

# Energy lower bound for the unitary $N + 1$ fermionic model

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We consider the stability problem for a unitary  $N + 1$  fermionic model, i.e., a system of  $N$  identical fermions interacting via zero-range interactions with a different particle, in the case of infinite two-body scattering length. Starting from the two-body boundary condition, we construct an explicit expression for the expectation value of the energy. Then we investigate its boundedness from below and exhibit a sufficient condition on the mass ratio, which guarantees the stability of the model.

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## I. INTRODUCTION AND MAIN RESULT

The study of the quantum mechanical many-body problem with pairwise zero-range interactions has received a considerable attention in recent years as an effective model describing the behavior of cold atoms near the BEC/BCS crossover ([2, 5] and references therein). The correct definition of the model, the occurrence of the Efimov effect and the analysis of the stability problem, i.e., the existence of a lower bound for the Hamiltonian, have been widely studied both in the physical [3, 4, 12, 13, 15–17] and in the mathematical (see, e.g., [6, 7, 10, 14]) literature. The typical approach to the problem is however quite different and it is not even clear a priori that the two strategies should lead to equivalent results. The investigation of all the relevant properties of the model (energy levels, bound states, scattering properties etc.) is indeed performed in the physics literature by imposing a suitable singular boundary condition on the admissible wave functions. The mathematical strategy to give a rigorous meaning to zero-range Schrödinger operator is on the other hand more involved and relies heavily on the theory of self-adjoint extensions: one introduces first a class of symmetric operators with domain containing functions which vanishes on the support of the interaction and then classifies all the possible self-adjoint extensions of such operators. It is well known that in the two-body case the two approaches are completely equivalent, since the whole class of Hamiltonians for a pair of particles with zero-range interaction can be explicitly constructed and it can be shown that the so obtained self-adjointness domains consist of wave functions satisfying the same boundary condition considered in the physics approach. In fact in this simple case all the spectral and scattering properties can be completely characterized [1]. On the opposite, when the number of particles exceeds two, the problem becomes much less trivial, since the class of self-adjoint extensions can be extremely wide and contains many operators without apparent physical meaning. It is then customary in mathematical physics to

focus the attention onto a much smaller class of Hamiltonians, which typically go under the name of Skornyakov Ter-Martirosyan (STM) operators. The heuristics behind this apparently arbitrary choice is just an analogy with the two-body case, which has however not a sound physics motivation. One of the goals of this note is precisely to show that starting from the physics boundary condition one can recover the expectation value of the STM operators, so establishing an equivalence between the two approaches. In fact the explicit expression of the energy obtained here (see (27)) is also simpler and easier to handle than the one studied, e.g., in [6], although perfectly equivalent from the mathematical point of view.

Once the physics motivations of the model have been discussed, we then turn our attention to a more specific question, i.e., the stability of the model under investigation. Indeed it is known that STM operators are symmetric but (in general) are neither self-adjoint nor bounded from below. This happens, for instance, in the case of three identical bosons, where it was shown in [9] that the STM operator admits self-adjoint extensions which can be explicitly constructed but they are all unbounded from below and therefore the system is unstable.

Here we are interested in the stability problem in the fermionic case, that is when fermions of different species interact among themselves. For the most general system composed by a mixture of  $N$  identical fermions of one type (with mass  $m_1$ ) and  $K$  identical fermions of a different type (with mass  $m_0$ ), the stability problem for the corresponding STM operator is open and some results are available only in special cases. For instance, a system composed by two identical fermions plus a different particle is known to be stable if (and only if) the mass ratio

$$\alpha = \frac{m_1}{m_0} \quad (1)$$

is smaller than the critical value 13.607 (see, e.g., [2, 6]). Further results are available in the  $3 + 1$  case only [3].

In this note, following the ideas of [6], we present an alternative and cleaner proof of a sufficient condition for the stability of a system composed by  $N$ , with  $N \geq 2$ ,

identical fermions plus a different particle. As a matter of fact our result, showing positivity of the energy expectation value, also rules out the occurrence of the Efimov spectrum. The stability condition can be cast in the following form

$$\alpha \leq \alpha_c(N) \quad (2)$$

where  $\alpha_c(N)$  is the solution of the following equation

$$\Lambda(\alpha, N) = \frac{2}{\pi}(N-1) \left(\frac{1+\alpha}{\alpha}\right)^2 \left[ \frac{\alpha}{\sqrt{1+2\alpha}} - \arcsin\left(\frac{\alpha}{1+\alpha}\right) \right] = 1 \quad (3)$$

Notice that for each  $N$  the function  $\Lambda(\alpha, N)$  is increasing, goes to infinity for  $\alpha \rightarrow \infty$  and  $\Lambda(0, N) = 0$ . So there is exactly one solution  $\alpha_c(N) > 0$  of (3) and moreover  $\Lambda(\alpha, N) < 1$  for  $\alpha < \alpha_c(N)$ . We remark that only for  $N = 2$  the condition (2) is optimal, i.e.,  $\alpha_c(2) = 13.607$ , and therefore the result provides a rigorous proof of what is already known in the physical literature.

For  $N > 2$  the condition is surely not optimal since, as it will be clear from the proof, the role of the antisymmetry is only partially exploited. Nevertheless we believe that the result can be of some interest since (2) gives a first sufficient stability condition which, apparently, was not known before. Some numerical values of  $\alpha_c(N)$  are listed here:  $\alpha_c(3) = 5.291$ ,  $\alpha_c(8) = 1.056$ ,  $\alpha_c(9) = 0.823$ , etc.. In particular this means that, in the case of equal masses, the system is stable if  $N \leq 8$ .

In the exposition of the proof we also aim at pointing out the major steps where an alternative but not mathematical rigorous approach, e.g., a numerical simulation, could lead to an improvement of the result. This can be easily done within this new derivation of the stability condition and it is another motivation for presenting it in a separate note.

Let us consider the formal Hamiltonian for a system of  $N$  identical fermions with mass  $m_1$  and a different particle with mass  $m_0$  (we set  $\hbar = 1$ )

$$\tilde{H} = -\frac{1}{2m_1} \sum_{i=1}^N \Delta_{\mathbf{x}_i} - \frac{1}{2m_0} \Delta_{\mathbf{x}_0} + \gamma \sum_{i=1}^N \delta(\mathbf{x}_0 - \mathbf{x}_i). \quad (4)$$

The parameter  $\gamma \in \mathbb{R}$  is a coupling constant which must be properly renormalized in order to give a precise meaning to the expression (4).

The formal Hamiltonian (4) can be given a precise meaning as a, possibly self-adjoint, operator in the Hilbert space of square integrable functions on  $\mathbb{R}^{3N+3}$  antisymmetric under exchange of fermions. More precisely, it is by definition a non trivial (self-adjoint) extension of the free Hamiltonian  $H_0$  restricted to smooth functions vanishing on the set

$$\Omega = \bigcup_{i \in \{1, \dots, N\}} \left\{ \mathbf{X} \in \mathbb{R}^{3(N+1)} \mid \mathbf{x}_i = \mathbf{x}_0 \right\} \quad (5)$$

with  $\mathbf{X} = (\mathbf{x}_0, \dots, \mathbf{x}_N)$ . As we already mentioned, we want to select the STM operator starting from the physical boundary condition on  $\Omega$ . We proceed by analogy with the well known two-body case.

For two (different) particles, extracting the center of mass motion and denoting by  $\mathbf{x}$  the relative coordinate, the domain of the operator consists of functions  $\psi \in L^2(\mathbb{R}^3)$ , which are regular for  $\mathbf{x} \neq 0$  and satisfy the following boundary condition as  $|\mathbf{x}| \rightarrow 0$

$$\psi(\mathbf{x}) = \left( \frac{1}{|\mathbf{x}|} - \frac{1}{a} \right) q + o(1), \quad (6)$$

where  $q \in \mathbb{C}$  depends on  $\psi$  and  $a \in \mathbb{R}$  has the physical meaning of a scattering length. Moreover, the Hamiltonian acts as the free Hamiltonian for  $|\mathbf{x}| \neq 0$ .

The STM extension  $H_a$  in our fermionic  $N+1$ -particle system is defined in an analogous way. Extracting the center of mass motion, the domain  $D(H_a)$  is made of functions  $\psi$  defined on the set

$$\mathcal{M} = \{ \mathbf{X} \in \mathbb{R}^{3(N+1)} \mid \mathbf{x}_{\text{cm}} = 0 \} \quad (7)$$

antisymmetric under the exchange of any pair of fermions, regular for  $\mathbf{x}_0 \neq \mathbf{x}_i$ ,  $i = 1, \dots, N$ . The standard formulation of the boundary condition satisfied as  $|\mathbf{x}_0 - \mathbf{x}_i| \rightarrow 0$  is (see, e.g., [17])

$$\psi(\mathbf{X}) = \left( \frac{1}{|\mathbf{x}_0 - \mathbf{x}_i|} - \frac{1}{a} \right) (-1)^{i+1} Q(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) + o(1) \quad (8)$$

where  $Q : \mathbb{R}^{3N} \rightarrow \mathbb{C}$  is a function antisymmetric in  $\check{\mathbf{x}}_i$  and  $a \in \mathbb{R}$  is the two-body scattering length corresponding to the interaction of a fermion with the different particle. In the above expression we have denoted

$$\mathbf{r}_{0i} = \frac{m_0 \mathbf{x}_0 + m_1 \mathbf{x}_i}{m_0 + m_1}, \quad (9)$$

$$\check{\mathbf{x}}_i = (\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N). \quad (10)$$

Notice that in the limiting procedure defining the boundary condition (8) the vectors  $\mathbf{r}_{0i}$  and  $\check{\mathbf{x}}_i$  are kept fixed.

Furthermore,  $H_a$  acts as the free Hamiltonian outside the set  $\Omega$ , i.e.,

$$(H_a \psi)(\mathbf{X}) = (H_0 \psi)(\mathbf{X}), \quad \text{if } \mathbf{X} \in \mathcal{M} \setminus \Omega. \quad (11)$$

A special role is played by the parameter-free case of infinite scattering length, known as the unitary case. We shall denote by  $H$  the corresponding STM extension, i.e.,  $H := H_\infty$ .

The main result discussed in this note is the following

**Theorem.** *In the unitary case the energy form, i.e., the expectation value of the energy, is positive for  $\alpha \leq \alpha_c(N)$ . More precisely, for any  $\psi \in D(H)$ ,*

$$(\psi, H\psi) \geq 0, \quad \text{if } \alpha \leq \alpha_c(N). \quad (12)$$

*This in particular implies stability for the unitary  $N+1$  fermionic model.*

In the next Section we derive a suitable expression for the energy form. In Section III we start from such expression to explain the steps required to prove our result. In Section IV we briefly summarize the content of the paper. In the Appendix we collect some technical results useful to reformulate the domain and the boundary condition characterizing the Hamiltonian.

## II. DERIVATION OF THE ENERGY FORM

For the sake of simplicity of notation from now on we drop the restriction to  $\mathcal{M}$  or, equivalently, the condition on momenta  $\mathbf{p}_{\text{cm}} = \sum \mathbf{p}_i = 0$ ; we will however take it into account at the end of the computation. The key point is to represent the domain  $D(H)$  as the set of wave functions decomposing as

$$\psi = w + \mathcal{G}Q \quad (13)$$

where  $w$  is a smooth function and  $\mathcal{G}Q$  contains the singular behavior prescribed in (8). More precisely,  $\mathcal{G}Q$  is the ‘‘potential’’ produced by the ‘‘charge’’  $Q$  distributed on the planes  $\{\mathbf{x}_i = \mathbf{x}_0\}$ , i.e.,

$$\begin{aligned} (\mathcal{G}Q)(\mathbf{X}) &= \frac{1}{\mu\sqrt{2\pi}} \sum_{j=1}^N \frac{(-1)^{j+1}}{(2\pi)^{\frac{3}{2}(N+1)}} \times \\ &\times \int d\mathbf{P} e^{i\mathbf{P}\cdot\mathbf{X}} \frac{\hat{Q}(\mathbf{p}_0 + \mathbf{p}_j, \check{\mathbf{p}}_j)}{h_0(\mathbf{P})} \end{aligned} \quad (14)$$

with

$$\mu = \frac{m_0 m_1}{m_0 + m_1}, \quad h_0(\mathbf{P}) = \frac{p_0^2}{2m_0} + \sum_{i=1}^N \frac{p_i^2}{2m_1}. \quad (15)$$

It is straightforward to verify that as  $\mathbf{y}_i = \mathbf{x}_0 - \mathbf{x}_i \rightarrow 0$  with  $\mathbf{r}_{0i}$  fixed

$$(\mathcal{G}Q)(\mathbf{X}) \simeq \frac{(-1)^{i+1}}{|\mathbf{y}_i|} Q(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) - (\Gamma_i Q)(\mathbf{r}_{0i}, \check{\mathbf{x}}_i), \quad (16)$$

where, setting  $\check{\mathbf{p}}_i = (\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \dots, \mathbf{p}_N)$ ,

$$\begin{aligned} (\Gamma_i Q)(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) &= \frac{(-1)^{i+1}}{(2\pi)^{\frac{3}{2}N}} \int d\check{\mathbf{p}}_i d\mathbf{q}_i e^{i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i + i\mathbf{q}_i \cdot \mathbf{r}_{0i}} \times \\ &\times \sqrt{\frac{\alpha}{(1+\alpha)^2} \mathbf{q}_i^2 + \frac{1}{1+\alpha} \check{\mathbf{p}}_i^2} \hat{Q}(\mathbf{q}_i, \check{\mathbf{p}}_i) + \\ &\sum_{j \neq i} \frac{(-1)^{j+1}}{\mu(2\pi)^{\frac{3}{2}N+2}} \int d\mathbf{P} e^{i(\mathbf{p}_0 + \mathbf{p}_i) \cdot \mathbf{r}_{0i} + i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i} \frac{\hat{Q}(\mathbf{p}_0 + \mathbf{p}_j, \check{\mathbf{p}}_j)}{h_0(\mathbf{P})}. \end{aligned} \quad (17)$$

Moreover, it is useful to note that the potential  $\mathcal{G}Q$  satisfy the equation

$$(H_0 \mathcal{G}Q)(\mathbf{X}) = 4\pi \sum_i (-1)^{i+1} Q(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) \delta(\mathbf{x}_i - \mathbf{x}_0) \quad (18)$$

in distributional sense and then, in particular,

$$(H_0 \mathcal{G}Q)(\mathbf{X}) = 0, \quad \text{if } \mathbf{X} \in \mathbb{R}^{3N+3} \setminus \Omega. \quad (19)$$

The proof of (16) and (18) is postponed to the Appendix.

Using the decomposition (13) and the asymptotic behavior (16), the boundary condition (8) in the unitary case can be equivalently written as

$$\lim_{|\mathbf{y}_i| \rightarrow 0} w(\mathbf{y}_i, \mathbf{r}_{0i}, \check{\mathbf{x}}_i) = (\Gamma_i Q)(\mathbf{r}_{0i}, \check{\mathbf{x}}_i). \quad (20)$$

We can now derive the expression for the energy form. Taking into account (11), the decomposition (13) and equation (19) and setting  $\mathcal{D}_\varepsilon = \{\mathbf{X} \in \mathbb{R}^{3(N+1)} \mid |\mathbf{x}_i - \mathbf{x}_0| > \varepsilon, i = 1, \dots, N\}$ , we have

$$\begin{aligned} (\psi, H\psi) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X} \bar{\psi}(\mathbf{X}) (H_0 \psi)(\mathbf{X}) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X} (\bar{w} + \overline{\mathcal{G}Q})(\mathbf{X}) (H_0 w)(\mathbf{X}) \\ &= (w, H_0 w) + \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X} (\overline{\mathcal{G}Q})(\mathbf{X}) (H_0 w)(\mathbf{X}). \end{aligned} \quad (21)$$

In the last integral of (21) we apply Green’s identities. Denoting  $S_i^2 = \{\mathbf{y}_i \in \mathbb{R}^3 \mid |\mathbf{y}_i| = 1\}$ , as  $\varepsilon \rightarrow 0$  and changing variables from  $(\mathbf{x}_0, \mathbf{x}_i)$  to  $(\mathbf{r}_{0i}, \check{\mathbf{y}}_i)$  we obtain

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X} (\overline{\mathcal{G}Q})(\mathbf{X}) (H_0 w)(\mathbf{X}) \\ &= - \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sum_{i=1}^N \int_{\mathbb{R}^{3N}} d\mathbf{r}_{0i} d\check{\mathbf{x}}_i \int_{S_i^2} d\sigma(\boldsymbol{\omega}_i) \frac{\partial(\overline{\mathcal{G}Q})}{\partial|\mathbf{y}_i|} \Big|_{|\mathbf{y}_i|=\varepsilon} w \Big|_{|\mathbf{y}_i|=\varepsilon} \\ &= \sum_{i=1}^N (-1)^{i+1} \int_{\mathbb{R}^{3N}} d\mathbf{r}_{0i} d\check{\mathbf{x}}_i \overline{Q}(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) \int_{S_i^2} d\sigma(\boldsymbol{\omega}_i) \lim_{\varepsilon \rightarrow 0} w \Big|_{|\mathbf{y}_i|=\varepsilon} \\ &= 4\pi \sum_{i=1}^N (-1)^{i+1} \int_{\mathbb{R}^{3N}} d\mathbf{r}_{0i} d\check{\mathbf{x}}_i \overline{Q}(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) (\Gamma_i Q)(\mathbf{r}_{0i}, \check{\mathbf{x}}_i), \end{aligned} \quad (22)$$

where we have used equation (19), the asymptotics (16) and the boundary condition (20). Taking into account (21) and (22), we find

$$\begin{aligned} (\psi, H\psi) &= (w, H_0 w) \\ &+ 4\pi \sum_{i=1}^N (-1)^{i+1} \int_{\mathbb{R}^{3N}} d\mathbf{r}_{0i} d\check{\mathbf{x}}_i \overline{Q}(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) (\Gamma_i Q)(\mathbf{r}_{0i}, \check{\mathbf{x}}_i). \end{aligned} \quad (23)$$

Inserting in (23) the explicit expression (17) of  $\Gamma_i Q$  and exploiting the antisymmetry property of the charge  $Q$ , the second term on the r.h.s. of (23) equals

$$\begin{aligned} &4\pi N \int d\mathbf{q} d\check{\mathbf{p}}_1 \sqrt{\frac{\alpha}{(1+\alpha)^2} \mathbf{q}^2 + \frac{1}{1+\alpha} \check{\mathbf{p}}_1^2} |\hat{Q}(\mathbf{q}, \check{\mathbf{p}}_1)|^2 + \\ &+ \frac{N(N-1)}{\mu\pi} \int d\mathbf{P} \frac{\overline{\hat{Q}(\mathbf{p}_0 + \mathbf{p}_1, \check{\mathbf{p}}_1)} \hat{Q}(\mathbf{p}_0 + \mathbf{p}_2, \check{\mathbf{p}}_2)}{h_0(\mathbf{P})}. \end{aligned} \quad (24)$$

Now we perform a change of variables: in the off-diagonal term we change coordinates from  $\mathbf{P}$  to  $(\mathbf{p}_{\text{cm}}, \mathbf{k}_1, \dots, \mathbf{k}_N)$  with

$$\mathbf{p}_{\text{cm}} = \mathbf{p}_0 + \sum \mathbf{p}_i, \quad \mathbf{k}_j = \frac{\alpha}{1+N\alpha} (\mathbf{p}_0 + \sum \mathbf{p}_i) - \mathbf{p}_j,$$

while in the diagonal term we replace  $(\mathbf{q}, \check{\mathbf{p}}_i)$  with

$$\mathbf{q} = \frac{1+\alpha}{1+N\alpha} \mathbf{p}_{\text{cm}} + \sum_{i=1}^{N-1} \mathbf{k}_i, \quad \mathbf{k}_j = \frac{\alpha}{1+N\alpha} \mathbf{p}_{\text{cm}} - \mathbf{p}_{j+1},$$

for  $j = 1, \dots, N-1$ , so that (24) can be rewritten

$$\frac{2N}{\pi} \left[ \frac{2\pi^2 \sqrt{1+2\alpha}}{1+\alpha} \int d\mathbf{p}_{\text{cm}} d\check{\mathbf{k}}_1 L(\check{\mathbf{k}}_1, \mathbf{p}_{\text{cm}}) |\hat{\Xi}(\check{\mathbf{k}}_1, \mathbf{p}_{\text{cm}})|^2 + (N-1) \int d\mathbf{p}_{\text{cm}} d\mathbf{k} G(\mathbf{k}, \mathbf{p}_{\text{cm}}) \overline{\hat{\Xi}(\check{\mathbf{k}}_1, \mathbf{p}_{\text{cm}})} \hat{\Xi}(\check{\mathbf{k}}_2, \mathbf{p}_{\text{cm}}) \right]$$

with  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_N)$ ,

$$\hat{\Xi}(\check{\mathbf{k}}_1, \mathbf{p}_{\text{cm}}) = \hat{Q} \left( \frac{m_0 + m_1}{M} \mathbf{p}_{\text{cm}} + \sum_{i=2}^N \mathbf{k}_i, \frac{m_1}{M} \mathbf{p}_{\text{cm}} - \mathbf{k}_2, \dots \right)$$

and

$$L(\check{\mathbf{k}}_1, \mathbf{p}_{\text{cm}}) = \left( \sum_{i=2}^N \mathbf{k}_i^2 + \frac{2\alpha}{1+2\alpha} \sum_{\substack{i,j=2 \\ i < j}}^N \mathbf{k}_i \cdot \mathbf{k}_j + \frac{(1+\alpha)^2}{1+2\alpha} \frac{\mu}{M} \mathbf{p}_{\text{cm}}^2 \right)^{1/2}, \quad (25)$$

$$G(\mathbf{k}, \mathbf{p}_{\text{cm}}) = \frac{1}{\sum_i \mathbf{k}_i^2 + \frac{2\alpha}{1+\alpha} \sum_{i < j} \mathbf{k}_i \cdot \mathbf{k}_j + \frac{\mu}{M} \mathbf{p}_{\text{cm}}^2}. \quad (26)$$

Now we recall that we have to impose the center of mass condition  $\mathbf{p}_{\text{cm}} = 0$ , so that we finally obtain

$$(\psi, H\psi) = (w, H_0 w) + \frac{2N}{\pi} \Phi(\xi), \quad (27)$$

where  $\xi(\mathbf{K}) = \hat{\Xi}(\mathbf{K}, 0)$  and the quadratic form  $\Phi$  is defined by

$$\Phi(\xi) = \int d\check{\mathbf{K}} \left( \Phi_1(\xi; \check{\mathbf{K}}) + (N-1) \Phi_2(\xi; \check{\mathbf{K}}) \right), \quad (28)$$

with  $\check{\mathbf{K}} = (\mathbf{k}_1, \dots, \mathbf{k}_{N-2})$  and (for  $N=2$  the extra variables  $\check{\mathbf{K}}$  are absent)

$$\Phi_1(\xi; \check{\mathbf{K}}) = 2\pi^2 \frac{\sqrt{1+2\alpha}}{1+\alpha} \int ds L(\mathbf{s}, \check{\mathbf{K}}, 0) |\hat{\xi}(\mathbf{s}, \check{\mathbf{K}})|^2, \quad (29)$$

$$\Phi_2(\xi; \check{\mathbf{K}}) = \int ds dt G(\mathbf{s}, \mathbf{t}, \check{\mathbf{K}}, 0) \bar{\hat{\xi}}(\mathbf{s}, \check{\mathbf{K}}) \hat{\xi}(\mathbf{t}, \check{\mathbf{K}}). \quad (30)$$

Notice that the contribution to the energy due to the regular part  $w$  is

$$(w, H_0 w) = \int d\mathbf{k}_1 \dots d\mathbf{k}_N \left( \frac{1}{2\mu} \sum_{i=1}^N \mathbf{k}_i^2 + \frac{1}{m_0} \sum_{i < j} \mathbf{k}_i \cdot \mathbf{k}_j \right) |\hat{w}_0(\mathbf{k}_1, \dots, \mathbf{k}_N)|^2. \quad (31)$$

As already mentioned in the introduction, the expression derived here for the energy form (27) is different from the one used in [6], where a dependence on a parameter  $\lambda$  is introduced in order to have a square integrable ‘‘potential’’. In fact, such a parameter is not essential and the expression (27) used here is much easier to handle. This makes the stability proof presented here more direct and clear.

### III. POSITIVITY OF THE ENERGY

From the above expression (27) we see that the Theorem is proved if we can show positivity of  $\Phi$ . Since the term  $\Phi_1$  is positive, the problem is reduced to show that

$$(N-1)\Phi_2 \geq -c\Phi_1$$

for some constant  $c \leq 1$ . A proof of this fact will be given here and for the sake of clarity it will be divided in several, but elementary, steps. The strategy will be the reduction of the form  $\Phi_2$  to one which can be diagonalized. This first requires a suitable change of variables; then we exploit the rotational symmetry of  $\Phi_2$  to perform a partial wave decomposition; once the additional degrees of freedom are dropped, the problem reduces to bound from below a two-particle energy, which can be diagonalized by means of the Fourier transform; to conclude the proof it suffices then to go back to the original expression and show that, if the condition  $\alpha \leq \alpha_c$  is satisfied,  $\Phi$  is positive. It is worth stressing that at the last stage of the proof (see, e.g., (40)) the fermionic symmetry of the charges  $\xi$  is totally neglected, in order to diagonalize the expression. This is clearly not optimal and an improvement of the condition (2) would require a different approach. In fact the change of variables itself (see (34)), which is the starting point of our analysis, make the antisymmetric requirement not apparent and therefore should probably be avoided if one wants to track down the role of the fermionic antisymmetry.

#### A. Change of variables

We set

$$\mathbf{p} = \mathbf{s} + \frac{\alpha}{2+\alpha} \sum_{i=1}^{N-2} \mathbf{k}_i, \quad \mathbf{q} = \mathbf{t} + \frac{\alpha}{2+\alpha} \sum_{i=1}^{N-2} \mathbf{k}_i, \quad (32)$$

and therefore we obtain

$$\Phi_1(\xi; \check{\mathbf{K}}) = 2\pi^2 \int d\mathbf{p} \sqrt{\frac{1+2\alpha}{(1+\alpha)^2} \mathbf{p}^2 + D(\check{\mathbf{K}})} |\eta(\mathbf{p}, \check{\mathbf{K}})|^2, \\ \Phi_2(\xi; \check{\mathbf{K}}) = \int d\mathbf{p} d\mathbf{q} \frac{\bar{\eta}(\mathbf{p}, \check{\mathbf{K}}) \eta(\mathbf{q}, \check{\mathbf{K}})}{\mathbf{p}^2 + \mathbf{q}^2 + \frac{2\alpha}{1+\alpha} \mathbf{p} \cdot \mathbf{q} + D(\check{\mathbf{K}})}, \quad (33)$$

where

$$\eta(\mathbf{p}, \check{\mathbf{K}}) = \hat{\xi} \left( \mathbf{p} - \frac{\alpha}{2+\alpha} \sum \mathbf{k}_i, \check{\mathbf{K}} \right), \quad (34)$$

$$D(\check{\mathbf{K}}) = \frac{1+3\alpha}{(1+\alpha)(1+2\alpha)} \left( \sum k_i^2 + \frac{2\alpha}{1+3\alpha} \sum_{i < j} \mathbf{k}_i \cdot \mathbf{k}_j \right).$$

#### B. Expansion in spherical harmonics

For any  $f \in L^2(\mathbb{R}^3)$  we consider the expansion

$$f(\mathbf{p}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm}(p) Y_l^m(\theta_p, \phi_p) \quad (35)$$

where  $\mathbf{p} = (p, \theta_p, \phi_p)$  and  $Y_l^m$  denotes the spherical harmonics of order  $l, m$  with  $l = 0, 1, \dots$  and  $m = -l, \dots, l$ . Using the above expansion we derive the following decomposition of  $\Phi_2$  in each subspace of fixed angular momentum  $l$ :

$$\begin{aligned} \Phi_2(\xi; \check{\mathbf{K}}) &= 2\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \int_0^{\infty} dp \int_0^{\infty} dq \overline{\eta_{lm}}(p, \check{\mathbf{K}}) \eta_{lm}(q, \check{\mathbf{K}}) \\ &\quad \times \int_{-1}^1 dy \frac{p^2 q^2 P_l(y)}{p^2 + q^2 + \frac{2\alpha}{1+\alpha} p q y + D(\check{\mathbf{K}})} \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^l G_l(\eta_{lm}; \check{\mathbf{K}}). \end{aligned} \quad (36)$$

It turns out that (for details see [6, Lemma 3.2])

$$G_l(\eta_{lm}; \check{\mathbf{K}}) \geq 0, \quad \text{for } l \text{ even}, \quad (37)$$

$$0 \geq G_l(\eta_{lm}; \check{\mathbf{K}}) \geq G_l^0(\eta_{lm}), \quad \text{for } l \text{ odd}, \quad (38)$$

where  $G_l^0$  is defined by

$$\begin{aligned} G_l^0(\eta_{lm}) &= 2\pi \int_{-1}^1 dy P_l(y) \times \\ &\quad \times \int_0^{\infty} dp \int_0^{\infty} dq \frac{p^2 \overline{\eta_{lm}}(p, \check{\mathbf{K}}) q^2 \eta_{lm}(q, \check{\mathbf{K}})}{p^2 + q^2 + \frac{2\alpha}{1+\alpha} p q y}. \end{aligned} \quad (39)$$

From (37) and (38) we then get

$$\Phi_2(\xi; \check{\mathbf{K}}) \geq \sum_{l \text{ odd}} \sum_{m=-l}^l G_l^0(\eta_{lm}). \quad (40)$$

### C. Diagonalization

Let us define

$$g^\sharp(k) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} e^{2x} g(e^x). \quad (41)$$

Then

$$\begin{aligned} G_l^0(g) &= 2\pi \int_{-1}^1 dy P_l(y) \times \\ &\quad \times \int dx_1 dx_2 \frac{e^{3x_1} \overline{g}(e^{x_1}) e^{3x_2} g(e^{x_2})}{e^{2x_1} + e^{2x_2} + \frac{2\alpha}{1+\alpha} y e^{x_1+x_2}} \\ &= \pi \int_{-1}^1 dy P_l(y) \int dx_1 dx_2 \frac{e^{2x_1} \overline{g}(e^{x_1}) e^{2x_2} g(e^{x_2})}{\cosh(x_1 - x_2) + \frac{\alpha}{1+\alpha} y}. \end{aligned}$$

The last integral is a convolution and therefore can be diagonalized by means of Fourier transform. Using the explicit Fourier transform of the kernel (see, e.g., [8]) we find for  $l$  odd

$$G_l^0(\eta_{lm}) = \int dk S_l(k) |\eta_{lm}^\sharp(k, \check{\mathbf{K}})|^2, \quad (42)$$

$$S_l(k) = -\frac{\pi^2}{\sinh(\frac{\pi}{2}k)}, \int_{-1}^1 dy P_l(y) \frac{\sinh(k \arcsin \frac{\alpha}{1+\alpha} y)}{\cos(\arcsin \frac{\alpha}{1+\alpha} y)}, \quad (43)$$

and the estimate (40) becomes

$$\Phi_2(\xi; \check{\mathbf{K}}) \geq \sum_{l \text{ odd}} \sum_{m=-l}^l \int dk S_l(k) |\eta_{lm}^\sharp(k, \check{\mathbf{K}})|^2. \quad (44)$$

### D. Bound from below

We notice that, for any fixed  $l$ ,  $S_l(k)$  is an even,  $C^\infty$  function of  $k$  and  $\lim_{k \rightarrow \infty} S_l(k) = 0$ . Furthermore for  $l$  odd we can show that  $S_l(k)$  is an increasing function of  $l$  for any fixed  $k$  (for details see [6, Lemma 3.5]). Then  $S_l(k) \geq S_1(k)$ . Moreover it is easy to see that

$$S_1(k) \geq S_1(0) = 4\pi \frac{1+\alpha}{\alpha} \left[ \frac{\sqrt{1+2\alpha}}{\alpha} \arcsin\left(\frac{\alpha}{1+\alpha}\right) - 1 \right] \quad (45)$$

where  $S_1(0) < 0$ . Therefore from (44) we have

$$\Phi_2(\xi; \check{\mathbf{K}}) \geq -|S_1(0)| \sum_{l=0}^{\infty} \sum_{m=-l}^l \int dk |\eta_{lm}^\sharp(k, \check{\mathbf{K}})|^2, \quad (46)$$

which can be rewritten in such a way to reconstruct the term  $\Phi_1$ . Indeed

$$\begin{aligned} \int dk |\eta_{lm}^\sharp(k, \check{\mathbf{K}})|^2 &= \int_{\mathbb{R}} dx e^{2x} |\eta_{lm}(e^x, \check{\mathbf{K}})|^2 \\ &\leq \frac{1+\alpha}{\sqrt{1+2\alpha}} \int_0^{\infty} dp p^2 \sqrt{\frac{1+2\alpha}{(1+\alpha)^2} p^2 + D(\check{\mathbf{K}})} |\eta_{lm}(p, \check{\mathbf{K}})|^2. \end{aligned}$$

Using this estimate in (46) we find

$$\begin{aligned} \Phi_2(\xi; \check{\mathbf{K}}) &\geq -|S_1(0)| \frac{1+\alpha}{\sqrt{1+2\alpha}} \times \\ &\quad \times \sum_{l=0}^{\infty} \sum_{m=-l}^l \int_0^{\infty} dp p^2 \sqrt{\frac{1+2\alpha}{(1+\alpha)^2} p^2 + D(\check{\mathbf{K}})} |\eta_{lm}(p, \check{\mathbf{K}})|^2 \\ &= -|S_1(0)| \frac{1+\alpha}{\sqrt{1+2\alpha}} \int d\mathbf{p} \sqrt{\frac{1+2\alpha}{(1+\alpha)^2} p^2 + D(\check{\mathbf{K}})} |\eta(\mathbf{p}, \check{\mathbf{K}})|^2 \\ &= -|S_1(0)| \frac{1+\alpha}{2\pi^2 \sqrt{1+2\alpha}} \Phi_1(\xi; \check{\mathbf{K}}) = -\frac{\Lambda(\alpha, N)}{(N-1)} \Phi_1(\xi; \check{\mathbf{K}}). \end{aligned}$$

We are now in position to conclude the proof of the Theorem. From (27), (28) and the inequality above, we get

$$(\psi, H\psi) \geq \frac{2N}{\pi} \Phi(\xi) \geq \frac{2N}{\pi} (1 - \Lambda(\alpha, N)) \int d\check{\mathbf{K}} \Phi_1(\xi; \check{\mathbf{K}}),$$

and taking  $\alpha \leq \alpha_c(N)$  we obtain the desired result  $(\psi, H\psi) \geq 0$ .

## IV. CONCLUSIONS

We have reported on a derivation of a sufficient condition on the mass ratio for the stability for the unitary

$N + 1$  fermionic model. Such a condition, which is optimal only in the two-particle case, is nevertheless non trivial for generic  $N$ . For instance it provides stability in the case of equal masses up to  $N = 8$ . We have also described the main steps of the proof, enlightening the points to be improved to get a more refined stability condition.

## V. APPENDIX

We first prove the asymptotic expression (16) for the potential  $\mathcal{G}Q$  for  $\mathbf{y}_i = \mathbf{x}_0 - \mathbf{x}_i \rightarrow 0$  with  $\mathbf{r}_{0i}$  fixed. Let us consider first the diagonal term in the sum (14) and change variables from  $(\mathbf{x}_0, \mathbf{x}_i)$  to  $(\mathbf{y}_i, \mathbf{r}_{0i})$ :

$$\int d\mathbf{P} e^{i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i} \exp \left\{ i\mathbf{p}_0 \cdot \left( \mathbf{r}_{0i} + \frac{\alpha}{1+\alpha} \mathbf{y}_i \right) \right\} \times \\ \times \exp \left\{ i\mathbf{p}_i \cdot \left( \mathbf{r}_{0i} - \frac{1}{1+\alpha} \mathbf{y}_i \right) \right\} \frac{\hat{Q}(\mathbf{p}_0 + \mathbf{p}_i, \check{\mathbf{p}}_i)}{h_0(\mathbf{P})} = \\ \int d\check{\mathbf{p}}_i e^{i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i} \int d\mathbf{q}_i e^{i\mathbf{q}_i \cdot \mathbf{r}_{0i}} \hat{Q}(\mathbf{q}_i, \check{\mathbf{p}}_i) \int d\mathbf{v}_i \frac{e^{i\mathbf{v}_i \cdot \mathbf{y}_i}}{h_0(\mathbf{q}_i, \mathbf{v}_i, \check{\mathbf{p}}_i)},$$

where we have changed variables to

$$\mathbf{q}_i = \mathbf{p}_0 + \mathbf{p}_i, \quad \mathbf{v}_i = \frac{\alpha}{1+\alpha} \mathbf{p}_0 - \frac{1}{1+\alpha} \mathbf{p}_i, \quad (47)$$

and in the new variables

$$h_0(\mathbf{q}_i, \mathbf{v}_i, \check{\mathbf{p}}_i) = \frac{\mathbf{q}_i^2}{2(m_0+m_1)} + \frac{\mathbf{v}_i^2}{2\mu} + \frac{\check{\mathbf{p}}_i^2}{2m_1}. \quad (48)$$

The last integral can be computed explicitly using

$$\frac{1}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{y}} \frac{1}{\mathbf{k}^2 + \lambda} = \frac{e^{-\sqrt{\lambda}|\mathbf{y}|}}{4\pi|\mathbf{y}|}, \quad (49)$$

so that the diagonal term in (14) becomes

$$\frac{(-1)^{i+1}}{(2\pi)^{\frac{3}{2}N}} \int d\check{\mathbf{p}}_i e^{i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i} \int d\mathbf{q}_i e^{i\mathbf{q}_i \cdot \mathbf{r}_{0i}} \frac{\hat{Q}(\mathbf{q}_i, \check{\mathbf{p}}_i)}{4\pi|\mathbf{y}_i|} \times \\ \times \exp \left\{ -\sqrt{\frac{\alpha}{(1+\alpha)^2} \mathbf{q}_i^2 + \frac{1}{1+\alpha} \check{\mathbf{p}}_i^2} |\mathbf{y}_i| \right\}. \quad (50)$$

Expanding the exponential around  $\mathbf{y}_i = 0$ , we obtain

$$\frac{(-1)^{i+1}}{(2\pi)^{\frac{3}{2}N}} \int d\check{\mathbf{p}}_i e^{i\check{\mathbf{p}}_i \cdot \check{\mathbf{x}}_i} \int d\mathbf{q}_i e^{i\mathbf{q}_i \cdot \mathbf{r}_{0i}} \hat{Q}(\mathbf{q}_i, \check{\mathbf{p}}_i) \times \\ \times \left( \frac{1}{|\mathbf{y}_i|} - \sqrt{\frac{\alpha}{(1+\alpha)^2} \mathbf{q}_i^2 + \frac{1}{1+\alpha} \check{\mathbf{p}}_i^2} \right) + o(1). \quad (51)$$

The first term in the expression above reproduces the singular contribution  $\frac{(-1)^{i+1}}{|\mathbf{y}_i|} Q(\mathbf{r}_{0i}, \check{\mathbf{x}}_i)$  and we set the second term equal to the diagonal part of  $\Gamma_i Q$ . On the other hand the off-diagonal terms with  $j \neq i$  in (14) are finite and can be simply evaluated at  $\mathbf{y}_i = 0$ , so providing the off-diagonal part of  $\Gamma_i Q$ .

Let us verify that the potential  $\mathcal{G}Q$  satisfies equation (18). From the definition (14) and the expression of  $H_0$

$$(H_0 \mathcal{G}Q)(\mathbf{X}) \\ = \frac{1}{\mu} \sum_{j=1}^N \frac{(-1)^{j+1}}{(2\pi)^{\frac{3}{2}N+2}} \int d\mathbf{P} e^{i\mathbf{P} \cdot \mathbf{X}} \hat{Q}(\mathbf{p}_0 + \mathbf{p}_j, \check{\mathbf{p}}_j) \\ = 4\pi \sum_i (-1)^{i+1} Q(\mathbf{r}_{0i}, \check{\mathbf{x}}_i) \delta(\mathbf{x}_i - \mathbf{x}_0). \quad (52)$$

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