situation, using (4.1) we derive $\mu\check{U}(0) - m < 0$. In particular,

$$\inf U \leq \inf \check{U} < \frac{m}{\mu}.$$

Considering now the limit $\hat{U}$ of translations of U by (a subsequence of) one of its minimizing sequences, we reduce to the case where

$$\hat{U}(0) = \min \hat{U} = \inf U, \qquad -D\Delta_x \hat{U} \geq m - \mu \hat{U}.$$

Hence, evaluating the latter at $x = 0$, we get the following contradiction:

$$0 \leq D\Delta_x \hat{U}(0) \leq \mu \inf U - m < 0.$$

Case 2: there exists a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ with distance $r > 0$ from 0.
It follows from (4.2) that $m > 0$. The sequence of the translated functions $V_k(x, y) := V(x + x_{n_k}, y + y_{n_k})$ converges (up to subsequences) locally uniformly to a function $\check{V}$ satisfying

$$\Delta \check{V} < 0 \ \text{ in } B_\rho, \qquad \check{V}(0,0) = m = \min \check{V},$$

for some $0 < \rho < r$. This is impossible.
Step 3. $V \leq 1$.
Assume by contradiction that $M := \sup V > 1$. Analogously to the previous step, we can reduce to the case where $V(0,0) = \max V = M > 1$ and either (U, V) satisfies (4.1) or $-\Delta V = f(V)$ in a neighbourhood of $(0,0)$. Since $f(M) < 0$, the latter case is impossible and, by Hopf's lemma, we necessarily have that $-\partial_y V(0,0) > 0$. Whence (4.1) yields

$$\sup U \geq U(0) > \frac{M}{\mu}.$$

By usual arguments, the limit $\hat{U}$ of translations of U by (a subsequence of) one of its maximizing sequence attains its maximum $\sup U$ at 0 and satisfies

$$D\Delta_x \hat{U}(0) \geq \mu \sup U - M > 0.$$

This is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.1 *Let (u, v) be a solution of (1.1) with a nonnegative, not identically equal to zero, bounded initial datum. Then,*

$$\lim_{t \to +\infty} (u(x, t), v(x, y, t)) \to (1/\mu, 1),$$

locally uniformly in $(x, y) \in \overline{\Omega}$.

PROOF. Take $R > 0$ large enough in such a way that the Dirichlet principal eigenvalue of $-\Delta$ in $B_R \subset \mathbb{R}^2$ is less than $f'(0)/d$. Call φ the associated principal eigenfunction. Hence, for $\varepsilon > 0$ small enough, the function $\underline{V}$ defined by $\underline{V}(x, y) := \varepsilon\varphi(x, y - R - 1)$ satisfies $V \leq 1$ and $-d\Delta \underline{V} < f(\underline{V})$ in $B_R(0, R + 1)$. Extending $\underline{V}$ by 0 outside $B_R(0, R + 1)$, we have that $(0, \underline{V})$ is a generalised subsolution of (1.1) in the sense of Proposition 3.3. It is then easy to check that the

solution $(\underline{u}, \underline{v})$ of (1.1) with initial datum $(0, \underline{V})$ is nondecreasing in time. On the other hand, the pair $(\overline{U}, \overline{V})$ defined by

$$\overline{U} := \max \left(\sup_{\mathbb{R}} u_0, \frac{1}{\mu} \sup_{\Omega} v_0, \frac{1}{\mu} \right), \qquad \overline{V} := \max \left(\mu \sup_{\mathbb{R}} u_0, \sup_{\Omega} v_0, 1 \right).$$

is a supersolution of (1.1). It is now sufficient to use Proposition 3.2 to infer that the solution $(\overline{u}, \overline{v})$ with initial datum $(\overline{U}, \overline{V})$ is nonincreasing in time. Therefore, owing to Proposition 4.1, as $t \to +\infty$, they both converge to the unique positive, bounded solution of (4.1): $(U, V) \equiv (\frac{1}{\mu}, 1)$. This convergence is locally uniform in (x, y) by parabolic estimates. Comparing (u, v) with $(\overline{u}, \overline{v})$, we derive

$$\limsup_{t \to +\infty} (u(x, t), v(x, y, t)) \leq (1/\mu, 1),$$

locally uniformly in $(x, y) \in \overline{\Omega}$. Furthermore, by Proposition 3.2 we know that $v > 0$ in $\Omega \times \mathbb{R}_+$. Hence, up to choosing a smaller ε in the definition of $\underline{V}$ if need be, it is not restrictive to assume that $\underline{V}(x, y) < v(x, y, 1)$ for $(x, y) \in \Omega$. Thus,

$$\liminf_{t \to +\infty} (u(x, t+1), v(x, y, t+1)) \geq \lim_{t \to +\infty} (\underline{u}(x, t), \underline{v}(x, y, t)) = (1/\mu, 1),$$

locally uniformly in $(x, y) \in \overline{\Omega}$. The proof is thereby achieved. $\qquad \square$

Biological interpretation. Even though there is no birth rate on the "road", the presence of a logistic term in the "field" is sufficient to ensure global invasion (road and field) by the species modelled by (1.1). This is of course due to the exchange term between the field and the road. $\qquad \square$

5 Exponential solutions

The upper bound for the spreading speed will follow from the existence of exponential solutions for the linearised problem

$$\begin{cases} \partial_t u - D \partial_{xx} u = v(x, 0, t) - \mu u & x \in \mathbb{R}, \ t \in \mathbb{R} \\ \partial_t v - d \Delta v = f'(0) v & (x, y) \in \Omega, \ t \in \mathbb{R} \\ -d \partial_y v(x, 0, t) = \mu u(x, t) - v(x, 0, t) & x \in \mathbb{R}, \ t \in \mathbb{R}, \end{cases} \qquad (5.1)$$

because, by the KPP hypothesis, solutions to (5.1) provide entire super-solutions to (1.1). The lower bound will follow from the existence of solutions with complex exponent. So, we will be looking for solutions of the form

$$(u(t, x), v(t, x, y)) = (e^{\alpha(x+ct)}, \gamma e^{\alpha(x+ct) - \beta y}) \qquad (5.2)$$

where α and γ are positive constants and β is a real (not necessarily positive) constant. In this section, we analyse the condition on (α, β, γ) so that u and v given

by (5.2) solve (5.1). The system on (α, β, γ) reads

$$\begin{cases} -D\alpha^2 + c\alpha &= \gamma - \mu \\ -d\alpha^2 + c\alpha &= f'(0) + d\beta^2 \\ d\beta\gamma &= \mu - \gamma \end{cases} \tag{5.3}$$

The third equation of (5.3) gives $\gamma = \mu/(1 + d\beta)$. Hence, the first equation, with unknown α and parameters D, d, β, μ, has the roots

$$\alpha_D^{\pm}(c, \beta) = \frac{1}{2D}\left(c \pm \sqrt{c^2 + \frac{4\mu dD\beta}{1 + d\beta}}\right).$$

For $\alpha_D^{\pm}(c, \beta)$ to be real, the parameter β has to be chosen in $(-\infty, -1/d) \cup [\beta_D(c), +\infty)$ with

$$\beta_D(c) = -\frac{c^2}{d(c^2 + 4\mu D)}.$$

Note that if $\beta < -1/d$ we have $\gamma < 0$. This case has therefore to be discarded. Also note that $\lim_{c \to +\infty} \beta_D(c) = -1/d$. Denote $\Gamma_{c,D}^{\pm} := \{\alpha = \alpha_D^{\pm}(c, \beta), \ \beta \geq \beta_D(c)\}$; in the (β, α) plane, $\Gamma_{c,D} := \Gamma_{c,D}^{+} \cup \Gamma_{c,D}^{-}$ is a smooth curve with leftmost point $(\beta_D(c), c/2D)$. See Fig. 1.

Let us now deal with the second equation in (5.3); in the (β, α) plane, it is the equation of the circle $\Gamma_{c,d}$ of centre $(0, c/2d)$ and radius

$$\beta_{KPP}(c) = \frac{\sqrt{c^2 - c_{KPP}^2}}{2d}.$$

In the whole discussion below, it will be necessary to require

$$c \geq c_{KPP} = 2\sqrt{df'(0)}.$$

This will be assumed without any further mention. The circle $\Gamma_{c,d}$ is the union of the two half circles $\Gamma_{c,d}^{\pm} = \{\alpha = \alpha_d^{\pm}(c, \beta)\}$, where

$$\alpha_d^{\pm}(c, \beta) = \frac{c \pm \sqrt{c^2 - c_{KPP}^2 - 4d^2\beta^2}}{2d}.$$

The quantity

$$\alpha_d^{-}(c, 0) = \frac{c}{2d} - \beta_{KPP}(c)$$

is the exponent at which a travelling wave of (1) will decay in the x variable. So, (5.3) amounts to finding c such that $\Gamma_{c,D}$ and $\Gamma_{c,d}$ intersect. The following properties of $\alpha_D^{\pm}$ and $\alpha_d^{\pm}$ are easily checked:

- The function α_D^{-} satisfies $\partial_\beta \alpha_D^{-} < 0$ and $\partial_{\beta\beta}\alpha_D^{-} > 0$; moreover $\alpha_D^{-}(\beta, c) < 0$ if $\beta > 0$, and < 0 if $\beta \in [\beta_D(c), 0)$.

- The function α_D^{+} satisfies $\partial_c \alpha_D^{+} > 0$, $\partial_\beta \alpha_D^{+} > 0$ wherever it is defined; moreover, $\partial_{\beta\beta}\alpha_D^{+}(c, \beta) < 0$.

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- The function α_d^- is even; moreover we have $\partial_c \alpha_d^- < 0$, $\partial_{\beta\beta} \alpha_d^- > 0$, and $\partial_\beta \alpha_d^- > 0$ for $\beta > 0$.

- The function α_d^+ is even; moreover we have $\partial_c \alpha_d^+ > 0$, $\partial_{\beta\beta} \alpha_d^+ < 0$, and $\partial_\beta \alpha_d^+ < 0$ for $\beta > 0$.

Case 1. $D > 2d$. For c very large we have

$$\alpha_D^+(c,0) = \frac{c}{D} > \alpha_d^-(c,0) \sim \frac{f'(0)}{c}.$$

So, there is $c_* > c_{KPP}$ such that $\Gamma_{c_*,D}$ and $\Gamma_{c_*,d}$ are tangent. The tangency point belongs to $\Gamma_{c_*,d}^-$; moreover, the properties $\partial_c \alpha_D^+ > 0$ and $\partial_c \alpha_d^- < 0$ imply that $\Gamma_{c,D}$ and $\Gamma_{c,d}$ intersect as soon as $c > c_*$. Due to the convexity properties of $\Gamma_{c,D}$ and $\Gamma_{c,d}$, these two curves intersect at exactly two points denoted by $(\beta_\pm, \alpha_\pm)$ with $\beta_- < \beta_+$, $\alpha_- < \alpha_+$. The situation is summarized in Figure 1. We can be a little more explicit and tell to what part of $\Gamma_{c,D}$ the points belong. The point here is whether $\Gamma_{c,D}^+$ and $\Gamma_{c,d}^+$ intersect, and the condition is

$$D \leq 2d + \delta \quad \text{with } \delta := \left(\max_{t>0} \frac{t}{(t^2 + c_{KPP}^2)(t+2)} \right) 4\mu d^2.$$

Indeed $\alpha_D^+ = \alpha_d^+$ admits a solution iff $\alpha_D^+(c, \beta_{KPP}(c)) \geq \alpha_d^+(c, \beta_{KPP}(c))$. The latter reads

$$\frac{4\mu d \beta_{KPP}(c)}{c^2(1 + d\beta_{KPP}(c))} \geq \frac{D - 2d}{d^2},$$

whence, calling $t := 2d\beta_{KPP}(c) = \sqrt{c^2 - c_{KPP}^2}$, we get

$$D - 2d \leq \frac{4\mu d^2 t}{(t^2 + c_{KPP}^2)(t+2)},$$

which admits solutions $t > 0$ iff $D \leq 2d + \delta$. Actually, if $D \leq 2d + \delta$, since

$$(D - 2d)(t^2 + c_{KPP}^2)(t+2) = 4\mu d^2 t$$

has two positive solutions (coinciding if $D = 2d + \delta$), it follows that there exist $c_* < \tilde{c}_1 \leq \tilde{c}_2$ such that, for all $\tilde{c}_1 \leq c \leq \tilde{c}_2$, the equation $\alpha_D^+(c, \beta) = \alpha_d^+(c, \beta)$ has a unique positive solution β. We have that $\delta < \mu d/f'(0)$, that $D \mapsto \tilde{c}_1(D)$ is increasing, $D \mapsto \tilde{c}_2(D)$ is decreasing and

$$\lim_{D \to 2d^-} \tilde{c}_1(D) = 0, \qquad \lim_{D \to 2d^-} \tilde{c}_2(D) = +\infty.$$

Case 2. $0 < D = 2d$. The centre of $\Gamma_{c,d}$ sits on $\Gamma_{c,D}$, exactly at the point $(0, c/D)$. So, for all $c > c_{KPP}$, $\Gamma_{c,D}$ and $\Gamma_{c,d}$ intersect exactly twice. In this case, we set $c_* := c_{KPP}$.

Case 3. $0 \leq D < 2d$. Similarly to the discussion of Case 1, there is a largest $c' > c_{KPP}$ such that, for all $c < c'$, $\Gamma_{c,d}$ and $\Gamma_{c,D}$ do not intersect. The situation
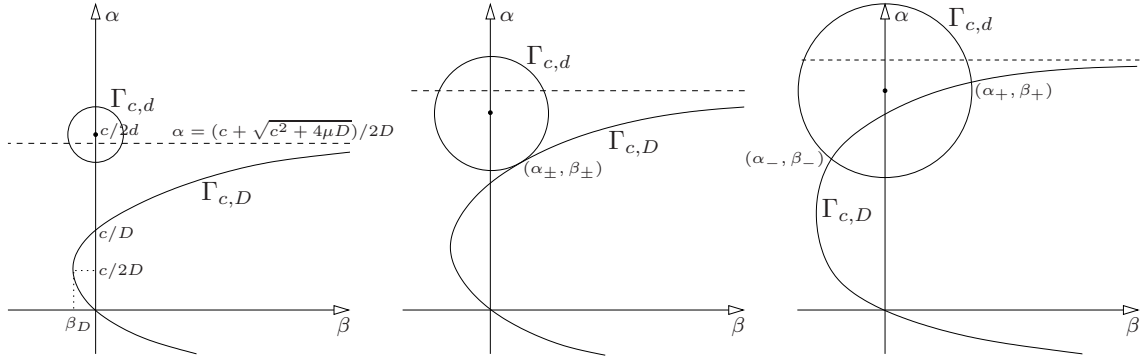
Figure 1: Case $D > 2d$; $c < c_*$ (left), $c = c_*$ (middle), $c > c_*$ (right).

here is more intricate than in Case 1, and also less interesting as far as spreading is concerned: we will indeed see that here, spreading occurs at velocity c_{KPP}. This study is therefore left for future investigation. Setting $c_* := c_{KPP}$, we see that for $c \geq c_{KPP}$, the centre of $\Gamma_{c,d}$ lies on the right of $\Gamma_{c,D}$. This shows that the exponential function associated with $\alpha = c/2d$, $\beta = 0$ is a supersolution of (5.1).

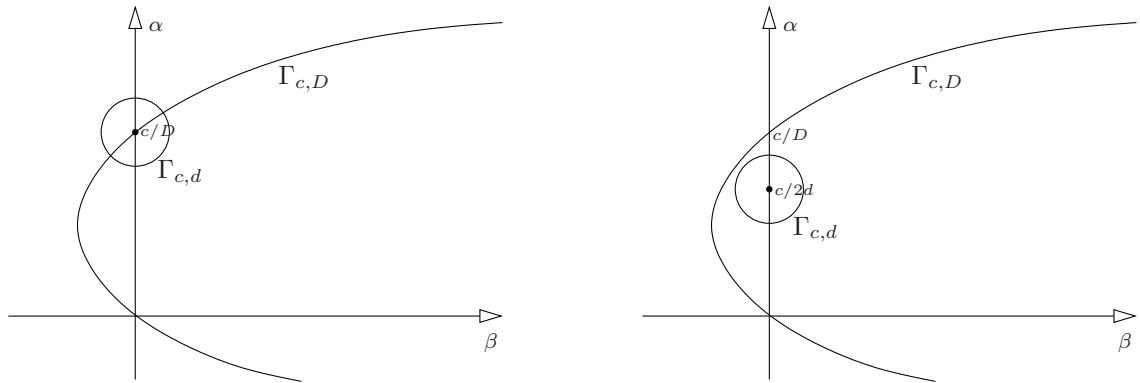Cases 2 and 3 are illustrated in Figure 2.

Figure 2: Cases $D = 2d$ (left) and $D < 2d$ (right).

Biological interpretation. There is a threshold value for D here, at $D = 2d$. At first sight, this might appear somewhat surprising as one might have expected for instance $D = d$ to be the threshold (identical diffusion on the road and on the field). It turns out that this threshold value is directly related to the fact that in our model there is no reaction on the road. As a matter of fact we may consider a related model where one allows reproduction on the road with the same rate as in the field. In this case, the new system reads:

$$\begin{cases} \partial_t u - D\partial_{xx} u = \nu v(x,0,t) - \mu u + f(u) & x \in \mathbb{R},\ t \in \mathbb{R} \\ \partial_t v - d\Delta v = f(v) & (x,y) \in \Omega,\ t \in \mathbb{R} \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \quad (5.4)$$

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Adapting the arguments of the next section one can show that then, the threshold occurs at $D = d$. A detailed study of the system with a reaction term on the road too will be done elsewhere. $\square$

6 Asymptotic spreading

Because of the comparison principle (Proposition 3.2), the standard way to prove that solutions spread at least at speed c_* is to find a compactly supported generalised stationary subsolution in the moving framework at velocity $c < c_*$. Here, generalised subsolution is in the sense of Proposition 3.3. We consider the linearised system penalised by $\delta > 0$ in the moving framework:

$$\begin{cases} \partial_t u - D\partial_{xx}u + c\partial_x u = v(x,0,t) - \mu u & x \in \mathbb{R},\ t \in \mathbb{R} \\ \partial_t v - d\Delta v + c\partial_x v = (f'(0) - \delta)v & (x,y) \in \Omega,\ t \in \mathbb{R} \\ -d\partial_y v(x,0,t) = \mu u(x) - v(x,0,t) & x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \tag{6.1}$$

For $D > 2d$, the main lemma is the following.

Lemma 6.1 *Assume $D > 2d$ and let c_* be as in Section 5. For $c < c_*$ close enough to c_*, there exists $\delta > 0$ such that (6.1) admits a nonnegative, compactly supported, generalised stationary subsolution $(\underline{u}, \underline{v}) \not\equiv (0,0)$.*

PROOF. In order to keep the notation as light as possible, we will carry out the discussion with $f'(0)$ instead of $f'(0) - \delta$ in (6.1), because all the results will perturb for small $\delta > 0$.

The first step is to devise a stationary solution of (6.1), not in Ω anymore, but in the horizontal strip $\Omega^L := \mathbb{R} \times (0, L)$ with $L > 0$. So, we are solving

$$\begin{cases} -DU'' + cU' = V(x,0) - \mu U & x \in \mathbb{R} \\ -d\Delta V + c\partial_x V = f'(0)V & (x,y) \in \Omega^L \\ -d\partial_y V(x,0) = \mu U(x) - V(x,0) & x \in \mathbb{R} \\ V(x,L) = 0 & x \in \mathbb{R}. \end{cases} \tag{6.2}$$

In a similar fashion as in the preceding section, we look for exponential solutions of the form $(1, \gamma(y))e^{\alpha x}$. An easy computation shows that such a solution exists if and only if the system with unknowns α and β (look for γ under the form $\gamma_1 e^{-\beta y} + \gamma_2 e^{\beta y}$)

$$\begin{cases} -D\alpha^2 + c\alpha + \dfrac{(1 + e^{-2\beta L})d\beta\mu}{1 - e^{-2\beta L} + (1 + e^{-2\beta L})d\beta} & = & 0 \\ -d(\alpha^2 + \beta^2) + c\alpha & = & f'(0) \\ d\beta(\gamma_1 - \gamma_2) & = & \mu - (\gamma_1 + \gamma_2) \\ \gamma_1 e^{-\beta L} + \gamma_2 e^{\beta L} & = & 0 \end{cases} \tag{6.3}$$

has a solution. The first equation of (6.3) defines, in the upper part of the plane (β, α), a curve $\Gamma_{c,D}^L$, which is symmetric with respect to the vertical axis. Moreover,

the set $\Gamma^L_{c,D} \cap \{\alpha > 0, \beta > 0\}$ tends to $\Gamma_{c,D} \cap \{\alpha > 0, \beta > 0\}$ as $L \to \infty$. In fact, it is the graph $\alpha = \alpha^{+,L}_D(c, \beta)$ with

$$\alpha^{+,L}_D(c, \beta) = \frac{1}{2D}\left(c \pm \sqrt{c^2 + \frac{4(1 + e^{-2\beta L})\mu d D \beta}{1 - e^{-2\beta L} + (1 + e^{-2\beta L})d\beta}}\right).$$

The curve $\Gamma^L_{c,D}$ is strictly above $\Gamma_{c,D}$; moreover, as $L \to \infty$, $\alpha^{+,L}_D$ converges to α^+_D, together with all its derivatives, in every compact subset in (c, β) avoiding $\{\beta \leq 0\}$. Thus, due to the implicit function theorem, for large L, the picture is analogous to the case 1 of the previous section: there exists a unique $c^L_* \in (c_{KPP}, c_*)$ such that $\Gamma^L_{c,D} \cap \{\beta > 0\}$ intersects $\Gamma_{c,d}$ twice if $c > c^L_*$ once if $c = c^L_*$ and never if $c < c^L_*$. Moreover we have $\lim_{L \to \infty} c^L_* = c_*$.

To obtain the compactly supported subsolution, fix $L > 0$ suitably large and set

$$h^L(c, \beta) := \alpha^{+,L}_D(c, \beta) - \alpha^-_d(c, \beta).$$

Call (β^L_*, α^L_*) the tangent point between $\Gamma^L_{c^L_*,D} \cap \{\beta > 0\}$ and $\Gamma_{c^L_*,d}$. We have

$$\partial_\beta h^L(c^L_*, \beta^L_*) = 0, \qquad -2a := \partial_{\beta\beta} h^L(c^L_*, \beta^L_*) < 0.$$

Also, set $e := \partial_c h^L(c^L_*, \beta^L_*) > 0$, $b := \partial_{c\beta} h^L(c^L_*, \beta^L_*)$, and, because we are working in a vicinity of (c^L_*, β^L_*):

$$\xi := c^L_* - c, \qquad \tau := \beta - \beta^L_*.$$

The equation $h^L(c, \beta) = 0$ becomes, for (c, β) in a neighbourhood of (c^L_*, β^L_*):

$$a\tau^2 + b\xi\tau + e\xi = \varphi(\tau, \xi) \tag{6.4}$$

where φ is analytic in τ in a neighbourhood of 0, vanishing at $(0,0)$ like $|\tau|^3 + \xi^2$. For small $\xi > 0$, the trinomial $a\tau^2 + d\xi\tau + e\xi$ has two roots $\tau_\pm = \pm i\sqrt{(e/a)\xi} + O(\xi)$; by (an adaptation of) Rouché's theorem (see Appendix B), equation (6.4) has two roots, conjugate up to the order ξ, still called $\tau_\pm$. They also satisfy $\tau_\pm = \pm i\sqrt{(e/a)\xi} + O(\xi)$. Reverting to the full notations, we see that, for c strictly less than and suitably close to c^L_*, equation (6.3) has a solution (β, α) with the following properties: β has real part ξ-close to β^L_*, hence positive, and it also has nonzero imaginary part of order $\xi^{1/2}$. It follows from the second equation of (6.3) that α has nonzero imaginary part too. Therefore, one can write

$$(\beta, \alpha) = (\beta_1 + i\beta_2, \alpha_1 + i\alpha_2),$$

with $\beta_1 > 0$ and $\alpha_2, \beta_2 \neq 0$. We thus obtain a solution $(U(x), V(x, y)) = (1, \gamma(y))e^{(\alpha_1 + i\alpha_2)x}$ to (6.2), with

$$\gamma(y) = \gamma_1(e^{-\beta y} - e^{\beta(-2L+y)}), \quad \gamma_1 = \frac{\mu}{1 - e^{-2\beta L} + d\beta(1 + 2e^{-2\beta L})}.$$

We have $\arg(\gamma_1) = O(\sqrt{\xi})$. Furthermore:

- $\mathrm{Re}(U) > 0$ on $(-\pi/2\alpha_2, \pi/2\alpha_2)$ and vanishes at the ends;
- $\mathrm{Re}(V) > 0$ if and only if

$$\cos(\alpha_2 x + \arg(\gamma_1) - \beta_2 y) > e^{-2\beta_1(L-y)} \cos(\alpha_2 x + \arg(\gamma_1) - \beta_2(2L - y)).$$

That is,

$$\cos(\alpha_2 x + \arg(\gamma_1) - \beta_2 y) > \frac{\sin(2\beta_2(L-y))}{e^{2\beta_1(L-y)} - \cos(2\beta_2(L-y))} \sin(\alpha_2 x + \arg(\gamma_1) - \beta_2 y).$$

Thus, the set where $\mathrm{Re}(V) > 0$ is periodic in the direction e_1, with period $\frac{2\pi}{\alpha_2}$. Its connected components intersecting the strip $\mathbb{R} \times (0, L)$ are bounded. Since $\beta_2 = O(\sqrt{\xi})$, by decreasing ξ we can make one of them, denoted by F, satisfy the property that $\{\alpha_2 x + \arg(\gamma_1) : (x, 0) \in \overline{F}\}$ is arbitrarily close to the set $[-\pi/2, \pi/2]$. Since $\arg(\gamma_1) = O(\sqrt{\xi})$, we have that $\{\alpha_2 x : (x, 0) \in \overline{F}\}$ is close to $[-\pi/2, \pi/2]$ for ξ small. We define the following functions:

$$\underline{u}(x) := \begin{cases} \max(\mathrm{Re}(U(x)), 0) & \text{if } |x| \leq \frac{\pi}{2\alpha_2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{v}(x, y) := \begin{cases} \max(\mathrm{Re}(V(x, y)), 0) & \text{if } (x, y) \in \overline{F} \\ 0 & \text{otherwise.} \end{cases}$$

The choice of F implies that the couple $(\underline{u}, \underline{v})$ is a generalised subsolution to (6.1) in the sense of Proposition 3.3. $\qquad\square$

Turn to the case $0 \leq D \leq 2d$. The main lemma is here the following:

Lemma 6.2 *Assume $0 \leq D \leq 2d$. For $-c_{KPP} < c < c_{KPP}$, the conclusion of Lemma 6.1 holds.*

PROOF. Let $|c| < c_{KPP}$. For $0 < \delta < f'(0) - c^2/4d$, the equation

$$-d\Delta V + c\partial_x V = (f'(0) - \delta/2)V, \quad (x, y) \in \mathbb{R}^2 \tag{6.5}$$

has a compactly supported subsolution. The construction is classical: setting $\omega := d^{-1/2}\sqrt{f'(0) - \delta - c^2/4d}$, we see that the Dirichlet problem

$$-d\phi'' + c\phi' = (f'(0) - \delta)\phi \quad \text{in } (-\pi/2\omega, \pi/2\omega), \qquad \phi(\pm\pi/2\omega) = 0$$

admits the positive solution $\phi(x) = e^{(c/2d)x} \cos(\omega x)$. Then, let $\psi_R(y)$ be the first eigenfunction of $-\partial_{yy}$ in $(-R, R)$; choosing R such that the associated eigenvalue is $\delta/2d$, it is readily seen that $V(x, y) := \phi(x)\psi_R(y - R - 1)$ is a solution to (6.5) in the rectangle $(-\pi/2\omega, \pi/2\omega) \times (1, 2R + 1)$, vanishing on the boundary. Extending it by 0 outside, we find that $(0, V)$ is a generalized subsolution to (6.1) (cf. Remark 3.4). $\square$

PROOF OF THEOREM 1.1: THE 'SPREADING' PART. Let c_* be as in Section 5 and let $(e^{\alpha(x+c_*t)}, \gamma e^{\alpha(x+c_*t)-\beta y})$ be the exponential entire supersolution of (5.1) - and

then of (1.1) - constructed there. By symmetry, $(e^{\alpha(-x+c_*t)}, \gamma e^{\alpha(-x+c_*t)-\beta y})$ is still a supersolution. The comparison principle - Proposition 3.2 - then yields

$$\forall c > c_*, \quad \lim_{t \to +\infty} \sup_{|x| \geq ct} (u(x,t), v(x,y,t)) = (0,0).$$

Fix now $0 < c < c_*$ (close enough to c_* if $D > 2d$) and consider the pair $(\underline{u}, \underline{v})$ given by Lemmas 6.1, 6.2. There exists $\gamma_0 > 0$ such that, for $0 < \gamma \leq \gamma_0$, $\gamma(\underline{u}, \underline{v})$ is a subsolution of the problem in the moving framework:

$$
\begin{cases}
\partial_t u - D\partial_{xx} u + c\partial_x u = v(x,0,t) - \mu u & x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d\Delta v + c\partial_x v = f(v) & (x,y) \in \Omega, \ t > 0 \\
-d\partial_y v(x,0,t) = \mu u(x,t) - v(x,0,t) & x \in \mathbb{R}, \ t > 0.
\end{cases}
\tag{6.6}
$$

For $0 < \gamma \leq \gamma_0$, let (u_γ, v_γ) be the solution of (6.6) with initial datum $\gamma(\underline{u}, \underline{v})$. Using the comparison principle for generalised subsolutions - Proposition 3.3 - we see that (u_γ, v_γ) is nondecreasing in t. Moreover, since $\gamma(\underline{u}, \underline{v})$ is not a solution of (6.6), it follows from the strong comparison principle given by Proposition 3.2 that (u_γ, v_γ) is strictly increasing in t. Thus, as $t \to +\infty$, (u_γ, v_γ) converges locally uniformly to a stationary solution (U_γ, V_γ) of (6.6) which is strictly larger than $\gamma(\underline{u}, \underline{v})$. There exists then $k > 0$ such that (U_γ, V_γ) is above the translated by any $h \in (-k, k)$ in the x-direction of $\gamma(\underline{u}, \underline{v})$. By comparison with the translated by h of (u_γ, v_γ), we infer that (U_γ, V_γ) is above the translated by h of itself, that is, it does not depend on x. Therefore, U_γ is constant and $V_\gamma = V_\gamma(y)$ satisfies

$$V_\gamma(0) = \mu U_\gamma, \qquad -dV_\gamma'' = f(V_\gamma) \ \text{ for } y > 0, \qquad V_\gamma'(0) = 0.$$

This easily implies $V_\gamma \equiv 1$ and $U_\gamma \equiv 1/\mu$. We will now conclude by comparing the solution (u,v) with a combination of the functions

$$(u_\gamma(x+ct,t), v_\gamma(x+ct,y,t)), \qquad (u_\gamma(-x+ct,t), v_\gamma(-x+ct,y,t)),$$

which are also solutions of (1.1). Notice that, up to waiting until time 1, it is not restrictive to assume that, at initial time, (u,v) is positive and then it is above $\gamma(\underline{u},\underline{v})$ for some $\gamma \leq \gamma_0$. Take $T > 2$ and $0 \leq \xi \leq c(T-2)$. Let $\tau \in [1, T/2]$ be such that $\xi = c(T - 2\tau)$. By comparison, we get

$$\forall (x,y) \in \Omega, \ 0 < t < \tau, \quad (u(x,t), v(x,y,t)) \geq (u_\gamma(x+ct,t), v_\gamma(x+ct,y,t)),$$

whence, in particular,

$$(u(x,\tau), v(x,y,\tau)) \geq (u_\gamma(x+c\tau,\tau), v_\gamma(x+c\tau,y,\tau)) \geq (u_\gamma(x+c\tau,1), v_\gamma(x+c\tau,y,1)).$$

Taking $0 < \gamma' \leq \gamma_0$ small enough (independently on T) in such a way that $\gamma'(\underline{u}(-x), \underline{v}(-x,y)) \leq (u_\gamma(x,1), v_\gamma(x,y,1))$, we find that, at $t = \tau$, (u,v) is larger than the pair

$$(u_{\gamma'}(-x+c(t-2\tau), t-\tau), v_{\gamma'}(-x+c(t-2\tau), y, t-\tau)).$$

17

Applying once again the comparison principle we get, for given $y > 0$,

$$(u(\xi, T), v(\xi, y, T)) \geq (u_{\gamma'}(0, T - \tau), v_{\gamma'}(0, y, T - \tau)) \geq (u_{\gamma'}(0, T/2), v_{\gamma'}(0, y, T/2)).$$

From this, we eventually infer

$$\lim_{T \to +\infty} \inf_{0 \leq \xi \leq c(T-2)} (u(\xi, T), v(\xi, y, T)) \geq (1/\mu, 1).$$

The reverse inequality follows from Theorem 4.1. This concludes the proof, as the negative values of ξ can be handled by reflection with respect to the y axis. $\qquad\square$

Biological interpretation. The result we have derived in this section is a remarkable effect of this model. Essentially, it says that propagation on the road can be accelerated by a large diffusion, provided that there is an exchange between the road and an outside region where reproduction and some diffusion occurs. For lower values of the diffusion on the road, namely, $D < 2d$, the propagation on the road is purely driven by that in the field. The invasion velocity then is the standard KPP invasion speed. However, when the critical value $D = 2d$ is crossed, a transition to a new regime takes place. Namely, the *combination* of reaction in the field and fast diffusion on the line yields an invasion speed strictly larger than the KPP invasion speed. One may view this regime as driven by the diffusion on the line. Note that it is rather remarkable that even though no reproduction takes place there (leading to dispersion if it were an isolated line), the large diffusion drives the overall phenomenon.

The precise value of the threshold diffusion here is $D = 2d$. In the previous section, we saw that when the reaction (effective reproduction) also takes place on the road, one is led to system (5.4). For that system, the threshold value is $D = d$. It is interesting to note that while in both cases a large diffusion on the road always enhances the spreading speed, when there is pure diffusion and no reproduction on the road, this effect starts to be felt at higher values of the diffusion when compared to the model in which the reaction holds everywhere. $\qquad\square$

7 The large diffusion limit of c_*

In this section we investigate the behaviour of the critical speed c_*, introduced in Section 5, as D goes to ∞. Hence, c_* is now understood as a function of the variable D. We will first show that

$$\sqrt{4\mu^2 + (f'(0))^2} - 2\mu \leq \liminf_{D \to \infty} \frac{c_*^2}{D} \leq \limsup_{D \to \infty} \frac{c_*^2}{D} \leq f'(0). \tag{7.1}$$

The proof of the last statement of Theorem 1.1 will follow easily.

For $D > 2d$, we know that $c_* > c_{KPP} = 2\sqrt{df'(0)}$. Moreover, it follows from geometrical considerations (see figure 1) that

$$\frac{c_*}{D} < \frac{c_*}{2d} - \beta_{KPP}(c_*) < \frac{1}{2D}(c_* + \sqrt{c_*^2 + 4\mu D}),$$

where, we recall, $\beta_{KPP}(c) = \frac{\sqrt{c^2 - c_{KPP}^2}}{2d}$ is the radius of the circle $\Gamma_{c,d}$. Whence,

$$\frac{1}{D} < \frac{1}{2d}\left(1 - \sqrt{1 - \frac{c_{KPP}^2}{c_*^2}}\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{c_*^2}}\right).$$

From the second inequality above we infer that, as $D \to \infty$, c_* tends to ∞ and then

$$\frac{1}{D} < \frac{1}{2d}c_*^{-2}\left(\frac{c_{KPP}^2}{2} + o(1)\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{c_*^2}}\right).$$

Calling $\tau := \frac{c_*}{\sqrt{D}}$, the first inequality above yields $\tau^2 < f'(0) + o(1)$, and then, by the second one,

$$2f'(0) + o(1) < \tau^2 + \tau\sqrt{\tau^2 + 4\mu} < f'(0) + \tau\sqrt{\tau^2 + 4\mu} + o(1).$$

Property (7.1) then follows.

PROOF OF THEOREM 1.1: THE LIMIT OF c_*. Revert to the algebraic system (5.3). Estimate (7.1) enables us to rescale the unknowns c and α as

$$c = \sqrt{D}\tilde{c}, \qquad \alpha = \tilde{\alpha}/\sqrt{D},$$

with $\tilde{c}$ bounded from below away from 0 and from above independently on D. It then follows from the first equation of (5.3) that $\tilde{\alpha}$ is bounded from above too. Drop the tildes. Because c and α are now to be found bounded, we drop the term $-d\alpha^2/D$ produced by the normalisation in (5.3), and the limiting system to solve is

$$\begin{cases} -\alpha^2 + c\alpha &= -\dfrac{d\beta\mu}{1 + d\beta} \\ \alpha &= \dfrac{1}{c}(f'(0) + d\beta^2). \end{cases} \tag{7.2}$$

The last equation of (7.2) is that of a parabola in the (β, α) plane, whose bottom is the point $(0, f'(0)/c)$. Call it $\Gamma_{c,d}^{\infty}$ - just to be consistent with the notations of Section 5. The first equation just represents the set $\Gamma_{c,D}$ for $D = 1$. When c is close to 0, the parabola $\Gamma_{c,d}^{\infty}$ is above this set - which lies below the line $\{\alpha = 2\sqrt{\mu}\}$. When c is close to infinity, because of their respective convexity properties, the curves intersect at two points (see Figure 3 below). Thus (same argument as in Section 5) there is $c_* > 0$ such that (7.2) has exactly one solution. This is the sought for limit. $\qquad\square$

Biological interpretation. When the diffusion on the road is very large, the spreading along the road also becomes very large. The invasion speed can be made as large as one wishes provided the diffusion on the single line is taken sufficiently large. This model therefore accounts for the enhancement of invasion speeds in the presence of lines with large diffusion coefficients even though no growth of population takes place there. The precise quantification we derive here states that the invasion speed grows like the square root of the pure diffusion on the road. It seems interesting to go further into this analysis and determine how the prefactor of $\sqrt{D}$ depends on the various parameters. $\qquad\square$
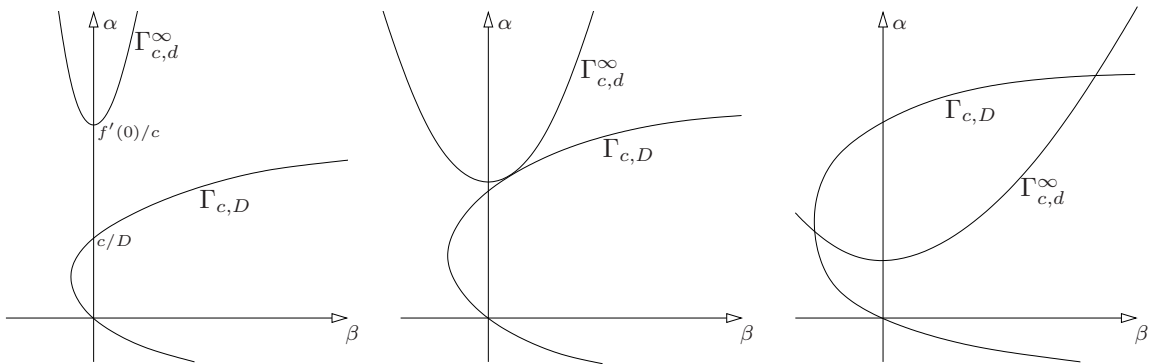
Figure 3: c close to 0 (left), $c = c_*$ (middle), c close to ∞ (right).

8   Conclusion

We have introduced a model that describes the effect of a line on which there is fast diffusion on the overall propagation of a species that diffuses with another constant and reproduces outside this line in a two dimensional framework.

We have found that this model conserves the population in absence of reproduction and mortality and preserves order. Then, we have shown that owing to the exchanges taking place between the line and the plane, there is an asymptotic speed of spreading which is the invasion velocity along the line.

We have computed the global asymptotic speed of spreading along the line. This is achieved with exponential solutions of the linearised system and compactly supported sub-solutions. The asymptotic speed is derived from an algebraic system.

When D, the diffusion on the road, is less than or equal to $2d$, where d is the diffusion in the field, there is no effect at all of the road: the propagation takes place at the classical KPP invasion speed. In contradistinction with this case, when D is larger than $2d$, there is an enhancement effect of the diffusion on the road leading to a speed higher than KPP. Lastly, this invasion speed is shown to behave like $\sqrt{D}$ for large values of D.

Appendix

A   Existence result for the Cauchy problem

PROOF OF THE EXISTENCE PART OF PROPOSITION 3.1. We prove the result for an initial datum (u_0, v_0) which is locally Hölder continuous, together with its derivatives up to order 2, and satisfies the compatibility condition

$$-d\partial_y v_0(x, y, 0) = \mu u_0(x, y) - v_0(x, y, 0).$$

The regularity of the initial datum is therefore inherited by the solution of the Cauchy problem for all time $t \geq 0$. The case of a merely continuous initial datum can then be handled by a standard regularization technique (see, e.g., [13]).

We will obtain a solution to (3.1)-(3.2) as the limit of a subsequence of solutions $((u_n, v_n))_n$ of the following problems:

$$
\begin{cases}
\partial_t u_n - D\Delta_x u_n - q \cdot \nabla_x u_n + \mu u_n = v_{n-1}(x, 0, t) & x \in \mathbb{R}^N, \ t > 0 \\
u_n|_{t=0} = u_0 & \text{in } \mathbb{R}^N,
\end{cases}
\tag{A.1}
$$

$$
\begin{cases}
\partial_t v_n - d\Delta v_n - r \cdot \nabla v_n = f(v_n) & (x, y) \in \Omega, \ t > 0 \\
v_n(x, 0, t) - d\partial_y v_n(x, 0, t) = \mu u_n(x, t) & x \in \mathbb{R}^N, \ t > 0 \\
v_n|_{t=0} = v_0 & \text{in } \Omega,
\end{cases}
\tag{A.2}
$$

starting from v_0.

Step 1. *Solvability of* (A.1), (A.2).

We say that a function $w(z, t)$ has admissible growth in z if it satisfies $|w(z, t)| \leq \beta e^{\sigma|z|^2}$, for some $\sigma, \beta > 0$. It is well known that the linear Cauchy problem is uniquely solvable in the class of functions with admissible growth in the space variable. If v_{n-1} is a continuous function with admissible growth, then problem (A.1) admits a unique classical solution u_n with admissible growth. In order to solve (A.2), notice that it can be reduced to a homogeneous system by replacing v_n with $v_n - v_0 - \mu(u_n - u_0)$. It then follows from the standard parabolic theory that it admits a unique classical solution with admissible growth. Let $((u_n, v_n))_n$ denote the family of solutions constructed in this way, starting from v_0.

Step 2. L^∞ *estimates.*

We show, with a recursive argument, that

$$
\forall n \in \mathbb{N}, \quad 0 \leq u_n \leq \frac{1}{\mu} H, \quad 0 \leq v_n \leq H, \quad \text{with } H := \max\left(\mu\|u_0\|_\infty, \|v_0\|_\infty, 1\right).
$$

The property trivially holds for $n = 0$. Assume that it holds for some value $n - 1$. Since 0 and $\frac{1}{\mu} H$ are respectively a sub and a supersolution of (A.1), the comparison principle yields $0 \leq u_n \leq \frac{1}{\mu} H$. It then follows that H is a supersolution of (A.2), whence $0 \leq v_n \leq H$.

Step 3. $W_p^{2,1}$ *estimates.*

By step 1 we know that $0 \leq v_{n-1} \leq H$. Thus, applying the local boundary estimates to (A.1) we infer that, for any given $\rho, T > 0$ and $1 < p < \infty$,

$$
\|u_n\|_{W_p^{2,1}(B_{\rho+1} \times (0,T))} \leq CH,
$$

where B_ρ denotes the N-dimensional ball of radius ρ and centre 0 and C is a constant only depending on N, D, q, μ, ρ, T, p and $\|u_0\|_{W_p^2(B_{\rho+2})}$ (and not on n). Set $Q_\rho := B_\rho \times (0, \rho)$. Since $0 \leq v_n \leq H$ too, the estimates yield

$$
\begin{aligned}
\|v_n\|_{W_p^{2,1}(Q_\rho \times (0,T))} &\leq C'\left(1 + \|v_n\|_{L^\infty(Q_{\rho+1} \times (0,T))} + \|u_n\|_{W_p^{2,1}(B_{\rho+1} \times (0,T))}\right) \\
&\leq C'(1 + H + CH),
\end{aligned}
$$

with C' only depending on N, d, r, μ, ρ, T, p, $\|f\|_\infty$, $\|u_0\|_{W_p^2(B_{\rho+1})}$ and $\|v_0\|_{W_p^2(Q_{\rho+1})}$. This shows that the $(u_n)_n$ and $(v_n)_n$ are uniformly bounded in $W_p^{2,1}(B_\rho \times (0,T))$ and $W_p^{2,1}(Q_\rho \times (0,T))$ respectively.

Step 4. *Existence of a solution.*

Now that we know that $(u_n)_n$ and $(v_n)_n$ are uniformly bounded in compact sets with respect to the $W_p^{2,1}$ norm, taking $p > N + 1$ and using the Morrey inequality, we infer that this is also true with respect to the C^α norm, for some $0 < \alpha < 1$. Then, by the Schauder estimates, the time derivative and the space derivatives up to order 2 are uniformly Hölder continuous in compact sets too. As a consequence, $((u_n, v_n))_n$ converges (up to subsequences) in $C_{loc}^{2,1}$ to some (u, v). Passing to the limit as $n \to \infty$ in (A.1), (A.2) we eventually find that (u, v) satisfies (3.1)-(3.2). Form step 1 we know that u and v are bounded and nonnegative. $\qquad\square$

B The equation $h^L(c, \beta) = 0$

In this section we describe in detail how, for $\xi > 0$ small enough, equation (6.4) admits two solutions close to τ_+ and τ_- respectively. We recall that $\tau_\pm = \pm i\sqrt{(e/a)\xi} + O(\xi)$ are the roots of the trinomial $g(\tau) := a\tau^2 + d\xi\tau + e\xi$. Let us focus on τ_+, the other case being analogous.

Let B be the ball of radius $A\xi$, centred at τ_+; A large and to be adjusted. For $\tau \in \partial B$, we have

$$|g(\tau)| = a|\tau - \tau_+|\,|\tau - \tau_-| \geq aA\xi(|\tau_+ - \tau_-| - A\xi) = 2aA\sqrt{e/a}\,\xi^{3/2} + O(\xi^2).$$

On the other hand we have $|\varphi(\tau, \xi)| \leq C\xi^{3/2} + O(\xi^2)$. We can therefore choose A large enough and then ξ small enough in such a way that $|g| > |\varphi|$ on ∂B. Since g and φ are holomorphic, by Rouché's theorem the equation (6.4) has the same number of solutions in B as $g = 0$, that is 1. Notice that such a solution has positive imaginary part proportional to $\sqrt{\xi}$ and real part of order ξ.

Acknowledgements

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