Sharp large time behaviour in N-dimensional Fisher-KPP equations

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Dedicated to L. Caffarelli, as a sign of friendship, admiration and respect.

Abstract

We study the large time behaviour of the Fisher-KPP equation $\partial_t u = \Delta u + u - u^2$ in spatial dimension N, when the initial datum is compactly supported. We prove the existence of a Lipschitz function s^∞ of the unit sphere, such that $u(t,x)$ converges, as t goes to infinity, to $U_{c_*}\left(|x| - c_* t + \frac{N+2}{c_*}\ln t + s^\infty\left(\frac{x}{|x|}\right)\right)$, where U_{c*} is the 1D travelling front with minimal speed $c_* = 2$. This extends an earlier result of Gärtner.

1 Introduction

The paper is devoted to the large time behaviour of the solution of the reaction-diffusion equation

$$\partial_t u = \Delta u + f(u), \quad t > 0\,,\ x \in \mathbb{R}^N \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^N \tag{2}$$

We will take

$$f(u) = u(1-u);$$

thus f is, in reference to the pioneering paper [19], said to be of the Fisher-KPP type. The initial datum u_0 is smooth and there exist $0 < R_1 < R_2$ such that

$$\forall x \in \mathbb{R}^N\,, \quad \mathbf{1}_{B_{R_1}}(x) \le u_0(x) \le \mathbf{1}_{B_{R_2}}(x), \tag{3}$$

where $\mathbf{1}_A$ is the indicator of the set A and B_R is the ball of $\mathbb{R}^N$ of radius R centered at the origin. By the maximum principle and the standard theory of parabolic equations (see for instance [17]), equation (1) has a unique classical solution $u(t,x)$ in $\mathcal{C}^\infty([0,+\infty[\times\mathbb{R}^N,[0,1])$ emanating from u_0. The first and most general result is due to Aronson and Weinberger [1]. The solution u spreads at the speed $c^* = 2\sqrt{f'(0)} = 2$ in the sense that

$$\min_{|x|\le ct} u(t,x) \to 1 \text{ as } t \to +\infty\,, \text{ for all } 0 \le c < c^*$$

and

$$\sup_{|x|\ge ct} u(t,x) \to 0 \text{ as } t \to +\infty\,, \text{ for all } c > c^*.$$

The goal of this paper is to sharpen this result.

Let us briefly recall what happens in the case $N = 1$. Equation (1) with $N = 1$ reads

$$\partial_t u = \partial_{xx} u + f(u), \quad t > 0\,,\, x \in \mathbb{R}. \tag{4}$$

It admits one-dimensional travelling fronts $U(x-ct)$ if and only if $c \ge c^* = 2$ where the profile U, depending on c, satisfies

$$U'' + c\,U' + f(U) = 0, \quad x \in \mathbb{R}, \tag{5}$$

together with the conditions at infinity

$$\lim_{x\to-\infty} U(x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} U(x) = 0. \tag{6}$$

Any solution U to (5)-(6) is a shift of a fixed profile U_c: $U(x) = U_c(x+s)$ with some fixed $s \in \mathbb{R}$. The profile U_{c^*} at minimal speed $c^* = 2$ satisfies, up to translation,

$$U_{c^*}(x) = (x+K)\,e^{-x} + O(e^{-(1+\gamma_0)x})\,, \text{ as } x \to +\infty$$

for some universal constants $K \in \mathbb{R}$ and $\gamma_0 > 0$. The large time behaviour of (4) has a history of important contributions, we only list two lasting ones. The first is the paper of Kolmogorov, Petrovskii and Piskunov [19]. They proved that the solution of (4) starting from the initial datum $\mathbf{1}_{(-\infty,0]}$ converges to U_{c*} in shape: there is a function

$$\sigma^\infty(t) = 2t + o_{t\to+\infty}(t),$$

such that

$$\lim_{t\to+\infty} u(t, x+\sigma^\infty(t)) = U_{c^*}(x) \quad \text{uniformly in } x \in \mathbb{R}.$$

The second contribution makes precise the $\sigma^\infty(t)$: in [5], Bramson proves the existence of a constant x_∞, depending on u_0, such that

$$\sigma^\infty(t) = 2t - \frac{3}{2}\ln t - x_\infty + o_{t\to+\infty}(1). \tag{7}$$

Formula (7) was proved through elaborate probabilistic arguments. As said before, the problem, as well as more complex variants of it, are currently the subject of intense investigations. See for instance [20] for an account of them.

In several space dimensions, the asymptotics have been pushed less far. In the framework of the Fisher-KPP equation that we are studying, the Aronson-Weinberger result is made precise up to $O(1)$ terms in Gärtner [12]. If N is the space dimension, the main result of [12] is that, for every $\lambda \in (0,1)$, the level set $\{u = \lambda\}$ is trapped, for large times, between two spheres of radius

$$R(t) = c^*t - \frac{N+2}{c^*}\ln t + O_{t\to+\infty}(1).$$

The $O_{t\to+\infty}(1)$ terms are not studied. It is shown by the second author in [25] that one cannot get rid of these terms, in the sense that generally the difference between the radii of the spheres does not tend to zero as $t \to +\infty$.

Gärtner's contribution is probabilistic, and a PDE proof of his result is provided by Ducrot [8], adapting to higher dimension the proof of (a weaker version of) Bramson's formula (7), given by F. Hamel, J. Nolen, L. Ryzhik and the first author in [15].

When the coefficients of the equation actually depend on x in a periodic fashion, as for instance for the equation

$$\partial_t u = \Delta u + \mu(x)u - u^2, \quad t > 0,\ x \in \mathbb{R}^N,$$

with μ periodic and positive (actually, more general assumptions on μ can be allowed, as well as inhomogeneous diffusion terms, or the presence of advection), a lot is now known on the spreading speed, or, in other words, the position of the level sets up to $O_{t\to+\infty}(1)$ terms. The first result in this direction is Freidlin-Gärtner [13], which gives, through a probabilistic approach, an almost explicit expression (the Freidlin-Gärtner formula) of the spreading speed in each direction. Several proofs and generalisations of this formula have been given, by various approaches: viscosity solutions [10], abstract dynamical systems [28], PDE approach [2], [24]. Let us mention an important contribution [27], which generalises Gärtner's result to periodic functions $\mu(x)$, by computing the relevant logarithmic shift. This work also generalises [16], a contribution that computes the shift for periodic μ, but in one space dimension.

Coming back to (1), the goal of the present paper is to show that it is actually possible to make precise the $O_{t\to+\infty}(1)$ in Gärtner's expansion in terms of a function s^∞ depending on the spherical variable. Our result is the

thm 1.1 *Let u_0 satisfy assumption (3). There is a Lipschitz function s^∞, defined on the unit sphere of $\mathbb{R}^N$, such that the solution u of (1) emanating from u_0 satisfies*

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^N}\left|u(t,x) - U_{c_*}\left(|x| - c_*t + \frac{N+2}{c_*}\ln t + s^\infty\Big(\frac{x}{|x|}\Big)\right)\right| = 0$$

with $c_* = 2$.

This completes the result of [12]. At this stage, let us anticipate the proof of the theorem, and let us give a brief explanation of the logarithmic shift observed here: it can be decomposed into two shifts having different origins. One is due to the curvature term $\frac{N-1}{c_*}\ln t$, it systematically arises in this type of large time issues for reaction-diffusion equations, the nonlinearity f does not need to be of the KPP type. See for instance [26], [29]. The other is the one-dimensional shift $\frac{3}{c_*}\ln t$, it is typical of the KPP nonlinearity. All this will be made clearer in Section 2.

Theorem 1.1 is in contrast with a recent paper [23] of the first and third authors, which studies (1) when the initial datum is trapped between two planar travelling waves. In this setting, the logarithmic shift is $\frac{3}{c^*}\ln t$, as in the one-dimensional case. However, the dynamics beyond the logarithmic shift is given by that of the heat equation on the whole line. This last equation, though extremely well-behaved as far as the regularity of its solutions is concerned, exhibits solutions that do not converge, as time goes to infinity, to anything. However, this last feature holds for reaction-diffusion that need not be of the KPP type, see [22].

Before starting the proof of our results, let us make a few remarks. The first one concerns the assumption (3), which does not encompass, strictly speaking, all compactly supported initial data. For a general (nontrivial) compactly supported initial datum, there exist $K > 1$ such that

$$\forall x \in \mathbb{R}^N, \quad \frac{1}{K}\mathbf{1}_{B_{R_1}}(x) \le u_0(x) \le K\mathbf{1}_{B_{R_2}}(x).$$

The left inequality is in fact inconsequential, the whole paper would hold under this assumption without any modification. The right inequality would not alter our conclusions, either; to obtain the compactness of the solutions in the area $|x| \sim c_*t - \dfrac{N+2}{2}\ln t$ one should simply work with the nonlinearity $\overline{f}(u) = Ku - u^2$, which is obviously larger than $u - u^2$.

Let us also mention that it would be certainly interesting to understand sharper asymptotics of $u(t,x)$. In one space dimension, a full expansion has been proposed, in the formal style, in [9], or with another approach in [4]. The next term in the expansion of the shift is computed, in a mathematically rigorous way, in [21]. The expansion is pushed even further in [14].

Let us finally say that the observed behaviour is quite typical of Fisher-KPP equations with second-order linear diffusion. Another important class of nonlinearities $f(u)$ in (1) satisfies $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, with $\int_0^1 f(u)du > 0$. A typical example is

$$f(u) = u(u-\theta)(1-u), \quad 0 < \theta < \frac{1}{2}.$$

In such case, a statement of the same type as Theorem 1.1 is contained in [26], with the important difference that the logarithmic delay is solely due to the curvature terms; the dynamics beyond the shift is the same as the one presented in Theorem 1.1. And, although the phenomenon does not look so remote to the one displayed in [26], it is quite different in nature, as the convergence to the wave is dictated by what happens in the region where the solution takes intermediate values. A similar, and recent contribution [7] treats the porous medium equation with Fisher-KPP nonlinearity; although the nonlinearity is the same as in the present paper, the result is of the type of [26] (although the dynamics beyond the shift is not made precise when the initial datum is nonradial), this is due to the fact that the solution does not have a tail that would govern the overall dynamics. We end this series of remarks by recalling a result of Jones [18], stating that the level sets of the solution of (1), whatever the nonlinearity is, will have oscillations only of the size $O_{t\to+\infty}(1)$. This is a consequence of the following fact: if λ is a regular value of u, the normal to the λ-level set of u meets the convex hull of the support of the initial datum. A very simple proof of this fact is given by Berestycki in [3].

In the next section, we transform the equations so as to uncover the basic mechanism at work, namely the fact that the whole phenomenon is dictated by the tail of the solution. The subsequent sections are different steps of the proof of Theorem 1.1, this will be explained in more detail in Section 2.

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2 Preparation of the equations, strategy of the proof, plan of the paper

There is a sequence of transformations that bring equation (1) to a form that will make clear that the region $|x| \sim \sqrt{t}$ in the moving frame, that we will subsequently call the diffusive zone, dictates the whole dynamics.

From now on, we take $t = 1$ as initial time and (2) is replaced by $u(1, x) = u_0(x)$. This will be handier in view of the following transformations and, since equation (1) is invariant by translation in time, there is no loss of generality.

1. We first use the polar coordinates

$$x \mapsto (r = |x| > 0, \Theta = \frac{x}{|x|} \in \mathbb{S}^{N-1})$$

   then (1) becomes

$$\partial_t u = \partial_{rr} u + \frac{N-1}{r}\partial_r u + \frac{\Delta_\Theta u}{r^2} + u - u^2, \quad t > 1,\ r > 0,\ \Theta \in \mathbb{S}^{N-1}.$$

   Here, Δ_Θ is the Laplace-Beltrami operator on the unit sphere of $\mathbb{R}^N$. Its precise expression will not be needed in the sequel. The initial condition reads $u(1, r, \Theta) = u_0(r, \Theta)$.

2. Let us believe that the transition zone where u is neither close to 1 nor 0 is located around $R(t) = 2t - k \ln t$ (k to be chosen later) and choose the change of variables $r' = r - R(t)$ and $u(t, r, \Theta) = u_1(t, r - R(t), \Theta)$. We drop the primes and indexes, and (1) becomes

$$\partial_t u = \partial_{rr} u + \frac{N-1}{r + 2t - k\text{ln}t}\partial_r u + (2 - \frac{k}{t})\partial_r u + \frac{\Delta_\Theta u}{(r + 2t - k\text{ln}t)^2} + u - u^2. \tag{8}$$

   The equation is valid for $t > 1$, $r > -2t + k\text{ln}t$, and $\Theta \in \mathbb{S}^{N-1}$ and the initial condition becomes $u(1, r, \Theta) = u_0(r + 2, \Theta)$.

3. To unreveal the mechanisms at work in the tail of the solution, we take out the exponential decay of the wave U_{c^*}, and set $u(t, r, \Theta) = e^{-r}v(t, r, \Theta)$; the equation transforms into

$$\partial_t v = \partial_{rr} v + (\frac{N-1}{r + 2t - k\text{ln}t} - \frac{k}{t})(\partial_r v - v) + \frac{\Delta_\Theta v}{(r + 2t - k\text{ln}t)^2} - e^{-r}v^2, \tag{9}$$

   with $t > 1$, $r > -2t + k \ln t$, $\Theta \in \mathbb{S}^{N-1}$ and initial datum $v(1, r, \Theta) = e^r u_0(r + 2, \Theta)$.

4. We now choose k. Our first guess is that the term in $\Delta_\Theta v$ will not matter too much, because it decays like t^{-2} (an integrable power of t), except in the zone $r \sim -2t$, where we know (for instance [1]) that $u(t,r,\Theta)$ goes to 1 as $t \to +\infty$. Hence we expect the dynamics to be like that of the one-dimensional equation. On the other hand, in the advection term, the quantity $\dfrac{N-1}{r+2t-k\ln t}$ is nonintegrable in t, except for extremely large r. Thus we wish to balance it with the $\dfrac{k}{t}$ term. However, instructed by the large time behaviour in one space dimension, we keep in mind that we should keep the quantity $-\dfrac{3}{2t}$ factoring $\partial_r v - v$. Hence we choose

$$\frac{N-1}{2} - k = -\frac{3}{2}, \tag{10}$$

hence

$$k = \frac{N+2}{2} = \frac{N+2}{c_*}.$$

In the sequel, we will keep the notation k, keeping in mind that k is given by the above formula.

5. In order to study (9) in the diffusive zone, that is, the region $r \sim \sqrt{t}$, we use the self-similar variables $\xi = \dfrac{r}{\sqrt{t}}$, $\tau = \ln t$. The variable Θ is unchanged:

$$\hat{w}(\tau,\xi,\Theta) = \hat{w}\left(\ln t, \frac{r}{\sqrt{t}}, \Theta\right) = \frac{1}{\sqrt{t}} v(t,r,\Theta). \tag{11}$$

Then (9) becomes

$$\partial_\tau \hat{w} + \mathcal{L}\hat{w} = \frac{e^\tau \Delta_\Theta \hat{w}}{(2e^\tau + \xi e^{\tau/2} - k\tau)^2} + h(\tau,\xi) e^{-\frac{\tau}{2}} \partial_\xi \hat{w} - \left(h(\tau,\xi) + \frac{3}{2}\right)\hat{w} - e^{\frac{3}{2}\tau - \xi e^{\frac{\tau}{2}}} \hat{w}^2, \tag{12}$$

where

$$\mathcal{L}w = -\partial_{\xi\xi} w - \frac{\xi}{2}\partial_\xi w - w,$$

and

$$h(\tau,\xi) = \frac{N-1}{2 + \xi e^{-\tau/2} - k\tau e^{-\tau}} - k.$$

Equation (12) is valid for $\tau > 0$, $\xi > -2e^{\frac{\tau}{2}} + k\tau e^{-\frac{\tau}{2}}$ and $\Theta \in \mathbb{S}^{N-1}$. The lower bound on ξ is a very negative quantity if τ is very large. As the range of negative ξ that are relevant will turn out to be extremely modest (we will always have $\xi \geq -e^{-(\frac{1}{2}-\delta)\tau}$, that is, $r \geq -t^\delta$ for some $\delta \in (0, \frac{1}{4})$), we will not mention this constraint on ξ in the sequel. Finally, the initial datum at $\tau = 0$ is

$$\hat{w}_0(\xi,\Theta) = e^\xi u_0(\xi + 2, \Theta),$$

therefore still compactly supported. Since $u \in (0,1)$, we also have the upper bound

$$\hat{w}(\tau,\xi,\Theta) \leq \exp\left(\xi e^{\tau/2} - \frac{\tau}{2}\right). \tag{13}$$

Since $k = \frac{N-1}{2} + \frac{3}{2}$, we have for all $\delta \in (0, \frac{1}{2})$:

$$h(\tau,\xi) = \begin{cases} -\dfrac{3}{2} + O(\xi e^{-\tau/2}) \text{ for } \xi \leq e^{(1/2-\delta)\tau}, \text{ that is, } r \leq t^{1-\delta}, \\ O(1) \text{ for } \xi \geq e^{(1/2-\delta)\tau}, \text{ that is, } r \geq t^{1-\delta}. \end{cases}$$

The information on h that we are going to retain is however the followig weaker version for $\delta \in (0, \frac{1}{4})$:

$$h(\tau,\xi) = \begin{cases} -\dfrac{3}{2} + O(e^{-(\frac{1}{2}-\delta)\tau}) \text{ for } \xi \leq e^{\delta\tau}, \text{ that is, } r \leq t^{1/2+\delta}, \\ O(1) \text{ for } \xi \geq e^{\delta\tau}, \text{ that is, } r \geq t^{1/2+\delta}. \end{cases}$$

6. To construct sub and super solutions, we will need to translate the solution $\hat{w}$. So let us set

$$\xi_\delta^\pm(\tau) = \pm e^{-(\frac{1}{2}-\delta)\tau},$$

we will often use the notation $\xi_\delta^\pm$ and not mention the dependence in τ, as things will - hopefully - be clear from the context. The constant $\delta > 0$ will be suitably small and, in any case, less that 1/4. The point $\xi = \xi_\delta^+(\tau)$ corresponds, in the (t, r, Θ) variables, to $r = t^\delta$ in the moving frame, that is, far ahead of the supposed location of the front $(r = O(1))$, but not quite as far as the diffusive zone $(r \sim \sqrt{t})$. The point $\xi = \xi_\delta^-(\tau)$ therefore corresponds to $r = -t^\delta$, that is, far at the back of the front location, but, again, not quite as far as $-\sqrt{t}$.

In order to consider at once the different zones involved, we let $\xi_\delta(\tau)$ denote one of the following three functions:

$$\xi_\delta^-(\tau), \quad \xi_\delta^+(\tau), \quad 0,$$

and define the translations

$$\hat{w}(\tau,\xi) = \tilde{w}(\tau, \xi - \xi_\delta(\tau)), \tag{14}$$

Equation (12) transforms into the following three equations (depending on the translation we made):

$$\begin{aligned} \partial_\tau \tilde{w} + \mathcal{L}\tilde{w} =& \big(\delta\xi_\delta + h(\tau, \xi+\xi_\delta)e^{-\frac{\tau}{2}}\big)\partial_\xi \tilde{w} - \left(h(\tau,\xi+\xi_\delta) + \frac{3}{2}\right)\tilde{w} \\ &+ \frac{\Delta_\Theta \tilde{w}}{(2e^{\frac{\tau}{2}} + \xi + \xi_\delta - k\tau e^{-\tau/2})^2} - e^{\frac{3}{2}\tau - (\xi+\xi_\delta)e^{\frac{\tau}{2}}}\tilde{w}^2, \end{aligned} \tag{15}$$

which are valid for $\tau > 0$, $\xi > -\xi_\delta - 2e^{\frac{\tau}{2}} + k\tau e^{-\frac{\tau}{2}}$ and $\Theta \in \mathbb{S}^{N-1}$. As before, the lower bound for ξ is in all cases very negative when τ is large, thus negligible. The initial datum at $\tau = 0$ is

$$\tilde{w}_0(\xi,\Theta) = e^{\xi+\xi_\delta(0)}u_0(\xi+\xi_\delta(0)+2,\Theta),$$

still compactly supported.

7. The last transformation turns $\mathcal{L}$ into the self-adjoint operator

$$\mathcal{M}w = -\partial_{\xi\xi}w + (\frac{\xi^2}{16} - \frac{3}{4})w.$$

This amounts to setting

$$\tilde{w}(\tau,\xi,\Theta) = e^{-\frac{\xi^2}{8}} w(\tau,\xi,\Theta). \tag{16}$$

The equation for w is

$$\partial_\tau w + \mathcal{M}w = l_1(\tau,\xi)\partial_\xi w + l_2(\tau,\xi)w + \frac{\Delta_\Theta w}{\left(\xi+\xi_\delta+2e^{\frac{\tau}{2}}-k\tau e^{-\frac{\tau}{2}}\right)^2} - e^{\frac{3\tau}{2}-\frac{\xi^2}{8}-(\xi+\xi_\delta)e^{\frac{\tau}{2}}}w^2 \tag{17}$$

which is valid for $\tau > 0$, $\xi > -\xi_\delta - 2e^{\frac{\tau}{2}} + k\tau e^{-\frac{\tau}{2}}$ (as usual very negative for τ large) and $\Theta \in \mathbb{S}^{N-1}$. The initial datum at $\tau = 0$ is

$$w_0(\xi,\Theta) = e^{\xi+\xi_\delta(0)+\frac{\xi^2}{8}} u_0(\xi+\xi_\delta(0)+2,\Theta), \tag{18}$$

still compactly supported. The functions l_1 and l_2 depend on h and are given by

$$l_1(\tau,\xi) = \delta\xi_\delta + h(\tau,\xi+\xi_\delta)e^{-\tau/2}, \qquad l_2(\tau,\xi) = -\frac{3}{2} - h(\tau,\xi+\xi_\delta) - \frac{\xi}{4}l_1(\tau,\xi).$$

They satisfy, for $\xi \geq 0$ and whatever ξ_δ is, the following estimates:

$$\begin{aligned} |l_1(\tau,\xi)| \leq&\ Ce^{-(\frac{1}{2}-\delta)\tau} \\ |l_2(\tau,\xi)| \leq&\ C\Big(\xi e^{-(\frac{1}{2}-\delta)\tau} + \mathbf{1}_{\xi+\xi_\delta\geq e^{\delta\tau}} + e^{-(\frac{1}{2}-\delta)\tau}\mathbf{1}_{\xi+\xi_\delta\leq e^{\delta\tau}}\Big), \end{aligned} \tag{19}$$

the constant C only depending on N and δ.

Equipped with all these transformations, we are now able to explain the core of the proof of Theorem 1.1. It is inspired by the ideas of [20] in one space dimension, with some novelties due to the transverse variable. Our main step will be to prove the

thm 2.1 *Let $\hat{w}$ be the solution of (12) with compactly supported initial datum $\hat{w}_0$. There exists a positive Lipschitz function α^∞ on the unit sphere such that*

$$\lim_{\tau\to+\infty} \sup_{\xi\in\mathbb{R}^+,\Theta\in\mathbb{S}^{N-1}} e^{\frac{3\xi^2}{16}}|\hat{w}(\tau,\xi,\Theta) - \alpha^\infty(\Theta)\phi_0(\xi)| = 0,$$

where $\phi_0(\xi) = \xi e^{-\xi^2/4}$ satisfies $\mathcal{L}\phi_0 = 0$.

The function α^∞ is possibly more regular than Lipschitz. Proving some additional regularity would entail nontrivial additional technicalities, we will explain this when it comes to studying the regularity in Θ.

The parallel step in [20] for $N = 1$ was to prove, for the equation

$$\partial_\tau\hat{w} + \mathcal{L}\hat{w} = -\frac{3}{2}e^{-\frac{\tau}{2}}\partial_\xi\hat{w} - e^{\frac{3}{2}\tau-\xi e^{\frac{\tau}{2}}}\hat{w}^2, \quad \tau > 0, \quad \xi\in\mathbb{R},$$

the existence of a constant $\alpha^\infty > 0$ such that

$$\hat{w}(\tau,\xi) \xrightarrow[\tau\to+\infty]{} \alpha^\infty\xi e^{-\xi^2/4}, \quad \text{in } \{\xi \geq e^{-(\frac{1}{2}-\delta)\tau}\}.$$

The main effort was to prove the compactness of the trajectories $(\hat{w}(\tau+T,\xi))_{T>0}$ as $T\to+\infty$; because the limiting trajectories satisfied the Dirichlet heat equation in self-similar variables, this entailed the convergence to a single Gaussian. To prove the compactness, we used a pair of sub/super solutions very much in the spirit of Fife-McLeod [11]; that one could actually use ideas from the analysis of bistable equations came as a surprise to us.

However, the barriers devised in [20] rely on the good sign of the disturbances (that is, the exponential correction in the function h) which allowed them to be sub and super solutions all the way down to $\xi = 0$. Because we are now dealing with a more complex equation, we can no longer rely on sign considerations, and we devise a pair of barriers that are sub and super solutions for more robust reasons than in [20]. While still being radial, these barriers rely on a technical innovation in the vicinity of $\xi = 0$, that is, if one thinks very much about the Fife-McLeod sub/super solutions, quite in the spirit of [11] once again.

Once this is achieved, an additional issue will be to deal with the variable Θ: as $\tau \to +\infty$, the Laplace-Beltrami operator will disappear from the asymptotic equations. That is, asymptotic regularity in Θ will have to be retrieved with bare hands.

Once convergence in the diffusive area is under control, the next step is to fix the translation $\sigma^\infty(t,\Theta)$. We choose it such that

$$U_{c_*}(r+\sigma^\infty(t,\Theta))\Big|_{r=t^\delta} = e^{-r}v(t,r,\Theta)\Big|_{r=t^\delta}.$$

That is,

$$\sigma^\infty(t,\Theta) = -\ln\alpha^\infty(\Theta) + O(t^{-\delta}).$$

We then prove the uniform convergence to $U_{c_*}(r-\ln\alpha^\infty(\Theta))$ by examining the difference

$$\tilde{v}(t,r,\Theta) = \left|v(t,r,\Theta) - e^r U_{c_*}(r+\sigma^\infty(t,\Theta))\right|$$

in the region $\{r < t^\delta\}$. For $N = 1$, it turned out in [20] that $\tilde{v}(t,x)$ was a sub-solution of (a perturbation of) the heat equation

$$\begin{aligned} V_t &= V_{xx} + O(t^{1-\delta}), && t>0,\ -t^\delta < x < t^\delta \\ V(t,-t^\delta) &= e^{-t^\delta}, && t>0 \\ V(t,t^\delta) &= 0, && t>0. \end{aligned} \qquad (20)$$

The condition at $x = -t^\delta$ simply comes from the fact that $v(t,x)$ decays, by definition, like e^x at $-\infty$. Although the domain might look very large, its first Dirichlet eigenvalue is of the order $t^{-2\delta}$, hence a much larger quantity than the right hand side of (20). Thus $V(t,x)$ could be proved to go to 0 uniformly in x as $t \to +\infty$, which implied the sought for convergence result. The same idea will work here again, up to the caveat that α^∞ is only Lipschitz in Θ, something that does not go very well with taking a Laplace-Beltrami operator. A simple regularisation argument will settle the issue.

Our experience with working with multi-dimensional reaction-diffusion equations is that the main additional difficulty is the transverse diffusion, which, in a very paradoxical way, does not help. This is not a rhetorical argument: its presence is really what prevented convergence in the earlier paper [23]. This explains why we have to be extra careful with the estimates.

The plan of the rest of the paper is the following: in Section 3, we construct the announced radial barriers for the solutions of (17) with $\xi_\delta = 0$ that are initially compactly supported. In Section 4, we successively prove Theorem 2.1, then Theorem 1.1. The paper ends with a discussion.

3 Radial barriers

As announced in the previous sections, we wish to construct a radial sub-solution in the region $\{\xi \geq \xi_\delta^+(\tau)\}$ (that is, r starting far ahead of the front) and a radial super-solution in the region $\{\xi \geq \xi_\delta^-(\tau)\}$, (that is, r starting far at the back of the front). From now on we drop the Θ variable; in the (τ, ξ) variables, the two end points $\xi_\delta^\pm(\tau)$ will rejoin at $\xi = 0$ as $\tau \to +\infty$: this will provide an estimate of the solution in the self-similar variables at $\xi \sim 0$, whereas the main body of the sub and super solutions will estimate w in the diffusive zone. For radial functions, equations (17) reduce to

$$\partial_\tau w + \mathcal{M}w = l_1(\tau,\xi)\partial_\xi w + l_2(\tau,\xi)w - e^{\frac{3\tau}{2}-\frac{\xi^2}{8}-(\xi+\xi_\delta)e^{\frac{\tau}{2}}}w^2 \tag{21}$$

This part is the most technical of the paper, we will try to keep the computations as light as possible. Many of them are in the spirit of those of [20] or [23]. Let us introduce some auxiliary quantities.

First, let $\phi_0(\xi) = \xi e^{-\xi^2/4}$, it solves

$$\mathcal{L}\phi = 0 \ (\xi > 0), \quad \phi(0) = \phi(+\infty) = 0. \tag{22}$$

Any solution of (22) is a multiple of ϕ_0. And its counterpart with respect to transformation (16) is $\varphi_0(\xi) = \xi e^{-\frac{\xi^2}{8}}$ which satisfies $\mathcal{M}\varphi_0 = 0$, $\varphi_0(0) = \varphi_0(+\infty) = 0$.

For $a > 0$, we call

$$\lambda_1(a) = \frac{\pi^2}{4a^2}, \quad \phi_{1,a}(\xi) = \cos\Big(\frac{\pi}{2a}\xi\Big).$$

Namely, $\lambda_1(a)$ is the first Dirichlet eigenvalue of $-\Delta$ on $(-a,a) \subset \mathbb{R}$ and $\phi_{1,a}$ is the associated eigenfunction with maximum equal to 1. We choose $a_0 \in (0,1)$ small enough to have $\lambda_1(a_0) \geq 100$. It will be suitably decreased in the sequel, independently of all other coefficients and variables. We set

$$\lambda_1 = \lambda_1(a_0), \quad \phi_1(\xi) = \phi_{1,a_0}(\xi).$$

Moreover, let $\gamma_1(\xi)$ be a nonnegative smooth function, equal to 1 if $\xi \leq \frac{a_0}{2}$, and zero if $\xi \geq \frac{2}{3}a_0$, and $\gamma_2(\xi)$ be a nonnegative smooth function, equal to zero if $\xi \leq 1$ and to 1 if $\xi \geq 2$.

We are now in a position to construct the sought for super and sub-solutions for (21):

Proposition 3.1 *There exist six functions $q_1^\pm(\tau)$, $q_2^\pm(\tau)$, $\zeta^\pm(\tau)$ which satisfy:*

-
$$\frac{e^{-\tau}}{C} \leq q_i^\pm(\tau) \leq Ce^{-(\frac{1}{2}-\delta)\tau}, \tag{23}$$
for some $C \geq 1$,
- *there are $\overline{\zeta}_0 \geq \underline{\zeta}_0 > 0$ such that, for all $\tau \geq 0$, $\zeta^\pm(\tau) \in [\underline{\zeta}_0, \overline{\zeta}_0]$ and $\dot{\zeta}^+(\tau) > 0$, $\dot{\zeta}^-(\tau) < 0$.*

In addition, the function

$$\overline{w}(\tau,\xi) := \zeta^+(\tau)\varphi_0(\xi) + q_1^+(\tau)\phi_1(\xi)\gamma_1(\xi) + q_2^+(\tau)\gamma_2(\xi)e^{-\xi^2/16} \tag{24}$$

is a super-solution to (21) with $\xi_\delta = \xi_\delta^-$ in the range $\tau > 0$, $\xi > 0$, whereas

$$\underline{w}(\tau,\xi) := \zeta^-(\tau)\varphi_0(\xi) - q_1^-(\tau)\phi_1(\xi)\gamma_1(\xi) - q_2^-(\tau)\gamma_2(\xi)e^{-\xi^2/16} \tag{25}$$

is a sub-solution to (21) with $\xi_\delta = \xi_\delta^+$ in the range $\tau \geq \tau_1$, $\xi > 0$, for some $\tau_1 > 0$.

Before proving this proposition, we need to state two ODE statements.

3.1 Two elementary ODE statements

Consider the system

$$\begin{cases} \dot{q} + q = & C_0(\zeta + \gamma)e^{-(\frac{1}{2}-\delta)\tau} \\ \dot{\zeta} = & C_0(q + \zeta e^{-(\frac{1}{2}-\delta)\tau}) \end{cases} \tag{26}$$

By elementary Cauchy theory, this system has a unique solution for any given pair of initial data.

Proposition 3.2 *Let* (q, ζ) *be the solution to* (26) *with parameters* $C_0 > 0$, $\gamma \in [0, 1]$ *and initial conditions*

$$q(0) = q_0 > 0, \qquad \zeta(0) = \zeta_0 > 0$$

satisfying, for some $h_0 > 0$,

$$\frac{\zeta_0 + \gamma}{q_0} \le h_0, \qquad \frac{q_0}{\zeta_0} \le h_0. \tag{27}$$

Then, there is $K > 0$, *only depending on* C_0 *and* h_0, *such that*

$$\forall \tau \ge 0\,, \quad 0 < \zeta_0 \le \zeta(\tau) \le K\zeta_0, \qquad q_0 e^{-\tau} \le q(\tau) \le Kq_0 e^{-(\frac{1}{2}-\delta)\tau}. \tag{28}$$

Proof. From the first equation in (26) we see that the function $e^\tau q$ is increasing as long as ζ remains nonnegative. Then, by the second equation, this holds true for ζ. As a consequence both q and ζ remain positive throughout their evolution. The lower bounds for q and ζ then follow.

For the upper bound on q, we subtract the equations in (26) to get

$$\dot{\zeta} = \dot{q} + (C_0 + 1)q - C_0\gamma e^{-(\frac{1}{2}-\delta)\tau} \le \dot{q} + (C_0 + 1)q.$$

Then, the first equation yields

$$\dot{q} + q \le C_0 e^{-(\frac{1}{2}-\delta)\tau}\Big(\gamma + \zeta_0 + \int_0^\tau \big(\dot{q} + (C_0 + 1)q\big)\Big).$$

We derive from (27)

$$\dot{q} + q \le Ke^{-(\frac{1}{2}-\delta)\tau}\Big(h_0 q_0 + q + \int_0^\tau q\Big), \tag{29}$$

where K denotes some positive constant, only depending on C_0 and h_0, whose value can be possibly increased along the proof. From this we shall infer the upper exponential estimate on q. It is enough to prove the existence of some (possibly larger) K such that $q(\tau) \le Kq_0$ for all $\tau > 0$. Indeed, once this is at hand, we find that

$$\frac{d}{d\tau}\big(q(\tau)e^\tau\big) \le Ke^{(\frac{1}{2}+\delta)\tau} q_0(1 + \tau) \le Kq_0 e^{\frac{7}{8}\tau},$$

where we have used that $\delta < 1/4$, which implies that $q(\tau) \le Kq_0 e^{-\frac{1}{8}\tau}$. Plugging this information back into (29) yields $q(\tau) \le Kq_0 e^{-(\frac{1}{2}-\delta)\tau}$. So, let us concentrate on the global upper bound on q.

We first prove that q grows at most exponentially fast, namely, there exists $\Lambda > 0$ large enough such that $q(\tau) \leq 2q_0 e^{\Lambda\tau}$ for any $\tau \geq 0$. Indeed, if this property fails for some Λ, defining

$$\bar{\tau} := \sup\{\tau \geq 0 \,|\, \forall s \in [0,\tau]\,,\; q(s) \leq 2q_0 e^{\Lambda s}\},$$

we derive $q(\bar{\tau}) = 2q_0 e^{\Lambda\bar{\tau}}$ and

$$\dot{q}(\bar{\tau}) \geq \frac{d}{d\tau}\left(2q_0 e^{\Lambda\tau}\right)_{\tau=\bar{\tau}} = \Lambda q(\bar{\tau}).$$

Then, owing to (29), using that $q(s) \leq 2q_0 e^{\Lambda s}$ for $s \in [0,\tau]$ we deduce that

$$\Lambda \leq -1 + K\left(\frac{h_0 q_0}{q(\bar{\tau})} + 1 + \frac{\int_0^{\bar{\tau}} q}{q(\bar{\tau})}\right) \leq K\left(\frac{h_0}{2} + 1 + \frac{1}{\Lambda}\right).$$

This is a contradiction for Λ large enough, depending algebraically on K and h_0.

Let us now improve this exponential bound to a constant. From (29) we get

$$\dot{q} \leq K e^{-(\frac{1}{2}-\delta)\tau} q_0\left(h_0 + 2e^{\Lambda\tau} + \frac{2}{\Lambda}e^{\Lambda\tau}\right).$$

Using again the crude estimate $\frac{1}{2} - \delta \geq \frac{1}{4}$, we infer the existence of another positive constant K_1, depending on K, h_0 and Λ, such that

$$q \leq K_1 q_0\left(1 + e^{(\Lambda-1/4)\tau}\right).$$

Then, iterating n-times, we get

$$q \leq K_n q_0\left(1 + e^{(\Lambda-n/4)\tau}\right).$$

When $n \geq \Lambda/4$ we have obtained the desired upper bound.

Finally, for the upper bound on ζ, we derive from (26) and (27)

$$\dot{\zeta} \leq K e^{-(\frac{1}{2}-\delta)\tau}(\zeta_0 + \zeta),$$

where again K is some constant depending on C_0 and h_0. This implies that

$$\frac{d}{d\tau}\ln(\zeta + \zeta_0) \leq K e^{-\frac{1}{4}\tau},$$

from which we deduce the desired bounded. This concludes the proof. □

We now deal with the following system:

$$\begin{cases} \dot{q} + q = & C_0\,\zeta e^{-(\frac{1}{2}-\delta)\tau}, & \tau > \tau_1 \\ \dot{\zeta} = & -C_0(q + \zeta e^{-(\frac{1}{2}-\delta)\tau}), & \tau > \tau_1, \end{cases} \tag{30}$$

where the initial time τ_1 is a parameter to be chosen. This system has a unique global solution and we have the

Proposition 3.3 *There is $\tau_1 > 0$ depending on C_0 for which the solution to* (30) *with initial data $q(\tau_1) = 0$ and $\zeta(\tau_1) = 1$ satisfies:*

- $1/2 \le \zeta(\tau) < 1$ *for all* $\tau > \tau_1$,
- $0 < q(\tau) \le 2C_0 e^{-(\frac{1}{2}-\delta)\tau}$ *for all* $\tau > \tau_1$.

Proof. Let $\tau_2 > \tau_1$ be such that $\zeta > 0$ in $[\tau_1, \tau_2)$. We infer from (30) that in the interval (τ_1, τ_2) the function q is positive and thus $\dot{\zeta} < 0$. Then, in such interval,

$$\frac{d}{d\tau}(q(\tau)e^{\tau}) = C_0\zeta(\tau)e^{(\frac{1}{2}+\delta)\tau} \le C_0 e^{(\frac{1}{2}+\delta)\tau},$$

which implies that

$$q(\tau) \le \frac{C_0}{1/2+\delta} e^{-(\frac{1}{2}-\delta)\tau} \le 2C_0 e^{-(\frac{1}{2}-\delta)\tau}.$$

Plugging this bound as well as $\zeta \le 1$ into the equation for ζ and integrating on (τ_1, τ_2) we obtain

$$\zeta(\tau) \ge 1 - \frac{C_0(2C_0+1)}{1/2-\delta} e^{-(\frac{1}{2}-\delta)\tau_1}.$$

Recalling that $\delta < 1/4$, we can therefore choose τ_1 large enough, only depending on C_0, in such a way that, say, $\zeta(\tau_2) \ge 1/2$ in $[\tau_1, \tau_2]$. This means that, with this choice, $1/2 \le \zeta(\tau) < 1$ for all $\tau > \tau_1$. From this, the bounds for q are readily derived. ☐

3.2 Super-solution

We want to prove Proposition 3.1 for super-solutions. So we look for a super-solution to (21) with $\xi_\delta = \xi_\delta^-$ of the form (24) and q_1, q_2, ζ are positive functions that will be suitably chosen, with $\dot{\zeta} \ge 0$. (We drop the + exponents on q_1, q_2 and ζ in this sub-section for simplicity). Let us set

$$\mathcal{N}w = \partial_\tau w + \mathcal{M}w - l_1(\tau,\xi)\partial_\xi w - l_2(\tau,\xi)w + e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi+\xi_\delta^-)e^{\frac{\tau}{2}}} w^2.$$

We want $\overline{w}$ to satisfy $\mathcal{N}\overline{w} \ge 0$ for $\tau > 0$, $\xi > 0$. A sufficient condition for that is

$$\overline{\mathcal{N}}\overline{w} \ge 0,$$

with

$$\overline{\mathcal{N}}w = \partial_\tau w + \mathcal{M}w - l_1(\tau,\xi)\partial_\xi w - l_2(\tau,\xi)w;$$

in other words we have dropped the positive nonlinear term.

We remark here that we are in the spirit of Fife-McLeod [11]: because the null space of $\mathcal{M}$ is not empty, the best we can do with a bare hand computation is estimating the solution, but not proving its convergence, as we have no idea of what multiple of φ_0 will be eventually picked. In [11], a similar computation estimated the position of the front, but did not prove convergence to a wave, as the translation invariance would not permit to guess the correct translate of the wave.

1. The region $0 \le \xi \le \frac{a_0}{2}$. This is, in comparison to [20] and [23], the newest part. Here, we have $\gamma_2 = 0$ since $a_0 < 1$, so that, using $\mathcal{M}\varphi_0 = 0$ and $-\phi_1'' = \lambda_1\phi_1$, we have:

$$\begin{aligned}\overline{\mathcal{N}}\overline{w} = &\Big(\dot{q}_1 + (\lambda_1 + \frac{\xi^2}{16} - \frac{3}{4})q_1\Big)\phi_1 - \Big(l_1(\tau,\xi)\phi_1' + l_2(\tau,\xi)\phi_1\Big)q_1 \\ &+\dot{\zeta}\varphi_0 - \zeta\Big(l_1(\tau,\xi)\varphi_0' + l_2(\tau,\xi)\varphi_0\Big).\end{aligned}$$

Recall that

$$\phi_1(\xi) = \cos\left(\frac{\pi}{2a_0}\xi\right),$$

so that

$$|\phi_1'(\xi)| \leq \sqrt{\lambda_1}\,\phi_1(\xi) \text{ on } [0, \frac{a_0}{2}].$$

The functions φ_0 and φ_0' are bounded from above by a universal constant C, whereas ϕ_1 stays above $\sqrt{2}/2$ on the interval $[0, a_0/2]$. The term l_1 is estimated in (19) by $Ce^{-(\frac{1}{2}-\delta)\tau}$, where, here and in the rest of this proof, C will be a suitably large constant independent of q_1, q_2 and ζ. In the range that we consider for ξ, the first indicator function appearing in the estimate (19) of l_2 vanishes after a (controlled) finite time and therefore l_2 is estimated by the same term as l_1. As a consequence, because we look for ζ satisfying $\dot{\zeta} \geq 0$, we infer that

$$\frac{\overline{\mathcal{N}w}}{\phi_1} \geq \dot{q}_1 + (\lambda_1 - C\sqrt{\lambda_1} - C)q_1 - C\zeta e^{-(\frac{1}{2}-\delta)\tau}.$$

We choose $a_0 > 0$ such that λ_1 is large enough to have

$$\lambda_1 - C\sqrt{\lambda_1} - C \geq 1,$$

this will fix a_0 once and for all. And so, a sufficient condition to have $\overline{\mathcal{N}w} \geq 0$ in this region is

$$\dot{q}_1 + q_1 \geq C\zeta e^{-(\frac{1}{2}-\delta)\tau}. \tag{31}$$

2. The region ξ large. By this, we mean that ξ will be larger than a constant $\xi_0 \geq 2$ that we will fix in the course of this section. In any case we have $\gamma_1(\xi) = 0$ and $\gamma_2(\xi) = 1$. And so, using $\dot{\zeta} \geq 0$, we find that

$$e^{\xi^2/16}\overline{\mathcal{N}w} \geq \dot{q}_2 + \left(\frac{3}{64}\xi^2 - \frac{|l_1(\tau,\xi)|\xi}{8} - |l_2(\tau,\xi)| - \frac{5}{8}\right)q_2 - \zeta\Big(l_1(\tau,\xi)\varphi_0' + l_2(\tau,\xi)\varphi_0\Big)e^{\xi^2/16}.$$

We estimate $l_i(\tau,\xi)$ from (19) as

$$|l_1(\tau,\xi)| \leq C \text{ and } |l_2(\tau,\xi)| \leq C(\xi+1)\,.$$

Thus, the term in factor of q_2 can be bounded from below by $\frac{3}{64}\xi^2 - \frac{9}{8}C\xi - (C + \frac{5}{8})$. Now, we fix ξ_0 large enough so that

$$\frac{3}{64}\xi_0^2 - \frac{9}{8}C\,\xi_0 - (C + \frac{5}{8}) \geq 1\,.$$

Finally, recalling that $\varphi_0(\xi) = \xi e^{-\xi^2/8}$, whence φ_0' decays as $\xi^2 e^{-\xi^2/8}$, we derive from (19)

$$\begin{aligned}
|l_1(\tau,\xi)\varphi_0'(\xi) + l_2(\tau,\xi)\varphi_0(\xi)|\,e^{\xi^2/16} &\leq C|l_1\xi^2 + l_2\xi|e^{-\xi^2/16}\\
&\leq C\left(\xi^2 e^{-(\frac{1}{2}-\delta)\tau} + \xi\mathbf{1}_{\xi+\xi_\delta^- \geq e^{\delta\tau}} + \xi\mathbf{1}_{\xi+\xi_\delta^- \leq e^{\delta\tau}}e^{-(\frac{1}{2}-\delta)\tau}\right)e^{-\xi^2/16}\\
&\leq Ce^{-(\frac{1}{2}-\delta)\tau}
\end{aligned}$$

Indeed, when $\xi + \xi_\delta^- \leq e^{\delta\tau}$, we use the boundedness of $(\xi + \xi^2)e^{-\xi^2/16}$ on $\mathbb{R}$ and when $\xi + \xi_\delta^- \geq e^{\delta\tau}$, we bound $\xi e^{-\xi^2/16}$ by $e^{-\xi^2/32}$ which decreases at least as $e^{-(\frac{1}{2}-\delta)\tau}$ as $\xi \geq e^{\delta\tau} - \xi_\delta^-$. Then, the inequality $\overline{\mathcal{N}w} \geq 0$ is satisfied if we have the sufficient condition

$$\dot{q}_2 + q_2 \geq C\zeta e^{-(\frac{1}{2}-\delta)\tau}. \tag{32}$$

3. The region $\frac{a_0}{2} \leq \xi \leq \xi_0$. Notice that, in this range, the functions l_i may be estimated by $Ce^{-(\frac{1}{2}-\delta)\tau}$. The functions γ_i may take all values between 0 and 1, and their derivatives are bounded. Thus we have

$$\begin{aligned}
\overline{\mathcal{N}}\overline{w} = \; & \dot{\zeta}\varphi_0 - \zeta\Big(l_1(\tau,\xi)\varphi_0' + l_2(\tau,\xi)\varphi_0\Big) \\
& + \dot{q}_1\gamma_1\phi_1 + \dot{q}_2\gamma_2 e^{-\xi^2/16} + \Big(\mathcal{M} - l_1(\tau,\xi)\partial_\xi - l_2(\tau,\xi)\Big)(q_1\gamma_1\phi_1 + q_2\gamma_2 e^{-\xi^2/16}) \\
\geq \; & \dot{\zeta}\varphi_0 - \zeta\Big(l_1(\tau,\xi)\varphi_0' + l_2(\tau,\xi)\varphi_0\Big) + \dot{q}_1\gamma_1\phi_1 + \dot{q}_2\gamma_2 e^{-\xi^2/16} - C(q_1 + q_2).
\end{aligned}$$

To render $\overline{\mathcal{N}}\overline{w}$ nonnegative in this range, a sufficient condition is to assume that (31) and (32) are satisfied, so that $\dot{q}_1 \geq -q_1$, $\dot{q}_2 \geq -q_2$. Moreover, φ_0 is bounded away from 0 in this range, so that the final sufficient condition is

$$\dot{\zeta} \geq C(q_1 + q_2 + \zeta e^{-(\frac{1}{2}-\delta)\tau}). \tag{33}$$

4. Proof of Proposition 3.1 for super-solutions. From Proposition 3.2 there exist positive functions $q_1 = q_2 = \frac{q}{2}$ and ζ satisfying the equalities in (31)-(33). Moreover, $\dot{\zeta} \geq 0$ and the bounds (23) hold. Define $\overline{w}$ as in (24). By the three above steps, $\overline{w}$ is a super-solution to (21) with $\xi_\delta = \xi_\delta^-$ in the range $\tau > 0$, $\xi > 0$. This proves Proposition 3.1 for super-solutions. □

3.3 Sub-solution

Proof of Proposition 3.1 for sub-solutions. We proceed as in the preceding section for super-solutions. The nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}w = \partial_\tau w + \mathcal{M}w - l_1(\tau,\xi)\partial_\xi w - l_2(\tau,\xi)w + e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi+\xi_\delta^+)e^{\frac{\tau}{2}}} w^2.$$

and we want $\underline{w}$, defined by (25) (we again drop the superscript $^-$), to satisfy $\mathcal{N}\underline{w} \leq 0$ for τ possibly large and $\xi > 0$. On the contrary to the previous section, we may not drop the nonlinear term as it does not have the right sign. Moreover, the nonlinear term is quadratic, thus a possible source of trouble. However, let us anticipate that the solution $w(\tau,\xi,\Theta)$ of (21) with $\xi_\delta = 0$ will be dominated by a super-solution $\overline{w}(\tau,\xi)$ of the type (24). We use the (quite non-optimal) estimate

$$\overline{w} \leq C,$$

C once again possibly huge. Let us also notice that, for $\xi \geq 0$, we have

$$\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi + \xi_\delta^+)e^{\frac{\tau}{2}} \leq \frac{3\tau}{2} - \xi_\delta^+ e^{\frac{\tau}{2}} = \frac{3\tau}{2} - e^{\delta\tau},$$

so that, all in all, we have for $\tau > 0$ and $\xi \geq 0$,

$$e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi+\xi_\delta^+)e^{\frac{\tau}{2}}} w^2 \leq Ce^{-(\frac{1}{2}-\delta)\tau} w.$$

The nonlinear term may therefore be included in $l_2(\tau,\xi)$, and a sufficient condition for $\mathcal{N}\underline{w} \leq 0$ is

$$\underline{\mathcal{N}}\,\underline{w} \leq 0,$$

with $\underline{\mathcal{N}}$ having the same form as $\overline{\mathcal{N}}$ before:

$$\underline{\mathcal{N}}w = \partial_\tau w + \mathcal{M}w - l_1(\tau,\xi)\partial_\xi w - l_2(\tau,\xi)w,$$

but with l_2 now incorporating an additional $Ce^{-(\frac{1}{2}-\delta)\tau}$. From then on, the computations proceed in a similar fashion as before, yielding the following conditions for q_i and ζ:

$$\begin{aligned}
\dot{q}_1 + q_1 &\geq C\zeta e^{-(\frac{1}{2}-\delta)\tau}, \quad \text{for the region } 0 \leq \xi \leq \frac{a_0}{2}.\\
\dot{q}_2 + q_2 &\geq C\zeta e^{-(\frac{1}{2}-\delta)\tau}, \quad \text{for the region } \xi \geq \xi_0.\\
\dot{\zeta} &\leq -C\big(q_1 + q_2 + \zeta e^{-(\frac{1}{2}-\delta)\tau}\big), \quad \text{for the region } \frac{a_0}{2} \leq \xi \leq \xi_0.
\end{aligned}$$

From Proposition 3.3, there exist positive functions $q_1 = q_2 = \frac{q}{2}$ and ζ solutions to the three equations above in some interval $(\tau_1, +\infty)$ only depending on C. Moreover, $\dot{\zeta} \leq 0$ and the right handside of (23) holds. Then, the function $\underline{w}$ defined by (25) is a sub-solution to (21) with $\xi_\delta = \xi_\delta^+$ in the range $\tau > \tau_1$, $\xi > 0$. This proves Proposition 3.1 for sub-solutions. $\square$

3.4 A sharp version of Proposition 3.1 with small parameters

Pick $\varepsilon \in (0,1]$, $\bar{\tau} \geq 0$ and consider the solution $w(\tau,\xi)$ of

$$\begin{aligned}
\partial_\tau w + \mathcal{M}w - l_1(\tau,\xi)\partial_\xi w - l_2(\tau,\xi)w =&\ f(\tau,\xi) \quad (\tau \geq \bar{\tau},\ \xi > 0)\\
w(\tau,0) =&\ \exp(-e^{\delta\tau}) \quad (\tau \geq \bar{\tau})\\
|w(\bar{\tau},\xi) - \varepsilon\varphi_0(\xi)| \leq&\ \varepsilon e^{-\xi^2/16}.
\end{aligned} \tag{34}$$

with

$$|f(\tau,\xi)| \leq \frac{Ce^{-\tau}}{\varepsilon^2} e^{-\xi^2/16}. \tag{35}$$

Notice the inhomogeneous term $f(\tau,\xi)$ that was not present in the equation covered by Proposition 3.1. This inhomogeneous term is, however, harmless, as its treatment will not require any new idea. The estimate on $w(\tau,\xi)$ is the following

Proposition 3.4 *For every $\varepsilon > 0$, there exists $\bar{\tau}_\varepsilon > 0$ such that, for all $\tau \geq \bar{\tau}_\varepsilon$, and for some universal $C > 0$ and $\delta \in (0, \frac{1}{4})$, we have*

$$|w(\tau,\xi)| \leq C\varepsilon e^{-\xi^2/16}(\xi + e^{-(\frac{1}{2}-\delta)(\tau-\bar{\tau}_\varepsilon)}).$$

Proof. In view of (35), we choose

$$\frac{Ce^{-\bar{\tau}_\varepsilon}}{\varepsilon^2} = \varepsilon, \quad \text{that is, } \bar{\tau}_\varepsilon = 3\ln\frac{1}{\varepsilon} + O(1).$$

We make a translation in time. Set $\tau' = \tau - \bar{\tau}_\varepsilon$, with $\tau' \geq 0$ and $W(\tau',\xi) = w(\tau' + \bar{\tau}_\varepsilon, \xi) = w(\tau,\xi)$. Given the assumption on f, we have $W(\tau',\xi) \leq \tilde{W}(\tau',\xi)$ with

$$\begin{aligned}
\partial_\tau \tilde{W} + \mathcal{M}\tilde{W} - l_1(\tau' + \bar{\tau}_\varepsilon, \xi)\partial_\xi \tilde{W} - l_2(\tau' + \bar{\tau}_\varepsilon, \xi)\tilde{W} =&\ \varepsilon e^{-\tau' - \xi^2/16} \quad (\tau' \geq 0,\ \xi > 0)\\
\tilde{W}(\tau',0) =&\ \exp\left(-\frac{e^{\delta\tau'}}{C^\delta\varepsilon^{3\delta}}\right)\\
\tilde{W}(0,\xi) =&\ \varepsilon\varphi_0(\xi) + \varepsilon e^{-\xi^2/16}
\end{aligned} \tag{36}$$

Moreover, due to (19), the functions l_i satisfy for $\tau' \geq 0$ and $\xi \geq 0$:

$$\begin{aligned} |l_1(\tau' + \bar{\tau}_\varepsilon, \xi)| \leq & \; C\varepsilon^{3(\frac{1}{2}-\delta)} e^{-(\frac{1}{2}-\delta)\tau'} \\ |l_2(\tau' + \bar{\tau}_\varepsilon, \xi)| \leq & \; C\varepsilon^{3(\frac{1}{2}-\delta)} \xi e^{-(\frac{1}{2}-\delta)\tau'} + \mathbf{1}_{\xi \geq \frac{e^{\delta\tau'}}{\varepsilon^{3\delta}} + \varepsilon^{3(\frac{1}{2}-\delta)} e^{-(\frac{1}{2}-\delta)\tau'}} \\ & + C\varepsilon^{3(\frac{1}{2}-\delta)} e^{-(\frac{1}{2}-\delta)\tau'} \mathbf{1}_{\xi \leq \frac{e^{\delta\tau'}}{\varepsilon^{3\delta}} + C\varepsilon^{3(\frac{1}{2}-\delta)} e^{-(\frac{1}{2}-\delta)\tau'}} \end{aligned}$$

the constant C only depending on N and δ. A super-solution to (36) is then sought for under the form

$$\overline{w}(\tau', \xi) = \zeta(\tau')\varphi_0(\xi) + q_1(\tau')\phi_1(\xi)\gamma_1(\xi) + q_2(\tau')\gamma_2(\xi)e^{-\xi^2/16}.$$

The equations for the q_i and ζ are

$$\begin{aligned} \dot{q}_1 + q_1 \geq C(\zeta + \varepsilon)e^{-(\frac{1}{2}-\delta)\tau'}, & \quad \text{for the region } \; 0 \leq \xi \leq \frac{a_0}{2}. \\ \dot{q}_2 + q_2 \geq C(\zeta + \varepsilon)e^{-(\frac{1}{2}-\delta)\tau'}, & \quad \text{for the region } \; \xi \geq \xi_0. \\ \dot{\zeta} = C\big(q_1 + q_2 + \zeta e^{-(\frac{1}{2}-\delta)\tau'}\big), & \quad \text{for the region } \; \frac{a_0}{2} \leq \xi \leq \xi_0, \end{aligned}$$

a possible admissible set of initial data being $q_i(0) = \varepsilon$, $\zeta(0) = \varepsilon$. Application of Proposition 3.2 yields the desired upper bound for W and hence for w. We proceed in the same way for a sub-solution leads to Proposition 3.4: a sub-solution is sought for under the form

$$\underline{w}(\tau', \xi) = \zeta(\tau')\varphi_0(\xi) - q_1(\tau')\phi_1(\xi)\gamma_1(\xi) - q_2(\tau')\gamma_2(\xi)e^{-\xi^2/16}.$$

with, this time, $\dot{\zeta} < 0$. The equations for the q_i and ζ , as well as their initial data, are the same as above, except in the equation for ζ, where the right handside comes with a minus sign. □

4 Convergence

As announced in Section 2, this section is divided in two parts: in the first subsection, we prove Theorem 2.1, namely, what happens for $r \sim \sqrt{t}$. We use the barriers constructed before, and the fact that the solutions of the limiting problem are quite simple. In the second subsection, we derive sharper information at the border of the domain, that is, $r \sim t^\delta$. We use this to control the behaviour of the solution for finite r, thus proving Theorem 1.1.

4.1 Convergence to an angle-depending self-similar solution

We want to prove theorem 2.1. Let $\hat{w}$ be the solution of (12) with compactly initial datum $\hat{w}_0$. *In particular, in agreement with* (14) *we take* $\xi_\delta(\tau) = 0$. The main effort in this section is to derive the compactness of the trajectories $(\hat{w}(T+\tau, \xi, \Theta))_{T>0}$ in a weighted L^∞ norm. As the asymptotic problem will simply be the heat equation in the variables (τ, ξ), convergence will follow. Let us first translate the radial barriers into an effective control of the solution.

Proposition 4.1 (1. Control of $\hat{w}$ from above and from the back of the front) *There is a pair of positive functions $(q_+(\tau), \zeta_+(\tau))$ such that ζ_+ is bounded and bounded away from 0 by constants that depend only on the initial datum and the constants appearing in the* *equation, whereas there is* $\delta \in (0, \frac{1}{4})$ *such that* $q_+(\tau) = O(e^{-(\frac{1}{2}-\delta)\tau})$ *as* $\tau \to +\infty$*. Moreover, for* $\tau \geq 0$*,* $\xi \geq \xi_\delta^-(\tau)$ *and* $\Theta \in \mathbb{S}^{N-1}$*, we have*

$$\hat{w}(\tau, \xi, \Theta) \leq \zeta_+(\tau)\phi_0(\xi - \xi_\delta^-(\tau)) + q_+(\tau)\left((\mathbf{1}_{[0,a_0/2]}\phi_1)(\xi - \xi_\delta^-(\tau)) + e^{-\frac{(\xi-\xi_\delta^-)^2}{16}}\right)e^{-\frac{(\xi-\xi_\delta^-)^2}{8}} \quad (37)$$

(2. Control of $\hat{w}$ from below and from the head of the front.)
There is a pair of positive functions $(q_-(\tau), \zeta_-(\tau))$ *that satisfy the same estimates as for* q_+ *and* ζ_+ *in item 1 above, and such that, for* $\xi \geq \xi_\delta^+(\tau)$*,* $\tau \geq \tau_1$ *and* $\Theta \in \mathbb{S}^{N-1}$*, we have*

$$\hat{w}(\tau, \xi, \Theta) \geq \zeta_-(\tau)\phi_0(\xi - \xi_\delta^+(\tau)) - q_-(\tau)\left((\mathbf{1}_{[0,a_0/2]}\phi_1)(\xi - \xi_\delta^+(\tau)) + e^{-\frac{(\xi-\xi_\delta^+)^2}{16}}\right)e^{-\frac{(\xi-\xi_\delta^+)^2}{8}} \quad (38)$$

Proof. Let us prove Point 1. Let $\hat{w}$ be the solution of (12) with compactly initial datum $\hat{w}_0$. Perform transformations 6. and 7. in section 2 with $\xi_\delta = \xi_\delta^-$. Then, the new function w defined by (14) and (16) satifies equation (17) with $\xi_\delta = \xi_\delta^-$ and initial datum w_0 defined in (18). This is the same equation as (21), up to the Laplace-Beltrami term. Moreover, $w_0(\xi, \Theta)$ is still compactly supported and the upper bound (13) yields

$$w(\tau, 0, \Theta) \leq \exp(-e^{\delta\tau} - \frac{\tau}{2}), \quad (39)$$

that is, the Dirichlet condition is doubly exponentially small.

Applying Proposition 3.1, there exist three functions q_1^+, q_2^+ and ζ^+ such that $\overline{w}$ defined by (24) is a super-solution to (21) and therefore to (17) with $\xi_\delta = \xi_\delta^-$ in the range $\tau > 0$, $\xi > 0$ and $\Theta \in \mathcal{S}^{N-1}$.

On one hand, if we choose $q_1^+(0)$, $q_2^+(0)$ and $\zeta^+(0)$ large enough, we have $\overline{w}(0, \xi) \geq w_0(\xi, \Theta)$ for all $\xi \geq 0$ and $\Theta \in \mathcal{S}^{N-1}$ because w_0 is compactly supported. On the other hand, we have from (23) that $q_1^+(\tau) \geq \frac{e^{-\tau}}{C}$. Thus, from (39), we have, at the expense of choosing C even smaller:

$$w(\tau, 0, \Theta) \leq \overline{w}(\tau, 0).$$

The comparison principle then yields $w \leq \overline{w}$ for $\tau \geq 0$, $\xi \geq 0$ and $\Theta \in \mathcal{S}^{N-1}$. Reverting to the original function $\hat{w}$, we infer that the desired upper estimate holds with $q_+ = q_1^+ + q_2^+$, $\zeta_+ = \zeta^+$, which, by Proposition 3.1, satisfy the properties stated in point 1. of Proposition 4.1

Let us prove Point 2. Let $\hat{w}$ be the solution of (12) with compactly initial datum $\hat{w}_0$. Perform transformations 6. and 7. in section 2 with $\xi_\delta = \xi_\delta^+$. Then, the new function w defined by (14) and (16) satifies equation (17) with $\xi_\delta = \xi_\delta^+$ and initial datum w_0 defined in (18). Applying Proposition 3.1, there exist three functions q_1^-, q_2^- and ζ^- such that $\underline{w}$ defined by (25) is a sub-solution to (21) and therefore to (17) with $\xi_\delta = \xi_\delta^+$ in the range $\tau > \tau_1$, $\xi > 0$ and $\Theta \in \mathcal{S}^{N-1}$.

At $\tau = \tau_1$, we have that $\underline{w} < 0$ for ξ sufficiently large and then, up to multiplying it by a small positive constant (which preserves the inequality $\underline{\mathcal{N}}\,\underline{w} \leq 0$ because the operator $\underline{\mathcal{N}}$ is linear), we can fit it below the positive solution w for all $\xi \geq 0$. Moreover, at $\xi = 0$, $\underline{w} = 0 < w$. We can therefore apply the comparison principle, concluding the proof of point 2. of Proposition 4.1. □

Proposition 4.1 has the following corollary.

Corollary 4.2 *For $\tau \geq 1$, $\xi > 0$ and $\Theta \in \mathcal{S}^{N-1}$, we have*

$$|\partial_\tau \hat{w}(\tau,\xi,\Theta)| + |\partial_\xi \hat{w}(\tau,\xi,\Theta)| \leq Ce^{-3\xi^2/16}.$$

Moreover, there are two constants $0 < \underline{k} \leq \overline{k}$, and $k_1 > 0$ such that, for $\tau \geq 1$, $\xi \leq 1$ and $\Theta \in \mathcal{S}^{N-1}$, we have

$$\underline{k}(\xi - k_1 e^{-(\frac{1}{2}-\delta)\tau}) \leq \hat{w}(\tau,\xi,\Theta) \leq \overline{k}(\xi + e^{-(\frac{1}{2}-\delta)\tau}). \tag{40}$$

Proof. Parabolic regularity yields the boundedness of $\partial_\tau \hat{w}$, $\partial_\xi \hat{w}$ and $\partial_{\xi\xi}\hat{w}$ in terms of the supremum of $\hat{w}$ on the product of $(\tau-1,\tau+1)\times(\xi-1,\xi+1)\times\mathcal{S}^{N-1}$. Of course the diffusion in Θ is degenerate, but it suffices to rescale Θ by the square root of the diffusion at the point under consideration, and drop the useless estimate in Θ. Inequality (40) just comes from the analysis of $\underline{w}$ and $\overline{w}$ in the vicinity of $\xi = 0$. □

As far as the variable Θ is concerned, we need an additional argument.

Proposition 4.3 *There is $C > 0$, depending only on the data, such that, for $\tau > 0$, $\xi \geq 0$ and $\Theta \in \mathcal{S}^{N-1}$, we have*

$$|\nabla_\Theta \hat{w}(\tau,\xi,\Theta)| \leq Ce^{-3\xi^2/16}.$$

Proof. Let $\hat{w}$ be the solution of (12) with compactly initial datum $\hat{w}_0$. Perform transformations 6. and 7. in Section 2 with $\xi_\delta = \xi_\delta^-$. Then, the new function w defined by (14) and (16) satifies equation (17) with $\xi_\delta = \xi_\delta^-$ and initial datum w_0 defined in (18).

Let Θ_i be any coordinate on the unit sphere, and

$$w_i(\tau,\xi,\Theta) = \partial_{\Theta_i} w(\tau,\xi,\Theta).$$

As there is no dependence with respect to Θ in the coefficients of (17), the equation for w_i is very similar to that for w:

$$\begin{cases} \partial_\tau w_i + \mathcal{M}w_i = & l_1(\tau,\xi)\partial_\xi w_i + l_2(\tau,\xi)w_i \\ & + \dfrac{\Delta_\Theta w_i}{\left(\xi + \xi_\delta^- + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2} - 2e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi+\xi_\delta^-)e^{\frac{\tau}{2}}} w w_i \\ w_i(0,\xi,\Theta) = & \partial_{\Theta_i} w_0(\xi,\Theta) \text{ compactly supported.} \end{cases}$$

Multiplying the equation for w_i by the sign of w_i and using Kato's inequality, as well as $w \geq 0$, we find out that $|w_i|$ solves the inequation

$$\partial_\tau |w_i| + \mathcal{M}|w_i| - l_1(\tau,\xi)\partial_\xi |w_i| - l_2(\tau,\xi)|w_i| - \frac{\Delta_\Theta |w_i|}{\left(\xi + \xi_\delta^- + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2} \leq 0.$$

If now $\overline{w}_{i,0}(\xi)$ is the supremum of $|w_i(0,\xi,.)|$ over the unit sphere, then we have $|w_i(\tau,\xi,\Theta)| \leq \overline{w}_i(\tau,\xi)$ with

$$\begin{cases} \partial_\tau \overline{w}_i + \mathcal{M}\overline{w}_i = & l_1(\tau,\xi)\partial_\xi \overline{w}_i + l_2(\tau,\xi)\overline{w}_i \\ \overline{w}_i(0,\xi) = & \overline{w}_{i,0}(\xi) \text{ compactly supported.} \end{cases}$$

Moreover, parabolic regularity yields, for the solution $u(t,r,\Theta)$ of (8):

$$|\nabla_\Theta u(t,-t^\delta,\Theta)| \leq C(1+t);$$

this translates into

$$|\nabla_\Theta v(t,-t^\delta,\Theta)| \leq C(1+t)e^{-t^\delta},$$

thus $|w_i(\tau,0,\Theta)| \leq Ce^{\frac{\tau}{2} - e^{\delta\tau}}$. Hence, $\overline{w}_i$ may be controlled by a super-solution similar to that constructed in Section 3.2, which proves the proposition. □

Remark 4.4 *The referee pointed out to us that an argument of the same sort could successively control the second derivatives, then the third derivatives, and so on. It may be so, but this is not a completely trivial fact which, in any case, requires additional arguments. Indeed, using the notations of the above proof, let* $w_{ii} := \partial_{\Theta_i\Theta_i} w$ *be the pure second angular derivative of* w *in the direction* e_i*. Let* w_i *the* i^{th} *component of* $\nabla_\Theta w$*. The equation for* w_{ii} *is*

$$\begin{aligned}\partial_\tau w_{ii} + \mathcal{M} w_{ii} - &\frac{\Delta_\Theta w_{ii}}{\left(\xi + \xi_\delta^- + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2} - l_1(\tau,\xi)\partial_\xi w_{ii} - l_2(\tau,\xi) w_{ii} \\ = &\ -2e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - (\xi+\xi_\delta^-)e^{\frac{\tau}{2}}}(w w_{ii} + w_i^2).\end{aligned}$$

The trouble is that the Kato's inequality process will not work here, as the term

$$e^{\frac{3\tau}{2} - (\xi+\xi_\delta^-)e^{\frac{\tau}{2}} - \frac{\xi^2}{8}} \mathrm{sgn}(w_{ii}) w_i^2$$

will not have a definite sign, and it may be huge in the region $\xi \sim e^{-\tau/2}$*. The best that can be expected at this stage, with no other ingredients, is a bound from above for* w_{ii}*, but it is not clear to us how to use it.*

In order to bound w_{ii}*, one can probably try to use the fact that the term is large only on a very small support - as in [21] - to infer the boundedness of* w_{ii}*, but it would add quite a few technicalities. Therefore we decide to limit ourselves to the Lipschitz regularity of* w *in* Θ*, which is enough for the sequel. This will force us to use arguments that are a little more abstract, but much less technical. In any case we thank the referee for giving us the occasion to clarify this issue.*

Proof of Theorem 2.1

Let $\hat{w}$ be the solution of (12) with compactly initial datum $\hat{w}_0$. Perform transformations 6. and 7. in Section 2 with $\xi_\delta = 0$. Then, the new function w defined by (14) and (16) satifies equation (17) with $\xi_\delta = 0$ and initial datum w_0 defined in (18).

Propositions 4.2 and 4.3 yield the compactness of the trajectory $(w(T + ., ., .))_{T>0}$ in the $L^\infty_{\tau,\xi,\Theta}$ norm, weighted by $e^{\xi^2/16}$. Therefore, there is a function w^∞ and a sequence $(T_n)_n$ going to infinity such that

$$\lim_{n\to+\infty} e^{\xi^2/16} |w(T_n + \tau, \xi, \Theta) - w^\infty(\tau, \xi, \Theta)| = 0, \tag{41}$$

the limit being locally uniform in τ, and uniform in (ξ, Θ). Moreover, w^∞ is Lipschitz-continuous in all its variables, and (40) entails $w^\infty(\tau, 0, \Theta) = 0$.

On the other hand, for any smooth function φ over the unit sphere, consider the integral

$$w_\varphi(\tau, \xi) = \int_{\mathbb{S}^{N-1}} w(\tau, \xi, \Theta)\varphi(\Theta) d\Theta.$$

The equation for w_φ is this time:

$$\begin{cases} \partial_\tau w_\varphi + \mathcal{M} w_\varphi = & l_1(\tau,\xi)\partial_\xi w_\varphi + l_2(\tau,\xi) w_\varphi - e^{\frac{3\tau}{2} - \frac{\xi^2}{8} - \xi e^{\frac{\tau}{2}}} \displaystyle\int_{\mathbb{S}^{N-1}} w^2 \varphi d\Theta \\ w_\varphi(0, \xi) = & \displaystyle\int_{\mathbb{S}^{N-1}} w_0(\xi, \Theta)\varphi(\Theta) d\Theta \text{ compactly supported.} \end{cases}$$

Consider first $\varphi \geq 0$ on $\mathbb{S}^{N-1}$. The integral term $\displaystyle\int_{\mathbb{S}^{N-1}} w^2 \varphi d\Theta$ is nonnegative, so the same type of super-solution as in Section 3.2 may be constructed for w_φ, just by discarding this term. Moreover, the same type of subsolution for w_φ can also be constructed, as we may simply estimate w_φ by a constant, and as the exponential factor $e^{\frac{3\tau}{2}-\xi e^{\frac{\tau}{2}}}$ is exponentially decaying in τ, as soon as ξ is just a little larger than $e^{-\tau/2}$. This yields the compactness of w_φ in the weighted L^∞ norm, but w_φ additionally satisfies a standard parabolic equation in the (τ, ξ) variables. Therefore, parabolic estimates hold, and a subsequence of $(w_\varphi(T + ., .))_{T>0}$ converges, locally in τ, and in the weighted L^∞ norm in ξ, to a solution w_φ^∞ of

$$\begin{cases} \partial_\tau w_\varphi^\infty + \mathcal{M} w_\varphi^\infty &= 0, \quad \tau \in \mathbb{R}, \quad \xi \geq 0 \\ w_\varphi^\infty(\tau, 0) = & 0. \end{cases} \tag{42}$$

The same argument as in [20], Lemma 5.1, yields the convergence of the full trajectory $(w_\varphi(T + ., .))_{T>0}$ to a steady state solution of (42), namely, a multiple of φ_0. This multiple has to be positive, because of Proposition 4.1, Point 2. We name it $\alpha_\varphi \varphi_0$. If now the function φ is allowed to change sign, the result persists because φ may be decomposed into $\varphi^+ - \varphi^-$.

The functional $\varphi \mapsto \alpha_\varphi$ is a nonnegative functional acting on the set of all continuous functions of the unit sphere. On the other hand, (41) yields, for all $\tau \in \mathbb{R}$:

$$\alpha_\varphi \varphi_0(\xi) = \int_{\mathbb{S}^{N-1}} w^\infty(\tau, \xi, \Theta)\varphi(\Theta) d\Theta.$$

This implies the following cascade of facts. First, the function w^∞ does not depend on τ, we call it $w^\infty(\xi, \Theta)$. Second, the functional $\varphi \mapsto \alpha_\varphi$ is linear, so, combined with positivity, it is a measure that we call μ. Third, we have, for all $\xi > 0$:

$$\int_{\mathbb{S}^{N-1}} \varphi(\Theta) d\mu(\Theta)\varphi_0(\xi) = \int_{\mathbb{S}^{N-1}} w^\infty(\xi, \Theta)\varphi(\Theta) d\Theta.$$

This entails that $\dfrac{w^\infty(\xi, \Theta)}{\varphi_0(\xi)}$ does not depend on ξ, call it $\alpha^\infty(\Theta)$. So, we have

$$\int_{\mathbb{S}^{N-1}} \varphi(\Theta) d\mu(\Theta) = \int_{\mathbb{S}^{N-1}} \alpha^\infty(\Theta)\varphi(\Theta) d\Theta.$$

so that μ is absolutely continuous with respect to the Lebesgue measure, $d\mu(\Theta) = \alpha^\infty(\Theta)d\Theta$. Because w^∞ is Lipschitz in Θ, this implies that α^∞ is Lipschitz by its above definition.

As a conclusion, we obtained the convergence of $w(\tau, \xi, \Theta)$ as τ goes to infinity towards $\alpha^\infty(\Theta)\varphi_0(\xi)$ in the $L^\infty_{\xi,\Theta}$ norm, weighted by $e^{\xi^2/16}$. Reverting to the original function $\hat{w}$, we get the desired convergence. □

4.2 Convergence to the shifted wave

The challenge is now to transmit the information given by Theorem 2.1 from the diffusive zone to the area of bounded x. To achieve that goal, we need to estimate the solution precisely in the transition zone, namley, $x \sim t^\delta$, i.e. to understand the behaviour of the solution $\hat{w}(\tau, \xi, \Theta)$ of (12) in the area $\xi \sim e^{-(\frac{1}{2}-\delta)\tau}$.

We would like to write an equation for $\hat{w}(\tau, \xi, \Theta) - \alpha^\infty(\Theta)\varphi_0(\xi)$ and infer from the analysis of the equation that this difference converges to 0 as $\tau \to +\infty$. The trouble is that we deliberately stopped investigating the regularity of α^∞, and that a term of the form

$$\frac{\phi_0(\xi)\Delta_\Theta \alpha^\infty}{\left(\xi + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2}$$

will be present in the equation for the difference, something that is not so easy to study as α^∞ is only known to be Lipschitz. So, we use a regularisation. If $(\rho_\varepsilon)_{\varepsilon>0}$ is an approximation of the identity on the unit sphere, we set

$$\alpha_\varepsilon^\infty(\Theta) = (\rho_\varepsilon * \alpha^\infty)(\Theta).$$

Because α^∞ is Lipschitz and positive, we have $\alpha_\varepsilon^\infty - C\varepsilon \leq \alpha^\infty \leq \alpha_\varepsilon^\infty + C\varepsilon$. We start with the following proposition.

Proposition 4.5 *Let $\hat{w}$ be the solution of* (12) *with compactly initial datum $\hat{w}_0$. Then, for every $\varepsilon > 0$, there are $\tau_\varepsilon > 0$ (possibly depending also on δ) and $\eta_\varepsilon > 1$ such that, for all $\tau \geq \tau_\varepsilon$ and $\xi \in [\xi_\delta^-(\tau), \eta_\varepsilon]$ we have:*

$$(\alpha_\varepsilon^\infty(\Theta) - C\varepsilon)\left(\xi - Ce^{-(\frac{1}{2}-\delta)(\tau-\tau_\varepsilon)}\right) \leq \hat{w}(\tau,\xi,\Theta) \leq (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)\left(\xi + Ce^{-(\frac{1}{2}-\delta)(\tau-\tau_\varepsilon)}\right). \quad (43)$$

Proof. We shall prove the upper estimate. For every $\varepsilon \in (0,1]$, there is, from Theorem 2.1, a time $\tau_\varepsilon > 0$ such that

$$(\alpha_\varepsilon^\infty(\Theta) - C\varepsilon)\phi_0(\xi) - \varepsilon e^{-3\xi^2/16} \leq \hat{w}(\tau_\varepsilon,\xi,\Theta) \leq (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)\phi_0(\xi) + \varepsilon e^{-3\xi^2/16}.$$

Perform transformations 6. and 7. in Section 2 with $\xi_\delta = \xi_\delta^-$. Then, the new function w defined by (14) and (16) satifies equation (17) with $\xi_\delta = \xi_\delta^-$ and initial datum w_0 defined in (18). Then,

$$(\alpha_\varepsilon^\infty(\Theta) - C\varepsilon)\varphi_0(\xi) - \varepsilon e^{-\xi^2/16} \leq w(\tau_\varepsilon,\xi,\Theta) \leq (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)\varphi_0(\xi) + \varepsilon e^{-\xi^2/16}.$$

Thus,

$$w(\tau,\xi,\Theta) \leq w^+(\tau,\xi,\Theta)$$

where

$$\partial_\tau w^+ + \mathcal{M}w^+ = l_1(\tau,\xi)\partial_\xi w^+ + l_2(\tau,\xi)w^+ + \frac{\Delta_\Theta w^+}{\left(\xi + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2},$$

for $\xi \geq 0$, $\theta \in \mathcal{S}^{N-1}$ and $\tau \geq \tau_\varepsilon$, with datum

$$w^+(\tau_\varepsilon,\xi,\Theta) = (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)\varphi_0(\xi) + \varepsilon e^{-\xi^2/16}$$

and Dirichlet condition

$$w^+(\tau,0,\theta) = O(e^{-e^{\delta\tau}}).$$

Consider now the difference

$$z(\tau,\xi,\Theta) = w^+(\tau,\xi,\Theta) - (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)\varphi_0(\xi).$$

Then $z(\tau,\xi,\Theta)$ solves an equation of the type

$$\partial_\tau z + \mathcal{M}z = l_1(\tau,\xi)\partial_\xi z + l_2(\tau,\xi)z + \frac{\Delta_\Theta z}{\left(\xi + 2e^{\frac{\tau}{2}} - k\tau e^{-\frac{\tau}{2}}\right)^2} + f(\tau,\xi,\Theta),$$

with the force terme being estimated by

$$|f(\tau,\xi,\Theta)| \leq C\frac{e^{-\xi^2/16}e^{-\tau}}{\varepsilon^2}.$$

Moreover, the initial datum is $z(\tau_\varepsilon, \xi, \Theta) = \varepsilon e^{-\xi^2/16}$ and the Dirichlet condition $z(\tau, 0, \Theta) = O(e^{-e^{\delta\tau}})$. The $1/\varepsilon^2$ factor comes from the Laplacian of $\alpha_\varepsilon^\infty$, that is certainly no more than a multiple of $1/\varepsilon^2$. By Kato's inequality, we have $|z(\tau, \xi, \Theta)| \leq \overline{w}_\varepsilon(\tau, \xi)$, with

$$\partial_\tau \bar{w}_\varepsilon + \mathcal{M}\bar{w}_\varepsilon = l_1(\tau, \xi)\partial_\xi \bar{w}_\varepsilon + l_2(\tau, \xi)\bar{w}_\varepsilon + C\frac{e^{-\tau}e^{-\xi^2/16}}{\varepsilon^2}$$

Application of Proposition 3.4 yields the right handside of (43) since ξ is bounded.

As for the left handside, we work with a sub-solution defined for $\xi_\delta = \xi_\delta^+(\tau)$, and replace the nonlinear term w^2 by a constant, due to the boundedness of w. The proof follows the same pattern as above. □

Proof of Theorem 1.1. We revert to the (t, r, Θ) variables, and to the function $v(t, r, \Theta)$ defined in Section 2. Recall that the equation for v is

$$\partial_t v = \partial_{rr} v + \left(\frac{N-1}{r + 2t - k\ln t} - \frac{k}{t}\right)(\partial_r v - v) + \frac{\Delta_\Theta v}{(r + 2t - k\ln t)^2} - e^{-r}v^2. \tag{44}$$

Also recall that the initial unknown $u(t, r, \Theta)$ in the moving frame satisfies $u(t, r, \Theta) = e^{-r}v(t, r, \Theta)$. We apply inequalities (43) in the following range of parameters: we first pick $\varepsilon > 0$. Then, fix $\delta_0 \in (0, 1/100)$ and set $\delta = \frac{\delta_0}{2}$. Consider $r = t^{\delta_0}$, so that $\xi = e^{-(\frac{1}{2}-\delta_0)\tau}$. There is $t_\varepsilon = e^{\tau_\varepsilon} > 0$ such that, for $t \geq t_\varepsilon$, $\Theta \in \mathcal{S}^{N-1}$, we have

$$(\alpha_\varepsilon^\infty(\Theta) - C\varepsilon)(t^{\delta_0} - C\sqrt{t_\varepsilon}^{1-\delta_0} t^{\delta_0/2}) \leq v(t, t^{\delta_0}, \Theta) \leq (\alpha_\varepsilon^\infty(\Theta) + C\varepsilon)(t^{\delta_0} + C\sqrt{t_\varepsilon}^{1-\delta_0} t^{\delta_0/2}), \tag{45}$$

We set

$$\psi_\varepsilon^\pm(r, \Theta) = (\alpha_\varepsilon^\infty(\Theta) \pm C\varepsilon)(r \pm C\sqrt{t_\varepsilon}^{1-\delta_0}\sqrt{r}),$$

then, inequation (45) becomes for $t \geq t_\varepsilon$, $\Theta \in \mathcal{S}^{N-1}$,

$$\psi_\varepsilon^-(t^{\delta_0}, \Theta) \leq v(t, t^{\delta_0}, \Theta) \leq \psi_\varepsilon^+(t^{\delta_0}, \Theta).$$

Taking ε even smaller and t_ε larger, we may assume that those functions are nonnegative. In the similar spirit as [23] Section 3, we define the upper and lower shifts as

$$\forall t \geq t_\varepsilon\,,\ \Theta \in \mathcal{S}^{N-1}\,, \quad U_{c_*}(r + s_\varepsilon^\pm(t, \Theta))\Big|_{r=t^{\delta_0}} = \psi_\varepsilon^\pm(r, \Theta)e^{-r}\Big|_{r=t^{\delta_0}}.$$

Note that $s_\varepsilon^\pm$ are both well-defined. Moreover, recall the equivalent

$$U_{c_*}(r) = (r + K)e^{-r} + O_{r\to+\infty}(e^{-(1+\gamma_0)r});$$

the implicit functions theorem yields, therefore for $t \geq t_\varepsilon$ and $\Theta \in \mathcal{S}^{N-1}$,

$$s_\varepsilon^\pm(t, \Theta) = -\ln(\alpha_\varepsilon^\infty(\theta) \pm C\varepsilon) + O(\frac{1}{t^{\delta_0}}), \quad \partial_t s_\varepsilon^\pm(t, \Theta) = O(\frac{1}{t^{1+\delta_0}}).$$

Moreover, the L^∞ norm of $\Delta_\Theta s_\varepsilon^\pm$ is bounded by a constant that may blow up as $\varepsilon \to 0$. Let us define $v_\varepsilon^\pm$ as the solutions of (44) for $t \geq t_\varepsilon$, $r \in (-t^{\delta_0}, t^{\delta_0})$, $\Theta \in \mathcal{S}^{N-1}$, that have $v(t_\varepsilon, r, \Theta)$ as initial datum at $t = t_\varepsilon$, and that satisfy the Dirichlet conditions:

$$v_\varepsilon^\pm(t, t^{\delta_0}, \Theta) = \psi_\varepsilon^\pm(t^{\delta_0}, \Theta), \quad v_\varepsilon^+(t, -t^{\delta_0}, \Theta) = e^{-t^{\delta_0}},\ v_\varepsilon^-(t, -t^{\delta_0}, \Theta) = 0,$$

for $t \geq t_\varepsilon$, $r \in (-t^{\delta_0}, t^{\delta_0})$, $\Theta \in \mathcal{S}^{N-1}$, we have

$$v_\varepsilon^-(t,r,\Theta) \leq v(t,r,\Theta) \leq v_\varepsilon^+(t,r,\Theta).$$

The last step of the proof is to prove that the functions $v_\varepsilon^\pm(t,r,\Theta)$ converge to $e^r U_{c_*}(r+s_\varepsilon^\pm)$, uniformly in r and Θ in their domains. Because ε is arbitrary, this will imply the convergence of v. We set

$$V_\varepsilon^\pm(t,r,\Theta) = v_\varepsilon^\pm(t,r,\Theta) - e^r U_{c_*}(r+s_\varepsilon^\pm);$$

we have

$$\begin{aligned} \partial_t V_\varepsilon^\pm = \ & \partial_{rr} V_\varepsilon^\pm + \left(\frac{N-1}{r+2t-k\mathrm{lnt}} - \frac{k}{t}\right)\left(\partial_r V_\varepsilon^\pm - V_\varepsilon^\pm\right) \\ & + \frac{\Delta_\Theta V_\varepsilon^\pm}{(r+2t-k\mathrm{lnt})^2} - (e^{-r}v_\varepsilon^\pm + U_{c_*})V_\varepsilon^\pm + O(\frac{1}{t^{1-\delta_0}}). \end{aligned}$$

We use one last time the process consisting in multiplying the equation by the sign of $V_\varepsilon^\pm$, then using Kato's inequality and the positivity of $U_{c_*} + e^{-r}v_\varepsilon^\pm$. This yields for $t \geq t_\varepsilon$, $r \in (-t^{\delta_0}, t^{\delta_0})$, $\Theta \in \mathcal{S}^{N-1}$, $|V_\varepsilon^\pm(t,r,\Theta)| \leq \overline{V}_\varepsilon^\pm(t,r)$, where $\overline{V}_\varepsilon^\pm$ satifies for $t \geq t_\varepsilon$ and $r \in (-t^{\delta_0}, t^{\delta_0})$

$$\partial_t \overline{V}_\varepsilon^\pm = \partial_{rr}\overline{V}_\varepsilon^\pm + (\frac{N-1}{r+2t-k\mathrm{lnt}} - \frac{k}{t})\left(\partial_r \overline{V}_\varepsilon^\pm - \overline{V}_\varepsilon^\pm\right) + O(\frac{1}{t^{1-\delta_0}})$$

$$\overline{V}_\varepsilon^\pm(t,-t^{\delta_0}) = e^{-t^{\delta_0}}, \quad \overline{V}_\varepsilon^\pm(t,t^{\delta_0}) = 0 \quad \overline{V}_\varepsilon^\pm(t_\varepsilon, r) = C\varepsilon.$$

We infer that both functions $\overline{V}_\varepsilon^\pm(t,.)$ converge to 0 as $t \to +\infty$. The reason is that the equation has lower order coefficients and right handside of order less than $\frac{1}{t}$, whereas the first eigenvalue of the Dirichlet Laplacian on $(-t^{\delta_0}, t^{\delta_0})$ is of order $t^{-2\delta_0}$. This heuristics in mind, we may find a barrier: the function

$$\bar{z}(t,r) = \frac{A}{t^{\delta_0}}\cos\left(\frac{r}{t^{2\delta_0}}\right)$$

is a super-solution to the equation for $\overline{V}_\varepsilon^\pm$, for t larger than some (possibly quite large) $t_\varepsilon > 0$. It is also larger than the values of $\overline{V}_\varepsilon^\pm$ at the boundary $\{-t^{\delta_0}, t^{\delta_0}\}$, and, for A large enough, can be put above $\overline{V}_\varepsilon^\pm$ at time t_ε. This concludes the proof of theorem 1.1 with $s^\infty(\Theta) = -\ln(\alpha^\infty(\Theta))$ which is at least Lipschitz. □

5 Discussion

Let us first mention that our result remains valid for more general nonlinearities. For an equation of the form

$$\partial_t u = \Delta u + f(u), \quad t > 0, \ x \in \mathbb{R}^N,$$

it suffices to assume that f is concave and positive on $(0,1)$, with $f(0) = f(1) = 0$. Thus $f'(0) > 0$ and the bottom speed is given by $c_* = 2\sqrt{f'(0)}$. Our result becomes the existence of a Lipschitz function s^∞ defined on the unit sphere such that

$$u(t,x) = U_{c_*}\left(|x| + c_* t - \frac{N+2}{c_*}\mathrm{ln}t + s^\infty(\frac{x}{|x|})\right) + o_{t\to+\infty}(1).$$

uniformly in $x \in \mathbb{R}^N$. In the course of the proof, the nonlinear term is no more u^2 but $g(u) = f'(0)u - f(u)$, which is positive and nondecreasing on $(0,1)$. It is not clear to us whether the result would subsist by merely assuming $f(u) \le f'(0)u$. What would probably be true is a statement of the form

$$u(t,x) = U_{c_*}\Big(|x| + c_*t - \frac{N+2}{c_*}\mathrm{ln}t + s^\infty(t, \frac{x}{|x|})\Big) + o_{t\to+\infty}(1),$$

with $s^\infty(t,\Theta) = O(1)$. Let us also mention that we could have given a slightly different version of Theorem 1.1 by stating that, for every direction $e \in \mathcal{S}^{N-1}$, then

$$\{u(t,x) = \lambda\} \cap \{x = re,\ r > 0\} \subset \{r = c_*t - \frac{N+2}{c_*}\mathrm{ln}t - s^\infty(e) + U_{c_*}^{-1}(\lambda) + o_{t\to+\infty}(1)\}.$$

The analysis of the solution on the diffusive zone would have been slightly simpler, in the sense that we would not have had to handle an asymptotically degenerate diffusion in e. On the other hand, recovering the convergence at the $O(1)$ spatial scale would have been more delicate. Additionally, this would not have proved the Lipschitz regularity of s in e. This last approach is, sometimes, better tailored to the geometric situation, where the front has a preferered direction of propagation. This is the case in the forthcoming paper [6], where the Fisher-KPP invasion occurs orthogonally to a line of fast diffusion.

We may adapt the preceding ideas to asymptotically homogeneous models of the form

$$\partial_t u = \Delta u + \mu(x)u - u^2, \quad (t > 0,\ x \in \mathbb{R}^N) \tag{46}$$

where the function $\nu(x) := \mu(x) - 1$ satisfies

$$\nu(x) = \frac{\lambda}{|x|^\alpha} + O_{|x|\to+\infty}(\frac{1}{|x|^{\alpha+\delta}}),\ |\nabla\nu(x)| = \frac{\alpha\lambda}{|x|^{1+\alpha}} + O_{|x|\to+\infty}(\frac{1}{|x|^{\alpha+1+\delta}}).$$

Theorem 1.1 becomes

thm 5.1 *Let u_0 satisfy assumption* (3). *There is a Lipschitz function s^∞, defined on the unit sphere of $\mathbb{R}^N$, such that the solution u of* (46) *emanating from u_0 satisfies*

$$u(t,x) = U_{c_*}\Big(|x| - c_*t + \frac{N+2}{c_*}\mathrm{ln}t + s^\infty(\frac{x}{|x|})\Big) + o_{t\to+\infty}(1),$$

if $\alpha > 1$, and

$$u(t,x) = U_{c_*}\Big(|x| - c_*t + \frac{N+2-\lambda}{c_*}\mathrm{ln}t + s^\infty(\frac{x}{|x|})\Big) + o_{t\to+\infty}(1),$$

if $\alpha = 1$.

The shift $\frac{N+2-\lambda}{c_*}$ has already been identified by Ducrot [8], up to $O(1)$ terms. His assumptions are more general than ours, in the sense that he neither requires the gradient estimate on ν, nor the quantitative estimate for $\nu(x) - \frac{\lambda}{|x|^\alpha}$. However, our result goes one step further. Theorem 5.1 would probably hold without the error estimate on $\nu(x)$, one would simply need to be more careful in the construction of sub and super solutions. On the other hand, we have not tried to push the limits of validity of Theorem 5.1, and this might well be quite an interesting question.

The proof of Theorem 5.1 goes exactly along the same lines as that of Theorem 1.1 for $\alpha > 1$, the term $\nu(x)$ being thrown into the perturbative terms $l_i(\tau, \xi)$. Of course they now depend on Θ, but in a smooth and exponentially small in time fashion, so they do not require any additional arguments. When $\alpha = 1$, the same algebraic steps as in Section 2 reveal the presence of a nonperturbative term in equation (9). More precisely, this equation becomes

$$\begin{aligned} \partial_t v = \ & \partial_{rr} v + (\frac{N-1}{r + c_* t - k \mathrm{ln} t} - \frac{k}{t}) \partial_r v - (\frac{N-1}{r + c_* t - k \mathrm{ln} t} - \frac{k}{t} - \frac{\lambda}{r + c_* t - k \ln t}) v \\ & + \left(\nu(r + c_* t - k \mathrm{ln} t, \Theta) - \frac{\lambda}{r + c_* t - l \ln t} \right) v + \frac{\Delta_\Theta v}{(r + c_* t - k \mathrm{ln} t)^2} - e^{-r} v^2. \end{aligned} \tag{47}$$

To identify k we simply have to make sure that equation (47) behaves like the Dirichlet heat equation, perturbed by higher order terms; thus the formula (10) becomes

$$\frac{N-1}{c_*} - \frac{\lambda}{c_*} - k = -\frac{3}{2},$$

hence the shift. The remaining terms will be, in the self-similar variables, exponentially decreasing terms. The method used to prove a gradient estimate in Θ for v will then work exactly as in Proposition 4.3, thanks to the estimate on $|\nabla \nu|$.

We finally mention that we leave open the question of higher order expansion of the shift, which is also quite an interesting question.

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