# $L^{p}-L^{q}$ decay estimates for dissipative linear hyperbolic systems in 1D 

September 19, 2018

## Corrado MASCIA ${ }^{1}$ and Thinh Tien NGUYEN ${ }^{2}$


#### Abstract

Given $A, B \in M_{n}(\mathbb{R})$, we consider the Cauchy problem for partially dissipative hyperbolic systems having the form $$
\partial_{t} u+A \partial_{x} u+B u=0
$$ with the aim of providing a detailed description of the large-time behavior. Sharp $L^{p}{ }_{-}$ $L^{q}$ estimates are established for the distance between the solution to the system and a time-asymptotic profile, where the profile is the superposition of diffusion waves and exponentially decaying waves.


Keywords: Large-time behavior, Dissipative linear hyperbolic systems, Asymptotic expansions
MSC2010: 35L45, 35C20

## Contents

1. Introduction ..... 1
Notations and Definitions ..... 6
2. Asymptotic expansions ..... 6
3. Fundamental solution ..... 13
4. Multiplier estimates ..... 22
4.1. Case $|x| \leq C t$ ..... 22
4.2. Case $|x|>C t$ ..... 25
5. Symmetry ..... 26
6. Proof of main results ..... 29
Appendix ..... 30
Eigenprojection computation ..... 30
Perturbation theory for linear operators ..... 34
Acknowledgments ..... 38
References ..... 38

## 1. Introduction

This work is concerned with the Cauchy problem

$$
\begin{equation*}
\partial_{t} u+\mathcal{L} u:=\partial_{t} u+A \partial_{x} u+B u=0, \quad u(0, x)=u_{0}(x) \tag{1.1}
\end{equation*}
$$

[^0]Even if we are going to discuss properties of the system (1.1) on its own, we primarily regard at the system (1.1) as a linearization of the nonlinear hyperbolic system

$$
\begin{equation*}
\partial_{t} u+\partial_{x} F(u)+G(u)=0 \tag{1.2}
\end{equation*}
$$

at a constant stationary state $\bar{u}$ satisfying $G(\bar{u})=0$ (see [1, 7] and descendants). In addition, linear systems fitting in the class (1.1) emerge as models for velocity jump processes such as the Goldstein-Kac model [3, 5] (a generalization was introduced in [9]) and in other fields of application.

On the other hand, decay estimates for the system (1.1) have been accomplished for years. In [10], it is proved that the $L^{2}$-norm of the solution $u$ to (1.1) is bounded by the sum of the two terms: the first term, with respect to the $L^{2}$-norm of the initial datum of $u$, decays exponentially and the second one, with respect to the $L^{q}$-norm of the initial datum of $u$ for $q \in[1, \infty]$, decays at the rate $(1 / q-1 / 2) / 2$. In that work, the matrices $A$ and $B$ are symmetric and they satisfy the Kawashima-Shizuta condition: if $z$ is an eigenvector of $A$, then $z$ does not belong to ker $B$, which is required for designing a compensating matrix to capture the dissipation of the system (1.1) over the degenerate kernel space of $B$ since the symmetric structure is not enough to guarantee the decay. The result is then improved in [2], if (1.1) has a convex entropy and satisfies the Kawashima-Shizuta condition, $B$ can be written in the block-diagonal form diag $\left(O_{m \times m}, D\right)$ where $O_{m \times m}$ is the $m \times m$ null matrix and $D \in M_{n-m}(\mathbb{R})$ is positive definite, and by considering the parabolic equation given by applying the Chapman-Enskog expansion to the system (1.1), the $L^{p}$-norm of the difference between the solution $u$ and the solution $U$ to the parabolic equation decays $1 / 2$ faster than the rate $(1-1 / p) / 2$ in terms of the $L^{1} \cap L^{2}$-norm of the initial datum of $u$. Recently, [11] can be seen as a generalization of [10] for any non symmetric matrix $B$ under appropriate conditions.

More detailed descriptions of the asymptotic behavior have been provided for specific classes of equations by $L^{p}-L^{q}$ estimates e.g. the $L^{p}-L^{q}$ estimate for the Cauchy problem for the damped wave equation

$$
\begin{equation*}
\partial_{t} u+\partial_{t t} u-\Delta u=0 . \tag{1.3}
\end{equation*}
$$

It shows that the time-asymptotic profile of the solution $u$ to (1.3) includes the solution to a heat equation and the solution to a wave equation, and when measuring the initial datum of $u$ in $L^{q}$, the $L^{p}$ distance between $u$ and this profile decays $\varepsilon>0$ faster than the rate $\alpha(p, q):=d(1 / q-1 / p) / 2$ where $d$ is the spacial dimension (see [4, 8]).

We are not aware of any results on such kind of estimate for the general system (1.1) with general exponents $p$ and $q$. We start with the following assumptions on the matrices $A$ and $B$.

Condition A. [Hyperbolicity] The matrix $A$ is diagonalizable with real eigenvalues.
Condition B. [Partial dissipativity] 0 is a semi-simple eigenvalue of $B$ with algebraic multiplicity $m \geq 1$ and the spectrum $\sigma(B)$ of $B$ can be decomposed as $\{0\} \cup \sigma_{0}$ with $\sigma_{0} \subset\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda>0\}$.

Let $P_{0}^{(0)}$ be the unique eigenprojection associated with the eigenvalue 0 of $B$, then the reduced system is given by

$$
\begin{equation*}
\partial_{t} w+C \partial_{x} w \approx 0 \tag{1.4}
\end{equation*}
$$

where $w:=P_{0}^{(0)} u$ and $C:=P_{0}^{(0)} A P_{0}^{(0)}$. One assumes
Condition C. [Reduced hyperbolicity] The matrix $C$ is diagonalizable with real eigenvalues considered in $\operatorname{ker}(B)$.

On the other hand, the requisite condition for the decay of the solution to the system (1.1), which is related to the well-known Kawashima-Shizuta condition, is that the eigenvalues $\lambda(i \xi)$ of the operator $E(i \xi):=-(B+i \xi A)$ satisfy

Condition D. [Uniform dissipativity] There is $\theta>0$ such that

$$
\begin{equation*}
\operatorname{Re}(\lambda(i \xi)) \leq-\frac{\theta|\xi|^{2}}{1+|\xi|^{2}}, \quad \text { for } \xi \neq 0 \tag{1.5}
\end{equation*}
$$

In this framework, we show that under the assumptions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$, the timeasymptotic profile of the solution to the system (1.1) is the superposition of diffusion waves and exponentially decaying waves.

The diffusion waves are constructed as follows. Let $\Gamma_{0}$ be an oriented closed curve enclosing the eigenvalue 0 except for the nonzero eigenvalues of $B$ in the resolvent set $\rho(B)$, one sets

$$
\begin{equation*}
S_{0}^{(0)}:=\frac{1}{2 \pi i} \int_{\Gamma_{0}} z^{-1}(B-z I)^{-1} d z \tag{1.6}
\end{equation*}
$$

On the other hand, let

$$
\begin{equation*}
P_{0}^{(1)}:=-P_{0}^{(0)} A S_{0}^{(0)}-S_{0}^{(0)} A P_{0}^{(0)} \tag{1.7}
\end{equation*}
$$

one defines

$$
\begin{equation*}
D:=-\left(P_{0}^{(1)} B P_{0}^{(1)}+P_{0}^{(0)} A P_{0}^{(1)}+P_{0}^{(1)} A P_{0}^{(0)}\right) . \tag{1.8}
\end{equation*}
$$

Then, we consider the Cauchy problem with respect to $U$ in $\operatorname{ran}\left(P_{j}^{(0)}\right)$, such that

$$
\begin{equation*}
\partial_{t} U+c_{j} \partial_{x} U-P_{j}^{(0)} D \partial_{x x} U=0, \quad U(0, x)=P_{j}^{(0)} u_{0}(x) \tag{1.9}
\end{equation*}
$$

where $c_{j}$ and $P_{j}^{(0)}$ are the $j$-th element of the spectrum of $C$ considered in $\operatorname{ker}(B)$ and the eigenprojection associated with it for $j=1, \ldots, h$ and $h \leq m$ is the cardinality of the spectrum of $C$ considered in $\operatorname{ker}(B)$ and $m$ is the algebraic multiplicity of the eigenvalue 0 of $B$. Thus, one can choose $U:=\sum_{j=1}^{h} U_{j}$ where $U_{j}$ is the solution to the system (1.9) for $j=1, \ldots, h$.

On the other hand, the coefficients $P_{0}^{(0)}$ and $S_{0}^{(0)}$ can be computed by the formula

$$
\begin{equation*}
P_{0}^{(0)}=\mathbb{P}_{m-1}(B) \quad \text { and } \quad S_{0}^{(0)}=\mathbb{S}_{m-1}(B) \tag{1.10}
\end{equation*}
$$

where the matrix-valued functions $\mathbb{P}$ and $\mathbb{S}$ are introduced in (6.1) and (6.2) in the appendix section. Moreover, let $\alpha>\max \{|\lambda|: \lambda \in \sigma(C)\}$ and let $C^{\prime}:=C+\alpha P_{0}^{(0)}$, then $C^{\prime}$ has
$h$ distinct nonzero eigenvalues denoted by $c_{j}^{\prime}$ with algebraic multiplicities $m_{j} \geq 1$ for $j \in\{1, \ldots, h\}$. Thus, $P_{j}^{(0)}$ can be computed by the formula

$$
\begin{equation*}
P_{j}^{(0)}=\mathbb{P}_{m_{j}-1}\left(C^{\prime}-c_{j}^{\prime} I\right) \tag{1.11}
\end{equation*}
$$

for $j \in\{1, \ldots, h\}$. Noting that the shift from $C$ to $C^{\prime}$ is requisite since we consider only the eigenvalues of $C$ restricted to $\operatorname{ker}(B)$.

The exponentially decaying waves are constructed as follows. Due to the diagonalizable property of $A$, let $Q \in M_{n}(\mathbb{R})$ be the invertible matrix diagonalizing $A$. Then, one sets

$$
\begin{equation*}
\bar{A}:=\operatorname{diag}\left(a_{1}, \ldots, a_{n}\right), \quad \bar{B}:=Q^{-1} B Q \tag{1.12}
\end{equation*}
$$

where $a_{j} \in \mathbb{R}$ for $j=1, \ldots, n$ are the repeated eigenvalues of $A$. Let define a partition denoted by $\left\{\mathcal{S}_{j}: j=1, \ldots, s\right\}$ of $\{1, \ldots, n\}$ for some $s \leq n$ such that $h, k \in \mathcal{S}_{j}$ if $a_{h}=a_{k}$, it is easy to see that $s$ is the cardinality of the spectrum of $\bar{A}$. On the other hand, we also define the matrix

$$
\left(\Pi_{j}^{(0)}\right)_{h k}:= \begin{cases}1 & \text { if } h=k \in \mathcal{S}_{j}  \tag{1.13}\\ 0 & \text { otherwise }\end{cases}
$$

for $h, k=1, \ldots, n$. Then we consider the Cauchy problem with respect to $V \in \operatorname{ran}\left(\Pi_{j}^{(0)}\right)$ such that

$$
\begin{equation*}
\partial_{t} V+\alpha_{j} \partial_{x} V+\Pi_{j}^{(0)} \bar{B} V=0, \quad V(0, x)=\Pi_{j}^{(0)} Q^{-1} u_{0}(x) \tag{1.14}
\end{equation*}
$$

where $\alpha_{j}=a_{h}$ if $h \in \mathcal{S}_{j}$ for $j=1, \ldots, s$. Thus, we can choose $V:=Q \sum_{j=1}^{s} V_{j}$ where $V_{j}$ is the solution to the system (1.14) for $j=1, \ldots, s$.

Theorem $1.1\left(L^{p}-L^{q}\right)$. If $u$ is the solution to the system (1.1) with the initial datum $u_{0} \in L^{q}(\mathbb{R})$, the conditions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ imply that, for $1 \leq q \leq p \leq \infty$ and $t \geq 1$, there are constants $C:=C(p, q)>0$ and $\delta>0$ such that

$$
\begin{equation*}
\|u-U-V\|_{L^{p}} \leq C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\left\|u_{0}\right\|_{L^{q}} \tag{1.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\|U\|_{L^{p}} \leq C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\left\|u_{0}\right\|_{L^{q}} \quad \text { and } \quad\|V\|_{L^{2}} \leq C e^{-\delta t}\left\|u_{0}\right\|_{L^{2}} \tag{1.16}
\end{equation*}
$$

Going back to the $L^{p}-L^{q}$ decay estimate in [8], if the initial condition for the Cauchy problem for (1.3) is given by $\left.\left(u, \partial_{t} u\right)\right|_{t=0}=\left(u_{0}, u_{1}\right)$, then the following estimate holds

$$
\begin{equation*}
\left\|u-U-e^{-t / 2} V\right\|_{L^{p}} \leq C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\left\|u_{0}+u_{1}\right\|_{L^{q}}, \quad \forall t \geq 1 \tag{1.17}
\end{equation*}
$$

where respectively, $U$ and $V$ are the solutions to the Cauchy problems

$$
\left\{\begin{array} { l } 
{ \partial _ { t } U - \partial _ { x x } U = 0 , } \\
{ U ( x , 0 ) = u _ { 0 } ( x ) + u _ { 1 } ( x ) , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
\partial_{t t} V-\partial_{x x} V=0 \\
V(x, 0)=u_{0}(x), \quad \partial_{t} V(x, 0)=0
\end{array}\right.\right.
$$

Comparing (1.15) with (1.17), we recognize a difference of $1 / 2$ in the decay rates. The better decay, which is valid for the linear damped wave equation, is a consequence of an additional property, namely the invariance with respect to the transformation $x \mapsto$ $-x$. Indeed, in terms of the Goldstein-Kac system, such symmetry implies that the
eigenvalue curves of $E(i \xi)=-(B+i \xi A)$ which pass through 0 can be expanded as $\lambda(i \xi):=-d_{0} \xi^{2}+\mathcal{O}\left(|\xi|^{4}\right)$ for some $d_{0}>0$ as $|\xi| \rightarrow 0$, and the fact that the error terms are $\mathcal{O}\left(|\xi|^{4}\right)$ guarantees the gain of 1 instead of $1 / 2$ in the decay rate, where $1 / 2$ holds for general cases where the error terms are $\mathcal{O}\left(|\xi|^{3}\right)$.

Thus, we are also interested in systems fitting in the class (1.1) that have an analogous property, namely

Hypothesis S. [Symmetry] There is an invertible symmetric matrix $S$ such that

$$
A S=-S A \quad \text { and } \quad B S=S B
$$

When the above assumption holds, if $u:=u(x, t)$ is a solution to (1.1), then the reflection $v:=v(x, t)=u(-x, t)$ is a solution to the same system as well.

Let us consider a stronger assumption than the condition $\mathbf{C}$ on the reduced system, namely

Condition C'. [Reduced strictly hyperbolicity] The matrix $C$ is diagonalizable with $m$ real distinct eigenvalues considered in $\operatorname{ker}(B)$.

Let $U=\sum_{j=1}^{m} U_{j}$ where $U_{j}$ is the solution to (1.9) with the initial datum given by

$$
\begin{equation*}
U(0, x):=\left(P_{j}^{(0)}+P_{j}^{(1)} \partial_{x}\right) u_{0}(x), \tag{1.18}
\end{equation*}
$$

where $P_{j}^{(0)}$ is already introduced and $P_{j}^{(1)}$ is as follows for $j \in\{1, \ldots, m\}$.
Let $\Gamma_{j}$ be an oriented closed curve enclosing the nonzero eigenvalue $c_{j}^{\prime}$ except for the other eigenvalues of $C^{\prime}$ in the resolvent set $\rho\left(C^{\prime}\right)$ for $j \in\{1, \ldots, m\}$. One sets

$$
\begin{equation*}
S_{j}^{(0)}:=\frac{1}{2 \pi i} \int_{\Gamma_{j}} z^{-1}\left(C^{\prime}-z I\right)^{-1} d z \tag{1.19}
\end{equation*}
$$

and then, $P_{j}^{(1)}$ can be computed by

$$
\begin{equation*}
P_{j}^{(1)}:=P_{j}^{(0)} D S_{j}^{(0)}+S_{j}^{(0)} D P_{j}^{(0)}, \tag{1.20}
\end{equation*}
$$

for all $j \in\{1, \ldots, m\}$. Similarly to before, $S_{j}^{(0)}$ can be computed by

$$
\begin{equation*}
S_{j}^{(0)}=\mathbb{S}_{m_{j}-1}\left(C^{\prime}-c_{j}^{\prime} I\right) \tag{1.21}
\end{equation*}
$$

since $c_{j}^{\prime}$ is simple for $j \in\{1, \ldots, m\}$.
Let $V$ be the same as before, one has
Theorem 1.2 (Increased decay rate). With the same hypotheses as in Theorem 1.1, if the condition $\mathbf{C}$ is substituted by the condition $\mathbf{C}$ ' and if the condition $\mathbf{S}$ holds, then, for $1 \leq q \leq p \leq \infty$, there is a positive constant $C:=C(p, q)>0$ such that

$$
\begin{equation*}
\|u-U-V\|_{L^{p}} \leq C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\left\|u_{0}\right\|_{L^{q}}, \quad \forall t \geq 1 \tag{1.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\|U\|_{L^{p}} \leq C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\left\|u_{0}\right\|_{L^{q}} \quad \text { and } \quad\|V\|_{L^{2}} \leq C e^{-\delta t}\left\|u_{0}\right\|_{L^{2}} . \tag{1.23}
\end{equation*}
$$

Once relaxing from $\mathbf{C}^{\prime}$ to $\mathbf{C}$, the decay rate in the estimate (1.15) does not increase in general since the condition $\mathbf{S}$ cannot prevent the eigenvalues of $E$ which converge to 0 from exhibiting non zero terms $(i \xi)^{3+\alpha}$ for $\alpha \in[0,1)$ in their expansions, and thus, it does not permit to have the gain of 1 in the decay rate.

The paper is organized as follows. In order to study the behavior of the solution to the system (1.1), we introduce the asymptotic expansion of the operator $E(i \xi)=-(B+i \xi A)$ in Section 2. Then, Section 3 and Section 4 are devoted to give the a priori estimates of the solution to the system (1.1). Moreover, the symmetry property of the system (1.1) is also discussed in Section 5. Then, we prove the main theorems in Section 6. Finally, we let the appendix section for some useful facts of the perturbation theory for linear operators in finite dimensional space together with a tool for computing the eigenprojections.

Notations and Definitions. Given a matrix operator $A$, we denote $\operatorname{ker}(A), \operatorname{ran}(A)$, $\rho(A)$ and $\sigma(A)$ the kernel, the range, the resolvent set and the spectrum of $A$ respectively.

On the other hand, we call $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ considered in a domain $\mathcal{D}$ if there is $u \in \mathcal{D}$ such that $u \neq O_{n \times 1}$ and $A u=\lambda u$.

For $x \in \mathbb{C}$ small enough, if $A(x)=A^{(0)}+\mathcal{O}(|x|)$ and $\lambda \in \sigma(A)$ satisfying $\lambda(x) \rightarrow \lambda^{(0)}$ as $|x| \rightarrow 0$ where $\lambda^{(0)} \in \sigma\left(A^{(0)}\right)$, the set of all such eigenvalues of $A$ is called the $\lambda^{(0)}$-group. Moreover, $P$ is called the total projection of a group if $P$ is the sum of the eigenprojections associated with the eigenvalues belonging to that group.

Let $T: \mathbb{R} \rightarrow \mathcal{B}$ where $\mathcal{B}$ is a Banach space with some suitable norm $|\cdot|_{\mathcal{B}}$. Define the $L^{p}(\mathbb{R}, \mathcal{B})$-norm of $T$ as follows.

$$
\|T\|_{L^{p}}:=\left(\int_{-\infty}^{+\infty}|T(x)|_{\mathcal{B}} d x\right)^{1 / p}, \quad 1 \leq p<\infty
$$

and

$$
\|T\|_{L^{\infty}}:=\operatorname{ess} \sup _{-\infty<x<+\infty}|T(x)|_{\mathcal{B}} .
$$

From here, we use the notation $|\cdot|$ instead of $|\cdot|_{\mathcal{B}}$ to indicate the norm associated with $\mathcal{B}$.
Let $m$ be a tempered distribution, $m$ is called a Fourier multiplier on $L^{p}$, for $1 \leq p \leq \infty$, if

$$
\sup _{\|f\|_{L^{p}=1}}\left\|\mathcal{F}^{-1}(m) * f\right\|_{L^{p}}<+\infty .
$$

The $M_{p}$ space, for $1 \leq p \leq \infty$, is the space of Fourier multipliers endowed with the norm

$$
\|m\|_{M_{p}}=\sup _{\|f\|_{L^{p}}=1}\left\|\mathcal{F}^{-1}(m) * f\right\|_{L^{p}}
$$

## 2. Asymptotic expansions

We study the asymptotic expansions of the eigenvalues of the operator $E(i \xi)=-(B+$ $i \xi A$ ) by dividing the frequency domain $\xi \in \mathbb{R}$ into the low frequency as $|\xi| \rightarrow 0$, the intermediate frequency as $|\xi|$ away from 0 and $+\infty$ and the high frequency as $|\xi| \rightarrow+\infty$.

Primarily, we consider the low-frequency case. Due to the fact that the eigenvalues of $E$ converge to the eigenvalues of $B$ as $|\xi| \rightarrow 0$ in general and the condition $\mathbf{B}$, the eigenvalues
of $E$ are divided into two groups such that one among them contains the eigenvalues of $E$ converging to 0 as $|\xi| \rightarrow 0$. Thus, we will study these two groups separately for the low-frequency case. We also recall the matrices $C$ and $D$ in (1.4) and (1.8) respectively.

Proposition 2.1 (Low frequency 1). Let $h \in \mathbb{Z}^{+}$be the cardinality of the spectrum of the matrix $C$ considered in $\operatorname{ker}(B)$. If the condition $\mathbf{C}$ holds, then, for $j \in\{1, \ldots, h\}$, there is $h_{j} \in \mathbb{Z}^{+}$to be less than or equal to the algebraic multiplicity of the $j$-th eigenvalue of $C$ considered in $\operatorname{ker}(B)$ such that there are $h_{j}$ groups of the eigenvalues of $E$ and the approximation of the elements of the $\ell$-th group has the form

$$
\begin{equation*}
\lambda_{j \ell}(i \xi)=-i c_{j} \xi-d_{j \ell} \xi^{2}+\mathcal{O}\left(|\xi|^{2}\right), \quad|\xi| \rightarrow 0 \tag{2.1}
\end{equation*}
$$

where $c_{j} \in \sigma(C)$ considered in $\operatorname{ker}(B)$ and $d_{j \ell} \in \sigma\left(P_{j}^{(0)} D P_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $\ell=1, \ldots, h_{j}$ with $P_{j}^{(0)}$ the eigenprojection associated with $c_{j}$. In particular, if the condition $\mathbf{D}$ holds, then

$$
\begin{equation*}
\operatorname{Re}\left(d_{j \ell}\right) \geq \theta>0, \quad \text { for } \ell=1, \ldots, h_{j} \text { and } j=1, \ldots, h \tag{2.2}
\end{equation*}
$$

Moreover, the total projection associated with the $\ell$-th group is then approximated by

$$
\begin{equation*}
P_{j \ell}(i \xi)=P_{j \ell}^{(0)}+\mathcal{O}(|\xi|), \quad|\xi| \rightarrow 0 \tag{2.3}
\end{equation*}
$$

where $P_{j \ell}^{(0)}$ is the eigenprojection associated with $d_{j \ell}$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $\ell \in$ $\left\{1, \ldots, h_{j}\right\}$ and $j \in\{1, \ldots, h\}$.

Proof. This proof is dealt with the 0 -group of $E$ i.e. the group contains the eigenvalues of $E$ converging to 0 as $|\xi| \rightarrow 0$. On the other hand, we can consider $T(\zeta):=B+\zeta A$ where $\zeta=i \xi$ instead of $E$ in order to apply Proposition 6.5 and Proposition 6.6 since $E=-T$. The proof then includes three steps of approximation and reduction steps interlacing them.

Step 0: It is obvious that the approximation of the elements of the 0-group of $T$ has the form

$$
\lambda_{0}(\zeta)=\mathcal{O}(1), \quad|\zeta| \rightarrow 0
$$

On the other hand, by Proposition 6.6, the total projection associated with this group is approximated by

$$
P_{0}(\zeta)=P_{0}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0
$$

where $P_{0}^{(0)}$ is the eigenprojection associated with the eigenvalue 0 of $B$.
In particular, we can perform a more accurate expansion of $P_{0}$. Indeed, we have

$$
\begin{equation*}
P_{0}(\zeta)=P_{0}^{(0)}+\zeta P_{0}^{(1)}+\mathcal{O}\left(|\zeta|^{2}\right), \quad|\zeta| \rightarrow 0 \tag{2.4}
\end{equation*}
$$

where $P_{0}^{(1)}$ can be computed by the formula (1.7). We will prove the formula (1.7) in brief. As $|\zeta| \rightarrow 0$, for $z \in \Gamma$ any compact set contained in the resolvent set $\rho(B)$ of $B$, we have the uniformly convergent expansion

$$
\begin{equation*}
(T(\zeta)-z I)^{-1}=(B-z I)^{-1}-\zeta(B-z I)^{-1} A(B-z I)^{-1}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0, \tag{2.5}
\end{equation*}
$$

and we also have the expansion about 0 of the resolvent

$$
\begin{equation*}
(B-z I)^{-1}=\sum_{h=-\infty}^{-1}\left(N_{0}^{(0)}\right)^{h} z^{h}+P_{0}^{(0)}+\sum_{h=1}^{+\infty}\left(S_{0}^{(0)}\right)^{h} z^{h} \tag{2.6}
\end{equation*}
$$

where $P_{0}^{(0)}, N_{0}^{(0)}$ and $S_{0}^{(0)}$ are the eigenprojection, the nilpotent matrix and the reduced resolvent coefficient associated with the eigenvalue 0 of $B$ respectively. On the other hand, the formula for $S_{0}^{(0)}$ is introduced in (1.6). The expansions (2.5) and (2.6) can be obtained easily due to the properties of the resolvent (see [6]). Therefore, since the total projection $P_{0}$ deduced from Proposition 6.6 can be seen as the Cauchy integral

$$
P_{0}(\zeta)=-\frac{1}{2 \pi i} \int_{\Gamma_{0}}(T(\zeta)-z I)^{-1} d z
$$

where $\Gamma_{0}$ is an oriented closed curve enclosing 0 except for the other eigenvalues of $B$ in the resolvent set $\rho(B)$. Hence, since $\Gamma_{0}$ is a compact set of $\rho(B)$, one can apply (2.5) and (2.6) into the integral formula of $P_{0}$ and we thus obtain (2.4) by computing the residue.

Reduction step: From Proposition 6.5 and Proposition 6.6 , one also has

$$
\mathbb{C}^{n}=\operatorname{ran}\left(P_{0}\right) \oplus\left(\mathbb{C}^{n}-\operatorname{ran}\left(P_{0}\right)\right), \quad T=T P_{0}+T\left(I-P_{0}\right)
$$

Thus, the study of the 0 -group of $T$ considered in $\mathbb{C}^{n}$ is reduced to the study of the eigenvalues of $T P_{0}$ considered in $\operatorname{ran}\left(P_{0}\right)$.

Step 1: Under the condition $\mathbf{B}$, the eigenvalue 0 of $B$ is semi-simple i.e. $B P_{0}^{(0)}=O_{n \times 1}$ and $\operatorname{ran}\left(P_{0}^{(0)}\right)=\operatorname{ker}(B)$. Thus, based on the expansion (2.4) of $P_{0}$ and the fact that $T P_{0}=P_{0} T P_{0}$, one has

$$
\begin{aligned}
T(\zeta) P_{0}(\zeta) & =\left(P_{0}^{(0)}+\zeta P_{0}^{(1)}+\mathcal{O}\left(|\zeta|^{2}\right)\right)(B+\zeta A)\left(P_{0}^{(0)}+\zeta P_{0}^{(1)}+\mathcal{O}\left(|\zeta|^{2}\right)\right) \\
& =\zeta\left(C-\zeta D+\mathcal{O}\left(|\zeta|^{2}\right)\right), \quad|\zeta| \rightarrow 0
\end{aligned}
$$

where $C$ is in (1.4) and $D$ is in (1.8). It follows that $\lambda \in \sigma\left(T P_{0}\right)$ considered in $\operatorname{ran}\left(P_{0}\right)$ if and only if $\tilde{\lambda}:=\zeta^{-1} \lambda$ is an eigenvalue of $T_{0}(\zeta):=C-\zeta D+\mathcal{O}\left(|\zeta|^{2}\right)$ considered in $\operatorname{ran}\left(P_{0}\right)$. Therefore, it returns to the eigenvalue problem of $T_{0}$ considered in the domain $\operatorname{ran}\left(P_{0}\right)$ and one can apply again Proposition 6.6.

Let $c_{j}$ be the $j$-th element of $\sigma(C)$ considered in $\operatorname{ker}(B)=\operatorname{ran}\left(P_{0}^{(0)}\right)$ for $j \in\{1, \ldots, h\}$, then by Proposition $6.6, \tilde{\lambda} \in \sigma\left(T_{0}\right)$ considered in $\operatorname{ran}\left(P_{0}\right)$ if and only if $\tilde{\lambda} \rightarrow c_{j}$ as $|\zeta| \rightarrow 0$ for some $j \in\{1, \ldots, h\}$. Thus, $\lambda \in \sigma\left(T P_{0}\right)$ considered in $\operatorname{ran}\left(P_{0}\right)$ if and only if $\zeta^{-1} \lambda \rightarrow c_{j}$ as $|\zeta| \rightarrow 0$ for some $j \in\{1, \ldots, h\}$. One concludes that the eigenvalues of $T P_{0}$ considered in $\operatorname{ran}\left(P_{0}\right)$ are characterized by $c_{j}$ for $j=1, \ldots, h$ and thus they are divided into $h$ groups such that the approximation of the elements of the $j$-th group with respect to $c_{j}$ has the form

$$
\lambda_{j}(\zeta)=c_{j} \zeta+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0
$$

and on the other hand, by Proposition 6.6, the total projection associated with this group is approximated by

$$
\begin{equation*}
P_{j}(\zeta)=P_{j}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0 \tag{2.7}
\end{equation*}
$$

where $P_{j}^{(0)}$ is the eigenprojection associated with $c_{j}$ considered in $\operatorname{ker}(B)$ for $j=1, \ldots, h$.
Reduction step: By Proposition 6.6, $T_{0}$ commutes with $P_{j}$ for all $j=1, \ldots, h$ and one has

$$
\operatorname{ran}\left(P_{0}\right)=\bigoplus_{j=1}^{h} \operatorname{ran}\left(P_{j}\right), \quad T_{0}=\sum_{j=1}^{h}\left(T_{0} P_{j}\right)
$$

The study of the eigenvalues of $T_{0}$ considered in $\operatorname{ran}\left(P_{0}\right)$ is then reduced to the study of the eigenvalues of $T_{0} P_{j}$ considered in $\operatorname{ran}\left(P_{j}\right)$ for $j=1, \ldots, h$.

Final step: Under the condition $\mathbf{C}$, for $j \in\{1, \ldots, h\}$, the eigenvalue $c_{j}$ of $C$ is semisimple i.e. $\left(C-c_{j} I\right) P_{j}^{(0)}=O_{n \times 1}$ and $\operatorname{ran}\left(P_{j}^{(0)}\right)=\operatorname{ker}\left(C-c_{j} I\right)$. Thus, based on the expansion (2.7) of $P_{j}$ and the fact that $T_{0} P_{j}=P_{j} T_{0} P_{j}$, one has

$$
\begin{aligned}
\left(T_{0}(\zeta)-c_{j} I\right) P_{j}(\zeta) & =\left(P_{j}^{(0)}+\mathcal{O}(|\zeta|)\right)\left(C-c_{j} I-\zeta D+\mathcal{O}\left(|\zeta|^{2}\right)\right)\left(P_{j}^{(0)}+\mathcal{O}(|\zeta|)\right) \\
& =\zeta\left(-D_{j}+\mathcal{O}(|\zeta|)\right), \quad|\zeta| \rightarrow 0 .
\end{aligned}
$$

where $D_{j}:=P_{j}^{(0)} D P_{j}^{(0)}$. It follows that $\lambda \in \sigma\left(T_{0} P_{j}\right)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\tilde{\lambda}:=\zeta^{-1}\left(\lambda-c_{j}\right)$ is an eigenvalue of $T_{j}(\zeta):=-D_{j}+\mathcal{O}(|\zeta|)$ considered in $\operatorname{ran}\left(P_{j}\right)$. Therefore, it returns to the eigenvalue problem of $T_{j}$ considered in the domain $\operatorname{ran}\left(P_{j}\right)$ and one can apply again Proposition 6.6.

For $j \in\{1, \ldots, h\}$, let $h_{j}$ be the cardinality of the spectrum of $D_{j}$ considered in $\operatorname{ker}(C-$ $\left.c_{j} I\right)=\operatorname{ran}\left(P_{j}^{(0)}\right)$ and let $d_{j \ell}$ be the $\ell$-th element of the spectrum for $\ell=1, \ldots, h_{j}$. Then by Proposition 6.6, $\tilde{\lambda} \in \sigma\left(T_{j}\right)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\tilde{\lambda} \rightarrow-d_{j \ell}$ as $|\zeta| \rightarrow 0$ for some $\ell \in\left\{1, \ldots, h_{j}\right\}$. Thus, $\lambda \in \sigma\left(T_{0} P_{j}\right)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\zeta^{-1}\left(\lambda-c_{j}\right) \rightarrow-d_{j \ell}$ as $|\zeta| \rightarrow 0$ for some $\ell \in\left\{1, \ldots, h_{j}\right\}$. One concludes that the eigenvalues of $T_{0} P_{j}$ considered in $\operatorname{ran}\left(P_{j}\right)$ are characterized by $d_{j \ell}$ for $\ell=1, \ldots, h_{j}$ and thus they are divided into $h_{j}$ groups such that the approximation of the elements of the $\ell$-th group with respect to $d_{j \ell}$ has the form

$$
\lambda_{j \ell}(\zeta)=c_{j} \zeta-d_{j \ell} \zeta^{2}+\mathcal{O}\left(|\zeta|^{2}\right), \quad|\zeta| \rightarrow 0 .
$$

and on the other hand, by Proposition 6.6, the total projection associated with this group is approximated by

$$
\begin{equation*}
P_{j \ell}(\zeta)=P_{j \ell}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0 \tag{2.8}
\end{equation*}
$$

where $P_{j \ell}^{(0)}$ is the eigenprojection associated with $d_{j \ell}$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $\ell=$ $1, \ldots, h_{j}$.

We then deduce from the above steps of approximation for $E(i \xi)=-T(i \xi)$ by multiplying $\lambda_{j \ell}(i \xi)$ by -1 to obtain (2.1), and (2.3) is the same as $P_{j \ell}(i \xi)$ for each $j \in\{1, \ldots, h\}$ and $\ell=1, \ldots, h_{j}$.

Finally, we prove the estimate (2.2). For $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$, since $\lambda_{j \ell}$ in (2.1) can be seen as an eigenvalue of $E$ and since $c_{j}$ is real by the condition $\mathbf{C}$, if the condition $\mathbf{D}$ holds, then for $|\xi|$ small, one has

$$
\operatorname{Re}\left(\lambda_{j \ell}(i \xi)\right)=-\operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2}+\operatorname{Re}\left(\mathcal{O}\left(|\xi|^{2}\right)\right) \leq-\frac{\theta|\xi|^{2}}{1+|\xi|^{2}}
$$

Passing through the limit as $|\xi| \rightarrow 0$, one has the desired estimate. The proof is done.
Remark 2.2. As a consequence, for $|\xi|$ small, in $\operatorname{ran}\left(P_{j \ell}\right)$, the operator $E$ has the representation

$$
\begin{equation*}
E_{j \ell}(i \xi)=\left(-i c_{j} \xi-d_{j \ell} \xi^{2}\right) I-\xi^{2} N_{j \ell}^{(0)}+\mathcal{O}\left(|\xi|^{3}\right), \tag{2.9}
\end{equation*}
$$

where $N_{j \ell}^{(0)}$ is the nilpotent matrix associated with the eigenvalue $d_{j \ell}$ of $P_{j}^{(0)} D P_{j}^{(0)}$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$.

Proposition 2.3 (Low frequency 2). Let $k \in \mathbb{Z}^{+}$be the number of the nonzero distinct eigenvalues of $B$. If the condition $\mathbf{B}$ holds, then there are $k$ groups of the eigenvalues of $E$ such that the approximation of the elements of the $j$-th group has the form

$$
\begin{equation*}
\eta_{j}(i \xi)=-e_{j}+\mathcal{O}(1), \quad|\xi| \rightarrow 0, \tag{2.10}
\end{equation*}
$$

where $e_{j} \in \sigma(B)$ with $\operatorname{Re}\left(e_{j}\right)>0$ for all $j=1, \ldots, k$.
Moreover, the total projection associated with the $j$-th group is then approximated by

$$
\begin{equation*}
F_{j}(i \xi)=F_{j}^{(0)}+\mathcal{O}(|\xi|), \quad|\xi| \rightarrow 0 \tag{2.11}
\end{equation*}
$$

Proof. Similarly to the proof of Proposition 2.1, we consider the operator $T(\zeta)=B+\zeta A$ where $\zeta=i \xi$. However, in this case, we study the eigenvalues of $T$ such that they converge to the nonzero eigenvalues of $B$ as $|\zeta| \rightarrow 0$. Let $e_{j}$ be the $j$-th element of the spectrum of $B$ except for 0 for $j=1, \ldots, k$. Then by Proposition 6.6 , for any $\eta \in \sigma(T)$ does not converge to $0, \eta \rightarrow e_{j}$ for some $j \in\{1, \ldots, k\}$. Hence, the approximation of these eigenvalues of $T$ is

$$
\eta_{j}(\zeta)=e_{j}+\mathcal{O}(1), \quad|\zeta| \rightarrow 0,
$$

and also from Proposition 6.6, the total projection associated with this group is approximated by

$$
\begin{equation*}
F_{j}(\zeta)=F_{j}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0 \tag{2.12}
\end{equation*}
$$

where $F_{j}^{(0)}$ is the eigenprojection associated with $e_{j}$ for $j=1, \ldots, k$. In particular, $\operatorname{Re}\left(e_{j}\right)>0$ due to the condition B.

Finally, since $E(i \xi)=-T(i \xi)$, we obtain (2.10) by multiplying $\eta_{j}(i \xi)$ by -1 and (2.11) is the same as $F_{j}(i \xi)$ for all $j=1, \ldots, k$.

Remark 2.4. As a consequence, for $|\xi|$ small, in $\operatorname{ran}\left(F_{j}\right)$, the operator $E$ has the representation

$$
\begin{equation*}
E_{j}(i \xi)=-e_{j} I-M_{j}^{(0)}+\mathcal{O}(|\xi|), \tag{2.13}
\end{equation*}
$$

where $M_{j}^{(0)}$ is the nilpotent matrix associated with the eigenvalue $e_{j}$ of $B$ for $j \in\{1, \ldots, k\}$.
The intermediate-frequency case is obtained as follows.

Proposition 2.5 (Intermediate frequency). In the compact domain $\varepsilon \leq|\xi| \leq R$, there is only a finite number of the exceptional points at which the eigenprojections and the nilpotent parts associated with the eigenvalues of $E$ may have poles even the eigenvalues are continuous there.

On the other hand, in every simple domain excluded the exceptional points, the operator $E$ has $r$ (independent from $\xi$ ) distinct holomorphic eigenvalues denoted by $\nu_{j}$ with constant algebraic multiplicity together with holomorphic eigenprojections and nilpotent parts denoted by $\Psi_{j}$ and $\Xi_{j}$ associated with them respectively for $j \in\{1, \ldots, r\}$.

If the condition $\mathbf{D}$ holds, then $\operatorname{Re}(\nu)<0$ for any $\nu \in \sigma(E)$ in the domain $\varepsilon \leq|\xi| \leq R$. Proof. See [6].

For the high frequency, in order to analyze the eigenvalues of $E(i \xi)=-(B+i \xi A)$, one can analyze the eigenvalues of the operator $\bar{E}(i \xi):=Q^{-1} E(i \xi) Q=(-i \xi)\left(\bar{A}+(i \xi)^{-1} \bar{B}\right)$ where $\bar{A}$ and $\bar{B}$ are already introduced in (1.12).

Proposition 2.6 (High frequency). Let $s \in \mathbb{Z}^{+}$be the cardinality of the spectrum of the matrix $\bar{A}$. If the condition $\mathbf{A}$ holds, then, for $j \in\{1, \ldots, s\}$, there is $s_{j} \in \mathbb{Z}^{+}$to be less than or equal to the algebraic multiplicity of the $j$-th eigenvalue of $\bar{A}$ such that there are $s_{j}$ groups of the eigenvalues of $\bar{E}$ and the approximation of the elements of the $\ell$-th group has the form

$$
\begin{equation*}
\mu_{j \ell}(i \xi)=-i \alpha_{j} \xi-\beta_{j \ell}+\mathcal{O}\left(|\xi|^{-1}\right), \quad|\xi| \rightarrow+\infty \tag{2.14}
\end{equation*}
$$

where $\alpha_{j} \in \sigma(\bar{A})$ considered in $\mathbb{C}^{n}$ and $\beta_{j \ell} \in \sigma\left(\Pi_{j}^{(0)} \bar{B} \Pi_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)$ for $\ell=1, \ldots, s_{j}$ with $\Pi_{j}^{(0)}$ defined in (1.13) is the eigenprojection associated with $\alpha_{j}$. In particular, if the condition $\mathbf{D}$ holds, then

$$
\begin{equation*}
\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta>0, \quad \text { for } \ell=1, \ldots, s_{j} \text { and } j=1, \ldots, s \tag{2.15}
\end{equation*}
$$

Moreover, the total projection associated with the $\ell$-th group is then approximated by

$$
\begin{equation*}
\Pi_{j \ell}(i \xi)=\Pi_{j \ell}^{(0)}+\mathcal{O}\left(|\xi|^{-1}\right), \quad|\xi| \rightarrow+\infty \tag{2.16}
\end{equation*}
$$

where $\Pi_{j \ell}^{(0)}$ is the eigenprojection associated with $\beta_{j \ell}$ considered in $\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)$ for $\ell \in$ $\left\{1, \ldots, s_{j}\right\}$ and $j \in\{1, \ldots, s\}$.
Proof. Similarly to before, we can consider $T(\zeta):=\bar{A}+\zeta \bar{B}$ where $\zeta=(i \xi)^{-1}$ firstly in order to apply Proposition 6.5 and Proposition 6.6 since $|\zeta| \rightarrow 0$ as $|\xi| \rightarrow+\infty$. The proof then consists of two steps of approximation and one reduction step between them.

First step: The eigenvalues of $T$ are divided into several groups characterized by $\alpha_{j} \in \sigma(\bar{A})$ for $j=1, \ldots, s$. Moreover, for $j \in\{1, \ldots, s\}$, the approximation for the elements of the $\alpha_{j}$-group is

$$
\mu_{j}(\zeta)=\alpha_{j}+\mathcal{O}(1), \quad|\zeta| \rightarrow 0
$$

and the total projection associated with this group is then approximated by

$$
\begin{equation*}
\Pi_{j}(\zeta)=\Pi_{j}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0 \tag{2.17}
\end{equation*}
$$

where $\Pi_{j}^{(0)}$ is the eigenprojection associated with the eigenvalue $\alpha_{j}$ of $\bar{A}$. In particular, $\Pi_{j}^{(0)}$ is exactly the same as (1.13) since the eigenprojection $\Pi_{j}^{(0)}$ can be computed explicitly by the Cauchy integral

$$
\Pi_{j}^{(0)}=-\frac{1}{2 \pi i} \int_{\Gamma_{j}}(\bar{A}-z I)^{-1} d z=-\frac{1}{2 \pi i} \int_{\Gamma_{j}} \operatorname{diag}\left(a_{1}-z, \ldots, a_{n}-z\right)^{-1} d z
$$

where $\Gamma_{j}$ is an oriented closed curve enclosing $\alpha_{j}$ except for the other eigenvalues of $\bar{A}$ in the resolvent set $\rho(\bar{A})$. Hence, one obtains that

$$
\left(\Pi_{j}^{(0)}\right)_{h k}=\left\{\begin{array}{ll}
1 & \text { if } h=k, a_{h}=\alpha_{j} \\
0 & \text { otherwise }
\end{array}= \begin{cases}1 & \text { if } h=k \in \mathcal{S}_{j} \\
0 & \text { otherwise }\end{cases}\right.
$$

for all $h, k=1, \ldots, n$ due to the definition of $\mathcal{S}_{j}$ the $j$-th element of the partition $\left\{\mathcal{S}_{j}\right.$ : $j=1, \ldots, s\}$ of $\{1, \ldots, n\}$.

Reduction step: By Proposition $6.6, T$ commutes with $\Pi_{j}$ for all $j=1, \ldots, s$ and one has

$$
\mathbb{C}^{n}=\bigoplus_{j=1}^{s} \operatorname{ran}\left(\Pi_{j}\right), \quad T=\sum_{j=1}^{s}\left(T \Pi_{j}\right)
$$

It implies that the study of the eigenvalues of $T$ considered in $\mathbb{C}^{n}$ is reduced to the study of the eigenvalues of $T \Pi_{j}$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ for $j=1, \ldots, s$.

Final step: Under the condition $\mathbf{A}$, the eigenvalues of $\bar{A}$ are semi-simple i.e. $\bar{A} \Pi_{j}^{(0)}=$ $\alpha_{j} \Pi_{j}^{(0)}$ and $\operatorname{ran}\left(\Pi_{j}^{(0)}\right)=\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)$ for all $j=1, \ldots, s$. Thus, based on the expansion (2.17) of $\Pi_{j}$ and the fact that $T \Pi_{j}=\Pi_{j} T \Pi_{j}$, one has

$$
\begin{aligned}
\left(T(\zeta)-\alpha_{j} I\right) \Pi_{j}(\zeta) & =\left(\Pi_{j}^{(0)}+\mathcal{O}(|\zeta|)\right)\left(\bar{A}-\alpha_{j} I+\zeta \bar{B}\right)\left(\Pi_{j}^{(0)}+\mathcal{O}(|\zeta|)\right) \\
& =\zeta\left(\Pi_{j}^{(0)} \bar{B} \Pi_{j}^{(0)}+\mathcal{O}(|\zeta|)\right), \quad|\zeta| \rightarrow 0
\end{aligned}
$$

for $j=1, \ldots, s$. It follows that $\mu \in \sigma\left(T \Pi_{j}\right)$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ if and only if $\tilde{\mu}:=$ $\zeta^{-1}\left(\mu-\alpha_{j}\right)$ is an eigenvalue of $T_{j}(\zeta):=\Pi_{j}^{(0)} \bar{B} \Pi_{j}^{(0)}+\mathcal{O}(|\zeta|)$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ for $j=1, \ldots, s$. Therefore, it returns to the eigenvalue problem of $T_{j}$ considered in the domain $\operatorname{ran}\left(\Pi_{j}\right)$ for $j=1, \ldots, s$ and one can apply again Proposition 6.6.

For $j \in\{1, \ldots, s\}$, let $s_{j}$ be the cardinality of the spectrum of $\Pi_{j}^{(0)} \bar{B} \Pi_{j}^{(0)}$ considered in $\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)=\operatorname{ran}\left(\Pi_{j}^{(0)}\right)$ and let $\beta_{j \ell}$ be the $\ell$-th elements of the spectrum for $\ell=1, \ldots, s_{j}$. Then, by Proposition 6.6, $\tilde{\mu} \in \sigma\left(T_{j}\right)$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ if and only if $\tilde{\mu} \rightarrow \beta_{j \ell}$ as $|\zeta| \rightarrow 0$ for some $\ell \in\left\{1, \ldots, s_{j}\right\}$. Thus, $\mu \in \sigma\left(T \Pi_{j}\right)$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ if and only if $\zeta^{-1}\left(\mu-\alpha_{j}\right) \rightarrow \beta_{j \ell}$ as $|\zeta| \rightarrow 0$ for some $\ell \in\left\{1, \ldots, s_{j}\right\}$. It implies the eigenvalues of $T \Pi_{j}$ considered in $\operatorname{ran}\left(\Pi_{j}\right)$ are characterized by $\beta_{j \ell}$ such that the approximation of the elements of the $\ell$-th group with respect to $\beta_{j \ell}$ is

$$
\mu_{j \ell}(\zeta)=\alpha_{j}+\beta_{j \ell} \zeta+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0
$$

and also by Proposition 6.6 that the total projection associated with this group is approximated by

$$
\Pi_{j \ell}(\zeta)=\Pi_{j \ell}^{(0)}+\mathcal{O}(|\zeta|), \quad|\zeta| \rightarrow 0
$$

where $\Pi_{j \ell}^{(0)}$ is the eigenprojection associated with $\beta_{j \ell}$ considered in $\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)$ for $\ell=$ $1, \ldots, s_{j}$.

We deduce from the above steps of approximation for $\bar{E}(i \xi)=(-i \xi) T\left((i \xi)^{-1}\right)$ by multiplying $\lambda_{j \ell}\left((i \xi)^{-1}\right)$ by $(-i \xi)$ to obtain (2.14), and (2.16) is the same as $\Pi_{j \ell}\left((i \xi)^{-1}\right)$ for each $j \in\{1, \ldots, s\}$ and $\ell=1, \ldots, s_{j}$.

Finally, we prove the estimate (2.15). For $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$, since $\mu_{j \ell}$ in (2.14) can be seen as an eigenvalue of $\bar{E}$ and thus of $E=Q \bar{E} Q^{-1}$ and since $\alpha_{j}$ is real by the condition $\mathbf{A}$, if the condition $\mathbf{D}$ holds, then for $|\xi|$ large, one has

$$
\operatorname{Re}\left(\mu_{j \ell}(i \xi)\right)=-\operatorname{Re}\left(\beta_{j \ell}\right)+\operatorname{Re}(\mathcal{O}(1)) \leq-\frac{\theta|\xi|^{2}}{1+|\xi|^{2}}
$$

Passing through the limit as $|\xi| \rightarrow+\infty$, one has the desired estimate. We finish the proof.

Remark 2.7. As a consequence, for $|\xi|$ large, in $\operatorname{ran}\left(\Pi_{j \ell}\right)$, the operator $E$ has the representation

$$
\begin{equation*}
E_{j \ell}(i \xi)=\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) I-\Theta_{j \ell}^{(0)}+\mathcal{O}\left(|\xi|^{-1}\right) \tag{2.18}
\end{equation*}
$$

where $\Theta_{j \ell}^{(0)}$ is the nilpotent matrix associated with the eigenvalue $\beta_{j \ell}$ of $\Pi_{j}^{(0)} \bar{B} \Pi_{j}^{(0)}$ considered in $\operatorname{ker}\left(\bar{A}-\alpha_{j} I\right)$ for $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$.

## 3. Fundamental solution

The aim of this section is to introduce the estimates for the fundamental solution to (1.1) in the frequency space. Let consider the fundamental system

$$
\begin{equation*}
\partial_{t} \hat{G}-E \hat{G}=0, \quad \hat{G}_{t=0}=I \tag{3.1}
\end{equation*}
$$

where $E=E(i \xi)=-(B+i \xi A)$ with $\xi \in \mathbb{R}$.
One sets the following kernel

$$
\begin{equation*}
\hat{K}(\xi, t):=\sum_{j, \ell=1}^{h, h_{j}} e^{\left(-i c_{j} \xi-d_{j \ell} \xi^{2}\right) t} e^{-N_{j \ell}^{(0)} \xi^{2} t} P_{j \ell}^{(0)}, \tag{3.2}
\end{equation*}
$$

and the kernel

$$
\begin{equation*}
\hat{V}(\xi, t):=Q \sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell)}\right)} e^{-\Theta_{j \ell}^{(0)} t} \Pi_{j \ell}^{(0)} Q^{-1} \tag{3.3}
\end{equation*}
$$

where the coefficients are introduced in the previous section.
Moreover, we introduce the two useful lemmas used in this section as follows.
Lemma 3.1. If $X$ is a constant complex nilpotent matrix, then for all $\varepsilon^{\prime}>0$, there exists $C=C\left(\varepsilon^{\prime}\right)>0$ such that

$$
\left|e^{c X+Y}-e^{c X}\right| \leq C e^{\varepsilon^{\prime}|c|+C|Y|}|Y|
$$

and

$$
\left|e^{c X+Y}-e^{c X}-e^{c X} Y\right| \leq C e^{\varepsilon^{\prime}|c|+C|Y|}|Y|^{2}
$$

for every complex constant $c:=c(t)$ and matrix $Y:=Y(t)$ for $t>0$.
Proof. The proof is based on the existence of a basis of $\mathbb{C}^{n}$ such that $|X| \leq \varepsilon^{\prime}$ for any fixed $\varepsilon^{\prime}>0$ once written in this basis, then the constant $C\left(\varepsilon^{\prime}\right)$ can be chosen as the product of the norm of the changing basis matrix and the norm of its inverse for any matrix norm. The second inequality due to the fact that the first order derivative $\mathrm{d}_{\exp }$ at $X$ of the application $X \rightarrow e^{X}$ is $e^{X}$ and thus one has

$$
\left|e^{X+Y}-e^{X}-e^{X} Y\right| \leq C|Y|^{2} \sup _{s \in[0,1]}\left|\mathrm{d}_{\exp }^{2}(X+s Y)\right| \leq C|Y|^{2} e^{|X|+|Y|}
$$

where $\mathrm{d}_{\text {exp }}^{2}$ is the second order derivative of $X \rightarrow e^{X}$. Thus, under a change of basis, one obtains the desired estimiate. One can find a detailed proof in [2].

Lemma 3.2. For $0<\varepsilon<R<+\infty$, if $a, b \geq 0$ and $c, d>0$ and $r \in[1, \infty]$, there exists $C:=C(r)>0$ such that for all $t \geq 1$, one has

$$
\begin{equation*}
\left\||\cdot|{ }^{a} t^{b} e^{-c|\cdot|^{d} t}\right\|_{L^{r}} \leq C t^{-\frac{1}{d} \frac{1}{r}+b-\frac{a}{d}} \tag{3.4}
\end{equation*}
$$

Proof. By changing of variables.
Proposition 3.3 (Fundamental solution estimates). For $0<\varepsilon<R<+\infty$, for $r \in[1, \infty]$ and $t \geq 1$, there exists positive constants $C:=C(r)$ and $\delta$ such that if the conditions $\mathbf{A}$, $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are satisfied, the following hold.

1. For $|\xi|<\varepsilon$, one has

$$
\begin{equation*}
\|\hat{G}-\hat{K}\|_{L^{r}} \leq C t^{-\frac{1}{2} \frac{1}{r}-\frac{1}{2}}, \quad\|\hat{V}\|_{L^{r}} \leq C e^{-\delta t} \tag{3.5}
\end{equation*}
$$

2. For $\varepsilon \leq|\xi| \leq R$, one has

$$
\begin{equation*}
\|\hat{G}\|_{L^{r}},\|\hat{K}\|_{L^{r}},\|\hat{V}\|_{L^{r}} \leq C e^{-\delta t} \tag{3.6}
\end{equation*}
$$

3. For $|\xi|>R$, one has

$$
\begin{equation*}
\|\hat{G}-\hat{V}\|_{L^{r}} \leq C e^{-\delta t} \quad \text { for } \quad r>1, \quad\|\hat{K}\|_{L^{r}} \leq C e^{-\delta t} \tag{3.7}
\end{equation*}
$$

Moreover, we also have

$$
\begin{equation*}
\left\|\mathcal{F}^{-1}(\hat{G}-\hat{V})\right\|_{L^{\infty}} \leq C e^{-\delta t} \tag{3.8}
\end{equation*}
$$

Proof. For $|\xi|<\varepsilon$, by Remark 2.2 and Remark 2.4, the solution $\hat{G}$ to the system (3.1) is given by $\hat{G}=\hat{G}_{1}+\hat{G}_{2}$ where

$$
\begin{equation*}
\hat{G}_{1}(\xi, t):=\sum_{j, \ell=1}^{h, h_{j}} e^{\left(-i c_{j} \xi-d_{j \ell} \xi^{2}\right) t} e^{-N_{j \ell}^{(0)} \xi^{2} t+\mathcal{O}\left(|\xi|^{3}\right) t}\left(P_{j \ell}^{(0)}+\mathcal{O}(|\xi|)\right), \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{G}_{2}(\xi, t):=\sum_{j=1}^{k} e^{-e_{j} t} e^{-M_{j}^{(0)} t+\mathcal{O}(|\xi|) t}\left(F_{j}^{(0)}+\mathcal{O}(|\xi|)\right) . \tag{3.10}
\end{equation*}
$$

It follows that $\hat{G}-\hat{K}=\hat{G}_{1}-\hat{K}+\hat{G}_{2}=I_{1}+I_{2}+J$ where $J=\hat{G}_{2}$ and

$$
\begin{equation*}
I_{1}:=\sum_{j, \ell=1}^{h, h_{j}} e^{\left(-i c_{j} \xi-d_{j \ell} \xi^{2}\right) t}\left(e^{-N_{j \ell}^{(0)} \xi^{2} t+\mathcal{O}\left(|\xi|^{3}\right) t}-e^{-N_{j \ell}^{(0)} \xi^{2} t}\right) P_{j \ell}^{(0)} \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{2}:=\sum_{j, \ell=1}^{h, h_{j}} e^{\left(-i c_{j} \xi-d_{j \ell} \xi^{2}\right) t} e^{-N_{j \ell}^{(0)} \xi^{2} t+\mathcal{O}\left(|\xi|^{3}\right) t} \mathcal{O}(|\xi|) . \tag{3.12}
\end{equation*}
$$

Firstly, we estimate for $I_{1}$ with $|\xi|<\varepsilon$ small enough by taking the matrix norm both sides of (3.11). Since $c_{j} \in \mathbb{R}$ for all $j \in\{1, \ldots, h\}$, one has

$$
\left|I_{1}\right| \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t}\left|e^{-N_{j \ell}^{(0)} \xi^{2} t+\mathcal{O}\left(|\xi|^{3}\right) t}-e^{-N_{j \ell}^{(0)} \xi^{2} t}\right|
$$

On the other hand, from Proposition 2.1, $\operatorname{Re}\left(d_{j \ell}\right) \geq \theta>0$ and $N_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$. Thus, by choosing $\varepsilon^{\prime}=\frac{1}{4} \operatorname{Re}\left(d_{j \ell}\right)$ for each $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$, from Lemma 3.1, we have

$$
\begin{aligned}
\left|I_{1}\right| & \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j e}\right)|\xi|^{2} t} e^{\frac{1}{4} \operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t+C|\xi|^{3} t}|\xi|^{3} t \\
& \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j e}\right)|\xi|^{2} t} e^{\frac{1}{4} \operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t+C \varepsilon|\xi|^{2} t}|\xi|^{3} t \leq C e^{-\frac{1}{2} \theta|\xi|^{2} t}|\xi|^{3} t .
\end{aligned}
$$

Hence, by applying Lemma 3.2, we have

$$
\begin{equation*}
\left\|I_{1}\right\|_{L^{r}} \leq C t^{-\frac{11}{2} \frac{1}{r}-\frac{1}{2}}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1 \tag{3.13}
\end{equation*}
$$

Similarly, we also have the estimate for $I_{2}$ with $|\xi|<\varepsilon$ small enough. Indeed, from (3.12), one has

$$
\begin{aligned}
\left|I_{2}\right| & \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t} e^{\left|N_{j \ell}^{(0)}\right||\xi|^{2} t+C|\xi|^{3} t}|\xi| \\
& \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j \ell} \ell|\xi|^{2} t\right.} e^{C \varepsilon|\xi|^{2} t}|\xi| \leq C e^{-\frac{1}{2} \theta|\xi|^{2} t}|\xi|
\end{aligned}
$$

since one can assume that $\left|N_{j \ell}^{(0)}\right|$ is small for all $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$ based on the fact that they are nilpotent matrices. Hence, by applying Lemma 3.2, we have

$$
\begin{equation*}
\left\|I_{2}\right\|_{L^{r}} \leq C t^{-\frac{1}{2} \frac{1}{r}-\frac{1}{2}}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1 \tag{3.14}
\end{equation*}
$$

We estimate for $J$. From (3.10), we have

$$
|J| \leq C \sum_{j=1}^{k} e^{-\operatorname{Re}\left(e_{j}\right) t} e^{\left|M_{j}^{(0)}\right| t+C|\xi| t}(1+|\xi|) .
$$

Then, by Proposition 2.3, $\operatorname{Re}\left(e_{j}\right)>0$ and $M_{j}^{(0)}$ is a nilpotent matrix for $j \in\{1, \ldots, k\}$, we can assume that $\left|M_{j}^{(0)}\right|$ is small, and thus, since $|\xi|<\varepsilon$ small enough, we obtain

$$
\begin{equation*}
\|J\|_{L^{r}} \leq C \sum_{j=1}^{k} e^{-\operatorname{Re}\left(e_{j}\right) t} e^{C \varepsilon t}(1+\varepsilon) \leq C e^{-\delta t} \tag{3.15}
\end{equation*}
$$

for some $\delta>0$.
Therefore, from (3.13), (3.14) and (3.15), one obtains for $|\xi|<\varepsilon$ that

$$
\|\hat{G}-\hat{K}\|_{L^{r}} \leq\left\|I_{1}\right\|_{L^{r}}+\left\|I_{2}\right\|_{L^{r}}+\|J\|_{L^{r}} \leq C t^{-\frac{1}{2} \frac{1}{r}-\frac{1}{2}}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

We now estimate for $\hat{V}$ in (3.3) with $|\xi|<\varepsilon$. Since $\alpha_{j} \in \mathbb{R}$ for all $j \in\{1, \ldots, s\}$, one has

$$
|\hat{V}(\xi, t)| \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell)}\right)} e^{\Theta_{j \ell}^{(0)} \mid t}
$$

Thus, by Proposition 2.6, since $\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta>0$ and $\Theta_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$, one obtains

$$
\|\hat{V}\|_{L^{r}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

since we can assume that $\left|\Theta_{j \ell}^{(0)}\right|$ is small for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$ similarly to before.

In the compact domain $\varepsilon \leq|\xi| \leq R$, there are the exceptional points where the eigenprojections and the nilpotent parts associated with the eigenvalues of $E(i \xi)=B+i \xi A$ in this domain may not be defined even the eigenvalues are continuous there. However, the number of these exceptional points is always finite in $\varepsilon \leq|\xi| \leq R$ as introduced in Proposition 2.5, once integrating, for $\hat{G}=e^{E t}$ and for some $\delta>0$, from the condition $\mathbf{D}$, we still obtain

$$
\|\hat{G}\|_{L^{r}} \leq\left\|e^{-\frac{\theta|\cdot|^{2}}{1+|\cdot|^{2}} t}\right\|_{L^{r}} \leq C e^{-\delta t}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

For $\hat{K}$ in (3.2), similarly to the small frequency, for $\varepsilon \leq|\xi| \leq R$, one has

$$
\begin{equation*}
|\hat{K}(\xi, t)| \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t} e^{\left|N_{j \ell}^{(0)}\right||\xi|^{2} t} \leq C e^{-\frac{1}{2} \theta \varepsilon^{2} t} \tag{3.16}
\end{equation*}
$$

since $\operatorname{Re}\left(d_{j \ell}\right) \geq \theta$ and $N_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$. Thus, for $\varepsilon \leq|\xi| \leq R$, one obtains

$$
\|\hat{K}\|_{L^{r}} \leq C e^{-\frac{1}{2} \theta \varepsilon^{2} t}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

For $\hat{V}$ in (3.3), similarly to the small frequency, for $\varepsilon \leq|\xi| \leq R$, one has

$$
\begin{equation*}
|\hat{V}(\xi, t)| \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{\left|\Theta_{j \ell}^{(0)}\right| t} \leq C e^{-\frac{1}{2} \theta t} \tag{3.17}
\end{equation*}
$$

since $\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta$ and $\Theta_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$. Hence, for $\varepsilon \leq|\xi| \leq R$, we have

$$
\|\hat{V}\|_{L^{r}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

Finally, we study the case $|\xi|>R$. By Remark 2.7, the solution $\hat{G}$ to the system (3.1) is given by

$$
\begin{equation*}
\hat{G}(\xi, t):=\sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t} e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t}\left(\Pi_{j \ell}^{(0)}+\mathcal{O}\left(|\xi|^{-1}\right)\right) . \tag{3.18}
\end{equation*}
$$

Then, we have $\hat{G}-\hat{V}=I+J$ where

$$
\begin{equation*}
I:=\sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t}\left(e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t}-e^{-\Theta_{j \ell}^{(0)} t}\right) \Pi_{j \ell}^{(0)} \tag{3.19}
\end{equation*}
$$

and

$$
\begin{equation*}
J:=\sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t} e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t} \mathcal{O}\left(|\xi|^{-1}\right) \tag{3.20}
\end{equation*}
$$

We estimate for $I$ firstly and then for $J$. Since $\alpha_{j} \in \mathbb{R}$ for all $j \in\{1, \ldots, s\}$, we have

$$
|I| \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t}\left|e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t}-e^{-\Theta_{j \ell}^{(0)} t}\right| .
$$

On the other hand, from Proposition 2.6, $\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta>0$ and $\Theta_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$. Let $\varepsilon^{\prime}=\frac{1}{4} \operatorname{Re}\left(\beta_{j \ell}\right)$ and applying Lemma 3.1, for $|\xi|>R$ large enough, we obtain

$$
\begin{aligned}
|I| & \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j}\right) t} e^{\frac{1}{4} \operatorname{Re}\left(\beta_{j}\right) t+C|\xi|^{-1} t}|\xi|^{-1} t \\
& \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{\frac{1}{4} \operatorname{Re}\left(\beta_{j}\right) t+C R^{-1} t}|\xi|^{-1} t \leq C e^{-\frac{1}{2} \theta t}|\xi|^{-1} t
\end{aligned}
$$

Thus, for $|\xi|>R$ large enough, one has

$$
\begin{equation*}
\|I\|_{L^{r}} \leq C e^{-\frac{1}{4} \theta t}, \quad \text { for } r \in(1, \infty] \text { and } t \geq 1 \tag{3.21}
\end{equation*}
$$

Similarly, we estimate for $J$ for $|\xi|>R$ large enough. From (3.20), one has

$$
\begin{aligned}
|J| & \leq\left.\sum_{\substack{j, \ell=1 \\
s, s_{j}}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{s, \Theta_{j}}(0)|t+C| \xi\right|^{-1} t|\xi|^{-1} \\
& \leq \sum_{j, \ell=1}^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{C R^{-1} t}|\xi|^{-1} \leq C e^{-\frac{1}{2} \theta t}|\xi|^{-1}
\end{aligned}
$$

since one can assume that $\left|\Theta_{j \ell}^{(0)}\right|$ is small for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$. Thus, for $|\xi|>R$ large enough, one has

$$
\begin{equation*}
\|J\|_{L^{r}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } r \in(1, \infty] \text { and } t \geq 1 \tag{3.22}
\end{equation*}
$$

Therefore, from (3.21) and (3.22), there is a constant $\delta>0$ such that

$$
\|\hat{G}-\hat{V}\|_{L^{r}} \leq\|I\|_{L^{r}}+\|J\|_{L^{r}} \leq C e^{-\delta t}, \quad \text { for } r \in(1, \infty] \text { and } t \geq 1
$$

On the other hand, we estimate for $\hat{K}$ in (3.2) with $|\xi|>R$. We have

$$
\begin{align*}
|\hat{K}(\xi, t)| & \leq C \sum_{j, \ell=1}^{h, h_{j}} e^{-\operatorname{Re}\left(d_{j e}\right)|\xi|^{2} t} e^{\left|N_{j \ell}^{(0)}\right||\xi|^{2} t}  \tag{3.23}\\
& \leq C e^{-\frac{1}{2} \theta R^{2} t} \sum_{j, \ell=1}^{h, h_{j}} e^{-\frac{1}{2} \operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t} e^{\frac{1}{4} \operatorname{Re}\left(d_{j \ell}\right)|\xi|^{2} t} \leq C e^{-\frac{1}{2} \theta R^{2} t} e^{-\frac{1}{4} \theta|\xi|^{2} t}
\end{align*}
$$

since $\operatorname{Re}\left(d_{j \ell}\right) \geq \theta>0$ and $N_{j \ell}^{(0)}$ that is a nilpotent matrix can be assumed to have $\left|N_{j \ell}^{(0)}\right|$ small enough for all $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$ by Proposition 2.6. Thus, for $|\xi|>R$, we have

$$
\|\hat{K}\|_{L^{r}} \leq C e^{-\frac{1}{2} \theta R^{2} t} t^{-\frac{1}{2}} \leq C e^{-\frac{1}{4} \theta R^{2} t}, \quad \text { for } r \in[1, \infty] \text { and } t \geq 1
$$

We now estimate the $L^{\infty}$-norm of the function $\mathcal{F}^{-1}(\hat{G}-\hat{V})=\mathcal{F}^{-1}(I)+\mathcal{F}^{-1}(J)$ in $L^{\infty}$ for $|\xi|>R$ large enough where $I$ and $J$ are in (3.19) and (3.20) respectively. Primarily, from (3.19) and by applying the Taylor expansion to the application $X \rightarrow e^{X}$, we have $I=I_{1}+I_{2}+I_{3}$ where

$$
\begin{equation*}
I_{1}:=t \sum_{j, \ell=1}^{s, s_{j}} \frac{e^{-i \alpha_{j} \xi t}}{i \xi} e^{-\beta_{j \ell} t} e^{-\Theta_{j \ell}^{(0)} t} M \Pi_{j \ell}^{(0)}, \tag{3.24}
\end{equation*}
$$

where $M$ is the coefficient in $\mathcal{O}\left(|\xi|^{-1}\right)$ associated with $(i \xi)^{-1}$, and

$$
\begin{equation*}
I_{2}:=t \sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t} e^{-\Theta_{j \ell}^{(0)} t} \mathcal{O}\left(|\xi|^{-2}\right) \Pi_{j \ell}^{(0)}, \tag{3.25}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{3}:=\sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t}\left(e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t}-e^{-\Theta_{j \ell}^{(0)} t}-e^{-\Theta_{j \ell}^{(0)} t} \mathcal{O}\left(|\xi|^{-1}\right) t\right) \Pi_{j \ell}^{(0)} . \tag{3.26}
\end{equation*}
$$

We first estimate for $\mathcal{F}^{-1}\left(I_{1}\right)=\sum_{j=1}^{s} \mathcal{F}_{j}^{-1}\left(I_{1}\right)$ where

$$
\mathcal{F}_{j}^{-1}\left(I_{1}\right):=t \sum_{\ell=1}^{s_{j}} \mathcal{F}^{-1}\left(\frac{e^{-i \alpha_{j} \xi t}}{i \xi}\right) e^{-\beta_{j \ell} t} e^{-\Theta_{j \ell}^{(0)} t} M \Pi_{j \ell}^{(0)} .
$$

for $j \in\{1, \ldots, s\}$.
For each $j \in\{1, \ldots, s\}$, one has

$$
\begin{align*}
\left|\mathcal{F}_{j}^{-1}\left(I_{1}\right)(x, t)\right| & \leq C t \sum_{\ell=1}^{s_{j}}\left|\int_{-\infty}^{-R}+\int_{R}^{+\infty} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi\right| e^{-\operatorname{Re}\left(\beta_{j} \ell\right) t} e\left|\Theta_{j \ell}^{(0)}\right| t \\
& \leq C t \sum_{\ell=1}^{s_{j}}\left|2 \int_{R}^{+\infty} \frac{\sin \left(\left(x-\alpha_{j} t\right) \xi\right)}{\xi} d \xi\right| e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e\left|\Theta_{j \ell}^{(0)}\right| t  \tag{3.27}\\
& \leq C t e^{-\frac{1}{2} \theta t}\left|x-\alpha_{j} t\right|
\end{align*}
$$

since $\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta>0$ and $\Theta_{j \ell}^{(0)}$ that is a nilpotent matrix with norm can be chosen small enough for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$ by Proposition 2.6. Hence, if $|x| \leq C t$ where $C$ is a positive constant, then, for all $j \in\{1, \ldots, s\}$, we have

$$
\left\|\mathcal{F}_{j}^{-1}\left(I_{1}\right)\right\|_{L^{\infty}} \leq C t^{2} e^{-\frac{1}{2} \theta t} \leq C e^{-\frac{1}{4} \theta t}
$$

We now estimate for $\mathcal{F}_{j}^{-1}\left(I_{1}\right)$ in the case where $|x|>C t$ and $C$ large enough for $j \in\{1, \ldots, s\}$. Noting that in this case we have

$$
\begin{equation*}
e^{x \alpha_{j}} \leq e^{|x|\left|\alpha_{j}\right|} \leq e^{\frac{|x|^{2}}{t}\left|\alpha_{j}\right||x|^{-1} t} \leq e^{\varepsilon \frac{|x|^{2}}{t}} \tag{3.28}
\end{equation*}
$$

where $\varepsilon$ is small enough.
One has

$$
\begin{equation*}
\left|\mathcal{F}_{j}^{-1}\left(I_{1}\right)(x, t)\right| \leq C t \sum_{\ell=1}^{s_{j}}\left|\int_{-\infty}^{-R}+\int_{R}^{+\infty} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi\right| e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{\left|\Theta_{j \ell}^{(0)}\right| t} . \tag{3.29}
\end{equation*}
$$

We estimate for the integral

$$
\begin{equation*}
H:=\int_{-\infty}^{-R}+\int_{R}^{+\infty} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi=\lim _{K \rightarrow+\infty} \int_{-K}^{-R}+\int_{R}^{K} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi=H_{1}+H_{2} \tag{3.30}
\end{equation*}
$$

Due to the fact that the integrand is holomorphic, we can estimate $H_{2}$ by considering $\xi=$ $\zeta+i \eta \in \mathbb{C}$ and by changing the path $\{(\zeta, 0): \zeta$ from $R$ to $K\}$ to the path $\gamma:=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3}$ in the complex plane where

$$
\begin{align*}
& \gamma_{1}:=\left\{(\zeta, \eta): \zeta=R, \eta \text { from } 0 \text { to } \frac{x}{t}\right\},  \tag{3.31}\\
& \gamma_{2}:=\left\{(\zeta, \eta): \zeta \text { from } R \text { to } K, \eta=\frac{x}{t}\right\} \tag{3.32}
\end{align*}
$$

and

$$
\begin{equation*}
\gamma_{3}:=\left\{(\zeta, \eta): \zeta=K, \eta \text { from } \frac{x}{t} \text { to } 0\right\} . \tag{3.33}
\end{equation*}
$$

Then, by parameterizing $\gamma_{1}(s)=R+i \frac{x}{t} s$ for $s \in[0,1]$, since $|x|>C t$, we have

$$
\begin{align*}
\left|\lim _{K \rightarrow+\infty} \int_{\gamma_{1}} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi\right| & =\left|\int_{0}^{1} \frac{e^{i\left(x-\alpha_{j} t\right) R+x \alpha_{j} s-\frac{|x|^{2}}{t} s}}{R+i \frac{x}{t} s} \frac{x}{t} d s\right| \\
& \leq \frac{C}{R} \int_{0}^{1}\left(\frac{|x|}{t}+\frac{|x|^{2}}{t^{2}}\right) e^{\varepsilon \frac{|x|^{2}}{t} s} e^{-\frac{|x|^{2}}{t} s} d s  \tag{3.34}\\
& \leq \frac{C}{R}\left(\frac{1}{|x|}+\frac{1}{t}\right)\left(1-e^{-\frac{|x|^{2}}{2 t}}\right) \leq \frac{C}{R} t^{-1} .
\end{align*}
$$

On the other hand, noting that

$$
\begin{equation*}
\frac{1}{-\eta+i \zeta}=\frac{1}{i \zeta}-\eta\left(\frac{1}{\zeta^{2}+\eta^{2}}+\frac{1}{i \zeta} \frac{\eta}{\zeta^{2}+\eta^{2}}\right) \tag{3.35}
\end{equation*}
$$

Thus, since $|x|>C t$, we have

$$
\begin{align*}
\left.\lim _{K \rightarrow+\infty} \int_{\gamma_{2}} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi \right\rvert\, & =\left|\int_{R}^{+\infty} \frac{e^{i x \zeta-i \alpha_{j} \zeta t-\frac{|x|^{2}}{t}+x \alpha_{j}}}{-\frac{x}{t}+i \zeta} d \zeta\right|  \tag{3.36}\\
& \leq e^{-\frac{|x|^{2}}{2 t}}\left|\int_{R}^{+\infty} e^{i x \zeta}\left(\frac{1}{i \zeta}-\frac{x}{t}\left(\frac{1}{\zeta^{2}+\frac{|x|^{2}}{t^{2}}}+\frac{1}{i \zeta} \frac{\frac{x}{t}}{\zeta^{2}+\frac{|x|^{2}}{t^{2}}}\right)\right) d \zeta\right| \\
& \leq C e^{-\frac{|x|^{2}}{2 t}}\left(\left|\int_{R}^{+\infty} \frac{e^{i x \zeta}}{i \zeta} d \zeta\right|+\left(\frac{|x|}{t}+\frac{|x|^{2}}{t^{2}}\right) \int_{R}^{+\infty} \frac{1}{\zeta^{2}} d \zeta\right) \\
& \leq C e^{-\frac{|x|^{2}}{2 t}} \frac{|x|^{2}}{2 t}\left(\frac{t}{|x|}+\frac{1}{|x|}+\frac{1}{t}\right) \leq C e^{-\delta t}
\end{align*}
$$

Similarly, we consider $\gamma_{3}(s)=K+i \frac{x}{t}(1-s)$ for $s \in[0,1]$, we have

$$
\begin{equation*}
\left|\lim _{K \rightarrow+\infty} \int_{\gamma_{3}} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi\right|=\left|\lim _{K \rightarrow+\infty} \int_{0}^{1} \frac{e^{i\left(x-\alpha_{j} t\right) K+x \alpha_{j}(1-s)-\frac{|x|^{2}}{t}(1-s)}}{K+i \frac{x}{t}(1-s)} \frac{x}{t} d s\right| \tag{3.37}
\end{equation*}
$$

On the other hand, noting that for a fixed $K$, we have

$$
\begin{align*}
\left|\int_{0}^{1} \frac{e^{i\left(x-\alpha_{j} t\right) K+x \alpha_{j}(1-s)-\frac{|x|^{2}}{t}(1-s)}}{K+i \frac{x}{t}(1-s)} \frac{x}{t} d s\right| & \leq \frac{C}{K} \int_{0}^{1}\left(\frac{|x|}{t}+\frac{|x|^{2}}{t^{2}}\right) e^{-\frac{|x|^{2}}{2 t}(1-s)} d s  \tag{3.38}\\
& =\frac{C}{K}\left(\frac{1}{|x|}+\frac{1}{t}\right) e^{-\frac{|x|^{2}}{2 t}}\left(e^{\frac{|x|^{2}}{2 t}}-1\right) \leq \frac{C}{K} t^{-1}
\end{align*}
$$

One deduces that

$$
\begin{equation*}
\lim _{K \rightarrow+\infty} \int_{0}^{1} \frac{e^{i\left(x-\alpha_{j} t\right) K+x \alpha_{j}(1-s)-\frac{|x|^{2}}{t}(1-s)}}{K+i \frac{x}{t}(1-s)} \frac{x}{t} d s=0 \tag{3.39}
\end{equation*}
$$

Hence, it implies that

$$
\begin{equation*}
\left|\lim _{K \rightarrow+\infty} \int_{\gamma_{3}} \frac{e^{i\left(x-\alpha_{j} t\right) \xi}}{i \xi} d \xi\right|=0 \tag{3.40}
\end{equation*}
$$

Finally, one can estimate $H_{1}$ similarly by substituting $R$ and $K$ by $-R$ and $-K$ respectively. Therefore, from (3.29), (3.30), (3.34), (3.36) and (3.40), one obtains

$$
\left\|\mathcal{F}_{j}^{-1}\left(I_{1}\right)\right\|_{L^{\infty}} \leq C e^{-\frac{1}{2} \theta t}
$$

for $|x|>C t$ where $C$ large enough since $\operatorname{Re}\left(\beta_{j \ell}\right) \geq \theta>0$ and $\Theta_{j \ell}^{(0)}$ is a nilpotent matrix for all $j \in\{1, \ldots, s\}$ and $\ell \in\left\{1, \ldots, s_{j}\right\}$ by Proposition 2.6.

Therefore, it implies that

$$
\left\|\mathcal{F}^{-1}\left(I_{1}\right)\right\|_{L^{\infty}} \leq C \sum_{j=1}^{s}\left\|\mathcal{F}_{j}^{-1}\left(I_{1}\right)\right\|_{L^{\infty}} \leq C e^{-\frac{1}{2} \theta t}
$$

We estimate for $\mathcal{F}^{-1}\left(I_{2}\right)$ and $\mathcal{F}^{-1}\left(I_{3}\right)$ where $I_{2}$ and $I_{3}$ are in (3.25) and (3.26) respectively. Since $\mathcal{F}^{-1}: L^{1} \rightarrow L^{\infty}$, one has

$$
\begin{equation*}
\left\|\mathcal{F}^{-1}\left(I_{2,3}\right)\right\|_{L^{\infty}} \leq C\left\|I_{2,3}\right\|_{L^{1}} . \tag{3.41}
\end{equation*}
$$

Hence, we only need to estimate $I_{2}$ and $I_{3}$ in $L^{1}$.
From (3.25), we have

$$
\left|I_{2}\right| \leq C \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{\left|\Theta_{j \ell}^{(0)}\right| t}|\xi|^{-2} t
$$

Thus, we obtain

$$
\left\|I_{2}\right\|_{L^{1}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } t \geq 1
$$

From (3.26), we have

$$
\left|I_{3}\right| \leq \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t}\left|e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t}-e^{-\Theta_{j \ell}^{(0)} t}-e^{-\Theta_{j \ell}^{(0)} t} \mathcal{O}\left(|\xi|^{-1}\right) t\right|
$$

Then, by Lemma 3.1, we obtain

$$
\left|I_{3}\right| \leq \sum_{j, \ell=1}^{s, s_{j}} e^{-\operatorname{Re}\left(\beta_{j \ell}\right) t} e^{\varepsilon^{\prime} t+C|\xi|^{-1} t}|\xi|^{-2} t^{2}
$$

where $\varepsilon^{\prime}$ is small enough. Therefore, since $|\xi|$ large enough, we have

$$
\left\|I_{3}\right\|_{L^{1}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } t \geq 1
$$

Thus, we deduces

$$
\left\|\mathcal{F}^{-1}(I)\right\|_{L^{\infty}} \leq\left\|\mathcal{F}^{-1}\left(I_{1}\right)\right\|_{L^{\infty}}+\left\|\mathcal{F}^{-1}\left(I_{2}\right)\right\|_{L^{\infty}}+\left\|\mathcal{F}^{-1}\left(I_{3}\right)\right\|_{L^{\infty}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } t \geq 1
$$

We now estimate $\mathcal{F}^{-1}(J)$ where $J$ is given by (3.20). From (3.20), one has $J=J_{1}+J_{2}$ where

$$
\begin{equation*}
J_{1}:=\sum_{j, \ell=1}^{s, s_{j}} \frac{e^{-i \alpha_{j} \xi t}}{i \xi} e^{-\beta_{j \ell} t} e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t} M \tag{3.42}
\end{equation*}
$$

where $M$ is the coefficient associated with the term $(i \xi)^{-1}$ in $\mathcal{O}\left(|\xi|^{-1}\right)$, and

$$
J_{2}:=\sum_{j, \ell=1}^{s, s_{j}} e^{\left(-i \alpha_{j} \xi-\beta_{j \ell}\right) t} e^{-\Theta_{j \ell}^{(0)} t+\mathcal{O}\left(|\xi|^{-1}\right) t} \mathcal{O}\left(|\xi|^{-2}\right)
$$

Then, we can estimate $\mathcal{F}^{-1}\left(J_{1}\right)$ as the case of $\mathcal{F}^{-1}\left(I_{1}\right)$ and we can estimate $\mathcal{F}^{-1}\left(J_{2}\right)$ as the case of $\mathcal{F}^{-1}\left(I_{2}\right)$ and $\mathcal{F}^{-1}\left(I_{3}\right)$. Thus, we deduces

$$
\left\|\mathcal{F}^{-1}(J)\right\|_{L^{\infty}} \leq\left\|\mathcal{F}^{-1}\left(J_{1}\right)\right\|_{L^{\infty}}+\left\|\mathcal{F}^{-1}\left(J_{2}\right)\right\|_{L^{\infty}} \leq C e^{-\frac{1}{2} \theta t}, \quad \text { for } t \geq 1
$$

Therefore, we conclude

$$
\left\|\mathcal{F}^{-1}(\hat{G}-\hat{V})\right\|_{L^{\infty}} \leq\left\|\mathcal{F}^{-1}(I)\right\|_{L^{\infty}}+\left\|\mathcal{F}^{-1}(J)\right\|_{L^{\infty}} \leq C e^{-\frac{1}{2} \theta t}
$$

for $t \geq 1$ and the proof is done.

## 4. Multiplier estimates

This section provides some useful Fourier multiplier estimates by recalling the Young inequality.

Lemma 4.1 (Young's inequality). For all $(p, q, r) \in[1, \infty]^{3}$ such that $\frac{1}{q}-\frac{1}{p}=1-\frac{1}{r}$ and $(f, g) \in L^{r} \times L^{q}$, we have $f * g \in L^{p}$ and $\|f * g\|_{L^{p}} \leq\|f\|_{L^{r}}\|g\|_{L^{q}}$.

We thus obtain the follows.
4.1. Case $|x| \leq C t$. Let $\chi_{1}$ and $\chi_{3}$ be cutoff functions defined on $[-\varepsilon, \varepsilon]$ and $(-\infty,-R] \cup$ $[R,+\infty)$ respectively for $\varepsilon$ small and $R$ large such that $\left|\chi_{1,2}\right| \leq 1$. Let $\chi_{2}:=1-\chi_{1}-\chi_{3}$, we introduce the multipliers

$$
m_{j}:=\chi_{j}(\hat{G}-\hat{K}-\hat{V}), \quad j=1,2,3
$$

The following holds.
Proposition 4.2. For $r \in[1, \infty], m_{j} \in M_{r}$ with

$$
\begin{equation*}
\left\|m_{j}\right\|_{M_{r}} \leq C t^{-\frac{1}{2}}, \quad j=1,2,3 \text { and } t \geq 1 \tag{4.1}
\end{equation*}
$$

Proof. We begin with $m_{1}$. For $|\xi| \leq \varepsilon$, we have $\hat{G}-\hat{K}=I_{1}+I_{2}+J$ where $I_{1}, I_{2}$ are in (3.11), (3.12) respectively and $J=\hat{G}_{2}$ as in (3.10). We then have

$$
\mathcal{F}^{-1}\left(\chi_{1} I_{1}\right)(x, t)=\sum_{j, \ell=1}^{h, h_{j}} \int_{-\varepsilon}^{\varepsilon} \chi_{1}(\xi) e^{i\left(x-c_{j} t\right) \xi-d_{j \ell} \xi^{2} t}\left(e^{-N_{j \ell}^{(0)} \xi^{2} t+\mathcal{O}\left(|\xi|^{3}\right) t}-e^{-N_{j \ell}^{(0)} \xi^{2} t}\right) P_{j \ell}^{(0)} d \xi
$$

For $j \in\{1, \ldots, h\}$ and $\ell \in\left\{1, \ldots, h_{j}\right\}$, let $z=e^{i \phi / 2} \xi$ where $\phi=\arg \left(d_{j \ell}\right) \in(-\pi / 2, \pi / 2)$ since $\operatorname{Re}\left(d_{j \ell}\right)>0$, one obtains

$$
\begin{aligned}
& \mathcal{F}^{-1}\left(\chi_{1} I_{1}\right)(x, t)=\sum_{j, \ell=1}^{h, h_{j}} \int_{\gamma} \chi_{1}\left(e^{-i \phi / 2} z\right) e^{i\left(x-c_{j} t\right) e^{-i \phi / 2} z-\left|d_{j \ell}\right| z^{2} t} \\
& \cdot\left(e^{-N_{j \ell}^{(0)} e^{-i \phi} z^{2} t+\mathcal{O}\left(\left|e^{-i \phi / 2} z\right|^{3}\right) t}-e^{-N_{j \ell}^{(0)} e^{-i \phi} z^{2} t}\right) P_{j \ell}^{(0)} e^{-i \phi / 2} d z
\end{aligned}
$$

where $\gamma:=\left\{z \in \mathbb{C}: z=e^{i \phi / 2} \xi, \xi \in[-\varepsilon, \varepsilon]\right\}$. Then, we will estimate for each summand by letting $\eta:=\min \left\{\frac{\left|x-c_{j} t\right|}{2\left|d_{j}\right| t}, \frac{\varepsilon}{2}\right\}$. Since the integrand is holomorphic, we can change the path of the integral from $\gamma$ to $\gamma:=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3}$ in the complex plane where

$$
\begin{align*}
\gamma_{1} & :=\left\{-\varepsilon e^{i \phi / 2}+i \operatorname{sgn}\left(x-c_{j} t\right) \eta e^{-i \phi / 2} s: s \in[0,1]\right\}  \tag{4.2}\\
\gamma_{2} & :=\left\{\zeta e^{i \phi / 2}+i \operatorname{sgn}\left(x-c_{j} t\right) \eta e^{-i \phi / 2}: \zeta \in[-\varepsilon, \varepsilon]\right\} \tag{4.3}
\end{align*}
$$

and

$$
\begin{equation*}
\gamma_{3}:=\left\{\varepsilon e^{i \phi / 2}+i \operatorname{sgn}\left(x-c_{j} t\right) \eta e^{-i \phi / 2}(1-s): s \in[0,1]\right\} \tag{4.4}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
\left|e^{i\left(x-c_{j} t\right) e^{-i \phi / 2} z-\left|d_{j}\right| z^{2} t}\right|=e^{-\left(x-c_{j} t\right)(\cos (\phi / 2) \operatorname{Im} z-\sin (\phi / 2) \operatorname{Re} z)} e^{-\left|d_{j \ell}\right|(\operatorname{Re} z-\operatorname{Im} z)(\operatorname{Re} z+\operatorname{Im} z) t} \tag{4.5}
\end{equation*}
$$

Moreover, $\left|\chi_{1}\right| \leq 1$ and similarly to before, by Lemma 3.1, since $N_{j \ell}^{(0)}$ is nilpotent and since $|z|=|\xi| \leq \varepsilon$ small, we have

$$
\begin{align*}
\left|e^{-N_{j \ell}^{(0)} e^{-i \phi} z^{2} t+\mathcal{O}\left(\left|e^{-i \phi / 2} z\right|^{3}\right) t}-e^{-N_{j \ell}^{(0)} e^{-i \phi} z^{2} t}\right| & \leq C|z|^{3} t e^{\varepsilon^{\prime}|z|^{2} t+C|z|^{3} t}  \tag{4.6}\\
& \leq C(|\operatorname{Re} z|+|\operatorname{Im} z|)^{3} t e^{\varepsilon^{\prime \prime}(|\operatorname{Re} z|+|\operatorname{Im} z|)^{2} t}
\end{align*}
$$

where $\varepsilon^{\prime}, \varepsilon^{\prime \prime}$ can be chosen as small as one needs.
Thus, for $z \in \gamma_{1}$, we have

$$
\begin{aligned}
& \operatorname{Re} z=-\varepsilon \cos (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \sin (\phi / 2) s \\
& \operatorname{Im} z=-\varepsilon \sin (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \cos (\phi / 2) s
\end{aligned}
$$

We then obtain from $\cos (\phi)>0, \eta^{2} s^{2} \leq \varepsilon^{2} / 2$ for $s \in[0,1]$ and $|z| \leq \varepsilon$ that for some $\delta>0$, one has

$$
\begin{equation*}
\left|\int_{\gamma_{1}}\right| \leq C \int_{0}^{1} e^{-\left|x-c_{j} t\right| \eta \cos (\phi) s} e^{-\left|d_{j \ell}\right| \cos (\phi)\left(\varepsilon^{2}-\eta^{2} s^{2}\right) t} e^{\varepsilon^{\prime \prime} \varepsilon^{2} t} \varepsilon^{3} t d s \leq C e^{-\delta t} \tag{4.7}
\end{equation*}
$$

For $z \in \gamma_{2}$, we have

$$
\begin{aligned}
& \operatorname{Re} z=\zeta \cos (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \sin (\phi / 2) \\
& \operatorname{Im} z=\zeta \sin (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \cos (\phi / 2)
\end{aligned}
$$

Hence, one has

$$
\begin{equation*}
\left|\int_{\gamma_{2}}\right| \leq C \int_{-\varepsilon}^{\varepsilon} e^{-\left|x-c_{j} t\right| \eta \cos (\phi)} e^{-\left|d_{j}\right| \cos (\phi)\left(\zeta^{2}-\eta^{2}\right) t} e^{\varepsilon^{\prime \prime}\left(\zeta^{2}+2|\zeta||\eta|+|\eta|^{2}\right) t}(|\zeta|+|\eta|)^{3} t d \zeta . \tag{4.8}
\end{equation*}
$$

If $\eta=\frac{\left|x-c_{j} t\right|}{2\left|d_{j \ell}\right| t}$, then since $|\zeta| \leq \varepsilon$ small and $\varepsilon^{\prime \prime}$ small enough, for some $c>0$, we have

$$
\begin{aligned}
\left|\int_{\gamma_{2}}\right| & \leq C \int_{-\varepsilon}^{\varepsilon} e^{-\frac{\left|x-c_{j} t\right|^{2}}{\left|d_{j \ell}\right|^{t}} \cos (\phi)} e^{\frac{\left|x-c_{j} t\right|^{2}}{2\left|d_{j}\right| t} \cos (\phi)} e^{-\left|d_{j \ell}\right| \cos (\phi) \zeta^{2} t} e^{\varepsilon^{\prime \prime} \zeta^{2} t+\varepsilon^{\prime \prime}|\zeta| \frac{\left|x-c_{j} t\right|}{\left|d_{j}\right| t}+\varepsilon^{\prime \prime} \frac{\left|x-c_{j} t\right|^{2}}{4\left|d_{j \ell}\right|^{2} t}} \\
& \cdot\left(|\zeta|^{3} t+3|\zeta|^{2} \frac{\left|x-c_{j} t\right|}{2\left|d_{j \ell}\right|}+3|\zeta| \frac{\left|x-c_{j} t\right|^{2}}{4\left|d_{j \ell}\right|^{2} t}+\frac{\left|x-c_{j} t\right|^{3}}{8\left|d_{j \ell}\right|^{3} t^{2}}\right) d \zeta \\
& \leq C \sum_{k=0}^{3} e^{-\frac{\left|x-c_{j} t\right|^{2}}{8\left|d_{j \ell}\right| t}} \cos (\phi) \\
& \left(\frac{\left|x-c_{j} t\right|}{\sqrt{t}}\right)^{k} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}\left|d_{j \ell}\right| \cos (\phi) \zeta^{2} t}|\zeta|^{3-k} t^{1-\frac{k}{2}} d \zeta \\
& \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c\left|d_{j \ell}\right| t}}
\end{aligned}
$$

If $\eta=\varepsilon / 2$, then $\left|x-c_{j} t\right| \geq \varepsilon\left|d_{j \ell}\right| t$ by the definition of $\eta$ and we have

$$
\begin{align*}
\left|\int_{\gamma_{2}}\right| & \leq C \int_{-\varepsilon}^{\varepsilon} e^{-\left|x-c_{j} t\right| \eta \cos (\phi)} e^{-\left|d_{j \ell}\right| \cos (\phi)\left(\zeta^{2}-\eta^{2}\right) t} e^{\varepsilon^{\prime \prime}\left(\zeta^{2}+2|\zeta||\eta|+|\eta|^{2}\right) t}(|\zeta|+|\eta|)^{3} t d \zeta  \tag{4.10}\\
& \leq C e^{-\varepsilon^{2}\left|d_{j \ell}\right| \cos (\phi) t} e^{\frac{1}{4} \varepsilon^{2}\left|d_{j \ell}\right| \cos (\phi) t} e^{\varepsilon^{\prime \prime} \varepsilon^{2} t} \int_{-\varepsilon}^{\varepsilon} e^{-\left|d_{j \ell}\right| \cos (\phi) \zeta^{2} t}\left(|\zeta|+\frac{\varepsilon}{2}\right)^{3} t d \zeta \leq C e^{-\delta t},
\end{align*}
$$

for some $\delta>0$ since $\varepsilon^{\prime \prime}$ can be chosen small enough.
For $z \in \gamma_{3}$, we have

$$
\begin{aligned}
& \operatorname{Re} z=\varepsilon \cos (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \sin (\phi / 2)(1-s), \\
& \operatorname{Im} z=\varepsilon \sin (\phi / 2)+\operatorname{sgn}\left(x-c_{j} t\right) \eta \cos (\phi / 2)(1-s) .
\end{aligned}
$$

Thus, similarly to $\gamma_{1}$, for some $\delta>0$, one has

$$
\begin{equation*}
\left|\int_{\gamma_{3}}\right| \leq C \int_{0}^{1} e^{-\left|x-c_{j} t\right| \eta \cos (\phi)(1-s)} e^{-\left|d_{j}\right| \cos (\phi)\left(\varepsilon^{2}-\eta^{2}(1-s)^{2}\right) t} e^{\varepsilon^{\prime \prime} \varepsilon^{2} t} \varepsilon^{3} t d s \leq C e^{-\delta t} . \tag{4.11}
\end{equation*}
$$

Therefore, from (4.7), (4.9), (4.10), (4.11) and the fact that $e^{-\delta t} \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c \mid d_{j} \ell t}}$ since $|x| \leq C t$ and $t \geq 1$, we obtain

$$
\left|\mathcal{F}^{-1}\left(\chi_{1} I_{1}\right)(x, t)\right| \leq C t^{-1} e^{-\frac{\left|x-c_{j}\right|^{2}}{c \mid j_{j} \ell t}}, \quad t \geq 1 .
$$

By the same way for $\mathcal{F}^{-1}\left(\chi_{1} I_{2}\right)$ and $\mathcal{F}^{-1}\left(\chi_{1} J\right)$, we also have

$$
\left|\mathcal{F}^{-1}\left(\chi_{1}\left(I_{2}+J\right)\right)(x, t)\right| \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c \mid d_{j \ell} t}}, \quad t \geq 1 .
$$

Hence, for $r \in[1, \infty]$, by the Young inequality and since $m_{1}=\chi_{1}\left(I_{1}+I_{2}+J\right)$, it follows that

$$
\left\|m_{1}\right\|_{M_{r}}=\sup _{\|f\|_{L^{r}=1}}\left\|\mathcal{F}^{-1}\left(m_{1}\right) * f\right\|_{L^{r}} \leq\left\|\mathcal{F}^{-1}\left(m_{1}\right)\right\|_{L^{1}} \leq C t^{-\frac{1}{2}}, \quad t \geq 1
$$

We consider $m_{2}$. Since $\left|\chi_{2}(\xi)\right|,\left|e^{i x \xi}\right| \leq 1$ for $\xi \in \mathbb{R}$, we have

$$
\begin{equation*}
\left|\mathcal{F}^{-1}\left(m_{2}\right)(x, t)\right| \leq \int_{\varepsilon \leq|\xi| \leq R}(|\hat{G}(\xi, t)|+|\hat{K}(\xi, t)|+|\hat{V}(\xi, t)|) d \xi \leq C e^{-\delta t} \tag{4.12}
\end{equation*}
$$

for some $\delta>0$ due to (3.16), (3.17) and the fact that $|\hat{G}(\xi, t)| \leq e^{-\frac{\left.\theta|\xi|\right|^{2}}{1+\left.|\xi|\right|^{2}} t}$ on $\varepsilon \leq|\xi| \leq R$ for $\theta>0$.

Thus, one has

$$
\left|\mathcal{F}^{-1}\left(m_{2}\right)(x, t)\right| \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c d_{j} \ell t}}, \quad t \geq 1 .
$$

Hence, for $r \in[1, \infty]$, by the Young inequality, it follows that

$$
\left\|m_{2}\right\|_{M_{r}}=\sup _{\|f\|_{L^{r}=1}}\left\|\mathcal{F}^{-1}\left(m_{2}\right) * f\right\|_{L^{r}} \leq\left\|\mathcal{F}^{-1}\left(m_{2}\right)\right\|_{L^{1}} \leq C t^{-\frac{1}{2}}, \quad t \geq 1
$$

Finally, we consider $m_{3}$. Based on the decomposition $\hat{G}-\hat{V}=I+J$ where $I$ is defined as $I_{1}$ in (3.24) and $J$ is the remainder, from (3.27) and a same treatment for $J$, we obtain for $|x| \leq C t$ that

$$
\left|\mathcal{F}^{-1}\left(\chi_{3}(\hat{G}-\hat{V})\right)(x, t)\right| \leq C \sum_{j=1}^{s} t e^{-\delta t}\left|x-\alpha_{j} t\right|+C e^{-\delta t} \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c d_{j} \ell t}}, \quad t \geq 1
$$

for some $\delta>0$.
Moreover, from (3.23), there is a $\theta>0$ such that

$$
\left|\mathcal{F}^{-1}\left(\chi_{3} \hat{K}\right)(x, t)\right| \leq e^{-\frac{1}{2} \theta R^{2} t} \int_{|\xi| \geq R} e^{-\frac{1}{4} \theta|\xi|^{2} t} d \xi \leq C e^{-\delta t} \leq C t^{-1} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c\left|d_{j} \ell\right| t}}, \quad t \geq 1
$$

for some $\delta>0$.
Hence, for $r \in[1, \infty]$, by the Young inequality and $m_{3}=\chi_{3}(\hat{G}-\hat{V}-\hat{K})$, it follows that

$$
\left\|m_{3}\right\|_{M_{r}}=\sup _{\|f\|_{L^{r}=1}}\left\|\mathcal{F}^{-1}\left(m_{3}\right) * f\right\|_{L^{r}} \leq\left\|\mathcal{F}^{-1}\left(m_{3}\right)\right\|_{L^{1}} \leq C t^{-\frac{1}{2}}, \quad t \geq 1
$$

We finish the proof.
4.2. Case $|x|>C t$. We introduce the multipliers

$$
m^{1}:=\hat{G}-\hat{V}, \quad \text { and } \quad m^{2}:=\hat{K} .
$$

The following holds.
Proposition 4.3. For $r \in[1, \infty]$, we have $m^{j} \in M_{r}$ with

$$
\begin{equation*}
\left\|m^{j}\right\|_{M_{r}} \leq C t^{-\frac{1}{2}}, \quad j=1,2 \text { and } t \geq 1 . \tag{4.13}
\end{equation*}
$$

Proof. We estimate the $L^{1}$-norm of $\mathcal{F}^{-1}\left(m^{1}\right)$. We have

$$
\begin{equation*}
\mathcal{F}^{-1}\left(m^{1}\right)(x, t)=\lim _{R \rightarrow+\infty} \int_{-R}^{R} e^{i x \xi}(\hat{G}(\xi, t)-\hat{V}(\xi, t)) d \xi . \tag{4.14}
\end{equation*}
$$

On the other hand, noting that the solution $\hat{G}$ to (3.1) is written as $\hat{G}(\xi, t)=e^{E(i \xi) t}$ and thus $\hat{G}$ is an entire function on the complex plane since $E(i \xi)=-(B+i \xi A)$. Moreover, due to the formula of $\hat{V}$ in (3.3), $\hat{V}$ is also holomorphic on the complex plane. Thus, by considering $\xi=\zeta+i \eta \in \mathbb{C}$, one can change the path of the integral in (4.14) from $\{(\zeta, 0): \zeta$ from $-R$ to $R\}$ to the path $\gamma:=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3}$ in the complex plane where

$$
\begin{align*}
\gamma_{1} & :=\left\{(\zeta, \eta): \zeta=-R, \eta \text { from } 0 \text { to } \frac{x}{t}\right\},  \tag{4.15}\\
\gamma_{2} & :=\left\{(\zeta, \eta): \zeta \text { from }-R \text { to } R, \eta=\frac{x}{t}\right\} \tag{4.16}
\end{align*}
$$

and

$$
\begin{equation*}
\gamma_{3}:=\left\{(\zeta, \eta): \zeta=R, \eta \text { from } \frac{x}{t} \text { to } 0\right\} . \tag{4.17}
\end{equation*}
$$

Furthermore, since $R$ and $|x| / t$ large, along these curves, the solution $\hat{G}$ has the representation of the high frequency case (3.18). Therefore, by the same computation as in (3.34)-(3.40) and letting $R \rightarrow+\infty$, we obtain

$$
\begin{equation*}
\left|\mathcal{F}^{-1}\left(m^{1}\right)(x, t)\right| \leq C e^{-\frac{|x|^{2}}{c t}} \leq C t^{-1} e^{-\frac{|x|^{2}}{2 c t}} \tag{4.18}
\end{equation*}
$$

for some $c, C>0$ since $e^{-\frac{|x|^{2}}{2 c t}} \leq e^{-C^{2} t} \leq t^{-1}$ due to the fact that $|x|>C t$ with $C$ large enough. Hence, we obtain

$$
\begin{equation*}
\left\|\mathcal{F}^{-1}\left(m^{1}\right)\right\|_{L^{1}} \leq C t^{-\frac{1}{2}} \tag{4.19}
\end{equation*}
$$

Thus, by the Young inequality, for $r \in[1, \infty]$, we have

$$
\begin{equation*}
\left\|m^{1}\right\|_{M_{r}}=\sup _{\|f\|_{L^{r}=1}}\left\|\mathcal{F}^{-1}\left(m^{1}\right) * f\right\|_{L^{r}} \leq\left\|\mathcal{F}^{-1}\left(m^{1}\right)\right\|_{L^{1}} \leq C t^{-\frac{1}{2}} \tag{4.20}
\end{equation*}
$$

The estimate for $m^{2}$ are similar and the proof is done.

## 5. Symmetry

We will discuss about the conditions $\mathbf{C}^{\prime}$ and $\mathbf{S}$ in order to increase the decay rate of the solution to the system (1.1). Recalling the matrices $C$ and $D$ as in (1.4) and (1.8) respectively.

Lemma 5.1. If the condition $\mathbf{C}$ ' holds, then there are $m$ distinct eigenvalues of $E(i \xi)=$ $-(B+i \xi A)$ converging to 0 as $|\xi| \rightarrow 0$ and they are expanded analytically, where $m=$ dim $\operatorname{ker}(B)$. The approximation of the $j$-th eigenvalue has the form

$$
\begin{equation*}
\lambda_{j}(i \xi)=-i c_{j} \xi-d_{j} \xi^{2}+\mathcal{O}\left(|\xi|^{3}\right), \quad|\xi| \rightarrow 0 \tag{5.1}
\end{equation*}
$$

where $c_{j} \in \sigma(C)$ considered in $\operatorname{ker}(B)$ and $d_{j} \in \sigma\left(P_{j}^{(0)} D P_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ with $P_{j}^{(0)}$ the eigenprojection associated with $c_{j}$ for $j \in\{1, \ldots, m\}$.

Proof. This is just a consequence of Proposition 2.1. Indeed, from (2.1), the approximation of the eigenvalues of $E$ converging to 0 as $|\xi| \rightarrow 0$ is

$$
\lambda_{j \ell}(i \xi)=-i c_{j} \xi-d_{j \ell} \xi^{2}+\mathcal{O}\left(|\xi|^{2}\right), \quad|\xi| \rightarrow 0
$$

where $c_{j} \in \sigma(C)$ considered in $\operatorname{ker}(B)$ and $d_{j \ell} \in \sigma\left(P_{j}^{(0)} D P_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $\ell=1, \ldots, h_{j}$ with $P_{j}^{(0)}$ the eigenprojection associated with $c_{j}$ for $j \in\{1, \ldots, h\}$. Noting that $h_{j}$ is the cardinality of the spectrum of $P_{j}^{(0)} D P_{j}^{(0)}$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for $j \in\{1, \ldots, h\}$ and $h$ is the cardinality of the spectrum of $C$ considered in $\operatorname{ker}(B)$.

On the other hand, since the condition $\mathbf{C}^{\prime}$ holds, $h=m$ where $m=\operatorname{dim} \operatorname{ker}(B)$. Moreover, also by the condition $\mathbf{C}^{\prime}$, one deduces that $c_{j}$ is simple for all $j \in\{1, \ldots, m\}$. Thus, $\operatorname{dim} \operatorname{ker}\left(C-c_{j} I\right)=1$ and therefore $h_{j}=1$ for $j \in\{1, \ldots, m\}$. It implies that there is only one $d_{j}:=d_{j 1} \in\left(P_{j}^{(0)} D P_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ for each $j \in\{1, \ldots, m\}$. Moreover, $d_{j}$ is also simple, and thus, one can continue the reduction process as in the proof of Proposition 2.1. Furthermore, due to the simplicity of the coefficients in the expansion of $\lambda_{j}$ provided $c_{j}$ is simple and the reduction process, there is no splitting in the expansion of the eigenvalues $\lambda_{j}$ i.e. the eigenvalues $\lambda_{j}$ can be expanded analytically for $j \in\{1, \ldots, m\}$ and the proof is done.

Let $p(\lambda, \kappa):=\operatorname{det}(E(\kappa)-\lambda I)$ be the dispersion polynomial associated with $E(\kappa)=$ $-(B+\kappa A)$, where $\lambda, \kappa \in \mathbb{C}$.

Lemma 5.2. If the condition $\mathbf{S}$ holds, then $p(\lambda,-\kappa)=p(\lambda, \kappa)$ for any $\lambda, \kappa \in \mathbb{C}$.
Proof. For $q(\lambda, \kappa):=p(\lambda,-\kappa)$, there holds $p(\lambda, \kappa)=0$ if and only if $q(\lambda, \kappa)=0$. Indeed, if the couple $(\lambda, \kappa)$ is such that $p(\lambda, \kappa)=0$ if and only if there exist a nonzero vector $u$ such that

$$
(\lambda I+\kappa A+B) u=0
$$

In such a case, setting $v=S^{-1} u$, there also holds

$$
\begin{aligned}
0 & =S^{-1}(\lambda I+\kappa A+B) S v=S^{-1}(\lambda S+\kappa A S+B S) v \\
& =S^{-1}(\lambda S-\kappa S A+S B) v=(\lambda I-\kappa A+B) v
\end{aligned}
$$

Hence, $q(\lambda, \kappa)=0$. The other implication can be proved in the same way.
For fixed $\kappa$, the polynomials $p$ and $q$ have both degree $n$ in $\lambda$ with principal term $\lambda^{n}$. Hence, there exist $\lambda_{1}^{p}, \ldots, \lambda_{n}^{p}$ and $\lambda_{1}^{q}, \ldots, \lambda_{n}^{q}$ with $\lambda_{i}^{p, q}=\lambda_{i}(\kappa)$ such that

$$
p(\lambda, \kappa)=\prod_{k=1}^{n}\left(\lambda-\lambda_{k}^{p}(\kappa)\right) \quad \text { and } \quad q(\lambda, \kappa)=\prod_{k=1}^{n}\left(\lambda-\lambda_{k}^{q}(\kappa)\right)
$$

Since $p$ and $q$ have the same zero-set, for any $k \in\{1, \ldots, n\}$ there exists $j$ such that $\lambda_{k}^{q}=\lambda_{j}^{p}$. As a consequence, $p \equiv q$.

Corollary 5.3. If the conditions $\mathbf{C}$ ' and $\mathbf{S}$ hold, then there are $m$ analytic distinct eigenvalues of $E$ converging to 0 as $|\xi| \rightarrow 0$ and the $j$-th eigenvalue has the aproximation

$$
\begin{equation*}
\lambda_{j}(i \xi)=-i c_{j} \xi-d_{j} \xi^{2}+\mathcal{O}\left(|\xi|^{4}\right), \quad|\xi| \rightarrow 0 \tag{5.2}
\end{equation*}
$$

where $c_{j} \in \sigma(C)$ considered in $\operatorname{ker}(B)$ and $d_{j} \in \sigma\left(P_{j}^{(0)} D P_{j}^{(0)}\right)$ considered in $\operatorname{ker}\left(C-c_{j} I\right)$ with $P_{j}^{(0)}$ the eigenprojection associated with $c_{j}$ for $j \in\{1, \ldots, m\}$.

Proof. From the proof of Lemma 5.1, since $d_{j}$ is simple for all $j \in\{1, \ldots, m\}$, one can continue the reduction process in the proof of Proposition 2.1 and thus the formula (5.1) can be refined as

$$
\lambda_{j}(i \xi)=-i c_{j} \xi-d_{j} \xi^{2}-e_{j}(i \xi)^{3}+\mathcal{O}\left(|\xi|^{4}\right), \quad|\xi| \rightarrow 0
$$

where $e_{j} \in \sigma\left(M_{j}\right)$ considered in $\operatorname{ker}\left(P_{j}^{(0)} D P_{j}^{(0)}-d_{j} I\right)$ for some suitable matrix $M_{j}$ for $j \in\{1, \ldots, m\}$.

By recalling the proof of Proposition 2.1 and by Lemma 5.1 one more time, substituting $(-i \xi)$ into $i \xi$, there are $m$ analytic distinct eigenvalues of $E(-i \xi)$ converging to 0 as $|\xi| \rightarrow 0$ such that the

$$
\begin{equation*}
\lambda_{j}(-i \xi)=-i\left(-c_{j}\right) \xi-d_{j} \xi^{2}-\left(-e_{j}\right)(i \xi)^{3}+\mathcal{O}\left(|\xi|^{4}\right), \quad|\xi| \rightarrow 0 \tag{5.3}
\end{equation*}
$$

where $c_{j}, d_{j}$ and $e_{j}$ are already introduced as before.
On the other hand, since $\sigma(E(i \xi)) \equiv \sigma(E(-i \xi))$ due to Lemma 5.2, one deduces that $\sigma\left(M_{j}\right)$ contains both $e_{j}$ and $-e_{j}$. Moreover, since $\operatorname{dim} \operatorname{ker}\left(P_{j}^{(0)} D P_{j}^{(0)}-d_{j} I\right)=1$, one concludes that $e_{j}=-e_{j}=0$. The proof is done.

Remark 5.4. The nilpotent parts associated with $\lambda_{j}$ for $j \in\{1, \ldots, m\}$ are zero since these eigenvalues are distinct and simple.

Moreover, for each $j \in\{1, \ldots, m\}$, the total projection associated with $\lambda_{j}$ is itself the eigenprojection associated with $\lambda_{j}$ and has the expansion (2.7) with $\zeta=i \xi$ i.e. we have

$$
\begin{equation*}
P_{j}(i \xi)=P_{j}^{(0)}+i \xi P_{j}^{(1)}+\mathcal{O}\left(|\xi|^{2}\right), \quad|\xi| \rightarrow 0 \tag{5.4}
\end{equation*}
$$

where $P_{j}^{(1)}$ can be computed by the formula (1.20) for $j \in\{1, \ldots, m\}$. This is based on the fact that there is no splitting after the second step of the reduction process and the formula of $P_{j}^{(1)}$ is proved similarly to the proof of the formula (1.7) in the proof of Proposition 2.1.

One sets the kernel

$$
\begin{equation*}
\hat{K}^{*}(\xi, t):=\sum_{j=1}^{m} e^{\left(-i c_{j} \xi-d_{j} \xi^{2}\right) t}\left(P_{j}^{(0)}+i \xi P_{j}^{(1)}\right) \tag{5.5}
\end{equation*}
$$

Then, the first estimate in (3.5) of Proposition 3.3 can be modified by
Proposition 5.5. If the conditions $C^{\prime}$ and $\mathbf{S}$ hold, for $r \in[1, \infty]$, one has

$$
\begin{equation*}
\left\|\hat{G}-\hat{K}^{*}\right\|_{L^{r}} \leq C t^{-\frac{1}{2} \frac{1}{r}-1} \tag{5.6}
\end{equation*}
$$

for $|\xi|<\varepsilon$ small enough and $t \geq 1$.
Proof. For $|\xi|<\varepsilon$, by Remark 2.2, Remark 2.4 and Corollary 5.3, the solution to the system (3.1) is given by $\hat{G}=\hat{G}_{1}+\hat{G}_{2}$ where $\hat{G}_{2}$ is given by (3.10) and

$$
\hat{G}_{1}(\xi, t)=\sum_{j=1}^{m} e^{\left(-i c_{j} \xi-d_{j} \xi^{2}\right) t+\mathcal{O}\left(|\xi|^{4}\right) t}\left(P_{j}^{(0)}+i \xi P_{j}^{(1)}+\mathcal{O}\left(|\xi|^{2}\right)\right)
$$

Thus, similarly to the proof of the first estimate in (3.5), we have $\hat{G}-\hat{K}^{*}=I_{1}+I_{2}+J$ where $J=\hat{G}_{2}$ and

$$
\begin{align*}
& I_{1}:=\sum_{j=1}^{m} e^{\left(-i c_{j} \xi-d_{j} \xi^{2}\right) t}\left(e^{\mathcal{O}\left(|\xi|^{4}\right) t}-1\right)\left(P_{j}^{(0)}+i \xi P_{j}^{(1)}\right),  \tag{5.7}\\
& I_{2}:=\sum_{j=1}^{m} e^{\left(-i c_{j} \xi-d_{j} \xi^{2}\right) t+\mathcal{O}\left(|\xi|^{4}\right) t} \mathcal{O}\left(|\xi|^{2}\right) \tag{5.8}
\end{align*}
$$

Hence, similarly to before, there is a constant $c>0$ such that

$$
\left|I_{1}\right| \leq C e^{-c|\xi|^{2} t}|\xi|^{4} t \quad \text { and } \quad\left|I_{2}\right| \leq C e^{-c|\xi|^{2} t}|\xi|^{2}
$$

Thus, together with (3.15), it implies that

$$
\left\|\hat{G}-\hat{K}^{*}\right\|_{L^{r}} \leq\left\|I_{1}\right\|_{L^{r}}+\left\|I_{2}\right\|_{L^{r}}+\|\hat{J}\|_{L^{r}} \leq C t^{-\frac{1}{2} \frac{1}{r}-1}
$$

for $|\xi|<\varepsilon, t \geq 1$ and $r \in[1, \infty]$. We finish the proof.
Similarly, by recall the multipliers $m_{j}$ for $j=1,2,3$ and $m^{j}$ for $j=1,2$ with $\hat{K}$ is substituted by $\hat{K}^{*}$, we can also refine Proposition 4.2 for $|x| \leq C t$ and Proposition 4.3 for $|x|>C t$.

Proposition $5.6(|x| \leq C t)$. For $r \in[1, \infty], m_{j} \in M_{r}$ with

$$
\begin{equation*}
\left\|m_{j}\right\|_{M_{r}} \leq C t^{-1}, \quad j=1,2,3 \text { and } t \geq 1 \tag{5.9}
\end{equation*}
$$

Proof. Similarly to the proof of Proposition 4.2, we only need to consider $\mathcal{F}^{-1}\left(\chi_{1} I_{1}\right)$ and $\mathcal{F}^{-1}\left(\chi_{1} I_{2}\right)$ on $\gamma_{2}$ where $I_{1}, I_{2}$ are now given by $(5.7),(5.8)$ respectively and $\gamma_{2}$ is the same as (4.3). The others is bounded by $e^{-\delta t}$ for some $\delta>0$ and thus since $|x| \leq C t$, they are dominated by $t^{-\frac{3}{2}} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c\left|d_{j}\right| t}}$ for some $c>0$ and $t \geq 1$.

Hence, noting that since $\left|e^{\mathcal{O}\left(\left|e^{-i \phi / 2} z\right|^{4}\right) t}-1\right| \leq C(|\operatorname{Re} z|+|\operatorname{Im} z|)^{4} t e^{\varepsilon(|\operatorname{Re} z|+|\operatorname{Im} z|)^{2} t}$ for $z=e^{i \phi / 2} \xi$ where $\xi \in[-\varepsilon, \varepsilon]$ and $\phi=\arg \left(d_{j}\right) \in(-\pi / 2, \pi / 2)$ for $j \in\{1, \ldots, m\}$, on $\gamma_{2}$, we have

$$
\begin{aligned}
\left|\mathcal{F}^{-1}\left(\chi_{1} I_{1}\right)\right| & \leq C \sum_{j=1}^{m} \int_{-\varepsilon}^{\varepsilon} e^{-\left|x-c_{j} t\right| \eta \cos (\phi)} e^{-\left|d_{j}\right| \cos (\phi)\left(\zeta^{2}-\eta^{2}\right) t} e^{\varepsilon(\zeta+|\eta|)^{2} t}(|\zeta|+|\eta|)^{4} t d \zeta \\
& \leq C \sum_{j=1}^{m} \sum_{k=0}^{4} e^{-\frac{\left.\left|x-c_{j}\right|\right|^{2}}{c^{2} d_{j} \mid t} \cos (\phi)}\left(\frac{\left|x-c_{j} t\right|}{\sqrt{t}}\right)^{k} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}\left|d_{j}\right| \cos (\phi) \zeta^{2} t}|\zeta|^{4-k} t^{1-\frac{k}{2}} d \zeta \\
& \leq C \sum_{j=1}^{m} t^{-\frac{3}{2}} e^{-\frac{\left|x-c_{j} t\right|^{2}}{c^{\prime}\left|d_{j}\right| t}}
\end{aligned}
$$

for some $c, c^{\prime}>0$ and $t \geq 1$. The estimate for $\mathcal{F}^{-1}\left(\chi_{1} I_{2}\right)$ is similarly.
Therefore, taking the $L^{1}$-norm in $x$ variable and using the Young inequality, we finish the proof.

Proposition $5.7(|x|>C t)$. For $r \in[1, \infty], m_{j} \in M_{r}$ with

$$
\begin{equation*}
\left\|m^{j}\right\|_{M_{r}} \leq C t^{-1}, \quad j=1,2 \text { and } t \geq 1 \tag{5.10}
\end{equation*}
$$

Proof. The proof is the same as in Proposition 4.3 and the fact that $e^{-\frac{|x|^{2}}{t}}$ is dominated by $t^{-\frac{3}{2}} e^{-\frac{|x|^{2}}{2 t}}$ since $|x|>C t$ and $t \geq 1$. The proof is done.

## 6. Proof of main results

Recall the well-known inequality
Lemma 6.1 (Interpolation inequality). Let $\left(p_{j}, q_{j}\right)_{j \in\{0,1\}}$ be two elements of $[1, \infty]^{2}$. Consider a linear operator $T$ which continuously maps $L^{p_{j}}$ into $L^{q_{j}}$ for $j \in\{0,1\}$. For any $\theta \in[0,1]$, if

$$
\left(\frac{1}{p_{\theta}}, \frac{1}{q_{\theta}}\right):=(1-\theta)\left(\frac{1}{p_{0}}, \frac{1}{q_{0}}\right)+\theta\left(\frac{1}{p_{1}}, \frac{1}{q_{1}}\right)
$$

then $T$ continuous maps $L^{p_{\theta}}$ into $L^{q_{\theta}}$ and $\|T\|_{\mathcal{L}\left(L^{p_{\theta} ; L^{q_{\theta}}}\right)} \leq\|T\|_{\mathcal{L}\left(L^{\left.p_{0} ; L^{q_{0}}\right)}\right.}^{1-\theta}\|T\|_{\mathcal{L}\left(L^{\left.p_{1} ; L^{q_{1}}\right)}\right.}^{\theta}$.
We then introduce detailed proofs for Theorem 1.1 and Theorem 1.2.
Proof of Theorem 1.1. Let $u$ be the solution to (1.1), recalling that $U=\sum_{j=1}^{h} U_{j}$ where $U_{j}$ is the solution to (1.9) for $j \in\{1, \ldots, h\}$ and $V=Q \sum_{j=1}^{s} V_{j}$ where $V_{j}$ is the solution to (1.14) for $j \in\{1, \ldots, s\}$. Then, we have

$$
u-U-V=\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{V}) * u_{0}
$$

On the other hand, let $\chi$ be the characteristic function, we have

$$
\begin{aligned}
\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{V})= & \mathcal{F}^{-1}\left[(\hat{G}-\hat{K}-\hat{V})\left(\chi_{[0, \varepsilon)}+\chi_{[\varepsilon, R]}+\chi_{(R, \infty)}\right)(|\xi|)\right] \\
= & \mathcal{F}^{-1}\left[(\hat{G}-\hat{K}-\hat{V})\left(\chi_{[0, \varepsilon)}+\chi_{[\varepsilon, R]}\right)(|\xi|)\right] \\
& +\mathcal{F}^{-1}\left[(\hat{G}-\hat{V}) \chi_{(R, \infty)}(|\xi|)\right]-\mathcal{F}^{-1}\left[\hat{K} \chi_{(R, \infty)}(|\xi|)\right] .
\end{aligned}
$$

Thus, since $\mathcal{F}^{-1}: L^{1} \rightarrow L^{\infty}$, we have

$$
\begin{aligned}
\left\|\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{V})\right\|_{L^{\infty}} & \leq C\left[\left\|(\hat{G}-\hat{K}-\hat{V})\left(\chi_{[0, \varepsilon)}+\chi_{[\varepsilon, R]}\right)\right\|_{L^{1}}\right. \\
& \left.+\left\|\hat{K} \chi_{(R, \infty)}\right\|_{L^{1}}\right]+\left\|\mathcal{F}^{-1}\left[(\hat{G}-\hat{V}) \chi_{(R, \infty)}(|\xi|)\right]\right\|_{L^{\infty}}
\end{aligned}
$$

Hence, by the estimates (3.5), (3.6), (3.7) and (3.8) in Proposition 3.3, we obtain

$$
\|u-U-V\|_{L^{\infty}} \leq C t^{-1}\left\|u_{0}\right\|_{L^{1}}, \quad t \geq 1
$$

Furthermore, from Proposition 4.2 and Proposition 4.3, for all $r \in[1, \infty]$, we also have

$$
\|u-U-V\|_{L^{r}} \leq C t^{-\frac{1}{2}}\left\|u_{0}\right\|_{L^{r}}, \quad t \geq 1
$$

Therefore, by the interpolation inequality, we obtained the desired results.
The proof of (1.16) is similar and we finish the proof.
Proof of Theorem 1.2. Similarly to before and from Propositions 5.5, 5.6 and 5.7. The proof is done.

## Appendix

Eigenprojection computation. In this subsection, we introduce a useful tool in order to compute the eigenprojection associated with a semi-simple eigenvalue of a matrix based on the determinant of this matrix and its minors. We start with some definitions - a set of indices $\mathcal{I}$ is a set $\mathcal{I}=\left\{k_{1}<\cdots<k_{\ell}\right\}$ with $k_{p} \in\{1, \ldots, n\}$ for any $p$; - an index-transformation $\chi$ is an injective map from a set of indices $\mathcal{I}$ to $\{1, \ldots, n\}$.

Then, we introduce some additional notations:

- given two matrices $A, B \in M_{n}(\mathbb{R})$, a set of indices $\mathcal{I}=\{i\}$ and an index-transformation $\chi$, we denote by $\Phi(A, B ; \mathcal{I}, \chi)$ the matrix obtained by substituting the $i$-th column of $A$ by the $\chi(i)$-th column of $B$.
- given sets of indices $\mathcal{I}, \mathcal{K}, \mathcal{L}$ satisfying $\mathcal{K} \subseteq \mathcal{I}$ and $|\mathcal{K}|=|\mathcal{L}|$, where $|\mathcal{K}|$ and $|\mathcal{L}|$ are the cardinalities of $\mathcal{K}$ and $\mathcal{L}$ respectively. A map $\chi_{\mathcal{K} \rightarrow \mathcal{L}}$ is an injective map from $\mathcal{I}$ to $\{1, \ldots, n\}$ defined by $\chi_{\mathcal{K} \rightarrow \mathcal{L}}(k)=k$ if $k \notin \mathcal{K}$; and there is a unique $\ell \in \mathcal{L}$ such that $\chi_{\mathcal{K} \rightarrow \mathcal{L}}(k)=\ell$ for each $k \in \mathcal{K}$.

We then set $[A]^{k}$ the $n \times n$ matrix with components defined by

$$
[A]_{i j}^{k}:=\sum \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{i \rightarrow j}\right), \quad i, j \in\{1, \ldots, n\}
$$

where the sum is made on sets of indices $\mathcal{I}$ containing $i$ and with cardinality $|\mathcal{I}|=k+1$. If $k=0$, then $[A]^{0}=\operatorname{adj}(A)$. Hence, the above notation can be seen as an extended version of the adjunct of the matrix $A$.

One sets

$$
\begin{align*}
& \mathbb{P}_{k}(A):=\frac{(k+1)[A]^{k}}{\operatorname{Tr}[A]^{k}},  \tag{6.1}\\
& \mathbb{S}_{k}(A):=\frac{(k+1)(k+2)[A]^{k+1} \operatorname{Tr}[A]^{k}-(k+1)^{2}[A]^{k} \operatorname{Tr}[A]^{k+1}}{(k+2)\left(\operatorname{Tr}[A]^{k}\right)^{2}} . \tag{6.2}
\end{align*}
$$

Let $\Gamma$ be an oriented closed curve enclosing 0 except for the other eigenvalues of $A$ in the resolvent set $\rho(A)$, one defines

$$
\begin{equation*}
P:=-\frac{1}{2 \pi i} \int_{\Gamma}(A-z I)^{-1} d z \quad \text { and } \quad S:=\frac{1}{2 \pi i} \int_{\Gamma} z^{-1}(A-z I)^{-1} d z . \tag{6.3}
\end{equation*}
$$

The matrix $P \in M_{n}(\mathbb{R})$ is called the eigenprojection associated with 0 and the matrix $S \in M_{n}(\mathbb{R})$ is called the reduced resolvent coefficient associated with 0 .

Proposition 6.2. Let $A \in M_{n}(\mathbb{R})$ and $0 \in \sigma(A)$ is semi-simple with algebraic multiplicity $m \geq 1$, then the eigenprojection associated with a has the formula

$$
\begin{equation*}
P=\mathbb{P}_{m-1}(A) \quad \text { and } \quad S=\mathbb{S}_{m-1}(A) . \tag{6.4}
\end{equation*}
$$

Before going to the proof of Proposition 6.2, we introduce the following results.
Lemma 6.3. Let $f(x):=\operatorname{det} \Phi\left(A+x B, C ; \mathcal{I}, \chi_{\mathcal{K} \rightarrow \mathcal{L}}\right)$ for $x \in \mathbb{C}$. By setting

$$
\mathcal{J}:=\left\{h \notin \mathcal{I}: \exists j \in \chi \mathcal{K} \rightarrow \mathcal{L}(\mathcal{I}), c_{j}=b_{h} \in \operatorname{col}(B)\right\}
$$

where $\operatorname{col}(B)$ is the column space of $B$, for any non negative integer $m$ satisfying $m \leq$ $n-|\mathcal{I}|$, the following holds

$$
f^{(m)}(x)=m!\sum \operatorname{det} \Phi\left(\Phi\left(A+x B, C ; \mathcal{I}, \chi_{\mathcal{K} \rightarrow \mathcal{L}}\right), B ; \mathcal{M}, \chi_{\mathcal{M} \rightarrow \mathcal{M}}\right)
$$

where the sum is made on the set of indices $\mathcal{M}$ with cardinality $|\mathcal{M}|=m$ and $\mathcal{M} \cap(\mathcal{I} \cup \mathcal{J})=$ $\emptyset$.

Proof. By definition, we have

$$
f(x)=\bigwedge_{h=1}^{n} f_{h}(x) \quad \text { where } \quad f_{h}(x):= \begin{cases}\left(a_{h}+b_{h} x\right) & \text { if } h \notin \mathcal{I}, \\ c_{\chi_{\mathcal{K}} \rightarrow \mathcal{L}}(h) & \text { if } h \in \mathcal{I} .\end{cases}
$$

Hence, the derivative of order $m \in \mathbb{N}$ of $f$ satisfies

$$
f^{(m)}(x)=\sum \bigwedge_{h=1}^{n} f_{h}^{\left(s_{h}\right)}(x) \text { where } f_{h}^{\left(s_{h}\right)}(x):= \begin{cases}\left(a_{h}+b_{h} x\right)^{\left(s_{h}\right)} & \text { if } h \notin \mathcal{I}, \\ c_{\chi \kappa \rightarrow \mathcal{C}(h)}^{\left(s_{h}\right)} & \text { if } h \in \mathcal{I}\end{cases}
$$

where $s_{h}:=s_{h}^{1}+\cdots+s_{h}^{m} \in \mathbb{N}$ with $s_{h}^{\ell} \in\{0,1\}$ for all $\ell \in\{1, \ldots, m\}$ and if we denote by $S \in\{1,0\}^{n \times m}$ the matrix defined by $S_{h \ell}:=s_{h}^{\ell}$, the sum is made on the set

$$
\mathcal{S}:=\left\{S: s_{1}+\cdots+s_{n}=m\right\} .
$$

It means that $f^{(m)}(x)$ is the sum of a finite number of determinants, where the determinants are generated by the elements $S$ of $\mathcal{S}$ and they are given by $D_{S}:=\bigwedge_{h=1}^{n} f_{h}^{\left(s_{h}\right)}(x)$.

Moreover, for any matrix $S \in \mathcal{S}$, if $s_{h} \geq 2$ for $h \notin \mathcal{I}$, then $\left(a_{h}+b_{h} x\right)^{\left(s_{h}\right)}=O_{n \times 1}$ and if $s_{h} \geq 1$ for $h \in \mathcal{I}, c_{\chi \mathcal{\mathcal { K }} \rightarrow \mathcal{L}(h)}^{\left(s_{h}\right)}=O_{n \times 1}$; and thus, the determinants related to these cases are zero. Thus, due to the condition $s_{1}+\cdots+s_{n}=m$ where $m \leq n-|\mathcal{I}|$, we can introduce a partition for $\mathcal{S}$ such that its elements denoted by $\mathcal{S}_{\mathcal{M}}$ are associated with index-sets $\mathcal{M}:=\left\{h_{1}, \ldots, h_{m}\right\} \subset\{1, \ldots, n\} \backslash \mathcal{I}$ and they are given by

$$
\mathcal{S}_{\mathcal{M}}:=\left\{S: s_{h}=\delta_{\mathcal{M}}(h) \text { for all } h=1, \ldots, n\right\}
$$

where

$$
\delta_{\mathcal{M}}(h):= \begin{cases}1 & \text { if } h \in \mathcal{M}, \\ 0 & \text { if } h \notin \mathcal{M} .\end{cases}
$$

In particular, for any $\mathcal{M}$, if $S$ and $S^{\prime}$ belong to $\mathcal{S}_{\mathcal{M}}$, one has $D_{S}=D_{S^{\prime}}$ since $s_{h}=\delta_{\mathcal{M}}(h)=$ $s_{h}^{\prime}$ for all $h \in\{1, \ldots, n\}$ where $s_{h}$ and $s_{h}^{\prime}$ are the sum of the elements of the $h$-th rows of the matrices $S$ and $S^{\prime}$ respectively. On the other hand, we have

$$
\begin{aligned}
D_{S} & =\bigwedge_{h=1}^{n} f_{h}^{\left(s_{h}\right)}(x) \quad \text { where } \quad f_{h}^{\left(s_{h}\right)}(x):= \begin{cases}b_{h} & \text { if } h \in \mathcal{M} \\
\left(a_{h}+b_{h} x\right) & \text { if } h \notin \mathcal{M} \cup \mathcal{I}, \\
c_{\chi \mathcal{K} \rightarrow \mathcal{C}}(h) & \text { if } h \in \mathcal{I}\end{cases} \\
& =\operatorname{det} \Phi(\Phi(A+x B, C ; \mathcal{I}, \chi \mathcal{K} \rightarrow \mathcal{L}), B ; \mathcal{M}, \chi \mathcal{M} \rightarrow \mathcal{M}) .
\end{aligned}
$$

Moreover, let $\sigma$ be a permutation of the set $\left\{e_{1}, \ldots, e_{m}\right\}$ where $e_{\ell}$ the $\ell$-th row of the identity matrix $I_{m \times m}$. Then, by definition, the rows of any matrix $S \in \mathcal{S}_{\mathcal{M}}$ for $\mathcal{M}=$ $\left\{h_{1}, \ldots, h_{m}\right\}$ must be in the form of

$$
\left(\begin{array}{lll}
S_{h_{\ell} 1} & \ldots & S_{h_{\ell} m}
\end{array}\right)= \begin{cases}\sigma\left(e_{\ell}\right) & \text { if } \ell \in\{1, \ldots, m\}, \\
O_{1 \times m} & \text { if } \ell \in\{m+1, \ldots, n\} .\end{cases}
$$

Therefore, since the $D_{S}=D_{S^{\prime}}$ for $S, S^{\prime} \in \mathcal{S}_{\mathcal{M}}$ and since the number of $\sigma$ is $m$ !, we obtain

$$
\begin{aligned}
f^{(m)}(x) & =\sum_{\mathcal{M}} \sum_{S \in \mathcal{S}_{\mathcal{M}}} D_{S} \\
& =m!\sum_{\mathcal{M}} \operatorname{det} \Phi\left(\Phi\left(A+x B, C ; \mathcal{I}, \chi_{\mathcal{K} \rightarrow \mathcal{L}}\right), B ; \mathcal{M}, \chi_{\mathcal{M} \rightarrow \mathcal{M}}\right) .
\end{aligned}
$$

Assuming that there is $\mathcal{M}$ to satisfy $\mathcal{M} \cap \mathcal{J} \neq \emptyset$, then there exists $h \in \mathcal{M}$ such that $h \notin \mathcal{I}$ and $b_{h}=c_{j}$ for some $j \in \chi_{\mathcal{K} \rightarrow \mathcal{L}}(\mathcal{I})$. Hence, the determinants $D_{S}$ generated by $S \in \mathcal{S}_{\mathcal{M}}$ have

$$
f_{h}^{\left(s_{h}\right)}(x)=b_{h}=c_{j} .
$$

Furthermore, since $j \in \chi_{\mathcal{K} \rightarrow \mathcal{L}}(\mathcal{I})$, there is $i \in \mathcal{I}$ such that $\chi_{\mathcal{I} \rightarrow \mathcal{J}}(i)=j$. Thus, the determinants $D_{S}$ also have

$$
f_{i}^{\left(s_{i}\right)}(x)=c_{\chi \mathcal{K} \rightarrow \mathcal{L}(i)}=c_{j}
$$

as well. Since $h \notin \mathcal{I}$ and $i \in \mathcal{I}$, it implies that there are at least two columns are the same and the determinants $D_{S}$ are accordingly zero. Therefore, we can choose $\mathcal{M} \cap \mathcal{J}=\emptyset$ for all $\mathcal{M}$. The proof is done.

Corollary 6.4. Let $h \in\{0, \ldots, n-1\}$ and $A \in M_{n}(\mathbb{R})$, for $x \in \mathbb{C}$, the following hold

$$
[A]^{h}=\left.(h!)^{-1}(\operatorname{adj}(A+x I))^{(h)}\right|_{x=0}, \quad \operatorname{Tr}[A]^{h}=\left.(h!)^{-1}(\operatorname{det}(A+x I))^{(h+1)}\right|_{x=0}
$$

where $\operatorname{adj}(A+x I)$ is the adjoint matrix of $A+x I$.
Proof. By the definition of the adjoint matrix, the elements of the matrix are the minors of the matrix $A+x I$. On the other hand, one defines

$$
M_{j i}(x):=\operatorname{det} \Phi\left(A+x I, I ;\{i\}, \chi_{i \rightarrow j}\right), \quad i, j \in\{1, \ldots, n\} .
$$

Thus, by the definition of the minor and the definition of $\Phi$, we have $(\operatorname{adj}(A+x I))_{i j}=$ $M_{j i}(x)$ for all $i, j \in\{1, \ldots, n\}$. Moreover, by Lemma 6.3 , the following holds

$$
\begin{aligned}
M_{j i}^{(h)}(0) & =h!\sum_{\mathcal{H} \ngtr i, j} \operatorname{det} \Phi\left(\Phi\left(A, I ;\{i\}, \chi_{i \rightarrow j}\right), I ; \mathcal{H}, \chi_{\mathcal{H} \rightarrow \mathcal{H}}\right) \\
& =h!\sum_{\mathcal{H} \cup\{i\}} \operatorname{det} \Phi\left(A, I ; \mathcal{H} \cup\{i\}, \chi_{i \rightarrow j}\right)=h![A]_{i j}^{h}
\end{aligned}
$$

where $\mathcal{H}$ has the cardinality $|\mathcal{H}|=h$ for $h \in\{0, \ldots, n-1\}$. We finished proving the first equality in the statement.

By the definition of $\chi_{\mathcal{M} \rightarrow \mathcal{N}}: \mathcal{I} \rightarrow\{1, \ldots, n\}$ for any set of indices $\mathcal{I}$ containing $\mathcal{M}$, it follows that $\chi_{i \rightarrow i} \equiv \chi_{\mathcal{I} \rightarrow \mathcal{I}}$ for any $\mathcal{I}$ containing $i$. Thus, by the definition of $[A]^{h}$ for $h \in\{0, \ldots, n-1\}$, we have

$$
\begin{aligned}
\operatorname{Tr}[A]^{h} & =\sum_{i=1}^{n} \sum_{\mathcal{I} \ni i} \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{i \rightarrow i}\right) \\
& =\sum_{\mathcal{I} \ni 1} \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{\mathcal{I} \rightarrow \mathcal{I}}\right)+\cdots+\sum_{\mathcal{I} \ni n} \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{\mathcal{I} \rightarrow \mathcal{I}}\right)
\end{aligned}
$$

where $\mathcal{I}$ has the cardinality $|\mathcal{I}|=h+1$.
Moreover, for any fixed set of indices $\mathcal{I}$ satisfying $|\mathcal{I}|=h+1, \mathcal{I}$ must be considered in $h+1$ terms in the right hand side of the formula of $\operatorname{Tr}[A]^{h}$. In fact, each of $i \in \mathcal{I}$ belongs to $\{1, \ldots, n\}$. Thus, for any fixed $\mathcal{I}$, we can collect $h+1$ quantities that are the same and we have

$$
\operatorname{Tr}[A]^{h}=(h+1) \sum_{\mathcal{I}} \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{\mathcal{I} \rightarrow \mathcal{I}}\right)
$$

where $\mathcal{I}$ has the cardinality $|\mathcal{I}|=h+1$.
Furthermore, by Lemma 6.3, one has

$$
\left.(\operatorname{det}(A+x I))^{(h+1)}\right|_{x=0}=(h+1)!\sum_{\mathcal{I}} \operatorname{det} \Phi\left(A, I ; \mathcal{I}, \chi_{\mathcal{I} \rightarrow \mathcal{I}}\right)
$$

The proof is done.
We can now give a proof for Proposition 6.2.
Proof of Proposition 6.2. By definition, the resolvent of the matrix $A$ is given by

$$
R(z):=(A-z I)^{-1}=\frac{\operatorname{adj}(A-z I)}{\operatorname{det}(A-z I)}
$$

For $z$ small, the resolvent can be expanded as

$$
R(z)=\frac{1}{z^{m}} \frac{\left.\sum_{h=0}^{n-1}(-1)^{h}(h!)^{-1}(\operatorname{adj}(A+x I))^{(h)}\right|_{x=0} z^{h}}{\left.\sum_{h=m}^{n}(-1)^{h}(h!)^{-1}(\operatorname{det}(A+x I))^{(h)}\right|_{x=0} z^{h-m}}
$$

Thus, Corollary 6.4 implies that

$$
R(z)=\frac{1}{z^{m}} \frac{\sum_{h=0}^{n-1}(-1)^{h}[A]^{h} z^{h}}{\sum_{h=m}^{n}(-1)^{h} h^{-1}\left(\operatorname{Tr}[A]^{h-1}\right) z^{h-m}}
$$

On the other hand, by using the Laurent expansion of $R(z)$ (see [6]), we also have

$$
R(z)=-\sum_{h=-1}^{+\infty} z^{-h-1}(N)^{h}-z^{-1} P+\sum_{h=0}^{+\infty} z^{h}(S)^{h+1}
$$

where $P, S$ are in (6.3) and $N=A P$ is the nilpotent matrix associated with the eigenvalue 0 of $A$. Then equating two sides, we obtain the formulas. We finish the proof.

Perturbation theory for linear operators. In this subsection, we introduce some results from the perturbation theory for linear operators in finite dimensional space that we will use for this paper. Moreover, we will sketch the proofs of them. For whom is interested in, see [6] for more details.

Proposition 6.5. Assume that $T$ is a matrix operator considered in a domain $\mathcal{D}:=\operatorname{ran}(P)$ where $P$ is a matrix operator. Let $\left(P_{j}\right)$ for $j=1, \ldots, k$ be a sequence of matrix operators such that

$$
\begin{equation*}
P_{j}^{2}=P_{j}, \quad P_{j} P_{j^{\prime}}=O \text { for } j \neq j^{\prime}, \quad P=\sum_{j=1}^{k} P_{j} \quad \text { and } \quad \operatorname{ran}(P)=\bigoplus_{j=1}^{k} \operatorname{ran}\left(P_{j}\right) . \tag{6.5}
\end{equation*}
$$

If $T$ commutes with $P_{j}$ for $j=1, \ldots, k$, then one has

$$
\begin{equation*}
T P_{j}=P_{j} T=P_{j} T P_{j} \quad \text { and } \quad T P=P T=P T P=\sum_{j=1}^{k}\left(T P_{j}\right) . \tag{6.6}
\end{equation*}
$$

Moreover, $\lambda \in \sigma(T)$ considered in $\operatorname{ran}(P)$ if and only if there is $j_{0} \in\{1, \ldots, k\}$ such that $\lambda \in \sigma(T)$ considered in $\operatorname{ran}\left(P_{j_{0}}\right)$.

Proof. For $j=1, \ldots, k$, since $P_{j}^{2}=P_{j}$ by (6.5), one has $P_{j} T P_{j}=T P_{j}^{2}=T P_{j}=P_{j} T$ if $T$ commutes with $P_{j}$.

Also from (6.5), one has $P=\sum_{j=1}^{k} P_{j}$ and thus, we have

$$
T P=\sum_{j=1}^{k}\left(T P_{j}\right)=\sum_{j=1}^{k} P_{j} T=P T
$$

We now prove that $P$ is a projection. Indeed, since $P_{j} P_{j^{\prime}}=O$ for $j \neq j^{\prime}$ and $P_{j}^{2}=P_{j}$, we have

$$
P^{2}=\left(\sum_{j=1}^{k} P_{j}\right)^{2}=\sum_{j, j^{\prime}=1}^{k} P_{j} P_{j^{\prime}}=\sum_{j=1}^{k} P_{j}=P .
$$

Hence, we have $P T P=P^{2} T=P T=T P=\sum_{j=1}^{k}\left(T P_{j}\right)$.
Assume that there is $u \in \operatorname{ran}(P)$ such that $u \neq O_{n \times 1}$ and $T u=\lambda u$. Then, $u=P u$ and one has $T P u=\lambda P u$. Moreover, since $P P_{j}=\sum_{j^{\prime}=1}^{k}\left(P_{j^{\prime}} P_{j}\right)=P_{j}=\sum_{j^{\prime}=1}\left(P_{j} P_{j^{\prime}}\right)=P_{j} P$ and $T P_{j}=P_{j} T$, one obtains

$$
T P_{j} u=T\left(P_{j} P\right) u=\left(P_{j} T\right) P u=\lambda P_{j} P u=\lambda P_{j} u .
$$

On the other hand, since the direct sum $\sum_{j=1}^{k}\left(P_{j} u\right)=P u=u \neq O_{n \times 1}$, there is at least $j_{0} \in\{1, \ldots, k\}$ such that $P_{j_{0}} u \neq O_{n \times 1}$. Thus, let $v=P_{j_{0}} u \in \operatorname{ran}\left(P_{j_{0}}\right), v \neq O_{n \times 1}$ and $T v=\lambda v$.

For the inverse, let $v \in \operatorname{ran}\left(P_{j_{0}}\right)$ for some $j_{0} \in\{1, \ldots, k\}$ such that $v \neq O_{n \times 1}$ and $T v=\lambda v, \operatorname{since} \operatorname{ran}\left(P_{j_{0}}\right) \subset \operatorname{ran}(P)$ by (6.5), we finish the proof.

Proposition 6.6. For $x \in \mathbb{C}$ small enough, let $T(x)=T^{(0)}+\mathcal{O}(|x|)$ where $T^{(0)}$ is a matrix and $T$ is considered in the domain $\mathcal{D}:=\operatorname{ran}(P)$ where $P(x)=P^{(0)}+\mathcal{O}(|x|)$. Assume that there are $k \leq n$ distinct eigenvalues $\lambda_{j}^{(0)}$ of $T^{(0)}$ considered in $\operatorname{ran}\left(P^{(0)}\right)$ where $j=1, \ldots, k$. Then, there is a unique sequence $\left(P_{j}\right)$ satisfying (6.5) and (6.6) such that $P_{j}(x)=P_{j}^{(0)}+\mathcal{O}(|x|)$ where $P_{j}^{(0)}$ is the eigenprojection associated with $\lambda_{j}^{(0)}$ where $j=1, \ldots, k$.

In particular, for any $\lambda \in \sigma(T)$ considered in $\operatorname{ran}(P), \lambda \in \sigma(T)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\lambda(x) \rightarrow \lambda_{j}^{(0)}$ as $|x| \rightarrow 0$ for $j \in\{1, \ldots, k\}$.

Before going to the proof of Proposition 6.6, we have the following lemma. Let the resolvent of a matrix operator $T$ where $T$ depends on $x \in \mathbb{C}$ be

$$
\begin{equation*}
R(x, z):=(T(x)-z I)^{-1}, \quad z \in \rho(T) \tag{6.7}
\end{equation*}
$$

Lemma 6.7. The resolvent of the matrix operator $T(x):=T^{(0)}+\mathcal{O}(|x|)$ is holomorphic in any neighborhood of $(x, y) \in \mathbb{C}^{2}$ such that $y \in \rho\left(T^{(0)}\right)$. Moreover, if $\Gamma$ a compact subset of $\rho\left(T^{(0)}\right)$, then $R(x, y)$ is a convergent series as $|x| \rightarrow 0$ uniformly in $y \in \Gamma$ and thus one has the expansion

$$
\begin{equation*}
R(x, y)=R^{(0)}(y)+\mathcal{O}(|x|), \quad|x| \rightarrow 0 \tag{6.8}
\end{equation*}
$$

where $R^{(0)}(y):=\left(T^{(0)}-y I\right)^{-1}$.
As a consequence, there is no eigenvalue of $T$ included in $\Gamma$.
Proof of Lemma 6.7. For $z \in \rho(T)$ and $y \in \rho\left(T^{(0)}\right)$, we have

$$
\begin{aligned}
T(x)-z I & =\left(T^{(0)}-y I\right)-\left((z-y) I-\left(T(x)-T^{(0)}\right)\right) \\
& =\left(1-\left((z-y) I-\left(T(x)-T^{(0)}\right)\right)\left(T^{(0)}-\lambda^{(0)} I\right)^{-1}\right)\left(T^{(0)}-y I\right)
\end{aligned}
$$

Thus, taking the inverse and since $T(x)-T^{(0)}=\mathcal{O}(|x|)$ for $x$ small, we obtain

$$
R(x, z)=R^{(0)}(y)\left(1-((z-y) I-\mathcal{O}(|x|)) R^{(0)}(y)\right)^{-1}
$$

Furthermore, for any matrix norm $\|\cdot\|$, we also have

$$
\left\|((z-y) I-\mathcal{O}(|x|)) R^{(0)}(y)\right\| \leq(|z-y|+C|x|)\left\|R^{(0)}(y)\right\|<1
$$

for $x$ and $z-y$ small enough. Thus, it implies that $R(x, z)$ can be expanded as a convergent series and is holomorphic in any neighborhood of $(x, y)$.

On the other hand, for $x$ small and $y \in \rho\left(T^{(0)}\right)$, one has

$$
T(x)-y I=\left(T^{(0)}-y I\right)+\mathcal{O}(|x|)=\left(I+\mathcal{O}(|x|)\left(T^{(0)}-y I\right)^{-1}\right)\left(T^{(0)}-y I\right)
$$

Thus, one deduces

$$
R(x, y)=R^{(0)}(y)\left(I+\mathcal{O}(|x|) R^{(0)}(y)\right)^{-1}=R^{(0)}(y)(I+\mathcal{O}(|x|))=R^{(0)}(y)+\mathcal{O}(|x|)
$$

On the other hand, one notes that $R(x, y)$ is expressed based on $R^{(0)}(y)$. Since $\Gamma$ is a compact subset of $\rho\left(T^{(0)}\right)$, the norm $\left\|\mathcal{O}(|x|) R^{(0)}(y)\right\|$ can be bounded by 1 uniformly for all $y \in \Gamma$. As a consequence, since $R(x, y)$ exists for all $x$ small and $y \in \Gamma$, there is no eigenvalue of $T$ belongs to $\Gamma$.

We are now going back to the proof of Proposition 6.6.
Proof of Proposition 6.6. Primarily, we have the follows. Let $\lambda \in \sigma(T)$ considered in $\mathbb{C}^{n}$, $\lambda$ must be a solution of the dispersion polynomial $p:=\operatorname{det}(T-\lambda I)$ that is an analytic function in $x \in \mathbb{C}$ since $T$ is analytic in $x \in \mathbb{C}$. Moreover, it is known that $\lambda$ is continuous and converges to an eigenvalue of $T^{(0)}$ as $|x| \rightarrow 0$ since $T(x)=T^{(0)}+\mathcal{O}(|x|)$ as $|x| \rightarrow 0$. Thus, one can write

$$
\begin{equation*}
\lambda(x):=\lambda^{(0)}+\mathcal{O}(1), \quad|x| \rightarrow 0, \tag{6.9}
\end{equation*}
$$

where $\lambda^{(0)} \in \sigma\left(T^{(0)}\right)$ considered in $\mathbb{C}^{n}$ is the limit of $\lambda$ as $|x| \rightarrow 0$. In particular, due to the formula (6.9), the eigenvectors $u \in \mathbb{C}^{n}$ associated with $\lambda$ can be chosen such that

$$
\begin{equation*}
u(x):=u^{(0)}+\mathcal{O}(1), \quad|x| \rightarrow 0 \tag{6.10}
\end{equation*}
$$

where $u^{(0)} \in \mathbb{C}^{n}$ are the eigenvectors associated with $\lambda^{(0)}$. It follows that $u \in \operatorname{ran}(P)$ if and only if $u^{(0)} \in \operatorname{ran}\left(P^{(0)}\right)$. Indeed, one has

$$
P u=\left(P^{(0)}+\mathcal{O}(|x|)\right)\left(u^{(0)}+\mathcal{O}(1)\right)=P^{(0)} u^{(0)}+\mathcal{O}(1)
$$

and thus $P u=u$ if and only if $P^{(0)} u^{(0)}=u^{(0)}$. It implies that $\lambda \in \sigma(T)$ considered in $\operatorname{ran}(P)$ if and only if $\lambda^{(0)} \in \sigma\left(T^{(0)}\right)$ considered in $\operatorname{ran}\left(P^{(0)}\right)$. Therefore, if $\lambda_{j}^{(0)}$ for $j=1, \ldots, k$ are the $k$ distinct eigenvalues of $T^{(0)}$ considered in $\operatorname{ran}\left(P^{(0)}\right)$, then the above argument and the expansion (6.9) show that for any eigenvalue $\lambda$ of $T$ considered in the domain $\mathcal{D}=\operatorname{ran}(P)$, then $\lambda$ converges to an eigenvalue $\lambda_{j}^{(0)}$ of $T^{(0)}$ considered in $\operatorname{ran}\left(P^{(0)}\right)$ for some $j \in\{1, \ldots, k\}$ as $|x| \rightarrow 0$. In particular, for each $j \in\{1, \ldots, k\}$, the set of all eigenvalues $\lambda$ of $T$ considered in $\mathcal{D}$ such that $\lambda \rightarrow \lambda_{j}^{(0)}$ as $|x| \rightarrow 0$ is the $\lambda_{j}^{(0)}$-group of $T$. For easy, we consider the formal formula

$$
\begin{equation*}
\sigma(T) \text { considered in } \mathcal{D}=\bigcup_{j=1}^{k}\left(\lambda_{j}^{(0)} \text {-group }\right), \tag{6.11}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{j}^{(0)} \text {-group }:=\left\{\lambda \in \sigma(T) \text { considered in } \mathcal{D}: \lambda \rightarrow \lambda_{j}^{(0)} \text { as }|x| \rightarrow 0\right\} . \tag{6.12}
\end{equation*}
$$

We are going to prove the unique existence of a sequence $\left(P_{j}\right)$ satisfying (6.5) and (6.6) where $j=1, \ldots, k$. First of all, we consider the domain $\mathcal{D}=\mathbb{C}^{n}$ i.e. $P=I$ the identity matrix and since $P(x)=P^{(0)}+\mathcal{O}(|x|)$ as $|x| \rightarrow 0, P^{(0)}=I$ as well. Hence, the eigenvalues $\lambda$ of $T$ and $\lambda_{j}^{(0)}$ of $T^{(0)}$ in this case are considered in $\mathbb{C}^{n}$. Let $\lambda \in \sigma(T)$ and let $\Gamma_{\lambda}$ be a
closed curve enclosing $\lambda$ except for the other eigenvalues of $T$ in the complex plane, since $\lambda$ is singularity of the resolvent $R(z)=(T-z I)^{-1}$ of $T$, the Cauchy integral

$$
\begin{equation*}
P_{\lambda}(x):=-\frac{1}{2 \pi i} \int_{\Gamma_{\lambda}} R(x, z) d z \tag{6.13}
\end{equation*}
$$

is exactly the eigenprojection associated with $\lambda$. The matrix operator $N_{\lambda}:=(T-\lambda I) P_{\lambda}$ is then the nilpotent part associated with $\lambda$. Moreover, $T P_{\lambda}=\lambda P_{\lambda}+N_{\lambda}=P_{\lambda} T$. Nonetheless, the resolvent $R(x, z)$ for $x$ small and $z \in \rho(T)$ cannot be expanded explicitly in general except for the case $z$ belongs to a compact set contained in $\rho\left(T^{(0)}\right)$ provided Lemma 6.7.

Based on that, for $j \in\{1, \ldots, k\}$, let $\Gamma_{j}$ be a closed curve in $\rho\left(T^{(0)}\right)$ such that $\Gamma_{j}$ encloses the eigenvalue $\lambda_{j}^{(0)}$ except for the other eigenvalues of $T^{(0)}$. Then, by Lemma 6.7, there is no eigenvalue of $T$ to belong to $\Gamma_{j}$ and therefore, for $x$ small enough, the interior domain bounded by $\Gamma_{j}$ only encloses the eigenvalues of $T$ such that $\lambda \rightarrow \lambda_{j}^{(0)}$ as $|x| \rightarrow 0$ i.e. the $\lambda_{j}^{(0)}$-group is contained in this domain except for the other groups of $T$. Hence, for every $\lambda$ included in the $\lambda_{j}^{(0)}$-group, one can choose $\Gamma_{\lambda}$ such that they are strictly contained in the domain bounded by $\Gamma_{j}$ and one has

$$
P_{j}(x):=-\frac{1}{2 \pi i} \int_{\Gamma_{j}} R(x, y) d y=\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }}\left(-\frac{1}{2 \pi i} \int_{\Gamma_{\lambda}} R(x, z) d z\right)=\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }} P_{\lambda}
$$

Hence, $P_{j}$ is called the total projection associated with the $\lambda_{j}^{(0)}$-group of $T$.
The sequence of the total projections $P_{j}$ where $j \in\{1, \ldots, k\}$ satisfies the properties (6.5). Indeed, since, for $\lambda \in \lambda_{j}^{(0)}$-group, $P_{\lambda}$ is an eigenprojection, one has

$$
P_{j}^{2}=\left(\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }} P_{\lambda}\right)^{2}=\sum_{\lambda, \lambda^{\prime} \in \lambda_{j}^{(0)} \text {-group }} P_{\lambda} P_{\lambda^{\prime}}=\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }} P_{\lambda}=P_{j}
$$

and for $j \neq j^{\prime}$, since $\lambda_{j}^{(0)} \neq \lambda_{j^{\prime}}^{0}$ due to the distinct property, one has

$$
P_{j} P_{j^{\prime}}=\left(\sum_{\substack{ \\\lambda \in \lambda_{j}^{(0)} \text {-group }}} P_{\lambda}\right)\left(\sum_{\substack{\lambda^{\prime} \in \lambda_{j^{\prime}}^{(0)} \text {-group }}} P_{\lambda^{\prime}}\right)=\sum_{\substack{\lambda \in \lambda_{j}^{(0)} \text {-group } \\ \lambda^{\prime} \in \lambda_{j^{\prime}}^{(0)} \text {-group }}} P_{\lambda} P_{\lambda^{\prime}}=O
$$

since these two groups are distinct. Moreover, we have $\mathbb{C}^{n}=\bigoplus_{\lambda \in \sigma(T)} \operatorname{ran}\left(P_{\lambda}\right)$ and $I=$ $\sum_{\lambda \in \sigma(T)} P_{\lambda}$ and thus from (6.11) and (6.12), one deduces

$$
I=\sum_{j=1}^{k} \sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }} P_{\lambda}=\sum_{j=1}^{k} P_{j} \quad \text { and } \quad \mathbb{C}^{n}=\bigoplus_{j=1}^{k} \bigoplus_{\lambda \in \lambda_{j}^{(0)} \text {-group }} \operatorname{ran}\left(P_{\lambda}\right)=\bigoplus_{j=1}^{k} \operatorname{ran}\left(P_{j}\right)
$$

Then, the property (6.6) holds if one proves that $T$ commutes with $P_{j}$ for all $j \in$ $\{1, \ldots, k\}$ due to Proposition 6.5. Infact, for all $\lambda \in \sigma(T)$, since $T P_{\lambda}=P_{\lambda} T$, one obtain
that

$$
T P_{j}=\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }}\left(T P_{\lambda}\right)=\sum_{\lambda \in \lambda_{j}^{(0)} \text {-group }}\left(P_{\lambda} T\right)=P_{j} T
$$

for all $j \in\{1, \ldots, k\}$.
On the other hand, from (6.8), one has

$$
P_{j}(x):=-\frac{1}{2 \pi i} \int_{\Gamma_{j}} R(x, y) d y=-\frac{1}{2 \pi i} \int_{\Gamma_{j}} R^{(0)}(y) d y+\mathcal{O}(|x|), \quad|x| \rightarrow 0
$$

where $R^{(0)}(y)=\left(T^{(0)}-y I\right)^{-1}$ for $y \in \rho\left(T^{(0)}\right)$. Then, it is easy to see that $R^{(0)}$ is the resolvent of the matrix $T^{(0)}$ and thus by the definition of $\Gamma_{j}$, it implies that

$$
P_{j}(x)=P_{j}^{(0)}+\mathcal{O}(|x|), \quad|x| \rightarrow 0
$$

where $P_{j}^{(0)}$ is the eigenprojection associated with $\lambda_{j}^{(0)}$ since it is known that $P_{j}^{(0)}=$ $-\frac{1}{2 \pi i} \int_{\Gamma_{j}} R^{(0)}(y) d y$.

We already construct the desired sequence of $P_{j}$ where $j=1, \ldots, k$ if $\mathcal{D}=\mathbb{C}^{n}$. For the case $\mathcal{D}=\operatorname{ran}(P)$, it is enough to define the unique eigenprojection $\tilde{P}_{j}$ associated with the domain $\operatorname{ran}\left(P_{j}\right) \cap \operatorname{ran}(P)$ where $P_{j}$ is constructed as before for each $j \in\{1, \ldots, k\}$. One can denote $\tilde{P}_{j}$ again by $P_{j}$ where $j=1, \ldots, k$.

Finally, we prove that for any $\lambda \in \sigma(T)$ considered in $\operatorname{ran}(P), \lambda \in \sigma(T)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\lambda(x) \rightarrow \lambda_{j}^{(0)}$ as $|x| \rightarrow 0$ for $j \in\{1, \ldots, k\}$. Indeed, for each $j \in\{1, \ldots, k\}$, since $P_{j}(x)=P_{j}^{(0)}+\mathcal{O}(|x|)$ as $|x| \rightarrow 0$, similarly to the beginning of the proof of this Proposition, we already prove that $\lambda \in \sigma(T)$ considered in $\operatorname{ran}\left(P_{j}\right)$ if and only if $\lambda^{(0)} \in \sigma\left(T^{(0)}\right)$ considered in $\operatorname{ran}\left(P_{j}^{(0)}\right)$ where $\lambda^{(0)}$ is the limit of $\lambda$ as $|x| \rightarrow 0$. On the other hand, $\operatorname{ran}\left(P_{j}^{(0)}\right) \subset \operatorname{ran}\left(P^{(0)}\right)$ due to the fact that $\operatorname{ran}\left(P_{j}\right) \subset \operatorname{ran}(P)$. Thus, $\lambda^{(0)} \in \sigma\left(T^{(0)}\right)$ considered in $\operatorname{ran}\left(P^{(0)}\right)$ which is the definition of the eigenvalues $\lambda_{j}^{(0)}$ for $j \in\{1, \ldots, k\}$. Thus, there is a unique $j_{0} \in\{1, \ldots, k\}$ such that $\lambda^{(0)}=\lambda_{j_{0}}^{(0)}$ since $\lambda_{j}^{(0)}$ for all $j \in\{1, \ldots, k\}$ are distinct. Nonetheless, since $\lambda^{(0)}$ is considered in $\operatorname{ran}\left(P_{j}^{(0)}\right)$, one obtains $j_{0}=j$ since $P_{j}^{(0)} P_{j_{0}}^{(0)} \neq O$ if and only if $j=j_{0}$. The proof is done.

## Acknowledgments

The second author's research is supported by the doctoral grant of Gran Sasso Science Institute 2014-2017.

## References

[1] S. Benzoni-Gavage, D. Serre, Multi-dimensional hyperbolic partial differential equations: First-order Systems and Applications, Oxford University Press on Demand, 2007. 1
[2] S. Bianchini, B. Hanouzet, R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, Commun. Pure Appl. Math. 60 (11) (2007) 1559-1622. 1, 3
[3] S. Goldstein, On diffusion by discontinuous movements, and on the telegraph equation, Q. J. Mech. Appl. Math. 4 (2) (1951) 129-156. 1
[4] T. Hosono, T. Ogawa, Large time behavior and $L^{p}-L^{q}$ estimate of solutions of 2dimensional nonlinear damped wave equations, J. Differ. Equ. 203 (1) (2004) 82-118. 1
[5] M. Kac, A stochastic model related to the telegrapher's equation. Reprinting of an article published in 1956. Papers arising from a Conference on Stochastic Differential Equations (Univ. Alberta, Edmonton, Alta., 1972), Rocky Mt. J. Math 4 (1974) 497-509. 1
[6] T. Kato, Perturbation theory for linear operators, vol. 132, Springer Science \& Business Media, 2013. 2, 2, 6, 6
[7] T.-P. Liu, Hyperbolic conservation laws with relaxation, Commun. Math. Phys. 108 (1) (1987) 153-175. 1
[8] P. Marcati, K. Nishihara, The $L^{p}-L^{q}$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differ. Equ. 191 (2) (2003) 445-469. 1, 1
[9] C. Mascia, Exact representation of the asymptotic drift speed and diffusion matrix for a class of velocity-jump processes, J. Differ. Equ. 260 (1) (2016) 401-426. 1
[10] Y. Shizuta, S. Kawashima, Others, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, Hokkaido Math. J 14 (2) (1985) 249-275. 1
[11] Y. Ueda, R. Duan, S. Kawashima, Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application, Arch. Ration. Mech. Anal. 205 (1) (2012) 239-266. 1


[^0]:    ${ }^{1}$ Dipartimento di Matematica "G. Castelnuovo", Sapienza - Università di Roma, P.le Aldo Moro, 2 00185 Roma (ITALY), mascia@mat.uniroma1.it
    ${ }^{2}$ Department of Mathematics, Gran Sasso Science Institute - Istituto Nazionale di Fisica Nucleare, Viale Francesco Crispi, 7-67100 L'Aquila (ITALY), nguyen.tienthinh@gssi.infn.it

