Moduli of double EPW-sextics

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Abstract

We will study the GIT quotient of the symplectic grassmannian parametrizing lagrangian subspaces of $\bigwedge^3 \mathbb{C}^6$ modulo the natural action of SL_6, call it $\mathfrak{M}$. This is a compactification of the moduli space of smooth double EPW-sextics and hence birational to the moduli space of HK 4-folds of Type $K3^{[2]}$ polarized by a divisor of square 2 for the Beauville-Bogomolov quadratic form. We will determine the stable points. Our work bears a strong analogy with the work of Voisin, Laza and Looijenga on moduli and periods of cubic 4-folds. We will prove a result which is analogous to a theorem of Laza asserting that cubic 4-folds with simple singularities are stable. We will also describe the irreducible components of the GIT boundary of $\mathfrak{M}$. Our final goal (not achieved in this work) is to understand completely the period map from $\mathfrak{M}$ to the Baily-Borel compactification of the relevant period domain modulo an arithmetic group. We will analyze the locus in the GIT-boundary of $\mathfrak{M}$ where the period map is not regular. Our results suggest that $\mathfrak{M}$ is isomorphic to Looijenga's compactification associated to 3 specific hyperplanes in the period domain.

Key words and phrases: GIT quotient, period map, hyperkähler varieties.

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0 Introduction

A compact Kähler manifold X is *hyperkähler* if it is simply connected and it carries a holomorphic symplectic form whose cohomology class spans $H^{2,0}(X)$. Two-dimensional hyperkähler manifolds are nothing else but $K3$ surfaces. Beauville [1] has constructed two classes of examples in each even dimension $2n > 2$: the Douady space $S^{[n]}$ parametrizing length-n analytic subspaces of a $K3$ surface S, and the generalized Kummer $K^n(T) \subset T^{[n+1]}$ consisting of length-$(n+1)$ analytic subspaces Z of a 2-dimensional (compact) torus T such that the associated cycle $\sum_{p\in T} \ell(\mathcal{O}_{Z,p})p$ sums up to 0 in the additive group T. These two examples are not deformation-equivalent since their second Betti numbers are 23 and 7 respectively. The author has constructed two other classes of examples, in dimensions 6 and 10, which have second Betti numbers equal to 8 and 24 respectively [24, 25, 31]. Up to deformation there are no other known examples.

A compact Kähler manifold with torsion first Chern class has a finite étale cover which is a product of factors which are complex tori, hyperkähler manifolds or Calabi-Yau manifolds[1]: this is the "Beauville-Bogomolov decomposition Theorem"(see [1]), and it shows that hyperkähler manifolds are fundametal objects in Kähler geometry.

Projective hyperkähler manifolds (we call them *hyperkähler varieties*) are dense in each deformation class of hyperkähler manifolds and they have a very rich geometry as demonstrated by the case of $K3$ surfaces. Hyperkähler varieties belonging to a single deformation class break up into a countable family of polarized-deformation classes, indexed by the discrete invariants of the polarization such as the degree of its highest self-intersection, see [8] for more details.

In the present work we will study moduli of polarized hyperkähler varieties belonging to a specific class: they are deformations of $S^{[2]}$, and they are polarized by a class of Beauville-Bogomolov square 2 (this means that the 4-tuple intersection has degree 12). Why this particular class? The reason is that the generic such variety is a double EPW-sextic i.e. a double cover of a particular kind of sextic hypersurface (an EPW-sextic, first introduced by Eisenbud-Popescu-Walter in [5]) and hence it has an explicit description. We should point out that the generic double EPW-sextic is *not* isomorphic (nor birational) to the Hilbert square of a $K3$ surface, and that only a handful of explicit locally complete families of hyperkähler varieties of dimension greater than 2 have been constructed, see [2, 4, 12, 13, 16] for the other families.

Let V be a complex vector-space of dimension 6; an EPW-sextic in $\mathbb{P}(V)$ is determined by the choice of a lagrangian subspace of $\bigwedge^3 V$ (the symplectic form is given by wedge-product), see **Subsection 1.1** for details. Moreover the double covers of two EPW-sextics are polarized-isomorphic only if the EPW-sextics are projectively equivalent. Thus we will study the quotient of the symplectic grassmannian parametrizing lagrangian subspaces of $\bigwedge^3 V$, call it $\mathbb{LG}(\bigwedge^3 V)$, by the natural action of $\mathrm{PGL}(V)$. There is a unique linearization of this action; we let

$$\mathfrak{M} := \mathbb{LG}(\bigwedge^{3} V)//\,\mathrm{PGL}(V) \tag{0.0.1}$$

be the GIT quotient. The open dense subset of $\mathbb{LG}(\bigwedge^3 V)$ parametrizing smooth double EPW-sextics is contained in the stable locus. It follows that $\mathfrak{M}$ is a compactification of the moduli space of smooth double EPW-sextics, and that it is birational to the moduli space of polarized deformations of $S^{[2]}$ with a polarization of square 2.

In order to explain our approach in studying $\mathfrak{M}$ we must recall that there is a strong analogy between the family of cubic hypersurfaces in $\mathbb{P}^5$ and the family of double EPW-sextics. In fact Beauville and Donagi [2] proved that the variety of lines on a smooth cubic 4-fold is a hyperkähler variety deformation equivalent to the Hilbert square of a $K3$ and that the (Plücker) polarization has square 6 for the Beauville-Bogomolov quadratic form (and divisibility 2). By varying the cubic 4-fold one gets a locally complete family of projective deformations of such varieties, moreover the Hodge structure of the primitive H^4 of a smooth cubic 4-fold is isomorphic to the primitive H^2 of the variety of lines on the cubic. Summarizing: the GIT quotient of the space of cubic 4-folds is birational to the moduli space of polarized deformations of $S^{[2]}$ with a polarization of square 6 and divisibility 2, and the period map which associates to a smooth cubic Z the Hodge structure on $H^4_{prim}(Z)$ is identified with the period map which associates to Z the Hodge structure on H^2_{prim} of the variety of lines on Z. Our work was greatly influenced by the results of Voisin, Laza and Looijenga on moduli and periods of cubic 4-folds, see [34, 14, 15, 18]. Following are some of their results. First Voisin [34] (see also [18]) proved that the period map for cubic 4-folds is birational (but it is not an isomorphism, nor is it regular). After that Laza and Looijenga [14, 15, 18] analyzed the GIT quotient of the space of cubic 4-folds, and they examined the (birational) period map, in particular they proved that the GIT quotient is identified with Looijenga's compactification associated to a particular hypersurface in the relevant quotient of a bounded symmetric domain of Type IV.

[1]A compact Kähler manifold X is Calabi-Yau if it has trivial canonical bundle and no non-zero holomorphic p-form in the range $0 < p < \dim X$

The present work deals with the GIT side of the story for double EPW-sextics, with a view towards proving that $\mathfrak{M}$ is identified with Looijenga's compactification associated to a particular arrangement of hypersurfaces in the relevant quotient of a bounded symmetric domain of Type IV (the period map is birational by [33, 9, 20, 21]), namely the hyperfaces $\mathbb{S}_2'$, $\mathbb{S}_2''$ and $\mathbb{S}_4$ of [30], but let me stress that this is still far from being proved.

Now let us describe the main results of the present work. In **Section 2** we will give a description of stable points of $\mathbb{LG}(\bigwedge^3 V)$ in terms of linear algebra. More precisely we will prove that the locus of non-stable $A \in \mathbb{LG}(\bigwedge^3 V)$ is the union of 12 locally closed subsets of $\mathbb{LG}(\bigwedge^3 V)$ (the standard non-stable strata) defined by "flag conditions". Two examples of standard non-stable strata are the following: the set $\mathbb{B}^*_{\mathcal{A}}$ of A for which there exists $0 \neq v_0$ such that $\dim(A \cap (v_0 \wedge \bigwedge^2 V)) \geq 5$, the set $\mathbb{B}^*_{\mathcal{A}^\vee}$ of A for which there exists a codimension-1 subspace $V_0 \subset V$ such that $\dim(A \cap \bigwedge^3 V_0) \geq 5$. In order to show that the standard non-stable strata parametrize non-stable lagrangians it will suffice to express the numerical function $\mu(A, \lambda)$ of a lagrangian A with respect to a 1-PS $\lambda \colon \mathbb{C}^\times \to \mathrm{SL}(V)$ in terms of the dimension of the intersections of A with the isotypical summands of $\bigwedge^3 \lambda$. The proof that any non-stable lagrangian belongs to one of the standard non-stable strata requires more work. First we will prove the Cone Decomposition Algorithm: it applies whenever we have a linearly reductive group G acting on a product of Grassmannains $\mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$ via a representation $G \to \mathrm{GL}(U^0) \times \ldots \times \mathrm{GL}(U^r)$. The algorithm provides a finite list of 1-PS's of G (ordering 1-PS's) with the property that if $A_\bullet = (A_0, \ldots, A_r)$ is non-stable then it is destabilized by a 1-PS conjugated to one of the ordering 1-PS's; that result should be of independent interest because it is applicable to other GIT problems. We will apply the Cone Decomposition Algorithm to the case of interest to us: using a computer we will get the finite list of ordering 1-PS's of $\mathrm{SL}(V)$. Another computation will give the following result: if A is not stable then it is destabilized by a 1-PS conjugated to one among the simplest ordering 1-PS's, where simplicity is measured by the magnitude of the weights of the 1-PS. The "simplest"ordering 1-PS's are exactly those defining the 12 standard non-stable strata.

After having obtained a linear algebra description of stable lagrangians we will ask for a characterization of stable lagrangians via geometric properties of the corresponding double EPW-sextic. In order to explain the relevant results we must introduce some notation. Given $A \in \mathbb{LG}(\bigwedge^3 V)$ one defines a subscheme $Y_A \subset \mathbb{P}(V)$ which is either a sextic hypersurface - in this case it is an EPW-sextic - or all of $\mathbb{P}(V)$ if A is "pathological", see **Subsection 1.1**. If Y_A is an EPW-sextic then it comes equipped with a degree-2 cover $X_A \to Y_A$, loc. cit. There is a dense open subset $\mathbb{LG}(\bigwedge^3 V)^0 \subset \mathbb{LG}(\bigwedge^3 V)$ parametrizing A's such that X_A is a smooth deformations of $K^{[2]}$, see [29]. One consequence of the results of **Section 2** is that if $Y_A = \mathbb{P}(V)$ then A is unstable and hence every point of $\mathfrak{M}$ represents an equivalence class of double EPW-sextics. Ideally we would like a characterization of (semi)stability in terms of the geometry of X_A. What we will provide is a partial answer in terms of the period map

$$\mathcal{P} \colon \mathbb{LG}(\bigwedge^3 V) \dashrightarrow \mathbb{D}^{BB}$$

Here $\mathbb{D}$ is the quotient of a 20-dimensional bounded symmetric domain of Type IV by a suitable arithmetic group, see [30], and $\mathbb{D}^{BB}$ is its Baily-Borel compactification. On the open $\mathbb{LG}(\bigwedge^3 V)^0$ parametrizing smooth double EPW-sextics the map $\mathcal{P}$ associates to A the Hodge structure on the primitive $H^2(X_A)_{\mathrm{pr}}$ modulo Hodge isometries. The map $\mathcal{P}$ induces the period map of the moduli space:

$$\mathfrak{p} \colon \mathfrak{M} \dashrightarrow \mathbb{D}^{BB}. \tag{0.0.2}$$

There is a prime divisor $\Sigma \subset \mathbb{LG}(\bigwedge^3 V)$ which is analogous to the prime divisor parametrizing singular cubic 4-folds: it is the locus of A which contain a non-zero decomposable vector $w_0 \wedge w_1 \wedge w_2$. Away from Σ the map $\mathcal{P}$ is regular and it lands into $\mathbb{D}$, the interior of the Baily-Borel compactification, see [27] and [29]. Let $A \in \Sigma$ be semistable: one may analyze the behaviour of $\mathcal{P}$ at A as follows [30]. Let $0 \neq w_0 \wedge w_1 \wedge w_2 \in A$ and $W \subset V$ be the span of w_0, w_1, w_2: one defines a Lagrangian degeneracy locus $C_{W,A} \subset \mathbb{P}(W)$ which is generically a sextic curve and in pathological cases is all of $\mathbb{P}(W)$, see **Subsection 3.2** for details. Notice that if $C_{W,A}$ is smooth the double cover of $\mathbb{P}(W)$ branched over $C_{W,A}$ is a $K3$ surface of degree 2. Let $\mathbb{D}^{BB}_{K3,2}$ be the Baily-Borel compactification of the period space for $K3$ surfaces of degree 2 and

$$|\mathcal{O}_{\mathbb{P}(W)}(6)| \dashrightarrow \mathbb{D}^{BB}_{K3,2} \tag{0.0.3}$$

be the compactified period map. Our (semi)stability geometric criteria will be guided by the following result.

Theorem 0.0.1 ([30])**.** *Let $A \in \Sigma$ be semistable with closed orbit and suppose that for all $W \in \mathrm{Gr}(3,V)$ such that $\bigwedge^3 W \subset A$ the following holds: $C_{W,A}$ is a sextic curve and it belongs to the regular locus of the compactified period map* (0.0.3)*. Then $\mathfrak{p}$ is regular at $[A]$. Moreover $\mathfrak{p}([A]) \in \mathbb{D}$ if and only if $C_{W,A}$ has simple singularities for all $W \in \mathrm{Gr}(3,V)$ such that $\bigwedge^3 W \subset A$.*

The above result motivates the following definition.

Definition 0.0.2. Let $\mathbb{LG}(\bigwedge^3 V)^{ADE} \subset \mathbb{LG}(\bigwedge^3 V)$ be the set of A such that for every $W \in \mathrm{Gr}(3,V)$ such that $\bigwedge^3 W \subset A$ we have that $C_{W,A}$ is a curve with simple singularities.

Below is the main result of **Section 3** - it is analogous to Proposition 3.2 of R. Laza [15] on periods of cubic 4-folds with simple singularities.

Theorem 0.0.3. *$\mathbb{LG}(\bigwedge^3 V)^{ADE}$ is contained in the stable locus $\mathbb{LG}(\bigwedge^3 V)^{st}$.*

Now let's pass to the contents of the remaining sections. First we will describe the results of **Section 5**, **Section 6**, **Section 7**, and then those of **Section 4**. One of the main results stated in **Section 5** and proved in **Section 6** and **Section 7** is the description of the irreducible components of the GIT-boundary $\partial\mathfrak{M} := (\mathfrak{M} \setminus \mathfrak{M}^{st})$ where $\mathfrak{M}^{st}$ is the open subset of $\mathfrak{M}$ parametrizing isomorphism classes of stable double EPW-sextics. The statement below is somewhat vague because in order to give a precise statement one first needs to define the standard non-stable strata - see **Theorem 5.1.1** for a complete statement.

Theorem 0.0.4. *Each irreducible component of $\partial\mathfrak{M}$ is the image of the semistable points of a standard non-stable strata, and viceversa the semistable points of each standard non-stable strata parametrize an irreducible component of $\partial\mathfrak{M}$. There are 8 irreducible components of $\partial\mathfrak{M}$ and their dimensions are given by the entries in the first row of Table* (8)*.*

A word of explanation regarding the number of irreducible components of $\partial\mathfrak{M}$. There are 12 standard non-stable strata but only 8 irreducible components of $\partial\mathfrak{M}$. The reason for this discrepancy is that the generic semistable point with closed orbit in one standard non-stable strata can be also the generic semistable point with closed orbit of a different standard non-stable strata: for example this happens for the non-stable strata $\mathbb{B}_{\mathcal{A}}$ and $\mathbb{B}_{\mathcal{A}^\vee}$ that we described above. The main tool that we will use in order to prove **Theorem 0.0.4** will be the Cone Decomposition Algorithm.

In order to explain the other main result stated in **Section 5** we give a definition.

Definition 0.0.5. Let $\mathfrak{I} \subset \mathfrak{M}$ be the subset of points represented by $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ for which the following hold:

(1) The orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$.

(2) There exists $W \in \Theta_A$ such that $C_{W,A}$ is either $\mathbb{P}(W)$ or a sextic curve in the indeterminacy locus of the period map (0.0.3).

By **Theorem 0.0.1** the indeterminacy locus of the period map (0.0.2) is contained in $\mathfrak{I}$ - an educated guess is that they are actually equal. The second main result proved in **Section 6** and **Section 7** is the following.

Theorem 0.0.6. *The intersection $\partial\mathfrak{M} \cap \mathfrak{I}$ has two irreducible components, $\mathfrak{X}_{\mathcal{V}}$ and $\mathfrak{X}_{\mathcal{Z}}$ of dimensions 3 and 1,defined in* **Subsubsection 7.2.1** *and* **Subsubsection 7.4.1** *respectively.*

Our results suggest that the period map (0.0.2) may be understood via Looijenga's compactifications of hyperplane arrangements [17] i.e. $\mathfrak{M}$ might be isomorphic to Looijenga's compactification of the complement of 3 specific hyperplanes in $\mathbb{D}$.

In order to give some details about this and explain the contents of **Section 4** we will go through some preliminaries. Let $A \in \Sigma$ and suppose that $W_1, W_2 \in \Theta_A$: then $W_1 \cap W_2 \neq \{0\}$ because A is lagrangian. Suppose that $W_1 \neq W_2$ and let $p \in \mathbb{P}(W_1 \cap W_2)$: then $p \in C_{W_i,A}$ for $i = 1, 2$ and a local equation of $C_{W_i,A}$ at p has vanishing linear term. Thus either $C_{W_i,A} = \mathbb{P}(W_i)$ or else every point of $\mathbb{P}(W_1 \cap W_2)$ is a singular point of $C_{W_i,A}$. This explains the relevance of those $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\dim \Theta_A > 0$ when determining $\mathfrak{I}$. Suppose that Θ is an irreducible component of Θ_A of strictly positive dimension. Since the planes $\mathbb{P}(W)$ for $W \in \Theta$ are pairwise incident Morin's Theorem [22] gives that Θ is contained in one of 6 families of pairwise incident planes, 3 elementary families defined by Schubert conditions and three more interesting families, namely one of the two rulings of a smooth quadric hypersurface $\mathcal{Q} \subset \mathbb{P}(V)$ by planes, the family of planes tangent to a Veronese surface $\mathcal{V}^2 \subset \mathbb{P}(V)$ and the family of planes which cut $\mathcal{V}^2$ in a conic. There are uniquely determined lagrangians A_+, A_k and A_h such that Θ_{A_+}, Θ_{A_k} and Θ_{A_h} are the three interesting families described above, see (2.4.11) and (3.3.20). In **Section 4** we will prove that A_+, A_k, A_h are semistable with closed orbits in $\mathbb{LG}(\bigwedge^3 V)^{ss}$; the corresponding points $\mathfrak{y} := [A_+]$, $\mathfrak{x} := [A_k]$, $\mathfrak{x}^\vee := [A_h]$ are distinct. (**Section 4** contains also results about other semistable lagrangians with large stabilizer.) We have $\mathfrak{y} = \mathfrak{X}_{\mathcal{V}} \cap \mathfrak{X}_{\mathcal{Z}}$ and $\mathfrak{x}, \mathfrak{x}^\vee \in \mathfrak{X}_{\mathcal{Z}}$.

Now we come to the relation with one of Looijenga's compactifications of complements of hyperplane arrangements. Suppose that A approaches A_+ generically: then X_A will approach the Hilbert square of a quartic $K3$ surface, see [6]. Similarly if A approaches A_k or A_h generically then X_A will approach the Hilbert square of a $K3$ of genus 2 or a moduli space of pure sheaves on such a $K3$. The corresponding periods will approach the divisor in $\mathbb{D}$ parametrizing points in the perpendicular to an element of square -4 in the relevant lattice in the first case and of square -2 (and divisibility 2) in the remaining two cases (there are two orbits of such elements under the action of the relevant arithmetic group): these are the hyperplanes that we mentioned above. What about the other points of $\mathfrak{X}_{\mathcal{V}} \cup \mathfrak{X}_{\mathcal{Z}}$? The picture that emerges from our results is the following: if A approaches generically a point in $(\mathfrak{X}_{\mathcal{W}} \setminus \{\mathfrak{y}\})$ ($\mathfrak{X}_{\mathcal{W}}$ is a curve in $\mathfrak{X}_{\mathcal{V}}$, see **Definition 4.4.3**) then X_A approaches the Hilbert square of a double cover of a smooth quadric surface, if A approaches generically a point in $(\mathfrak{X}_{\mathcal{V}} \setminus \mathfrak{X}_{\mathcal{W}})$ then X_A approaches the Hilbert square of a $K3$ which is a double cover of the Hirzebruch surface $\mathbb{F}_2$, if A approaches generically a point in $(\mathfrak{X}_{\mathcal{Z}} \setminus \{\mathfrak{y}, \mathfrak{x}, \mathfrak{x}^\vee\})$ then X_A approaches the Hilbert square of a $K3$ which is a double cover of the Hirzebruch surface $\mathbb{F}_4$.

Notation and conventions: Throughout the paper V is a complex vector-space of dimension 6. We choose a volume-form vol on V and we let $(,)_V$ be the corresponding symplectic form on $\bigwedge^3 V$ i.e.

$$(\alpha, \beta)_V := \mathrm{vol}(\alpha \wedge \beta).$$

Let W be a finite-dimensional complex vector-space. The span of a subset $S \subset W$ is denoted by $\langle S \rangle$. Let $S \subset \bigwedge^q W$: the smallest subspace $U \subset W$ such that $S \subset \mathrm{im}(\bigwedge^q U \longrightarrow \bigwedge^q W)$ is the *support of S*, we denote it by $\mathrm{supp}(S)$. If $S = \{\alpha\}$ is a singleton we let $\mathrm{supp}(\alpha) = \mathrm{supp}(\{\alpha\})$ (thus if $q = 1$ we have $\mathrm{supp}(\alpha) = \langle \alpha \rangle$).

Let U be a complex vector-space. Let $U_1, \ldots, U_\ell \subset U$ be a collection of subspaces and $i_1 + \cdots + i_\ell = d$ a partition of d; the associated *wedge subspace* of $\bigwedge^d U$ is defined to be

$$\bigwedge^{i_1} U_1 \wedge \cdots \wedge \bigwedge^{i_\ell} U_\ell := \langle \alpha_1 \wedge \cdots \wedge \alpha_\ell \mid \alpha_s \in \bigwedge^{i_s} U_s \rangle \tag{0.0.4}$$

Let W be a finite-dimensional complex vector-space. We will adhere to pre-Grothendieck conventions: $\mathbb{P}(W)$ is the set of 1-dimensional vector subspaces of W. Given a non-zero $w \in W$ we will denote the span of w by $[w]$ rather than $\langle w \rangle$; this agrees with standard notation. Given a non-empty subset $Z \subset \mathbb{P}(W)$ we let $\langle Z \rangle \subset \mathbb{P}(W)$ be the linear span of Z and $\langle\langle Z \rangle\rangle \subset W$ be the cone over $\langle Z \rangle$ i.e. the span of the set of $w \in W$ such that $[w] \in Z$.

Schemes are defined over $\mathbb{C}$, the topology is the Zariski topology unless we state the contrary, points are closed points. As customary we identify locally-free sheaves with vector-bundles.

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1 Preliminaries

1.1 EPW-sextics and their double covers

Let V be a complex vector-space of dimension 6. Choose a volume-form vol on V. Wedge-product followed by vol defines a symplectic form on $\bigwedge^3 V$: let $\mathbb{LG}(\bigwedge^3 V)$ be the symplectic grassmannian parametrizing lagrangian subspaces of $\bigwedge^3 V$ (of course $\mathbb{LG}(\bigwedge^3 V)$ is independent of the choice of vol). Let

$$F \subset \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.1.1}$$

be the locally-free subsheaf whose fiber at $[v] \in \mathbb{P}(V)$ is equal to

$$F_v := \{\alpha \in \bigwedge^3 V \mid v \wedge \alpha = 0\}. \tag{1.1.2}$$

Choose $A \in \mathbb{LG}(\bigwedge^3 V)$: we let

$$F \xrightarrow{\lambda_A} (\bigwedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.1.3}$$

be Inclusion (1.1.1) followed by the obvious quotient map, and Y_A be the degeneracy locus of λ_A. Thus $Y_A = V(\det \lambda_A)$ and since $\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$ it follows that Y_A is either a sextic hypersurface or $\mathbb{P}(V)$. As is easily checked Y_A is a sextic for A generic, on the other hand there do exist A such that $Y_A = \mathbb{P}(V)$, e.g. $A = F_w$. A sextic hypersurface in a 5-dimensional projective space is an *EPW-sextic* if it is projectively equivalent to Y_A for a certain $A \in \mathbb{LG}(\bigwedge^3 V)$. We let

$$\mathbb{N}(V) := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid Y_A = \mathbb{P}(V)\}. \tag{1.1.4}$$

It follows from the definitions that $\mathbb{N}(V)$ is a proper closed $\mathrm{PGL}(V)$-invariant subset of $\mathbb{LG}(\bigwedge^3 V)$. If $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ there is a double cover $f_A \colon X_A \to Y_A$. We will recall the definition of X_A, full details are in [29]. Since A is Lagrangian the symplectic form defines a canonical isomorphism $\bigwedge^3 V/A \cong A^\vee$; thus (1.1.3) defines a map of vector-bundles $\lambda_A \colon F \to A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}$. Let $i \colon Y_A \hookrightarrow \mathbb{P}(V)$ be the inclusion map: since a local generator of $\det \lambda_A$ annihilates $\mathrm{coker}(\lambda_A)$ there is a unique sheaf ζ_A on Y_A such that we have an exact sequence

$$0 \longrightarrow F \xrightarrow{\lambda_A} A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow i_*\zeta_A \longrightarrow 0. \tag{1.1.5}$$

Let $\xi_A := \zeta_A(-3)$. We will define a map $\xi_A \otimes \xi_A \to \mathcal{O}_{Y_A}$ that will equip $\mathcal{O}_{Y_A} \oplus \xi_A$ with the structure of a (commutative) $\mathcal{O}_{Y_A}$-algebra. Given this one sets

$$X_A := \mathrm{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A) \tag{1.1.6}$$

and we will let $f_A \colon X_A \to Y_A$ be the structure map. Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to A; the associated projection $\bigwedge^3 V \to A$ defines a map $\mu_{A,B} \colon F \to A \otimes \mathcal{O}_{\mathbb{P}(V)}$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \to & 0 \\ & & \downarrow{\scriptstyle \mu_{A,B}} & & \downarrow{\scriptstyle \mu^t_{A,B}} & & \downarrow{\scriptstyle \beta_A} & & \\ 0 & \to & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_A} & F^\vee & \longrightarrow & \mathcal{E}xt^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \to & 0\,. \end{array} \tag{1.1.7}$$

As is suggested by our notation the map β_A is independent of the choice of B. Composing the canonical isomorphism

$$\mathcal{E}xt^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_*\mathcal{H}om(\zeta_A, \mathcal{O}_{Y_A}(6)) \tag{1.1.8}$$

with β_A we get a homomorphism $\zeta_A \to \mathcal{H}om(\zeta_A, \mathcal{O}_{Y_A}(6))$. Equivalently we have defined a homomorphism $\zeta_A \otimes \zeta_A \to \mathcal{O}_{Y_A}(6)$; tensoring with $\mathcal{O}_{Y_A}(-3) \otimes \mathcal{O}_{Y_A}(-3)$ we get a homomorphism $\xi_A \otimes \xi_A \to \mathcal{O}_{Y_A}$. This homomorphism provides $\mathcal{O}_{Y_A} \oplus \xi_A$ with the structure of a (commutative)

$\mathscr{O}_{Y_A}$-algebra. A *double EPW-sextic* is given by the double cover $f_A\colon X_A \to Y_A$ for a certain $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$.

Below we list notation that will be used throughout this work. Given an isotropic subspace $A \subset \bigwedge^3 V$ (e.g. a lagrangian) we let

$$\Theta_A := \{W \in \mathrm{Gr}(3, V) \mid \bigwedge^3 W \subset A\}. \tag{1.1.9}$$

Let $\Sigma \subset \mathbb{LG}(\bigwedge^3 V)$ and $\widetilde{\Sigma} \subset \mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V)$ be defined by

$$\Sigma \ := \ \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \Theta_A \neq \emptyset\}, \tag{1.1.10}$$

$$\widetilde{\Sigma} \ := \ \{(W, A) \in \mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V) \mid W \in \Theta_A\}. \tag{1.1.11}$$

1.2 Double EPW-sextics modulo isomorphisms

Let $A_1, A_2 \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$. The double covers f_{A_1}, f_{A_2} are *isomorphic* if there exists a commutative diagram

$$\begin{array}{ccc} X_{A_1} & \xrightarrow{\sim} & X_{A_2} \\ {\scriptstyle f_{A_1}}\downarrow & & \downarrow{\scriptstyle f_{A_2}} \\ Y_{A_1} & \xrightarrow{\sim} & Y_{A_2} \end{array} \tag{1.2.1}$$

with horizontal isomorphisms. We will prove the following result.

Proposition 1.2.1. *Let $A_1, A_2 \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$. The double covers f_{A_1}, f_{A_2} are isomorphic if and only if A_1, A_2 are* $\mathrm{PGL}(V)$*-equivalent.*

Before proving the above proposition we go through a few preliminaries. Let F be the vector-bundle on $\mathbb{P}(V)$ given by (1.1.1): a straightforward computation involving the Euler sequence (see Proposition 5.11 of [26]) gives an isomorphism

$$F \cong \Omega^3_{\mathbb{P}(V)}(3). \tag{1.2.2}$$

Moreover (op. cit.) the transpose of Inclusion (1.1.1) induces an isomorphism

$$\bigwedge^3 V^\vee \cong H^0(F^\vee). \tag{1.2.3}$$

Claim 1.2.2. *The vector-bundle F is slope-stable.*

Proof. Since the (co)tangent bundle of a projective space is slope-stable [10] the vector-bundle $\Omega^3_{\mathbb{P}(V)}$ is poly-stable i.e. a direct sum of stable bundles of equal slope (op. cit.); by (1.2.2) it follows that F is poly-stable. The slope of F is $\mu(F) = -3/5$ and the rank is $r(F) = 10$; it follows that if F is not slope-stable then

$$F = \mathcal{A} \oplus \mathcal{B}, \qquad \mu(\mathcal{A}) = \mu(\mathcal{B}) = -3/5, \quad r(\mathcal{A}) = r(\mathcal{B}) = 5. \tag{1.2.4}$$

By (1.2.2) we have $\chi(F(-3)) = -1$; since it is odd we get that for any $g \in \mathrm{PGL}(V)$ we have $g^*\mathcal{A} \ncong \mathcal{B}$. The action of $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ lifts to an action on F and hence on $F^\vee$; this action is induced by $\mathrm{SL}(V)$-actions on $\mathcal{A}^\vee$ and $\mathcal{B}^\vee$ because $\mathcal{A}, \mathcal{B}$ are slope-stable and $g^*\mathcal{A} \ncong \mathcal{B}$ for any $g \in \mathrm{PGL}(V)$. Hence the induced $\mathrm{SL}(V)$-action on $H^0(F^\vee)$ is the direct-sum of representations $H^0(\mathcal{A}^\vee)$ and $H^0(\mathcal{B}^\vee)$. Since $F^\vee$ is globally generated each of $H^0(\mathcal{A}^\vee)$, $H^0(\mathcal{B}^\vee)$ is non-zero; that is a contradiction because by (1.2.3) the $\mathrm{SL}(V)$-representation $H^0(F^\vee)$ is the standard representation $\bigwedge^3 V^\vee$ and hence is irreducible. □

Proof of **Proposition 1.2.1**. It follows from the definition of double EPW-sextic that if A_1 and A_2 are $\mathrm{PGL}(V)$-equivalent then f_{A_1} and f_{A_2} are isomorphic. Let's prove the converse. Since Y_{A_k} is a hypersurface in $\mathbb{P}(V) \cong \mathbb{P}^5$ its Picard group is generated by the hyperplane class and moreover Y_{A_k} is linearly normal. It follows that Y_{A_1} is projectively equivalent to Y_{A_2} and hence by acting with a suitable element of $\mathrm{PGL}(V)$ we may assume that $Y_{A_1} = Y_{A_2} = Y$. We will prove that with this hypothesis $A_1 = A_2$. First notice that if $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ then ξ_A is the (-1)-eigensheaf of $f_A\colon X_A \to Y_A$. By (1.2.1) we get that there exists an isomorphism $\xi_{A_1} \xrightarrow{\sim} \xi_{A_2}$ and hence also an isomorphism $\phi\colon \zeta_{A_1} \xrightarrow{\sim} \zeta_{A_2}$. Isomorphism (1.2.2) and Bott vanishing give that $h^i(F) = 0$ for all i; by (1.1.5) we get an isomorphism $A_k^\vee \xrightarrow{\sim} H^0(\zeta_{A_k})$. Thus we have a commutative diagram with exact rows and vertical isomorphisms

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_{A_1}} & A_1^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_{A_1} & \to & 0 \\ & & \downarrow \psi & & \downarrow H^0(\phi)\otimes \mathrm{Id}_{\mathcal{O}} & & \downarrow \phi & & \\ 0 & \to & F & \xrightarrow{\lambda_{A_2}} & A_2^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_{A_2} & \to & 0 \end{array} \tag{1.2.5}$$

By (1.2.3) the transpose $\psi^t\colon F^\vee \to F^\vee$ induces an automorphism

$$H^0(\psi^t)\colon \bigwedge^3 V^\vee \xrightarrow{\sim} \bigwedge^3 V^\vee.$$

By (1.2.5) we have

$$H^0(\psi^t) \circ H^0(\lambda_{A_2}^t) = H^0(\lambda_{A_1}^t) \circ H^0(\phi)^t. \tag{1.2.6}$$

Let $s\colon \bigwedge^3 V \xrightarrow{\sim} \bigwedge^3 V^\vee$ be the isomorphism defined by the symplectic form $(,)_V$ i.e. $s(v)(w) := (v, w)_V$. Letting $j_k\colon A_k \hookrightarrow \bigwedge^3 V$ be inclusion we have

$$s \circ j_k = H^0(\lambda_{A_k}^t). \tag{1.2.7}$$

Let $\epsilon := s^{-1} \circ H^0(\psi^t) \circ s$; we claim that

$$\epsilon(A_2) = A_1. \tag{1.2.8}$$

In fact by (1.2.6) and (1.2.7) we have

$$\begin{aligned} \epsilon \circ j_2 = s^{-1} \circ H^0(\psi^t) \circ s \circ j_2 = s^{-1} \circ H^0(\psi^t) \circ H^0(\lambda_{A_2}^t) = \\ = s^{-1} \circ H^0(\lambda_{A_1}^t) \circ H^0(\phi)^t = j_1 \circ H^0(\phi)^t \end{aligned} \tag{1.2.9}$$

and this proves (1.2.8). By **Claim 1.2.2** the vector-bundle F is slope-stable and hence $\psi = c\,\mathrm{Id}_F$ for some $c \in \mathbb{C}^\times$. It follows that $H^0(\psi^t) = c\,\mathrm{Id}_{H^0(F^\vee)}$ and hence $\epsilon = c\,\mathrm{Id}_{\bigwedge^3 V}$ by (1.2.3). Thus $\epsilon(A_2) = A_2$ and therefore $A_2 = A_1$ by (1.2.8). □

1.3 The GIT quotient

Let $\mathrm{Pic}^{\mathrm{PGL}}(\mathbb{LG}(\bigwedge^3 V))$ be the group of $\mathrm{PGL}_6(\mathbb{C})$-linearized line-bundles on $\mathbb{LG}(\bigwedge^3 V)$. We have a homomorphism $\mathrm{Pic}^{\mathrm{PGL}}(\mathbb{LG}(\bigwedge^3 V)) \to \mathrm{Pic}(\mathbb{LG}(\bigwedge^3 V))$, which is injective because there are non non-trivial regular maps $\mathrm{PGL}_6(\mathbb{C}) \to \mathbb{C}^*$, see Prop. 1.4 of [23]. Now $\mathrm{Pic}(\mathbb{LG}(\bigwedge^3 V)) \cong \mathbb{Z}$, and the (non-trivial) canonical line-bundle of $\mathbb{LG}(\bigwedge^3 V)$ is $\mathrm{PGL}_6(\mathbb{C})$-linearized. It follows that up to multiples there is a unique $\mathrm{PGL}_6(\mathbb{C})$-linearized ample line-bundle on $\mathbb{LG}(\bigwedge^3 V)$ and hence (0.0.1) defines $\mathfrak{M}$ unambiguously. The unique linearized ample line-bundle defines the open subsets of $\mathbb{LG}(\bigwedge^3 V)$ of stable and semistable points, we will denote them by $\mathbb{LG}(\bigwedge^3 V)^{st}$ and $\mathbb{LG}(\bigwedge^3 V)^{ss}$ respectively. **Corollary 2.5.1** gives that if $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ then A is stable. The open dense subset $\mathbb{LG}(\bigwedge^3 V)^0 \subset \mathbb{LG}(\bigwedge^3 V)$ parametrizing EPW-sextics whose natural double cover is smooth is contained in $(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$, see Prop. 3.4 of [29]. Thus we get that the open dense $(\mathbb{LG}(\bigwedge^3 V)^0 // \mathrm{PGL}_6(\mathbb{C})) \subset \mathfrak{M}$ is the moduli space of smooth double EPW-sextics, and hence $\mathfrak{M}$ is a compactification of the moduli space of smooth double EPW-sextics.

We showed in [26] that there is a non-trivial involution $\delta\colon \mathfrak{M} \to \mathfrak{M}$; we will recall the definition. Let

$$\begin{array}{ccc} \bigwedge^3 V & \overset{\delta_V}{\overset{\sim}{\longrightarrow}} & \bigwedge^3 V^\vee \\ \alpha & \mapsto & \beta \mapsto \mathrm{vol}(\alpha \wedge \beta) \end{array} \tag{1.3.1}$$

be the isomorphism defined by $(,)_V$. We notice that δ_V sends isotropic subspaces of $\bigwedge^3 V$ to isotropic subspaces of $\bigwedge^3 V^\vee$; in particular it induces an isomorphism $\mathbb{LG}(\bigwedge^3 V) \overset{\sim}{\longrightarrow} \mathbb{LG}(\bigwedge^3 V^\vee)$. We notice the following: given $E \in \mathrm{Gr}(5, V)$

$$E \in Y_{\delta_V(A)} \text{ if and only if } (\bigwedge^3 E) \cap A \neq \{0\}. \tag{1.3.2}$$

Let $A \in \mathbb{LG}(\bigwedge^3 V)$ be generic: then $Y_{\delta_V(A)}$ is the classical dual $Y_A^\vee$ of Y_A, see [26]. The map δ_V induces a regular involution

$$\begin{array}{ccc} \mathfrak{M} & \overset{\delta}{\longrightarrow} & \mathfrak{M} \\ [A] & \mapsto & [\delta_V(A)] \end{array} \tag{1.3.3}$$

We showed in [26] that a generic EPW-sextic is not self-dual and hence δ is not the identity.

1.4 Moduli of plane sextics

The GIT quotient of the space of plane sextic curves (analyzed by Shah [32]) may be considered as a compactification of the moduli space of polarized $K3$ surfaces (S, H) where $\deg(H \cdot H) = 2$ and $|H|$ is free of base curves. In fact the double cover $f\colon S \to \mathbb{P}^2$ branched over a smooth sextic is such a $K3$ (with H any element of $|f^*\mathcal{O}_{\mathbb{P}^2}(1)|$), and conversely if $|H|$ is free of base curves then $|H|$ is free and the map $S \to |H|^\vee \cong \mathbb{P}^2$ is of degree 2 branched over smooth sextic. The period map

$$|\mathcal{O}_{\mathbb{P}^2}(6)| \dashrightarrow \mathbb{D}^{BB}_{K3,2} \tag{1.4.1}$$

associates to a smooth sextic the period point of the associated double cover. The GIT quotient $|\mathcal{O}_{\mathbb{P}^2}(6)|//\,\mathrm{PGL}_3(\mathbb{C})$ and the period map (1.4.1) should be viewed as simpler models of our moduli space $\mathfrak{M}$ and the period map $\mathcal{P}\colon \mathbb{LG}(\bigwedge^3 V) \dashrightarrow \mathbb{D}^{BB}$. In addition one may determine whether the period map $\mathcal{P}$ is regular at a semistable lagrangian with minimal orbit by examining plane sextic curves associated to the lagrangian, and hence it is useful to translate (semi)stability of lagrangians into geometric conditions on the associated plane sextics. Here we will recall Shah's results and terminology for semistable plane sextics with closed orbit. First let us recall that a curve C has a *simple singularity* at $p \in C$ if the following hold:

(i) p is a planar singularity i.e. $\dim \Theta_p C \leq 2$.

(ii) C is reduced in a neighborhood of p.

(iii) $\mathrm{mult}_p(C) \leq 3$ and if equality holds the blow-up of C at p does not have a point of multiplicity 3 lying over p.

Remark 1.4.1. Let $C \subset \mathbb{P}^2$ be a curve. Then C has simple singularities if and only if the double cover $S \to \mathbb{P}^2$ branched over C is a normal surface with DuVal singularities; in particular if C is a sextic then the minimal desingularization of S is a $K3$ surface with A-D-E curves lying over the singularities of S.

Theorem 1.4.2 (Shah [32]). *Let $C \subset \mathbb{P}^2$ be a sextic curve. Then C is* $\mathrm{PGL}(3)$*-semistable with minimal orbit (i.e. orbit closed in $|\mathcal{O}_{\mathbb{P}^2}(6)|^{ss}$) if and only if it belongs to one of the following classes:*

I. *C has simple singularities.*

II. *In suitable coordinates*

(1) $C = V((X_0X_2 + a_1X_1^2)(X_0X_2 + a_2X_1^2)(X_0X_2 + a_3X_1^2))$ *where a_1, a_2, a_3 are distinct.*

(2) $C = V(X_0^2 F(X_1, X_2))$ *where* F *has has no multiple factors.*

(3) $C = V((X_0X_2 + X_1^2)^2 F(X_0, X_1, X_2))$ *and* $V(X_0X_2 + X_1^2)$, $V(F)$ *intersect transversely.*

(4) $C = V(F(X_0, X_1, X_2)^2)$ *where* $V(F(X_0, X_1, X_2)$ *is a smooth cubic curve.*

III. *In suitable coordinates*

(1) $C = V((X_0X_2 + X_1^2)^2 (X_0X_2 + aX_1^2))$ *where* $a \neq 1$.

(2) $C = V(X_0^2 X_1^2 X_2^2)$.

IV. $C = 3D$ *where* D *is a smooth conic.*

Remark 1.4.3. The following will be useful in detecting sextic curves of Type II-1, II-2, III-1, III-2 or IV. Let $P \in \mathbb{C}[X_0, X_1, X_2]_6$. Suppose that $G < \mathrm{SL}_3(\mathbb{C})$ and $gP = P$ for all $g \in G$.

(1) Assume that (in the standard basis) $G = \{\mathrm{diag}(t^{-2}, t, t) \mid t \in \mathbb{C}^\times\}$. Then $P = X_0^2 F(X_1, X_2)$.

(2) Assume that (in the standard basis) $G = \{\mathrm{diag}(t, 1, t^{-1}) \mid t \in \mathbb{C}^\times\}$. Then

$$P = (b_1 X_0 X_2 + a_1 X_1^2)(b_2 X_0 X_2 + a_2 X_1^2)(b_3 X_0 X_2 + a_3 X_1^2). \tag{1.4.2}$$

(3) Assume that G is the maximal torus diagonal in the standard basis. Then $P = cX_0^2 X_1^2 X_2^2$.

Remark 1.4.4. The period map (1.4.1) is regular at C if and only if C is semistable and the unique semistable sextic with closed orbit PGL(3)-equivalent to C is not of Type IV. Equivalently: C is in the indeterminacy of (1.4.1) if and only if

(1) there exists $p \in C$ such that C has consecutive triple points[2] at p and moreover letting $\widetilde{C}$ be the strict transform of C in the blow-up of $\mathbb{P}^2$ at p, the tangent cone to $\widetilde{C}$ at its unique singular point lying over p is a triple line, or

(2) there exists $p \in C$ such that $\mathrm{mult}_p C \geq 4$ and if equality holds the tangent cone to C at p equals $3l_1 + l_2$ (l_1, l_2 are lines through p).

Remark 1.4.5. Let $C \in |\mathcal{O}_{\mathbb{P}^2}(6)|$ be semistable and assume that the period map (1.4.1) is regular at C. Let C_0 be the unique semistable sextic with closed orbit PGL(3)-equivalent to C; then C_0 is of Type I, II or III by **Remark 1.4.4**. The type of C_0 is related to the image of C under the period map. First recall that the boundary $\partial\mathbb{D}_{K3,2}^{BB} := (\mathbb{D}_{K3,2}^{BB} \setminus \mathbb{D}_{K3,2})$ is the union of boundary components of Type II, each of which is the quotient of the upper-half plane by an arithmetic group, and boundary components of Type III, i.e. points. There are four Type II boundary components and one Type III boundary component and the following hold (see [32] and Rmk. 5.6 of [7]):

(1) If C_0 is of Type I the period map takes it to a point of $\mathbb{D}_{K3,2}$.

(2) If C_0 is of Type II the period map takes it to a point of a Type II boundary component of $\mathbb{D}_{K3,2}^{BB}$, determined by the numbering (1),...,(4) in **Theorem 1.4.2**.

(3) If C_0 is of Type III the period map takes it to the unique Type III boundary component of $\mathbb{D}_{K3,2}^{BB}$.

[2] A plane singularity (C, p) has *consecutive triple points* if $\mathrm{mult}_p C = 3$ and the blow-up of C at p has one triple point lying over p.

2 One-parameter subgroups and stability

2.1 Outline of the section

Below is the main result of the present section.

Theorem 2.1.1. *The non-stable locus $\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{LG}(\bigwedge^3 V)^{st}$ is the union of the standard non-stable strata, which are defined in* **Subsection 2.4** *and are listed in Table* (1)*:*

$$\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{LG}(\bigwedge^3 V)^{st} = \mathbb{B}^*_{\mathcal{A}} \cup \mathbb{B}^*_{\mathcal{A}^\vee} \cup \mathbb{B}^*_{\mathcal{C}_1} \cup \mathbb{B}^*_{\mathcal{C}_2} \cup \mathbb{B}^*_{\mathcal{D}} \cup \mathbb{B}^*_{\mathcal{E}_1} \cup \mathbb{B}^*_{\mathcal{E}_2} \cup \mathbb{B}^*_{\mathcal{E}_1^\vee} \cup \mathbb{B}^*_{\mathcal{E}_2^\vee} \cup \mathbb{B}^*_{\mathcal{F}_1} \cup \mathbb{B}^*_{\mathcal{F}_2} \cup \mathbb{X}^*_{\mathcal{N}_3}.$$

Theorem 2.1.1 will be proved in **Subsection 2.5**. The standard non-stable strata are irreducible locally-closed subsets of $\mathbb{LG}(\bigwedge^3 V)$, and each is defined by imposing a certain "flag condition" on $A \in \mathbb{LG}(\bigwedge^3 V)$: by way of example $\mathbb{B}^*_{\mathcal{A}}$ is the set of A such that there exists $[v_0] \in \mathbb{P}(V)$ for which $\dim(A \cap F_{v_0}) \geq 5$. The section is organized as follows. In **Subsection 2.2** we will consider a linearly reductive group acting on a product of grassmannians and we will write out explicitely Mumford's numerical function associated to a 1-PS, then we will examine the numerical function when the group acts on the lagrangian subspaces of a symplectic vector space. In **Subsection 2.3** we will introduce the Cone Decomposition Algorithm: it applies to a linearly reductive group acting on a product of Grassmannians. The output of the algorithm is an explicit description of the non-stable locus as a finite union of translates of Schubert cells; it will be the key ingredient in the proof of **Theorem 2.1.1**, and also in the proof of many results of **Section 6** and **Section 7**. In **Subsection 2.4** we will define the standard non-stable strata that enter into the statement of **Theorem 2.1.1**, and also corresponding standard unstable strata; we will show that a lagrangian belonging to a standard non-stable (unstable) stratum is non-stable (unstable). **Subsection 2.5** is devoted to the proof of **Theorem 2.1.1**: by applying the Cone Decomposition Algorithm we will show that if A is non-stable then it belongs to one of the standard non-stable strata.

Let us fix our choice of sign for the numerical function associated to a 1-PS. Let W be a (finite-dimensional) complex vector space and $\lambda\colon \mathbb{C}^\times \to GL(W)$ a homomorphism. Let

$$W = \oplus_{a \in \mathbb{Z}} W_a, \qquad \lambda(t)|_{W_a} = t^a \operatorname{Id}_{W_a}, \tag{2.1.1}$$

be the decomposition into isotypical addends. Given $[w] \in \mathbb{P}(W)$ let $w = \sum_{a \in \mathbb{Z}} w_a$ be the decomposition according to (2.1.1); we set

$$\mu([w], \lambda) := \min\{a \mid w_a \neq 0\}. \tag{2.1.2}$$

(Warning: our μ is the opposite of Mumford's μ, see [23].)

Remark 2.1.2. Keep notation as above. Then $\mu([w], \lambda) \geq 0$ if and only if $\lim_{t \to 0} \lambda(t) w$ exists. Suppose that $\mu([w], \lambda) \geq 0$ and let $\overline{w} := \lim_{t \to 0} \lambda(t) w$. Then $\overline{w} = 0$ if and only if $\mu([w], \lambda) > 0$.

Below is the formulation of the Hilbert-Mumford Criterion that goes with our choice of μ.

Theorem 2.1.3 (Hilbert-Mumford's Criterion [23])**.** *Let G be a linearly reductive group acting on a projective variety $Z \subset \mathbb{P}(W)$ via a homomorphism $\rho\colon G \to \mathrm{SL}(W)$. Then*

(1) $[w]$ is stable if and only if $\mu([w], \rho \circ \lambda) < 0$ for all 1-PS's $\lambda\colon \mathbb{C}^\times \to G$.

(2) $[w]$ is semistable if and only if $\mu([w], \rho \circ \lambda) \leq 0$ for all 1-PS's $\lambda\colon \mathbb{C}^\times \to G$.

(3) $[w]$ is unstable if and only if there exists a 1-PS $\lambda\colon \mathbb{C}^\times \to G$ for which $\mu([w], \rho \circ \lambda) > 0$.

2.2 (Semi)stability and flags

Let $U^0, \ldots, U^r$ be finite-dimensional complex vector spaces. Let G be a linearly reductive group and

$$G \to \mathrm{GL}(U^0) \times \ldots \times \mathrm{GL}(U^r) \tag{2.2.1}$$

be a homomorphism. Let $m_p, n_p > 0$ be integers for $0 \le p \le r$; we assume that $n_p < \dim U^p$. Homomorphism (2.2.1) gives a representation ρ of G on $\mathrm{S}^{m_0}(\bigwedge^{n_0} U^0) \otimes \ldots \otimes \mathrm{S}^{m_r}(\bigwedge^{n_r} U^r)$: we assume that

$$\rho\colon G \to \mathrm{SL}\left(\mathrm{S}^{m_0}(\bigwedge^{n_0} U^0) \otimes \ldots \otimes \mathrm{S}^{m_r}(\bigwedge^{n_r} U^r)\right). \tag{2.2.2}$$

Let $\mathcal{L}_p$ be the Plücker ample line-bundle on $\mathrm{Gr}(n_p, U^p)$. We have the embedding

$$\mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r) \hookrightarrow \mathbb{P}\left(\mathrm{S}^{m_0}(\bigwedge^{n_0} U^0) \otimes \ldots \otimes \mathrm{S}^{m_r}(\bigwedge^{n_r} U^r)\right) \tag{2.2.3}$$

associated to $\mathcal{L}_0^{m_0} \otimes \ldots \otimes \mathcal{L}_r^{m_r}$. Homomorphism (2.2.1) induces an action of G on $\mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$. The main example for us is the action of $G = \mathrm{SL}(V)$ on $\bigwedge^3 V$ and the induced action on $\mathrm{Gr}(10, \bigwedge^3 V)$: we will be interested in the closed $\mathrm{SL}(V)$-invariant subset $\mathbb{LG}(\bigwedge^3 V) \subset \mathrm{Gr}(10, \bigwedge^3 V)$. On the other hand we will examine more general homomorphisms in **Section 6** and **Section 7**. Let $\lambda\colon \mathbb{C}^\times \to G$ be a 1-PS. Let $\mu^{\mathbf{m}}(\cdot, \rho \circ \lambda)$ be the Hilbert-Mumford numerical function defined by Embedding (2.2.3) - here $\mathbf{m} = (m_0, \ldots, m_r)$ and the input is a point $(A_0, \ldots, A_r) \in \mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$. One expands $\mu^{\mathbf{m}}$ as follows. Let $\pi_p\colon G \to \mathrm{GL}(U^p)$ be projection. Then $\pi_p \circ \lambda\colon \mathbb{C}^\times \to \mathrm{GL}(U^p)$ and we have the numerical function $\mu(A_p, \pi_p \circ \lambda)$ (relative to $\mathcal{L}_p$): abusing notation we will denote it by $\mu(A_p, \lambda)$. We have

$$\mu^{\mathbf{m}}((A_0, \ldots, A_r), \rho \circ \lambda) = \sum_{p=0}^{r} m_p \mu(A_p, \lambda). \tag{2.2.4}$$

Next we will write out explicitly $\mu(A_p, \lambda)$. First we must introduce the λ-type of A_p. To simplify notation we set $U = U^p$. Thus we suppose that $\lambda\colon \mathbb{C}^\times \to GL(U)$ is a homomorphism ($\pi_p \circ \rho \circ \lambda$ in the notation used above). Let

$$U = U_{e_0} \oplus \ldots \oplus U_{e_s} \tag{2.2.5}$$

be the decomposition into isotypical summands for the action of λ. We assume throughout that the weights are numbered in decreasing order:

$$e_0 > e_1 > \ldots > e_s. \tag{2.2.6}$$

For $0 \le i \le s$ we let

$$L_i := U_{e_0} \oplus \ldots \oplus U_{e_i}. \tag{2.2.7}$$

Definition 2.2.1. Let $\lambda\colon \mathbb{C}^\times \to \mathrm{GL}(U)$ be a homomorphism. Keep notation as above, in particular (2.2.5) and (2.2.6). Let $0 < n < \dim U$ and $A \in \mathrm{Gr}(n, U)$. We let

$$d_i^\lambda(A) := \dim(A \cap L_i / A \cap L_{i-1}) \qquad 0 \le i \le s. \tag{2.2.8}$$

The vector $d^\lambda(A) := (d_0^\lambda(A), \ldots, d_s^\lambda(A))$ is the λ-type of A. More generally let $\lambda\colon \mathbb{C}^\times \to \mathrm{GL}(U^0) \times \ldots \times \mathrm{GL}(U^r)$ be a homomorphism and $(A_0, \ldots, A_r) \in \mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$: the collection of vectors

$$(d^{\pi_0 \circ \lambda}(A_0), \ldots, d^{\pi_r \circ \lambda}(A_r))$$

is the λ-type of $(A_0, \ldots, A_r)$. Whenever possible we omit reference to λ i.e. we denote the λ-type of $(A^0, \ldots, A^r)$ by $(d(A^0), \ldots, d(A^r))$.

Let $\lambda\colon \mathbb{C}^\times \to GL(U)$ be a homomorphism - we assume that (2.2.5) and (2.2.6) hold. Let $A \in \mathrm{Gr}(n, U)$. Then $\mu(A, \lambda)$ is determined by the λ-type of A:

$$\mu(A, \lambda) = \sum_{i=0}^{s} e_i d_i^\lambda(A). \tag{2.2.9}$$

In order to examine $\lim_{t \to 0} \lambda(t)(A_0, \ldots, A_p)$ we introduce a definition.

Definition 2.2.2. Keep notation as in **Definition 2.2.1**. Let $0 < n < \dim U$ and $A \in \mathrm{Gr}(n, U)$. Then A is *λ-split* if $A = (A \cap U_{e_0}) \oplus (A \cap U_{e_1}) \oplus \ldots \oplus (A \cap U_{e_s})$.

Remark 2.2.3. Keep notation as above. Then $A \in \mathrm{Gr}(n, U)$ is λ-split if and only if $\lambda(t)A = A$ for all $t \in \mathbb{C}^\times$.

Next assume that λ is a 1-PS of G. Let $(A_0, \ldots, A_r) \in \mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$ and suppose that $\mu^{\mathbf{m}}((A_0, \ldots, A_r), \rho \circ \lambda) = 0$. Let ω be a generator of $(\bigwedge^{\max} A_0)^{m_0} \otimes \ldots \otimes (\bigwedge^{\max} A_r)^{m_r}$. Then $\lim_{t \to 0} \rho \circ \lambda(t)\omega$ exists and is non-zero by **Claim 2.2.7**: call it $\overline{\omega}$. Of course there exists a unique $(\overline{A}_0, \ldots, \overline{A}_r) \in \mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$ such that $(\bigwedge^{\max} \overline{A}_0)^{m_0} \otimes \ldots \otimes (\bigwedge^{\max} \overline{A}_r)^{m_r} = \mathbb{C}\overline{\omega}$. The result below follows directly from the definitions.

Claim 2.2.4. *For $0 \le p \le r$ the subspace $\overline{A}_p$ is λ-split of type equal to $d^\lambda(A_p)$.*

Next we consider the case in which we are given a symplectic form $\sigma \in \bigwedge^2 U^\vee$ and G acts via a homomorphism

$$G \longrightarrow \mathrm{Sp}(U, \sigma) := \{g \in \mathrm{GL}(U) \mid g^*\sigma = \sigma\}.$$

The main example for us is $G = \mathrm{SL}(V)$, $U = \bigwedge^3 V$ and $\sigma = (,)_V$. Let's go through some elementary facts regarding Decomposition (2.2.5). If a weight e occurs then so does $-e$: by (2.2.6) we get that

$$e_i + e_{s-i} = 0, \qquad 0 \le i \le s. \tag{2.2.10}$$

Moreover

$$U_{e_i} \bot U_{e_k} \text{ if } i + k \ne s$$

and

$$\begin{array}{ccc} U_{e_i} \times U_{e_{s-i}} & \longrightarrow & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \sigma(\alpha, \beta) \end{array}$$

is a perfect pairing - in particular $\dim U_{e_i} = \dim U_{e_{s-i}}$ and the restriction of $(,)_V$ to U_0 is a symplectic form. Now assume that $A \in \mathbb{LG}(U)$ where "lagrangian" refers to the symplectic form σ. Then the first half of the $d_i(A)$'s determine the remaining ones - this is a well-known fact, we recall the proof for the reader's convenience.

Claim 2.2.5. *Let U be a finite-dimensional complex vector-space and $\sigma \in \bigwedge^2 U^\vee$ a symplectic form. Let $\lambda\colon \mathbb{C}^\times \to \mathrm{Sp}(U, \sigma)$ be a homomorphism. Let* (2.2.5) *be the isotypical decomposition of λ and suppose that* (2.2.6) *holds. For $A \in \mathbb{LG}(U)$ we have that*

$$d_i^\lambda(A) + d_{s-i}^\lambda(A) = \dim U_{e_i}, \qquad 0 \le i \le s. \tag{2.2.11}$$

Proof. We have $L_i^\perp = L_{s-i-1}$ where orthogonality is with respect to the symplectic form σ. Thus σ induces a perfect pairing

$$(L_i/L_{i-1}) \times (L_{s-i}/L_{s-i-1}) \longrightarrow \mathbb{C}.$$

Intersecting A with L_i and with L_{s-i} we get that

$$d_i^\lambda(A) + d_{s-i}^\lambda(A) \le \dim U_{e_i} = \dim U_{e_{s-i}} \tag{2.2.12}$$

because projection defines an isomorphism $U_{e_i} \cong L_i/L_{i-1}$ and A is lagrangian. On the other hand

$$\sum_{i=0}^{s} \dim U_{e_i} = \dim U = 2\dim A = \sum_{i=0}^{s}(d_i^\lambda(A) + d_{s-i}^\lambda(A)) \le \sum_{i=0}^{s} \dim U_{e_i}.$$

It follows that (2.2.12) is an equality for $0 \le i \le s$. □

Definition 2.2.6. Keep assumptions as in **Claim 2.2.5**. The *reduced λ-type of A* is

$$d_{red}^\lambda(A) := (d_0^\lambda(A), \ldots, d_{[(s-1)/2]}^\lambda(A)).$$

(In other words we truncate the λ-type of A right before the middle.)

By **Claim 2.2.5** the reduced λ-type of A determines the λ-type of A.

Claim 2.2.7. *Let U be a finite-dimensional complex vector-space and $\sigma \in \bigwedge^2 U^\vee$ a symplectic form. Let $\lambda\colon \mathbb{C}^\times \to \mathrm{Sp}(U,\sigma)$ be a homomorphism. Let $A \in \mathbb{LG}(U)$. Then*

$$\mu(A,\lambda) = 2\left(\sum_{0\le i<s/2} e_i d_i^\lambda(A) - \sum_{i<s/2} \frac{e_i \dim U_{e_i}}{2}\right). \tag{2.2.13}$$

Proof. By (2.2.9), (2.2.10) and (2.2.11) we have

$$\mu(A,\lambda) = \sum_{i=0}^{s} e_i d_i^\lambda(A) = \sum_{0\le i<s/2} e_i d_i^\lambda(A) + \sum_{s/2<i\le s}^{s} e_i d_i^\lambda(A) =$$
$$= \sum_{0\le i<s/2} e_i d_i^\lambda(A) - \sum_{0\le i<s/2} e_i(\dim U_{e_i} - d_i^\lambda(A)).$$

The last term on the right is clearly equal to the right-hand side of (2.2.13). □

2.3 The Cone Decomposition Algorithm

We will study (semi)stability of points in $\mathrm{Gr}(n_0,U^0)\times\ldots\times\mathrm{Gr}(n_r,U^r)$ with respect to Embedding (2.2.3). Let $T<G$ be a maximal torus. Let $\check{X}(T)$ be the lattice of 1-PS of T (thus we include the trivial homomorphism) - the structure of free finitely generated group is given by pointwise multiplication in T. Let $\check{X}(T)_\mathbb{R} := \check{X}(T)\otimes_\mathbb{Z}\mathbb{R}$.

Notation 2.3.1. Let $C \subset \check{X}(T)_\mathbb{R}$ be a Weyl chamber for the action of the Weyl group $N_G(T)/T$.

Thus C is a closed convex cone in $\check{X}(T)_\mathbb{R}$. Let's be explicit in the case $G = \mathrm{SL}(V)$. Choose a basis $\mathsf{F} = \{v_0,\ldots,v_5\}$ of V. We have an associated maximal torus and corresponding $\check{X}(T)$:

$$T = \{\mathrm{diag}(t_0,\ldots,t_5) \mid t_0\cdots t_5 = 1\}, \qquad \check{X}(T) = \{\lambda(t) = \mathrm{diag}(t^{r_0},\ldots,t^{r_5}) \mid r_0+\ldots+r_5 = 0\}.$$

The choice of C corresponds to an ordering of the r_i's. Our choice will be the standard one:

$$C = \{(r_0,\ldots,r_5)\in\mathbb{R}^6 \mid r_0+\ldots+r_5 = 0, \quad r_0\ge r_1\ge\ldots\ge r_5\}. \tag{2.3.1}$$

Next let $T \to \mathrm{GL}(U^p)$ be the composition of the inclusion $T<G$, Homomorphism (2.2.1) and the projection $\mathrm{GL}(U^0)\times\ldots\times\mathrm{GL}(U^r)\to\mathrm{GL}(U^p)$. The T-module U^p decomposes as a weight spaces

$$U^p = \bigoplus_{\chi\in M^p} U_\chi^{\oplus a_\chi} \tag{2.3.2}$$

where the action on U_χ is given by χ and M^p is a (finite) set of characters of T. For $\chi_1 \ne \chi_2 \in M^p$ let

$$J_{\chi_1,\chi_2} := \{\lambda\in\check{X}(T) \mid \chi_1\circ\lambda = \chi_2\circ\lambda\}. \tag{2.3.3}$$

Then J_{χ_1,χ_2} is a subgroup of $\check{X}(T)$ and $\mathrm{rk}\, J_{\chi_1,\chi_2} = (\mathrm{rk}\,\check{X}(T)-1)$. Thus

$$H_{\chi_1,\chi_2} := J_{\chi_1,\chi_2}\otimes\mathbb{R}\subset\check{X}(T)_\mathbb{R} \tag{2.3.4}$$

is a codimension-1 vector subspace: we name it an *ordering hyperplane* for Homomorphism (2.2.1). Let $0\ne v\in\check{X}(T)_\mathbb{R}$: then

$$[v[:= \{xv \mid x\ge 0\} \tag{2.3.5}$$

is the *half-line* generated by v.

Definition 2.3.2. Let C be as in **Notation 2.3.1**. A half-line $[v[\subset C$ is an *ordering ray* for Homomorphism (2.2.1) if the subspace $\langle v\rangle$ is the intersection of a collection of ordering hyperplanes for Homomorphism (2.2.1). (We let $0\le p\le r$ be arbitrary.) A 1-PS $\lambda\colon\mathbb{C}^\times\to T$ contained in C is an ordering 1-PS for Homomorphism (2.2.1) if it generates an ordering ray.

The Cone Decomposition Algorithm states that if certain (weak) conditions hold then a point of $\mathrm{Gr}(U^0) \times \ldots \times \mathrm{Gr}(U^r)$ is non-stable (unstable) if and only if it is projectively equivalent to a point which is destabilized (desemistabilized) by an ordering 1-PS. Since the set of ordering rays is finite the algorithm allows us (in theory) to list all the non-stable (unstable) points. First we define a subdivision of C into chambers as follows. An open *ordering-chamber* is a connected component of

$$C \setminus \bigcup_{\chi_1 \neq \chi_2 \in M} H_{\chi_1,\chi_2}.$$

The closure (in C) of an open chamber is a *closed ordering-chamber*. Let $\mathbf{m} = (m_0, \ldots, m_r) \in \mathbb{N}_+^{r+1}$ correspond to a choice of very ample line-bundle on $\mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$ - see **Subsection 2.2**.

Lemma 2.3.3. *Let $(A_0, \ldots, A_r) \in \mathrm{Gr}(n_0, U^0) \times \ldots \times \mathrm{Gr}(n_r, U^r)$. Let $C_k \subset C$ be a closed ordering-chamber. There exists a* **linear** *function $\varphi_k \colon \check{X}(T)_{\mathbb{R}} \to \mathbb{R}$ such that*

$$\mu^{\mathbf{m}}((A_0, \ldots, A_r), \lambda) = \varphi_k(\lambda) \tag{2.3.6}$$

for all $\lambda \in C_k$.

Proof. Let $0 \leq p \leq r$. We may give an ordering $M^p = \{\chi_1, \ldots, \chi_u\}$ such that the following holds. For $1 \leq j \leq u$ let $\chi_j \circ \lambda(t) = t^{e_j(\lambda)}$. Then

$$\text{if } \lambda \in C_k \text{ and } i > j \text{ then } e_i(\lambda) \geq e_j(\lambda). \tag{2.3.7}$$

In fact the ordering-chambers have been defined so that (2.3.7) holds. Let $\lambda \in C_k$: then U^p is a $\mathbb{C}^\times$ module via the homomorphism $\lambda \colon \mathbb{C}^\times \to T$. We have the decomposition into sub-representations of $\mathbb{C}^\times$:

$$U^p = U_{\chi_1}^{\oplus a_{\chi_1}} \oplus \ldots \oplus U_{\chi_u}^{\oplus a_{\chi_u}}$$

where U_{χ_j} corresponds to the character $t^{e_j(\lambda)}$. For $1 \leq j \leq u$ let

$$L'_j := U_{\chi_1} \oplus \ldots \oplus U_{\chi_j}.$$

Let $d'_j := \dim(A \cap L'_j / A \cap L'_{j-1})$. We claim that

$$\mu(A_p, \lambda) = \sum_{j=1}^{u} d'_j e_j(\lambda). \tag{2.3.8}$$

In fact if λ is in the open ordering chamber whose closure is C_k then $d'_j = d^\lambda(A_p)$ and hence (2.3.8) holds by (2.2.9). One easily checks that (2.3.8) holds as well for λ in the boundary of C_k. The function from the set of 1-PS's in C_k to $\mathbb{Z}$ which assigns $e_j(\lambda)$ to λ is the restriction of a linear function on $\check{X}(T)_{\mathbb{R}}$. Thus the lemma follows from Equation (2.2.4). □

Before proving the key result we introduce some notation. Suppose first that $G = T_0 \times G_1$ where T_0 is a torus and G_1 is a semisimple group. Then $T = T_0 \times T_1$ where T_1 is a maximal torus of G_1. Thus we may define

$$P = \{H_{\chi_1,\chi_2} \mid \chi_1, \chi_2 \in \widehat{T_0}\}. \tag{2.3.9}$$

In general G is isogenous to a product of a torus T_0 and a semisimple group and the same definition makes sense.

Proposition 2.3.4. *Keep notation and assumptions as above, in particular choose a maximal torus $T < G$ and a cone C as in* **Notation 2.3.1**. *Suppose that the following hold:*

(1) Each face of C spans an ordering-hyperplane.

(2) Let P be as in (2.3.9): then the intersection $\cap_{H \in P} H$ is equal to $Z \times N(T_1)$ where $\dim Z \leq 1$.

Let $(A_0,\ldots,A_r)\in \mathrm{Gr}(n_0,U^0)\times\ldots\times\mathrm{Gr}(n_r,U^r)$. Then $(A_0,\ldots,A_r)$ is non-stable (unstable) if and only if its G-orbit contains $(A'_0,\ldots,A'_r)$ which is destabilized (desemistabilized) by an ordering 1-PS of G.

Proof. Suppose that $(A_0,\ldots,A_r)$ is non-stable (unstable): we must prove that its orbit contains an element which is destabilized (desemistabilized) by an ordering 1 PS. By the Hilbert-Mumford criterion there exists a 1-PS λ of G such that

$$\mu^{\mathbf{m}}((A_0,\ldots,A_r),\lambda_0)\ge 0 \quad (\mu^{\mathbf{m}}((A_0,\ldots,A_r),\lambda_0)>0). \tag{2.3.10}$$

Since T is a maximal torus there exists $g_1\in G$ such that $g_1\circ\lambda\circ g_1^{-1}\colon\mathbb{C}^\times\to T$. By our choice of cone C (see **Notation 2.3.1**) there exists $g_2\in G$ such that $\lambda':=g_2\circ g_1\circ\lambda\circ g_1^{-1}\circ g_2^{-1}\in C$. Let $\mathbf{a}:=g_2\circ g_1(A_0,\ldots,A_r)$: by (2.3.10) we have $\mu^{\mathbf{m}}(\mathbf{a},\lambda')\ge 0$ (respectively $\mu^{\mathbf{m}}(\mathbf{a},\lambda')>0$). Let's prove that there exists an ordering 1-PS $\overline{\lambda}$ such that $\mu^{\mathbf{m}}(\mathbf{a},\overline{\lambda})\ge 0$ (respectively $\mu^{\mathbf{m}}(\mathbf{a},\overline{\lambda})>0$). There exists a closed ordering cone C_k such that $\lambda'\in C_k$. Since C_k is a closed convex cone (with vertex 0) we may write $C_k=L\times K$ where $L\subset\check{X}(T)_{\mathbb{R}}$ is a vector subspace and K is a pointed cone with vertex 0 (i.e. it contains no lines). Thus K is the convex envelope of its extremal rays (see for example Prop. 1.35 of [3]); by Item (1) each extremal ray is spanned by an ordering 1-PS and hence K is the convex envelope of $[\lambda_1[,\ldots,[\lambda_c[$ where $\lambda_1,\ldots,\lambda_c$ are ordering 1-PS's. On the other hand all vector-subspaces of C are contained in $\mathfrak{t}_0$; thus $L\subset\mathfrak{t}_0$. It follows that $\dim L\le 1$. In fact suppose that $\dim L\ge 2$. By Item (2) there exists $f\in\check{X}(T)^\vee_{\mathbb{R}}$ such that $\ker f$ is an ordering hyperplane and f takes strictly positive and strictly negative valuse on L; that implies that C_k is not an ordering cone, contradiction. We have proved that $\dim L\le 1$. Thus $L=\{0\}$ or $L=\langle\lambda_0\rangle$ where λ_0 is an ordering 1-PS. Since $\lambda'\in C_k$ we have

$$0\ne\lambda'=x(\pm\lambda_0)+\sum_{i=1}^{c}z_i\lambda_i,\qquad x\ge 0,\ z_i\ge 0. \tag{2.3.11}$$

Now let $\varphi_k\in\check{X}(T)^\vee_{\mathbb{R}}$ be the linear function associated to $\mathbf{a}$ as in **Lemma 2.3.3**. By hypothesis $\varphi_k(\lambda')\ge 0$ (respectively $\varphi_k(\lambda')>0$) and hence (2.3.11) gives that there exists one of $\pm\lambda_0,\lambda_1,\ldots,\lambda_c$, say $\overline{\lambda}$ such that $\varphi_k(\overline{\lambda})\ge 0$ (respectively $\varphi_k(\overline{\lambda})>0$). Then $\overline{\lambda}$ is an ordering ray and $\mu^{\mathbf{m}}(\mathbf{a},\overline{\lambda})\ge 0$ (respectively $\mu^{\mathbf{m}}(\mathbf{a},\overline{\lambda})>0$) by **Lemma 2.3.3**. □

2.4 The standard non-stable strata

We will define the standard non-stable strata (and the standard unstable strata). In **Subsection 2.5** we will prove that $A\in\mathbb{LG}(\bigwedge^3 V)$ is stable if and only if it does not belong to one of the standard non-stable strata. Some of the standard non-stable and unstable strata have appeared in [28] as loci of lagrangians containing a strictly positive-dimensional set of decomposable elements - we will make the connection in **Subsubsection 2.4.2**. In **Section 3** we will give geometric meaning to all of the standard non-stable strata.

2.4.1 The definitions

Let λ be a 1-PS of $\mathrm{SL}(V)$ and

$$\mathsf{F}:=\{v_0,\ldots,v_5\} \tag{2.4.1}$$

be a basis of V which diagonalizes λ. Thus

$$\lambda(t)v_i=t^{r_i}v_i\qquad 0\le i\le 5\qquad \sum_{i=0}^{5}r_i=0. \tag{2.4.2}$$

Let

$$\bigwedge^3 V=U_{e_0}\oplus\ldots\oplus U_{e_s},\quad \bigwedge^3\lambda(t)|_{U_{e_i}}=t^{e_i}\,\mathrm{Id}_{U_{e_i}} \tag{2.4.3}$$

be the decomposition of $\bigwedge^3 \lambda$ into isotypical summands. Notation is as in (2.2.5) but notice the potential for confusion between λ and $\bigwedge^3 \lambda$. In particular the weights are in decreasing order - see (2.2.6). Let

$$\mathcal{P}_\lambda := \{(d_0, \ldots, d_{[(s-1)/2]}) \mid d_i \in \mathbb{N}, \quad d_i \leq \dim U_{e_i}\}.$$

The reduced λ-type of $A \in \mathbb{LG}(\bigwedge^3 V)$ belongs to $\mathcal{P}_\lambda$; viceversa every $[(s+1)/2]$-tuple in $\mathcal{P}_\lambda$ is the reduced λ-type of some A. Let $\mathbf{d} = (d_0, \ldots, d_{[(s-1)/2]}) \in \mathbb{P}_\lambda$; we let

$$\mu(\mathbf{d}, \lambda) := 2\left(\sum_{0 \leq i < s/2} e_i d_i - \sum_{i < s/2} \frac{e_i \dim U_{e_i}}{2}\right). \tag{2.4.4}$$

The above definition is motivated by (2.2.13).

Definition 2.4.1. Let $\succeq$ be the partial ordering on $\mathcal{P}_\lambda$ defined by $\mathbf{a} \succeq \mathbf{b}$ if

$$(a_0 + a_1 + \ldots + a_i) \geq (b_0 + b_1 + \ldots + b_i), \qquad 0 \leq i < s/2.$$

Claim 2.4.2. *Keep notation as above. Let* $\mathbf{a}, \mathbf{b} \in \mathcal{P}_\lambda$. *If* $\mathbf{a} \succeq \mathbf{b}$ *then* $\mu(\mathbf{a}, \lambda) \succeq \mu(\mathbf{b}, \lambda)$ *and equality holds if only if* $\mathbf{a} = \mathbf{b}$.

Proof. By (2.2.13) we need to show that

$$\sum_{0 \leq i < s/2} e_i(a_i - b_i) \geq 0$$

and that equality holds if and only if $\mathbf{a} = \mathbf{b}$. Let $x_i := (a_0 - b_0) + \ldots + (a_i - b_i)$. Since $\mathbf{a} \succeq \mathbf{b}$ we have $x_i \geq 0$ for $0 \leq i < s/2$, moreover $x_i = 0$ for all $0 \leq i < s/2$ if and only if $\mathbf{a} = \mathbf{b}$. A straightforward computation gives that

$$\sum_{0 \leq i < s/2} e_i(a_i - b_i) = \left(\sum_{0 \leq i \leq [(s-3)/2]} (e_i - e_{i+1}) x_i\right) + e_{[(s-1)/2]} x_{[(s-1)/2]}.$$

The claim follows because $e_0 > e_1 > \ldots > e_{[(s-1)/2]} > 0$. □

Let $\mathbf{r} = (r_0, \ldots, r_5)$ be the sequence (counted with multiplicities) of weights of λ. Given $\mathbf{d} \in \mathcal{P}_\lambda$ we let

$$\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}} := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid d^\lambda_{red}(A) \succeq \mathbf{d}\}. \tag{2.4.5}$$

Claim 2.4.3. *The Schubert variety* $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ *is closed and irreducible. If in addition* $\mu(\mathbf{d}, \lambda) \geq 0$ *(*$\mu(\mathbf{d}, \lambda) > 0$*) then* $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ *is contained in the non-stable locus (respectively the unstable locus) of* $\mathbb{LG}(\bigwedge^3 V)$.

Proof. $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ is closed by uppersemiconinuity of the dimension of the intersection of subspaces. One checks easily that the locus of $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\mathbf{d}^\lambda(A) = \mathbf{d}$ is open dense in $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ and irreducible; it follows that $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ is irreducible. The statement about non-stability (respectively instability) follows at once from **Claim 2.2.7** and **Claim 2.4.2**. □

Let

$$\mathbb{E}^*_{\mathbf{r},\mathbf{d}} := \bigcup_{\mathsf{F}} \mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}, \qquad \mathbb{E}_{\mathbf{r},\mathbf{d}} := \overline{\mathbb{E}^*_{\mathbf{r},\mathbf{d}}} \tag{2.4.6}$$

where F runs through the set of bases of V; thus $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ is locally closed and $\mathbb{E}_{\mathbf{r},\mathbf{d}}$ is (tautologically) closed. If $\mu(\mathbf{d}, \lambda) = 0$ then $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ and $\mathbb{E}_{\mathbf{r},\mathbf{d}}$ are contained in the non-stable locus by **Claim 2.4.3**. Similarly if $\mu(\mathbf{d}, \lambda) > 0$ then both $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ and $\mathbb{E}_{\mathbf{r},\mathbf{d}}$ are contained in the unstable locus of $\mathbb{LG}(\bigwedge^3 V)$. We will define non-stable (unstable) strata by choosing certain $\mathbf{r}$ and $\mathbf{d}$ such that $\mu(\mathbf{d}, \lambda) = 0$ ($\mu(\mathbf{d}, \lambda) > 0$). Table (1) defines the *standard* non-stable strata by defining the corresponding $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ where F is the basis (2.4.1). We explain the notation of that table. We let $(5, -1_5)$ stand for

$(5,-1,-1,-1,-1,-1)$ and similarly for the other rows in the first column. To a given row we associate the 1-PS λ given by (2.4.2) where $\mathbf{r}=(r_0,\ldots,r_5)$ is the entry in the first column. The second column contains $\mu(\mathbf{d},\lambda)$. The third column gives a $\mathbf{d}\in\mathcal{P}_\lambda$ such that $\mu(\mathbf{d},\lambda)=0$. The fourth column gives a flag condition on $A\in\mathbb{LG}(\bigwedge^3 V)$ which is equivalent to $A\in\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ - for $\mathbf{r}$ and $\mathbf{d}$ in the same row. In that column we adopt the notation

$$V_{ij}:=\langle v_i,v_{i+1},\ldots,v_j\rangle,\qquad 0\le i<j\le 5. \tag{2.4.7}$$

An entry in the last column is the name that we have chosen for $\mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ with $\mathbf{r}$ and $\mathbf{d}$ in the same row. We let

$$\mathbb{B}^*_{\mathcal{A}}:=\bigcup_{\mathsf{F}}\mathbb{B}^{\mathsf{F}}_{\mathcal{A}},\ \mathbb{B}_{\mathcal{A}}:=\overline{\mathbb{B}}^*_{\mathcal{A}},\ \ldots\ ,\mathbb{B}^*_{\mathcal{F}_2}:=\bigcup_{\mathsf{F}}\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2},\ \mathbb{B}_{\mathcal{F}_2}:=\overline{\mathbb{B}}^*_{\mathcal{F}_2},\quad \mathbb{X}^*_{\mathcal{N}_3}:=\bigcup_{\mathsf{F}}\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3},\ \mathbb{X}_{\mathcal{N}_3}:=\overline{\mathbb{X}}^*_{\mathcal{N}_3}. \tag{2.4.8}$$

Table (2) defines the *standard* unstable strata; notation is as in Table (1) except that we have $\mathbb{X}$'s everywhere - the rationale for the distinction between $\mathbb{B}$'s and $\mathbb{X}$'s will be explained in **Section 3**.

Remark 2.4.4. Let $\mathcal{X}\in\{\mathcal{A},\mathcal{A}^\vee,\ldots,\mathcal{F}_1\}$ be one of the indices of the standard non-stable strata with the exception of $\mathcal{N}_3$; by definition we have $\mathbb{X}_{\mathcal{X},+}\subset\mathbb{B}_{\mathcal{X}}$. Similarly $\mathbb{X}_{\mathcal{N}_3,+}\subset\mathbb{X}_{\mathcal{N}_3}$.

Duality. Given a 1-PS λ let λ^{-1} be the inverse 1-PS i.e. $\lambda^{-1}(t)=\lambda(t^{-1})$. The set of weights of $\bigwedge^3\lambda$ and of $\bigwedge^3\lambda^{-1}$ are the same and moreover $\dim U_e(\bigwedge^3\lambda)=\dim U_e(\bigwedge^3\lambda^{-1})$ for each weight e. Thus $\mathcal{P}_\lambda=\mathcal{P}_{\lambda^{-1}}$ and

$$\mu(\mathbf{d},\lambda)=\mu(\mathbf{d},\lambda^{-1}),\qquad \mathbf{d}\in\mathcal{P}_\lambda=\mathcal{P}_{\lambda^{-1}}.$$

This implies that the non-stable (or unstable) strata $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ come in couples, namely $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ and $\mathbb{E}^*_{-\mathbf{r},\mathbf{d}}$. Notice that if a non-stable (or unstable) stratum $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}$ appears in Table (1) then so does $\mathbb{E}^*_{-\mathbf{r},\mathbf{d}}$. The remarkable fact is that the mirror of a stratum may be identified with the image of the stratum when we apply the duality isomorphism $\mathbb{LG}(\bigwedge^3 V)\xrightarrow{\sim}\mathbb{LG}(\bigwedge^3 V^\vee)$ induced by (1.3.1): more precisely we have

$$\delta_V(\mathbb{E}^*_{\mathbf{r},\mathbf{d}}(V))=\mathbb{E}^*_{-\mathbf{r},\mathbf{d}}(V^\vee),$$

where $\mathbb{E}^*_{\mathbf{r},\mathbf{d}}(V)$ is the non-stable (or unstable) stratum in $\mathbb{LG}(\bigwedge^3 V)$ indicized by $\mathbf{r},\mathbf{d}$ and similarly for $\mathbb{E}^*_{-\mathbf{r},\mathbf{d}}(V^\vee)$. The above equation explains our notation for coupled non-stable (or unstable) strata in Tables (1) and (2).

2.4.2 Geometric significance of certain strata

Let

$$\Sigma_\infty:=\{A\in\mathbb{LG}(\bigwedge^3 V)\mid\dim\Theta_A>0\}. \tag{2.4.9}$$

Theorem 2.37 of [28] lists the irreducible components of Σ_∞, in particular it gives that

$$\mathbb{B}_{\mathcal{A}},\quad \mathbb{B}_{\mathcal{A}^\vee},\quad \mathbb{B}_{\mathcal{C}_2},\quad \mathbb{B}_{\mathcal{D}},\quad \mathbb{B}_{\mathcal{E}_2},\quad \mathbb{B}_{\mathcal{E}_2^\vee},\quad \mathbb{B}_{\mathcal{F}_1} \tag{2.4.10}$$

are irreducible components of Σ_∞, that they are pairwise distinct and that if A is generic in one of the above standard non-stable strata then Θ_A is an irreducible curve[3]. How do we distinguish geometrically the strata above? We consider a generic A in the stratum and we look at the curve Θ_A and the ruled 3-fold $R_{\Theta_A}\subset\mathbb{P}(V)$ swept out by $\mathbb{P}(W)$ for $W\in\Theta_A$. A few examples: if $A\in\mathbb{B}_{\mathcal{F}_1}$ then Θ_A is a line, if $A\in\mathbb{B}_{\mathcal{D}}$ then Θ_A is a conic, if $A\in\mathbb{B}_{\mathcal{E}_2}$ or $A\in\mathbb{B}_{\mathcal{E}_2^\vee}$ then Θ_A is a rational normal cubic curve, in the first case R_{Θ_A} is a cone in the second it is not, etc. - see Section 2 of [28] for a detailed discussion. In [28] we described also those A such that $\dim\Theta_A>1$; it will turn out that they are not stable, actually unstable with a few explicit exceptions - see **Lemma 6.1.8**. Below

[3] Writing $\Theta_A=\mathbb{P}(A)\cap\mathrm{Gr}(3,V)$ we may give Θ_A a structure of scheme: it is generically reduced but not reduced everywhere.

we will give a geometric consequence of the results of [28]. First we will recall the definition of a particular $\mathrm{PGL}(V)$-orbit in $\mathbb{LG}(\bigwedge^3 V)$, see Section 1.5 of [28]. We have embeddings

$$\begin{array}{ccc} \mathbb{P}(U) & \overset{i_+}{\hookrightarrow} & \mathrm{Gr}(3,\bigwedge^2 U) \\ [u] & \mapsto & \{u\wedge u' \mid u'\in U\} \end{array}, \qquad \begin{array}{ccc} \mathbb{P}(U^\vee) & \overset{i_-}{\hookrightarrow} & \mathrm{Gr}(3,\bigwedge^2 U) \\ [f] & \mapsto & \bigwedge^2(\ker f). \end{array} \tag{2.4.11}$$

The pull-back to $\mathbb{P}(U)$, $\mathbb{P}(U^\vee)$ of the Plücker line-bundle on $\mathrm{Gr}(3,\bigwedge^2 U)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(U)}(2)$, $\mathcal{O}_{\mathbb{P}(U^\vee)}(2)$ respectively and the map on global sections is surjective; it follows that each of $\mathrm{im}(i_+)$, $\mathrm{im}(i_-)$ spans a 9-dimensional subspace of $\bigwedge^3(\bigwedge^2 U)$. Now choose an isomorphism $V\cong\bigwedge^2 U$ where U is a complex vector-space of dimension 4. Let

$$A_+(U),\ A_-(U)\subset\bigwedge^3 V \tag{2.4.12}$$

be the affine cones over the linear spans of $\mathrm{im}(i_+)$, $\mathrm{im}(i_-)$; thus $\dim A_+(U)=\dim A_-(U)=10$. Since each of $A_+(U)$, $A_-(U)$ is spanned by decomposable vectors and the supports of any two of them intersect non-trivially it follows that $A_+(U),A_-(U)\in\mathbb{LG}(\bigwedge^3 V)$. Let $\mathcal{Q}:=\mathrm{Gr}(2,U)\subset\mathbb{P}(\bigwedge^2 U)$ be the Grassmannian embedded by Plücker: in Section 1.5 of [28] we proved

$$Y_{A_+(U)}=3\mathcal{Q}. \tag{2.4.13}$$

Of course $A_+(U),A_-(U)$ is well-defined up to $\mathrm{PGL}(V)$; we denote it by A_+,A_-. Moreover it is clear that the orbits $\mathrm{PGL}(V)A_+$ and $\mathrm{PGL}(V)A_-$ coincide (nonetheless it is useful to consider both lagrangians, see below). We notice that $\Theta_{A_+}\cong\mathbb{P}(U)$, $\Theta_{A_-}\cong\mathbb{P}(U^\vee)$, in particular $\dim\Theta_{A_+}=\dim\Theta_{A_-}=3$. Theorem 2.36 of [28] lists those $A\in\mathbb{LG}(\bigwedge^3 V)$ such that $\dim\Theta_A>2$: that classifiation together with Table (2) gives the following result.

Proposition 2.4.5. *Let $A\in\mathbb{LG}(\bigwedge^3 V)^{ss}$ and suppose that $\dim\Theta_A>2$; then A is projectively equivalent to A_+.*

Later we will prove that A_+ is actually semistable.

Corollary 2.4.6. *Let $A\in\mathbb{LG}(\bigwedge^3 V)^{ss}$. Then $Y_A\neq\mathbb{P}(V)$ and $Y_{\delta(A)}\neq\mathbb{P}(V^\vee)$.*

Proof. The isomorphism $\mathbb{LG}(\bigwedge^3 V)\xrightarrow{\sim}\mathbb{LG}(\bigwedge^3 V^\vee)$ induced by (1.3.1) maps semi-stable points to semi-stable points hence it suffices to prove that $Y_A\neq\mathbb{P}(V)$. Suppose that $A\in\mathbb{LG}(\bigwedge^3 V)^{ss}$ and that $Y_A=\mathbb{P}(V)$: by Claim 1.11 of [28] we have $\dim\Theta_A\geq 3$. By **Proposition 2.4.5** it follows that A is projectively equivalent to A_+. Claim 1.14 of [28] gives that Y_{A_+} is a triple quadric (in fact the Plücker quadric), in particular $Y_{A_+}\neq\mathbb{P}(V)$: that is a contradiction. □

Remark 2.4.7. Let U be as above i.e. $\dim U=4$. Then we have an isomorphism of $GL(U)$-modules

$$\bigwedge^3(\bigwedge^2 U)=\left(\mathrm{S}^2 U\otimes\det U\right)\oplus\left(\mathrm{S}^2 U^\vee\otimes(\det U)^2\right). \tag{2.4.14}$$

The direct summand $\mathrm{S}^2 U\otimes\det U$ is identified with $A_+(U)$ and $\mathrm{S}^2 U^\vee\otimes(\det U)^2$ is identified with $A_-(U)$.

2.5 The stable locus

Proof of **Theorem 2.1.1**. We will apply the Cone Decomposition Algorithm of **Subsection 2.3** to the action of $\mathrm{SL}(V)$ on $\mathbb{LG}(\bigwedge^3 V)\subset\mathrm{Gr}(10,\bigwedge^3 V)$. We choose a basis $\mathsf{F}=\{v_0,\ldots,v_5\}$ of V and we let $T<\mathrm{SL}(V)$ be the maximal torus of elements diagonal in the basis F. We make the standard choice of cone $C\subset\check{X}(T)_{\mathbb{R}}$ - see (2.3.1). First we list all ordering hyperplanes. Let

$$3=|\{i,j,k\}|=|\{l,m,n\}|,\qquad 0\leq i,j,k,l,m,n\leq 5 \tag{2.5.1}$$

and $\Phi^{i,j,k}_{l,m,n}\colon\check{X}(T)_{\mathbb{R}}\to\mathbb{R}$ be the linear function

$$\Phi^{i,j,k}_{l,m,n}(r_0,r_1,\ldots,r_5):=r_i+r_j+r_k-r_l-r_m-r_n. \tag{2.5.2}$$

Table 1: Standard non-stable strata.

$(r_0,\dots,r_5)$	$\mu(\mathbf{d},\lambda)$	reduced type $\mathbf{d}$	flag condition	name
$(5,-1_5)$	$2(3d_0-15)$	(5)	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{15})\geq 5$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{A}}$
$(1_5,-5)$	$2(3d_0-15)$	(5)	$\dim A\cap(\bigwedge^3 V_{04})\geq 5$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{A}^\vee}$
$(1_3,-1_3)$	$2(3d_0+d_1-6)$	$(1,3)$	$A\supset\bigwedge^3 V_{02}$ and $\dim A\cap(\bigwedge^2 V_{02}\wedge V_{35})\geq 3$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1}$
		$(0,6)$	$\dim A\cap(\bigwedge^3 V_{02}\oplus(\bigwedge^2 V_{02}\wedge V_{35}))\geq 6$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_2}$
$(1,0_4,-1)$	$2(d_0-3)$	(3)	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{14})\geq 3$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{D}}$
$(4,1_2,-2_3)$	$2(6d_0+3d_1-12)$	$(1,2)$	$A\supset[v_0]\wedge\bigwedge^2 V_{12}$ and $\dim A\cap([v_0]\wedge V_{12}\wedge V_{35})\geq 2$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}$
		$(0,4)$	$\dim A\cap([v_0]\wedge(\bigwedge^2 V_{12})\oplus([v_0]\wedge V_{12}\wedge V_{35}))\geq 4$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$
$(2_3,-1_2,-4)$	$2(6d_0+3d_1-12)$	$(1,2)$	$A\supset\bigwedge^3 V_{02}$ and $\dim A\cap(\bigwedge^2 V_{02}\wedge V_{34})\geq 2$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$
		$(0,4)$	$\dim A\cap(\bigwedge^3 V_{02}\oplus(\bigwedge^2 V_{02}\wedge V_{34}))\geq 4$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2^\vee}$
$(1_2,0_2,-1_2)$	$2(2d_0+d_1-4)$	$(2,0)$	$A\supset(\bigwedge^2 V_{01}\wedge V_{23})$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_1}$
		$(1,2)$	$\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23})\geq 1$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23}\oplus\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23})\geq 3$	$\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$
$(2,1,0_2,-1,-2)$	$2(3d_0+2d_1+d_2-7)$	$(1,1,2)$	$\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23})\geq 1$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23}\oplus\langle v_0\wedge v_1\wedge v_4, v_0\wedge v_2\wedge v_3\rangle)\geq 2$ and $\dim A\cap(\bigwedge^3 V_{03}\oplus[v_0]\wedge V_{13}\wedge[v_4]\oplus[v_0\wedge v_1\wedge v_5])\geq 4$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$

Table 2: Standard unstable strata.

$(r_0,\dots,r_5)$	$\mu(\mathbf{d},\lambda)$	reduced type $\mathbf{d}$	flag condition	name
$(5,-1_5)$	$2(3d_0-15)$	(6)	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{15})\geq 6$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{A}_+}$
$(1_5,-1_5)$	$2(3d_0-15)$	(6)	$\dim A\cap(\bigwedge^3 V_{04})\geq 6$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{A}^\vee_+}$
$(1_3,-1_3)$	$2(3d_0+d_1-6)$	$(1,4)$	$A\supset\bigwedge^3 V_{02}$ and $\dim A\cap(\bigwedge^2 V_{02}\wedge V_{35})\geq 4$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{C}_{1,+}}$
		$(0,7)$	$\dim A\cap(\bigwedge^3 V_{02}\oplus(\bigwedge^2 V_{02}\wedge V_{35}))\geq 7$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{C}_{2,+}}$
$(1,0_4,-1)$	$2(d_0-3)$	(4)	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{14})\geq 4$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{D}_+}$
$(4,1_2,-2_3)$	$2(6d_0+3d_1-12)$	$(1,3)$	$A\supset[v_0]\wedge\bigwedge^2 V_{12}$ and $\dim A\cap([v_0]\wedge V_{12}\wedge V_{35})\geq 3$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}_{1,+}}$
		$(0,5)$	$\dim A\cap([v_0]\wedge(\bigwedge^2 V_{12})\oplus([v_0]\wedge V_{12}\wedge V_{35}))\geq 5$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}_{2,+}}$
$(2_3,-1_2,-4)$	$2(6d_0+3d_1-12)$	$(1,3)$	$A\supset\bigwedge^3 V_{02}$ and $\dim A\cap(\bigwedge^2 V_{02}\wedge V_{34})\geq 3$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}^\vee_{1,+}}$
		$(0,5)$	$\dim A\cap(\bigwedge^3 V_{02}\oplus(\bigwedge^2 V_{02}\wedge V_{34}))\geq 5$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}^\vee_{2,+}}$
$(1_2,0_2,-1_2)$	$2(2d_0+d_1-4)$	$(2,1)$	$A\supset\bigwedge^2 V_{01}\wedge V_{23}$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23})\geq 1$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{F}_{1,+}}$
$(1_2,0_2,-1_2)$	$2(2d_0+d_1-4)$	$(1,3)$	$\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23})\geq 1$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23}\oplus\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23})\geq 4$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{F}_{2,+}}$
$(2,1,0_2,-1,-2)$	$2(3d_0+2d_1+d_2-7)$	$(1,1,3)$	$\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23})\geq 1$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{23}\oplus\langle v_0\wedge v_1\wedge v_4, v_0\wedge v_2\wedge v_3\rangle)\geq 2$ and $\dim A\cap(\bigwedge^3 V_{03}\oplus[v_0]\wedge V_{13}\wedge[v_4]\oplus[v_0\wedge v_1\wedge v_5])\geq 5$	$\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_{3,+}}$

Table 3: "Essential" functions $\Phi^{i,j,k}_{l,m,n}(\mathbf{x})$ with $\{i,j,k\} \cap \{l,m,n\} = \emptyset$.

$\Phi^{1,2,3}_{0,4,5}$	$\Phi^{2,3,4}_{0,1,5}$	$\Phi^{1,2,4}_{0,3,5}$	$\Phi^{0,3,4}_{1,2,5}$	$\Phi^{0,2,5}_{1,3,4}$
$-x_1 + x_3 + 2x_4 + x_5$	$-x_1 - 2x_2 - x_3 + x_5$	$-x_1 + x_3 + x_5$	$x_1 - x_3 + x_5$	$x_1 + x_3 - x_5$

It is clear that $H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and only if there exist i,j,k,l,m,n as above with $\{i,j,k\} \neq \{l,m,n\}$ such that $H = \ker(\Phi^{i,j,k}_{l,m,n})$. The faces of C span the hyperplanes $\ker(r_a - r_b)$ for $0 \leq a < b \leq 5$; since $r_a - r_b = \Phi^{a,j,k}_{b,j,k}$ we get that the hypotheses of **Proposition 2.3.4** are satisfied. Thus $A \in \mathbb{LG}(\bigwedge^3 V)$ is not stable if and only if there exist $A' \in \mathrm{SL}(V)A$ and an ordering 1-PS $\overline{\lambda}$ of $\mathrm{SL}(V)$ such that $\mu(A', \overline{\lambda}) \geq 0$. Next let us list all ordering 1-PS's of $\mathrm{SL}(V)$ i.e. those $\mathbf{r} \in C$ which span the zero-set of four linearly independent functions among the $\Phi^{i,j,k}_{l,m,n}$'s. It is convenient to work with the coordinates $(x_1, \ldots, x_5)$ given by

$$x_i := r_{i-1} - r_i, \qquad i = 1, \ldots, 5 \tag{2.5.3}$$

In the coordinates $x_1, \ldots, x_5$ the cone C is the set of vectors with non-negative coordinates. Following is the column of the linear functions $r_0, \ldots, r_5$ (restricted to $\check{X}(T)_{\mathbb{R}}$) in terms of the coordinates $(x_1, \ldots, x_5)$:

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} 5/6 & 2/3 & 1/2 & 1/3 & 1/6 \\ -1/6 & 2/3 & 1/2 & 1/3 & 1/6 \\ -1/6 & -1/3 & 1/2 & 1/3 & 1/6 \\ -1/6 & -1/3 & -1/2 & 1/3 & 1/6 \\ -1/6 & -1/3 & -1/2 & -2/3 & 1/6 \\ -1/6 & -1/3 & -1/2 & -2/3 & -5/6 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.5.4}$$

By definition the linear functions $(r_{i-1} - r_i)$ are equal to the new coordinate functions x_i. We will rewrite the linear functions $\Phi^{i,j,k}_{l,m,n}$ in the new coordinates. First notice that whenever $\Phi^{i,j,k}_{l,m,n}$ is a linear combination of a collection of the x_i's with coefficients of the same sign then it may be disregarded because its zero set is the zero set of a collection of the coordinate functions $x_1, \ldots, x_5$. If $|\{i,j,k\} \cap \{l,m,n\}| = 2$ then $\Phi^{i,j,k}_{l,m,n}$ is a sum of x_i's with coefficients of the same sign and hence we disregard it. Next let's consider the $\Phi^{i,j,k}_{l,m,n}$'s such that $|\{i,j,k\} \cap \{l,m,n\}| = 1$: up to ± 1 we get the following functions

$$(x_1-x_3), (x_1-x_4), (x_1-x_5), (x_2-x_4), (x_2-x_5), (x_3-x_5), (x_1+x_2-x_4), (x_1+x_2-x_5), (x_2+x_3-x_5),$$
$$(x_1 - x_3 - x_4), (x_1 - x_4 - x_5), (x_2 - x_4 - x_5), (x_1 + x_2 - x_4 - x_5). \tag{2.5.5}$$

Lastly assume that $\{i,j,k\} \cap \{l,m,n\} = \emptyset$; then $\Phi^{i,j,k}_{l,m,n}(\mathbf{r}) = 2(r_i + r_j + r_k)$. The functions

$$\Phi^{0,1,2}_{3,4,5}(\mathbf{x}) = x_1 + 2x_2 + 3x_3 + 2x_4 + x_5, \ \Phi^{0,1,3}_{2,4,5}(\mathbf{x}) = x_1 + 2x_2 + x_3 + 2x_4 + x_5,$$
$$\Phi^{2,3,5}_{0,1,4}(\mathbf{x}) = -(x_1 + 2x_2 + x_3 + x_5), \ \Phi^{1,4,5}_{0,2,3}(\mathbf{x}) = -(x_1 + x_3 + 2x_4 + x_5), \ \Phi^{0,2,4}_{1,3,5}(\mathbf{x}) = x_1 + x_3 + x_5$$

have all non-zero coefficients of the same sign and hence we may disregard them. Table (3) lists the remaining such functions (with $\{i,j,k\} \cap \{l,m,n\} = \emptyset$) modulo ± 1. It follows that in order to list all ordering 1-PS's we must find all non-zero solutions $(x_1, \ldots, x_5) \in C$ of 4 linearly independent linear functions among the union of the set of coordinate functions, the set given by (2.5.5) and that given by Table (3). In practice we consider the 5×23-matrix M whose columns are the coordinates of the linear functions listed above i.e.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \end{bmatrix}$$

and we proceed as follows. For each 5×4 minor M_I of M we compute (actually we ask a computer to compute) the vector in $\mathbb{R}^5$ whose coordinates are the determinants with alternating signs of 4×4 minors of M_I and discard all those vectors whose coordinates do not have the same sign. The remaining vectors are the **x**-coordinates of ordering 1-PS's (with many repetitions). Multiplying each such vector by the matrix appearing in (2.5.4) one gets the weights of all ordering 1-PS's. The outcome of the computations is as follows. First the 1-PS's appearing in Table (1) are among the ordering 1-PS's. For example the first three 1-PS's of Table (1) correspond in the **x**-coordinates to the extremal rays of C generated by $(1,0,0,0,0)$, $(0,0,0,0,1)$ and $(0,0,1,0,0)$ respectively. Tables (18), (19) and (20) in **Section B** list all the ordering 1-PS's up to rescaling and duality (ordering 1-PS's come in dual pairs $(r_0,\ldots,r_5)$ and $(-r_5,\ldots,-r_0)$). Tables (18), (19) and (20) give also the strictly-positive weight isotypical addends of $\bigwedge^3 \lambda$ for each ordering 1-PS in the list; abc denotes $v_a \wedge v_b \wedge v_c$ and an isotypical addend is determined via its monomial basis. Next one needs to examine, for each ordering 1-PS λ, the set of $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\mu(A,\lambda) \geq 0$. One finishes the proof of **Theorem 2.1.1** by checking that each such A belongs to one of the standard non-stable strata i.e. those listed in Table (1): details are in Tables (21), (22), (23) and (24) of **Section B**. One should read the tables as follows. The first column of each row gives the weights of an ordering 1 PS λ, the second column contains an explicit expression for $\mu(\mathbf{d},\lambda)$ (to get it use Tables (18), (19) and (20)), the third column contains a collection of subsets of $\mathcal{P}_\lambda$ (to be precise a condition on $\mathbf{d}$ determining such a subset) whose union is all of

$$\mathcal{P}_\lambda^{\geq 0} := \{\mathbf{d} \in \mathcal{P}_\lambda \mid \mu(\mathbf{d},\lambda) \geq 0\},$$

the last column gives for each such subset of $\mathcal{P}_\lambda^{\geq 0}$ a stratum (or union of strata) containing all $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\mathbf{d}^\lambda(A)$ belongs to the subset. We notice that since Table (1) is invariant under duality it suffices to examine one ordering 1-PS in each dual pair. Following are a few remarks on how to check that the last step of the proof has been carried out correctly. One first needs to make sure that every $\mathbf{d} \in \mathcal{P}_\lambda^{\geq 0}$ belongs to one of the sets defined by the conditions on the third column: that is time-consuming but completely straightforward. Secondly one needs to verify that each subset of $\mathbf{d} \in \mathcal{P}_\lambda^{\geq 0}$ listed in Tables (21), (22) and (23) is contained in the stratum (or union of strata) on the same row and on the last column: that is completely routine except in the two cases below.

$\boxed{\lambda(t) = (t^7, t^4, t, t, t^{-5}, t^{-8}),\ \mathbf{d} \in \mathcal{P}_\lambda \text{ such that } (d_0+d_1) \geq 1 \text{ and } d_2 \geq 2}$ We remark that the ordering 1-PS appears in Table (22). Suppose that $\mathbf{d}^\lambda(A) = (d_0, d_1, \ldots)$ is as above (notice that $d_2 = 2$ by Table (19) in **Section B**). Then A contains

$$0 \neq \alpha = v_0 \wedge w_1 \wedge w_2, \qquad \beta = v_0 \wedge w_1' \wedge w_2' + v_1 \wedge v_2 \wedge v_3, \quad w_1, w_2, w_1', w_2' \in \langle v_1, v_2, v_3 \rangle.$$

We distinguish two cases according to whether $w_1' \wedge w_2'$ is a multiple of $w_1 \wedge w_2$ or not. If the former holds then A contains $v_1 \wedge v_2 \wedge v_3$ and since $\langle w_1, w_2 \rangle \subset \langle v_1, v_2, v_3 \rangle$ it follows that $A \in \mathbb{B}^*_{\mathcal{F}_1}$. If the latter holds then we may complete w_1, w_2 to a basis $\{w_1, w_2, w_3\}$ of $\langle v_1, v_2, v_3 \rangle$ in such a way that

$$\beta = v_0 \wedge w_1 \wedge w_3 + w_1 \wedge w_2 \wedge w_3 = w_1 \wedge w_3 \wedge (v_0 - w_2).$$

Since

$$\dim(\operatorname{supp}\alpha \cap \operatorname{supp}\beta) = \dim(\langle v_0, w_1, w_2 \rangle \cap \langle w_1, w_3, (v_0 - w_2) \rangle) = 2$$

we get that $A \in \mathbb{B}^*_{\mathcal{F}_1}$.

$\boxed{\lambda(t) = (t^{10}, t^7, t, t^{-2}, t^{-5}, t^{-11}),\ \mathbf{d} = (0,0,1,1,3,0)}$ We remark that the ordering 1-PS appears in Table (23). Let $\mathsf{F} := \{v_0, v_1, v_2, z_3, z_4, v_5\}$ be a basis of V. Let $\mathbf{r}$ be the set of weights of λ in decreasing order and $\mathbf{d}$ be as above. Let λ' be the 1-PS corresponding to $\mathbb{X}_{\mathcal{N}_3}$ according to Table (1) and $\mathbf{r}'$ its set of weights in decreasing order. Let $\mathbf{d}' = (1,1,2)$ be the λ'-type defining $\mathbb{X}_{\mathcal{N}_3}$. Let $A \in \mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$: we will exhibit a basis F' of V (depending on A) such that

$$A \in \mathbb{E}^{\mathsf{F}'}_{\mathbf{r}',\mathbf{d}'} := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \mathbf{d}^{\lambda'}(A) \succeq (1,1,2)\}. \tag{2.5.6}$$

Since $A \in \mathbb{E}^{\mathsf{F}}_{\mathbf{r},\mathbf{d}}$ there exist $\alpha, \beta, \gamma, \delta \in A$ such that

$$\begin{aligned}\alpha =& v_0 \wedge v_1 \wedge \omega_1,\\ \beta =& v_0 \wedge (v_1 \wedge \omega_2 + v_2 \wedge z_3),\\ \gamma =& v_0 \wedge (v_1 \wedge \omega_3 + v_2 \wedge (az_3 + z_4)),\\ \delta =& v_0 \wedge (v_1 \wedge \omega_4 + bv_2 \wedge z_3 + v_1 \wedge v_5),\end{aligned}$$

where

$$\omega_1, \omega_2, \omega_3, \omega_4 \in \langle v_2, z_3, z_4 \rangle, \qquad \omega_1 \neq 0.$$

There exists $(x_0, y_0) \neq (0, 0)$ such that

$$\omega_1 \in \langle v_2, x_0 z_3 + y_0(az_3 + z_4) \rangle.$$

Let $v_3 := x_0 z_3 + y_0(az_3 + z_4)$. Notice that v_2, v_3 are linearly independent and they belong to $\langle v_2, z_3, z_4 \rangle$; thus there exists $v_4 \in \langle z_3, z_4 \rangle$ such that $\{v_2, v_3, v_4\}$ is a basis of $\langle v_2, z_3, z_4 \rangle$. We let $\mathsf{F}' := \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Let's prove that (2.5.6) holds. Let $\mathbf{d}'_{\lambda'}(A) = (d'_0(A), d'_1(A), d'_2(A))$. First $d'_0(A) \geq 1$ because $\alpha \neq 0$. Next

$$A \ni (x_0\beta + y_0\gamma) = v_0 \wedge (v_1 \wedge (x_0\omega_2 + y_0\omega_3) + v_2 \wedge v_3), \qquad (x_0\omega_2 + y_0\omega_3) \in \langle v_2, v_3, v_4 \rangle.$$

It follows that $d'_1(A) \geq 1$. Lastly let $L_0 \subset L_1 \subset \ldots \subset L_6 = \bigwedge^3 V$ be the filtration defined by the isotypical addends of $\bigwedge^3 \lambda'$ in decreasing order, see (2.2.7). Then $\beta, \gamma, \delta \in L_2$ and the image of $\langle \beta, \gamma, \delta \rangle$ in L_2/L_1 has dimension 2, thus $d'_2(A) \geq 2$. This finishes the proof that (2.5.6) holds. $\square$

For $d \geq 0$ let $\widetilde{\Sigma}[d] \subset \widetilde{\Sigma}$ be given by

$$\widetilde{\Sigma}[d] := \{(W, A) \in \widetilde{\Sigma} \mid \dim(A \cap (\bigwedge^2 W \wedge V)) \geq d + 1\}. \tag{2.5.7}$$

(Notice that $\widetilde{\Sigma} := \widetilde{\Sigma}[0]$.) Let $\Sigma[d] \subset \mathbb{LG}(\bigwedge^3 V)$ be the image of $\widetilde{\Sigma}[d]$ under the projection

$$\mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V) \longrightarrow \mathbb{LG}(\bigwedge^3 V).$$

Corollary 2.5.1. *If $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma_\infty \setminus \Sigma[2])$ then A is stable.*

Proof. By **Theorem 2.1.1** it suffices to prove that if A belongs to one of the standard non-stable strata then either $\dim \Theta_A > 0$ (i.e. $A \in \Sigma_\infty$) or $A \in \Sigma[2]$. By definition we may assume that $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ for $\mathcal{X}$ one of $\mathcal{A}, \mathcal{A}^\vee, \ldots, \mathcal{F}_2$, or $A \in \mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$, where F is the basis $\{v_0, \ldots, v_5\}$ of V. If

$$A \in (\mathbb{B}^{\mathsf{F}}_{\mathcal{A}} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{A}^\vee} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{C}_2} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{D}} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2^\vee} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{F}_1})$$

then $A \in \Sigma_\infty$, see **Subsubsection 2.4.2**. It remains to consider $A \in (\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2} \cup \mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3})$. Going through Table (1) one easily checks the following: If $A \in (\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee})$ then $\bigwedge^3 V_{02} \subset A$ and $\dim(A \cap (\bigwedge^2 V_{02} \wedge V)) \geq 3$, if $A \in (\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2} \cup \mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3})$ there exists a 3-dimensional subspace $W \subset V_{03}$ containing V_{01} such that $\bigwedge^3 W \subset A$ and $\dim(A \cap (\bigwedge^2 W \wedge V)) \geq 3$. $\square$

Theorem 2.1.1 provides an algorithm that decides whether a given $A \in \mathbb{LG}(\bigwedge^3 V)$ is stable or not: see **Remark 3.4.5** for details.

2.6 The GIT-boundary

Let $\mathfrak{M}^{st} \subset \mathfrak{M}$ be the (open) subset parametrizing $\mathrm{PGL}(V)$-orbits of stable points; the *GIT-boundary* of $\mathfrak{M}$ is $\partial\mathfrak{M} := (\mathfrak{M} \setminus \mathfrak{M}^{st})$. Let $\mathbb{B}^*_{\mathcal{A}}, \mathbb{B}^*_{\mathcal{A}^\vee}, \ldots, \mathbb{X}^*_{\mathcal{N}_3}$ be the standard non-stable strata. If $\mathbb{B}^*_{\mathcal{X}}$ (or $\mathbb{X}^*_{\mathcal{N}_3}$) is such a stratum we let

$$\mathfrak{B}_{\mathcal{X}} := \mathbb{B}^*_{\mathcal{X}} // \mathrm{PGL}(V), \qquad \mathfrak{X}_{\mathcal{N}_3} := \mathbb{X}^*_{\mathcal{N}_3} // \mathrm{PGL}(V). \tag{2.6.1}$$

By **Theorem 2.1.1** we have the equality

$$\partial\mathfrak{M} = \mathfrak{B}_{\mathcal{A}} \cup \mathfrak{B}_{\mathcal{A}^\vee} \cup \mathfrak{B}_{\mathcal{C}_1} \cup \mathfrak{B}_{\mathcal{C}_2} \cup \mathfrak{B}_{\mathcal{D}} \cup \mathfrak{B}_{\mathcal{E}_1} \cup \mathfrak{B}_{\mathcal{E}_2} \cup \mathfrak{B}_{\mathcal{E}_1^\vee} \cup \mathfrak{B}_{\mathcal{E}_2^\vee} \cup \mathfrak{B}_{\mathcal{F}_1} \cup \mathfrak{B}_{\mathcal{F}_2} \cup \mathfrak{X}_{\mathcal{N}_3}. \tag{2.6.2}$$

We will show that there exist equalities among some of the above sets. Let $\mathsf{F} = \{v_0, \ldots, v_5\}$ be a basis of V. Given a subscript $\mathcal{X} \in \{\mathcal{A}, \mathcal{A}^\vee, \ldots, \mathcal{N}_3\}$ we let $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ be the corresponding Schubert varieties appearing in Table (1) (if $\mathcal{X} = \mathcal{N}_3$ the Schubert variety is denoted $\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$). Let $\lambda_{\mathcal{X}} \colon \mathbb{C}^\times \longrightarrow \mathrm{SL}(V)$ be the standard ordering 1-PS which is diagonal in the basis F and whose weights appear on the first column of the row of Table (1) that contains $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ (or $\mathbb{X}_{\mathcal{N}_3}$). Let $U_{e_0}, \ldots, U_{e_i}, \ldots, U_{e_s}$ be the isotypical summands of $\bigwedge^3 \lambda_{\mathcal{X}}$ as in (2.4.3), with weights in decreasing order: $e_0 > e_1 > \ldots > e_s$. We have a $\lambda_{\mathcal{X}}$-type

$$\mathbf{d}_{\mathcal{X}} = (d_0, d_1, \ldots, d_{[(s-1)/2]}). \tag{2.6.3}$$

which appears in the third column of the row of Table (1) that contains $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ (or $\mathbb{X}_{\mathcal{N}_3}$) . Let $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}} \subset \mathbb{LG}(\bigwedge^3 V)$ be the set of A which are $\lambda_{\mathcal{X}}$-split of type $\mathbf{d}_{\mathcal{X}}$. **Claim 2.2.4** gives the following:

Claim 2.6.1. *Every point of $\mathfrak{B}_{\mathcal{X}}$ is represented by a point of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ and every point of $\mathfrak{X}_{\mathcal{N}_3}$ is represented by a point of $\mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$.*

Next let F' be the basis of V obtained by reading the vectors in F in reverse order: $\mathsf{F}' := \{v_5, v_4, v_3, v_2, v_1, v_0\}$. As is easily checked we have

$$\mathbb{S}^{\mathsf{F}}_{\mathcal{A}} = \mathbb{S}^{\mathsf{F}'}_{\mathcal{A}^\vee}, \qquad \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1} = \mathbb{S}^{\mathsf{F}'}_{\mathcal{C}_2}, \qquad \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1} = \mathbb{S}^{\mathsf{F}'}_{\mathcal{E}_2^\vee}, \qquad \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee} = \mathbb{S}^{\mathsf{F}'}_{\mathcal{E}_2} \tag{2.6.4}$$

and hence $\mathfrak{B}_{\mathcal{A}} = \mathfrak{B}_{\mathcal{A}^\vee}$, $\mathfrak{B}_{\mathcal{C}_1} = \mathfrak{B}_{\mathcal{C}_2}$, $\mathfrak{B}_{\mathcal{E}_1} = \mathfrak{B}_{\mathcal{E}_2^\vee}$ and $\mathfrak{B}_{\mathcal{E}_1^\vee} = \mathfrak{B}_{\mathcal{E}_2}$. Thus (2.6.2), **Claim 2.6.1** and the above equalities give the following:

$$\partial\mathfrak{M} = \mathfrak{B}_{\mathcal{A}} \cup \mathfrak{B}_{\mathcal{C}_1} \cup \mathfrak{B}_{\mathcal{D}} \cup \mathfrak{B}_{\mathcal{E}_1} \cup \mathfrak{B}_{\mathcal{E}_1^\vee} \cup \mathfrak{B}_{\mathcal{F}_1} \cup \mathfrak{B}_{\mathcal{F}_2} \cup \mathfrak{X}_{\mathcal{N}_3}. \tag{2.6.5}$$

Since each $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ is closed and irreducible each set on the right-hand side of (2.6.5) is closed and either irreducible or empty (this will hold if there is no semistable point of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$). The above discussion gives no answer to the following questions: are any of the sets appearing on the right-hand side of (2.6.5) empty? are there inclusion relations between those sets? what are their dimensions? The answers are in **Section 5**.

3 Plane sextics and stability of lagrangians

3.1 The main result of the section

Theorem 2.1.1 gives a description of non-stable elements $A \in \mathbb{LG}(\bigwedge^3 V)$ in terms of linear-algebra flag conditions on A. One would like to establish a link between non-stability of A and geometric properties of X_A (we may assume that $Y_A \neq \mathbb{P}(V)$ because if $Y_A = \mathbb{P}(V)$ then A is unstable by **Corollary 2.4.6**). A first hint of what an answer might be is given by **Corollary 2.5.1**: if $\Theta_A = \emptyset$ then A is stable. In the present section we will refine that result. Assume that $A \in \Sigma$ and let $W \in \Theta_A$. In **Subsection 3.2** we will define a determinantal locus $C_{W,A} \subset \mathbb{P}(W)$ with the property that

$$\operatorname{supp} C_{W,A} = \{[w] \in \mathbb{P}(W) \mid \dim(A \cap F_w) \geq 2\}. \tag{3.1.1}$$

Either $C_{W,A}$ is a sextic curve or (in pathological cases) it equals $\mathbb{P}(W)$. Below is the main result of the present section.

Theorem 3.1.1. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ be non-stable, and hence*

$$A \in (\mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_1} \cup \mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_1} \cup \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{B}_{\mathcal{E}_1^\vee} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{F}_2} \cup \mathbb{X}_{\mathcal{N}_3}) \tag{3.1.2}$$

by **Theorem 2.1.1**. *Then there exists $W \in \Theta_A$ such that $C_{W,A}$ is not a curve with simple singularities, more precisely either $C_{W,A} = \mathbb{P}(W)$ or else $C_{W,A}$ is a sextic curve and*

(1) there exists $[v_0] \in C_{W,A}$ such that $\operatorname{mult}_{[v_0]} C_{W,A} \geq 4$ if $A \in (\mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_1})$,

(2) $C_{W,A}$ is singular along a line (and hence non-reduced) if $A \in (\mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{F}_2})$,

(3) $C_{W,A}$ is singular along a conic (and hence non-reduced) if $A \in \mathbb{B}_{\mathcal{E}_1^\vee}$,

(4) $C_{W,A}$ is singular along a cubic (and hence equal to a double cubic)) if $A \in (\mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_1})$.

(5) $C_{W,A}$ has consecutive triple points if $A \in (\mathbb{B}_{\mathcal{E}_2} \cup \mathbb{X}_{\mathcal{N}_3})$.

The statement of **Theorem 3.1.1** is obtained by putting together the statements of **Proposition 3.3.1** and **Proposition 3.4.1**. Notice that **Theorem 0.0.3** follows at once from **Theorem 3.1.1**.

3.2 Plane sextics

Let $W \in \operatorname{Gr}(3, V)$. Let

$$\mathcal{E}_W := (\bigwedge^3 W)^{\perp} / \bigwedge^3 W \tag{3.2.1}$$

where $\bigwedge^3 W^{\perp}$ is the orthogonal of $\bigwedge^3 W$ with respect to $(,)_V$. The symplectic form $(,)_V$ induces a symplectic form on $\mathcal{E}_W$ that we will denote by $(,)_W$. Let $[w] \in \mathbb{P}(W)$; since F_w is a Lagrangian subspace of $\bigwedge^3 V$ containing $\bigwedge^3 W$ we have the lagrangian

$$G_w := F_w / \bigwedge^3 W \in \mathbb{LG}(\mathcal{E}_W). \tag{3.2.2}$$

Thus we have a Lagrangian sub-vector-bundle G of $\mathcal{E}_W \otimes \mathcal{O}_{\mathbb{P}(W)}$ defined by

$$G := F \otimes \mathcal{O}_{\mathbb{P}(W)} / \bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}(W)}. \tag{3.2.3}$$

We will associate to $B \in \mathbb{LG}(\mathcal{E}_W)$ a subscheme $C_B \subset \mathbb{P}(W)$ by mimicking the definition of EPW-sextic given in **Section 1**. Composing the inclusion $G \hookrightarrow \mathcal{E}_W \otimes \mathcal{O}_{\mathbb{P}(W)}$ and the quotient map $\mathcal{E}_W \otimes \mathcal{O}_{\mathbb{P}(W)} \to (\mathcal{E}_W/B) \otimes \mathcal{O}_{\mathbb{P}(W)}$ we get a map of vector-bundles

$$G \xrightarrow{\nu_B} (\mathcal{E}_W/B) \otimes \mathcal{O}_{\mathbb{P}(W)}. \tag{3.2.4}$$

We let $C_B = V(\det \nu_B)$; thus $\operatorname{supp} C_B = \{[w] \in \mathbb{P}(W) \mid G_w \cap B \neq \{0\}\}$. A straightforward computation gives that

$$\det G \cong \mathcal{O}_{\mathbb{P}(W)}(-6). \tag{3.2.5}$$

Thus C_B is a sextic curve unless it is equal to $\mathbb{P}(W)$. Next suppose that $(W, A) \in \widetilde{\Sigma}$. Since $\bigwedge^3 W \subset A \subset (\bigwedge^3 W)^\perp$ we have the lagrangian

$$B := (A/\bigwedge^3 W) \in \mathbb{LG}(\mathcal{E}_W). \tag{3.2.6}$$

Definition 3.2.1. Suppose that $(W, A) \in \widetilde{\Sigma}$. We let $C_{W,A} := C_B$ where B is given by (3.2.6).

Notice that (3.1.1) holds by definition. Let $B \in \mathbb{LG}(\mathcal{E}_W)$ and ν_B be given by (3.2.4): we will write out the first terms in the Taylor expansion of $\det \nu_B$ in a neighborhood of $[v_0] \in \mathbb{P}(W)$. Let $W_0 \subset W$ be complementary to $[v_0]$. We have an isomorphism

$$\begin{array}{ccc} W_0 & \xrightarrow{\sim} & \mathbb{P}(W) \setminus \mathbb{P}(W_0) \\ w & \mapsto & [v_0 + w] \end{array} \tag{3.2.7}$$

onto a neighborhood of $[v_0]$; thus $0 \in W_0$ is identified with $[v_0]$. We have

$$C_B \cap W_0 = V(g_0 + g_1 + \cdots + g_6), \qquad g_i \in \mathrm{S}^i W_0^\vee \tag{3.2.8}$$

where the g_i's are well-determined up to a common non-zero multiplicative factor. We will describe explicitly the g_i's for $i \leq \dim(B \cap G_{v_0})$. Given $w \in W$ we define the Plücker quadratic form $\psi_w^{v_0}$ on G_{v_0} as follows. Let $\overline{\alpha} \in G_{v_0}$ be represented by $\alpha \in F_{v_0}$. Thus $\alpha = v_0 \wedge \beta$ where $\beta \in \bigwedge^2 V$ is defined modulo $(\bigwedge^2 W + [v_0] \wedge V)$: we let

$$\psi_w^{v_0}(\overline{\alpha}) := \operatorname{vol}(v_0 \wedge w \wedge \beta \wedge \beta). \tag{3.2.9}$$

Proposition 3.2.2. *Keep notation and hypotheses as above. Let $\overline{K} := B \cap G_{v_0}$ and $\overline{k} := \dim \overline{K}$. Then*

(1) $g_i = 0$ for $i < \overline{k}$, and

(2) there exists $\mu \in \mathbb{C}^*$ such that

$$g_{\overline{k}}(w) = \mu \det(\psi_w^{v_0}|_{\overline{K}}), \quad w \in W_0. \tag{3.2.10}$$

Proof. Let $B_1 := B$ and $B_2 \in \mathbb{LG}(\mathcal{E}_W)$ be transversal both to B_1 and G_{v_0}. Then $\mathcal{E}_W = B_1 \oplus B_2$ and we have an isomorphism $B_2 \cong B_1^\vee$ such that $(,)_W$ is identified with the standard symplectic form on $B_1 \oplus B_1^\vee$. There exists an open $\mathcal{W} \subset W_0$ containing 0 such that G_{v_0+w} is transversal to B_2 for all $w \in \mathcal{W}$ and hence G_{v_0+w} is the graph of a map $\widetilde{q}(w)\colon B_1 \to B_2 = B_1^\vee$. Since G_{v_0+w} is Lagrangian the map $\widetilde{q}(w)$ is symmetric; we let $q(w)$ be the associated quadratic form. The map $\mathcal{W} \to \mathrm{S}^2 B_1^\vee$ mapping w to $q(w)$ is regular and there exists $\rho \in H^0(\mathcal{O}_{\mathcal{W}}^*)$ such that

$$g(w) = \rho \det q(w), \quad w \in \mathcal{W}. \tag{3.2.11}$$

We have $\ker q(0) = B_1 \cap G_{v_0}$; by **Proposition A.1.2** we get that $\det q \in \mathfrak{m}_0^{\overline{k}}$ where $\mathfrak{m}_0 \subset \mathcal{O}_{\mathcal{W},0}$ is the maximal ideal; thus Item (1) follows from (3.2.11). Let's prove Item (2). Let $(\det q)_{\overline{k}} \in \left(\mathfrak{m}_0^{\overline{k}}/\mathfrak{m}_0^{\overline{k}+1}\right) \cong \mathrm{S}^{\overline{k}} W_0^\vee$ be the class of $\det q$; by (3.2.11) we have

$$g_{\overline{k}}(w) = \rho(0)(\det q)_{\overline{k}}(w), \quad w \in V_0. \tag{3.2.12}$$

We have $\ker q(0) = \overline{K}$; by **Proposition A.1.2** there exists $\theta \in \mathbb{C}^*$ such that

$$(\det q)_{\overline{k}}(w) = \theta \det\left(\left.\frac{d\,(q(tw)|_{\overline{K}})}{dt}\right|_{t=0}\right), \quad w \in W_0. \tag{3.2.13}$$

Thus in order to finish the proof of Item (2) it suffices to show that

$$\left.\frac{d\,(q(tw)|_{\overline{K}})}{dt}\right|_{t=0} = \psi^{v_0}_w|_{\overline{K}}, \quad w \in W_0. \tag{3.2.14}$$

Let $\widetilde{B}_i \in \mathbb{LG}(\bigwedge^3 V)$ be such that $\widetilde{B}_i/\bigwedge^3 W = B_i$. Let $\overline{\alpha} \in \overline{K}$ be represented by $\alpha \in F_{v_0}$; thus we also have $\alpha \in \widetilde{B}_1$. Assume that $tw \in \mathcal{W}$ where $\mathcal{W}$ is as above; there exists $r(tw)(\alpha) \in \widetilde{B}_2$ well-defined modulo $\bigwedge^3 W$ such that $(\alpha + r(tw)(\alpha)) \in F_{v_0+tw}$. Thus

$$(\alpha + r(tw)(\alpha)) = (v_0 + tw) \wedge \zeta(tw). \tag{3.2.15}$$

By definition of $q(tw)$ we have

$$q(tw)(\alpha) = \mathrm{vol}(\alpha \wedge r(tw)(\alpha)). \tag{3.2.16}$$

Now multiply (3.2.15) on the left by α; since $\alpha \in F_{v_0}$ we have $v_0 \wedge \alpha = 0$ and hence

$$q(tw)(\alpha) = t \cdot \mathrm{vol}(\alpha \wedge w \wedge \zeta(tw)) \tag{3.2.17}$$

for $w \in W_0$. Differentiating with respect to t and setting $t = 0$ we get that

$$\left.\frac{d\,(q(tw)|_{\overline{K}})}{dt}\right|_{t=0} (\overline{\alpha}) = \mathrm{vol}(\alpha \wedge w \wedge \zeta(0)). \quad w \in W_0. \tag{3.2.18}$$

We may write $\alpha = v_0 \wedge \beta$ because $\alpha \in F_{v_0}$. Setting $t = 0$ in (3.2.15) we get that $v_0 \wedge \zeta(0) = v_0 \wedge \beta$. Thus (3.2.18) reads

$$\left.\frac{d\,(q(tw)|_{\overline{K}})}{dt}\right|_{t=0} (\overline{\alpha}) = \mathrm{vol}(v_0 \wedge w \wedge \beta \wedge \beta) = \psi^{v_0}_w(\overline{\alpha}), \quad w \in W_0. \tag{3.2.19}$$

This proves (3.2.14). □

Corollary 3.2.3. *Let $(W, A) \in \widetilde{\Sigma}$ and $[v_0] \in \mathbb{P}(W)$. Then either $C_{W,A} = \mathbb{P}(W)$ or*

$$\mathrm{mult}_{[v_0]} C_{W,A} \geq \dim(A \cap F_{v_0}) - 1.$$

Proof. Let B be given by (3.2.6). We apply **Proposition 3.2.2**: it suffices to notice that $\overline{k} = (\dim(A \cap F_{v_0}) - 1)$. □

Our last result will be useful when we will describe $C_{W,A}$ for properly semistable A with closed orbit in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ - we will use it repeatedly in **Section 5**. Choose a direct-sum decomposition $V = W \oplus U$; thus $\dim U = 3$ and we have an identification

$$\mathcal{E}_W \cong \mathcal{E}^U_W := \bigwedge^2 W \otimes U \oplus W \otimes \bigwedge^2 U. \tag{3.2.20}$$

Notice that $\mathcal{E}^U_W$ is the direct-sum of a vector-space and its dual (after the choice of volume-forms on W and on U) and hence it is equipped with a symplectic form (defined up to scalar). Under Isomorphism (3.2.20) the symplectic form on $\mathcal{E}^U_W$ is identified, up to a scalar, with the symplectic form on $\mathcal{E}_W$. We have the embedding

$$\begin{array}{ccc} \mathbb{P}(W) & \hookrightarrow & \mathbb{LG}(\mathcal{E}^U_W) \\ [w] & \mapsto & G^U_w := \{\alpha \in \mathcal{E}^U_W \mid w \wedge \alpha = 0\} \end{array} \tag{3.2.21}$$

and the pull-back map

$$\mathbf{\Phi}\colon |\mathcal{O}_{\mathbb{LG}(\mathcal{E}^U_W)}(1)| \dashrightarrow |\mathcal{O}_{\mathbb{P}(W)}(6)|. \tag{3.2.22}$$

Let $(W, A) \in \widetilde{\Sigma}$: thus $A = \bigwedge^3 W \oplus B$ where $B \in \mathcal{E}^U_W$. Then $\bigwedge^9 B$ corresponds (via wedge-multiplication) to a hyperplane $H_B \in |\mathcal{O}_{\mathbb{LG}(\mathcal{E}^U_W)}(1)|$ and

$$C_{W,A} = \mathbf{\Phi}(H_B). \tag{3.2.23}$$

(Notice that $C_{W,A} = \mathbb{P}(W)$ if and only if H_B in the indeterminacy locus of $\mathbf{\Phi}$.) Of course $\mathbf{\Phi}$ is the projectivization of the map Φ of global sections induced by (3.2.21). We will write out Φ as a $GL(W) \times GL(U)$-equivariant map. Write $G^U_w = G'_w \oplus G''_w$ where $G'_w = G^U_w \cap (\bigwedge^2 W \otimes U)$ and $G''_w = G^U_w \cap (W \otimes \bigwedge^2 U)$. We have embeddings

$$\begin{array}{ccccccc} \mathbb{P}(W) & \hookrightarrow & \mathrm{Gr}(6, \bigwedge^2 W \otimes U) & \quad & \mathbb{P}(W) & \hookrightarrow & \mathrm{Gr}(3, W \otimes \bigwedge^2 U) \\ {[w]} & \mapsto & G'_w & & {[w]} & \mapsto & G''_w \end{array}$$

They define $GL(W) \times GL(U)$-equivariant surjections

$$\bigwedge^6(\bigwedge^2 W^\vee \otimes U^\vee) = H^0(\mathcal{O}_{\mathrm{Gr}(6,\bigwedge^2 W \otimes U)}(1)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}(W)}(3)) \otimes (\det W)^{-3} \otimes (\det U)^{-2} = \mathrm{S}^3 W^\vee \otimes (\det W)^{-3} \otimes (\det U)^{-2}. \tag{3.2.24}$$

and

$$\bigwedge^3(W^\vee \otimes \bigwedge^2 U^\vee) = H^0(\mathcal{O}_{\mathrm{Gr}(3,W\otimes\bigwedge^2 U)}(1)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}(W)}(3)) \otimes (\det U)^{-2} = \mathrm{S}^3 W^\vee \otimes (\det U)^{-2}. \tag{3.2.25}$$

It follows from the definitions that Φ is identified with the composition of the following $GL(W) \times GL(U)$-equivariant maps

$$\begin{aligned} &\bigwedge^9 \mathcal{E}^U_W \xrightarrow{\sim} \bigwedge^9 (\mathcal{E}^U)^\vee_W \otimes (\det W)^9 \otimes (\det U)^9 \twoheadrightarrow \\ &\twoheadrightarrow (\mathrm{S}^3 W^\vee \otimes (\det W)^{-3} \otimes (\det U)^{-2}) \otimes (\mathrm{S}^3 W^\vee \otimes (\det U)^{-2}) \otimes (\det W)^9 \otimes (\det U)^9 \twoheadrightarrow \mathrm{S}^6 W^\vee \otimes (\det W)^6 \otimes (\det U)^5. \end{aligned} \tag{3.2.26}$$

(We get the first surjection by writing the exterior power of a direct-sum as direct-sum of tensors products of exterior powers, the second surjection follows from (3.2.24) and (3.2.25), the last surjection is defined by multiplication of polynomials.) We have

$$C_{W,A} = V(\Phi(\omega_0)), \qquad 0 \neq \omega_0 \in \bigwedge^9 B. \tag{3.2.27}$$

Claim 3.2.4. *Let $(W, A) \in \widetilde{\Sigma}$ and $\omega \in \bigwedge^{10} A$. Suppose that there exist a direct-sum decomposition $V = W \oplus U$ and $g = (g_W, g_U) \in (GL(W) \times GL(U)) \cap \mathrm{SL}(V)$ such that $g\omega = \omega$. Let $\overline{g}_W := (\det g_W)^{-1/3} g_W$ - thus $\overline{g}_W \in \mathrm{SL}(W)$. Write $C_{W,A} = V(P)$ where $P \in \mathrm{S}^6 W^\vee$; then $\overline{g}_W P = P$.*

Proof. The statement is equivalent to $g_W(P) = (\det g_W)^{-2} P$. Write $A = \bigwedge^3 W \oplus B$ where $B \in \mathbb{LG}(\mathcal{E}^U_W)$. Then $\omega = \alpha \wedge \omega_0$ where $\alpha \in \bigwedge^3 W$ and $\omega_0 \in \bigwedge^9 B$. We have $g\omega_0 = (\det g_W)^{-1}\omega_0$ because $g\omega = \omega$. The claim follows from (3.2.27) and the $GL(W) \times GL(U)$-equivariance of Φ - see (3.2.26). □

3.3 Non-stable strata and plane sextics, I

In the present subsection we will prove the following result.

Proposition 3.3.1. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that it belongs to*

$$\mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_1} \cup \mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{E}^\vee_1} \cup \mathbb{B}_{\mathcal{E}^\vee_2} \cup \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{F}_2}. \tag{3.3.1}$$

Then there exists $W \in \Theta_A$ such that $C_{W,A}$ is not a curve with simple singularities, more precisely either $C_{W,A} = \mathbb{P}(W)$ or else $C_{W,A}$ is a sextic curve and

(1) there exists $[v_0] \in C_{W,A}$ such that $\mathrm{mult}_{[v_0]} C_{W,A} \geq 4$ if $A \in \mathbb{B}_{\mathcal{A}}$,

(2) $C_{W,A}$ is singular along a line (and hence non-reduced) if $A \in (\mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{E}^\vee_2} \cup \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{F}_2})$,

(3) $C_{W,A}$ is singular along a conic (and hence non-reduced) if $A \in \mathbb{B}_{\mathcal{E}^\vee_1}$,

(4) $C_{W,A}$ is singular along a cubic (and hence equal to a double cubic)) if $A \in (\mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_1})$.

The proof will be given at the end of the subsection. First we will identify the bad points of $C_{W,A}$ for $(W,A) \in \widetilde{\Sigma}$. Let $[v_0] \in \mathbb{P}(W)$ and $W_0 \subset W$ be a subspace complementary to $[v_0]$. We choose $V_0 \in \mathrm{Gr}(5,V)$ such that

$$V = [v_0] \oplus V_0, \quad V_0 \cap W = W_0. \tag{3.3.2}$$

We have an isomorphism

$$\begin{array}{ccc} \bigwedge^2 V_0 / \bigwedge^2 W_0 & \xrightarrow{\sim} & G_{v_0} \\ \overline{\beta} & \mapsto & \overline{v_0 \wedge \beta}. \end{array} \tag{3.3.3}$$

Let $\psi_w^{v_0}$ be as in (3.2.9): we will view it as a quadratic form on $\bigwedge^2 V_0/\bigwedge^2 W_0$ via Isomorphism (3.3.3). Let $V(\psi_w^{v_0}) \subset \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0)$ be the zero-locus of $\psi_w^{v_0}$. **Proposition 3.2.2** suggests that in order to determine the local form of $C_{W,A}$ at $[v_0]$ we should examine the intersection of the $V(\psi_w^{v_0})$ for $w \in W_0$. Let

$$\widetilde{\mu}\colon \mathbb{P}(\bigwedge^2 V_0) \dashrightarrow \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0) \tag{3.3.4}$$

be projection with center $\bigwedge^2 W_0$. Let

$$\mathbb{G}r(2,V_0)_{W_0} := \widetilde{\mu}(\mathbb{G}r(2,V_0)). \tag{3.3.5}$$

(The right-hand side is to be interpreted as the closure of $\widetilde{\mu}(\mathbb{G}r(2,V_0) \setminus \{\bigwedge^2 W_0\})$.) Let μ be the restriction of $\widetilde{\mu}$ to $\mathbb{G}r(2,V_0)$. The rational map

$$\mu\colon \mathbb{G}r(2,V_0) \dashrightarrow \mathbb{G}r(2,V_0)_{W_0} \tag{3.3.6}$$

is birational because $\mathbb{G}r(2,V_0)$ is cut out by quadrics. We have

$$\dim \mathbb{G}r(2,V_0)_{W_0} = 6, \qquad \deg \mathbb{G}r(2,V_0)_{W_0} = 4. \tag{3.3.7}$$

Claim 3.3.2. *Keep notation as above. Then*

$$\bigcap_{w \in W_0} V(\psi_w^{v_0}) = \mathbb{G}r(2,V_0)_{W_0} \tag{3.3.8}$$

and the scheme-theoretic intersection on the left is reduced.

Proof. For $v_0, v \in V$ let $\phi_v^{v_0}$ be the Plücker quadratic form on F_{v_0} defined as follows. Let $\alpha \in F_{v_0}$; then $\alpha = v_0 \wedge \beta$ for some $\beta \in \bigwedge^2 V$. We set

$$\phi_v^{v_0}(\alpha) := \mathrm{vol}(v_0 \wedge v \wedge \beta \wedge \beta). \tag{3.3.9}$$

(The above equation gives a well-defined quadratic form on F_{v_0} because β is determined up to addition by an element of F_{v_0}.) Let

$$\begin{array}{cccc} \lambda_{V_0}^{v_0}\colon & \bigwedge^2 V_0 & \xrightarrow{\sim} & F_{v_0} \\ & \beta & \mapsto & v_0 \wedge \beta \end{array} \tag{3.3.10}$$

Now let $[v_0] \in \mathbb{P}(W)$ be as above; we will identify $\bigwedge^2 V_0$ and F_{v_0} via (3.3.10). If $w \in W_0$ then $V(\phi_w^{v_0}) \subset \mathbb{P}(F_{v_0}) = \mathbb{P}(\bigwedge^2 V_0)$ is a Plücker quadric containing $\mathbb{G}r(2,V_0)$ and singular at $\bigwedge^2 W_0$. The quadric $V(\psi_w^{v_0})$ is the projection of $V(\phi_w^{v_0})$ and hence it contains $\mathbb{G}r_{W_0}(2,V_0)$. Thus the left-hand side of (3.3.8) contains the right-hand side of (3.3.8). Since $V(\psi_w^{v_0})$ is an irreducible quadric for every $w \in W_0$ the left-hand side of (3.3.8) is of pure dimension 6, Cohen-Macaulay and of degree 4; thus the claim follows from (3.3.7). □

Next we will identify the points $[w] \in \mathbb{P}(W)$ such that $C_{W,A}$ is not as nice as possible - see **Proposition 3.3.6**. First we give a few definitions. Given a subspace $W \subset V$ we let

$$S_W := (\bigwedge^2 W) \wedge V. \tag{3.3.11}$$

Now suppose that $W \in \mathrm{Gr}(3,V)$; then $S_W \in \mathbb{LG}(\bigwedge^3 V)$ and $\mathbb{P}(S_W) \subset \mathbb{P}(\bigwedge^3 V)$ is the projective space tangent to $\mathrm{Gr}(3,V)$ at W.

Definition 3.3.3. Let $(W, A) \in \widetilde{\Sigma}$. We let $\mathcal{B}(W, A) \subset \mathbb{P}(W)$ be the set of $[w]$ such that

(1) there exists $W' \in (\Theta_A \setminus \{W\})$ with $[w] \in W'$, or

(2) $\dim(A \cap F_w \cap S_W) \geq 2$.

Remark 3.3.4. As is easily checked $\mathcal{B}(W, A)$ is a closed subset of $\mathbb{P}(W)$.

Let

$$\rho_{V_0}^{v_0} \colon F_{v_0} \xrightarrow{\sim} \bigwedge^2 V_0 \tag{3.3.12}$$

be the inverse of (3.3.10). Now let $[v_0] \in \mathbb{P}(W)$ be as above and let

$$K := \rho_{V_0}^{v_0}(A \cap F_{v_0}). \tag{3.3.13}$$

Then $K \supset \bigwedge^2 W_0$ and hence

$$\mathbb{P}(K/\bigwedge^2 W_0) \subset \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0). \tag{3.3.14}$$

Claim 3.3.5. *Keep notation as above. Then* $[v_0] \in \mathcal{B}(W, A)$ *if and only if*

$$\mathbb{P}(K/\bigwedge^2 W_0) \cap \mathbb{G}r(2, V_0)_{W_0} \neq \emptyset. \tag{3.3.15}$$

(The intersection above makes sense by (3.3.14).)

Proof. Let's prove that $[v_0] \in \mathcal{B}(W, A)$ if and only if

(a) $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is not equal to the singleton $\{\bigwedge^2 W_0\}$, or

(b) $\mathbb{P}(K) \cap \Theta_{\bigwedge^2 W_0}\mathrm{Gr}(2, V_0)$ is not equal to the singleton $\{\bigwedge^2 W_0\}$.

(Here $\Theta_{\bigwedge^2 W_0}\mathrm{Gr}(2, V_0) \subset \mathbb{P}(\bigwedge^2 V_0)$ is the projective tangent space to $\mathrm{Gr}(2, V_0)$ at $\bigwedge^2 W_0$.) In fact (a) holds if and only if Item (1) of **Definition 3.3.3** holds with $w = v_0$. On the other hand (b) holds if and only if Item (2) of **Definition 3.3.3** holds (with $w = v_0$) because

$$\Theta_{\bigwedge^2 W_0}\mathrm{Gr}(2, V_0) = \mathbb{P}(\rho_{V_0}^{v_0}(F_{v_0} \cap S_W)). \tag{3.3.16}$$

This proves that $[v_0] \in \mathcal{B}(W, A)$ if and only if one of Items (a), (b) above holds. Since $\mathbb{G}r(2, V_0)_{W_0}$ is obtained by projecting $\mathbb{G}r(2, V_0)$ from $\bigwedge^2 W_0$ the claim follows. ☐

Proposition 3.3.6. *Let* $(W, A) \in \widetilde{\Sigma}$ *and* $[v_0] \in \mathbb{P}(W)$. *Then* $[v_0] \notin \mathcal{B}(W, A)$ *if and only if one of the following holds:*

(1) $\dim(F_{v_0} \cap A) = 1$ *i.e.* $[v_0] \notin C_{W,A}$ *by* (3.1.1),

(2) $\dim(F_{v_0} \cap A) = 2$ *and* $C_{W,A}$ *is a smooth curve at* $[v_0]$,

(3) $\dim(F_{v_0} \cap A) = 3$ *and* $C_{W,A}$ *is a curve with an ordinary node at* $[v_0]$.

Proof. Suppose that $[v_0] \notin \mathcal{B}(W, A)$ - we will prove that one of Items (1), (2), (3) holds. First let's show that

$$\dim(F_{v_0} \cap A) \leq 3. \tag{3.3.17}$$

Let $K := \rho_{V_0}^{v_0}(F_{v_0} \cap A)$. Assume that (3.3.17) does not hold, i.e. that $\dim \mathbb{P}(K) \geq 3$. Since $\dim \mathrm{Gr}(2, V_0) = 6$ we get that

(α) $\dim(\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)) > 0$, or

(β) $\dim \mathbb{P}(K) = 3$ and the intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is zero-dimensional.

If (α) holds then $\mathbb{P}(K)\cap \mathrm{Gr}(2,V_0)$ is not equal to the singleton $\bigwedge^2 W_0$ and hence $[v_0]\in \mathcal{B}(W,A)$, contradiction. Now suppose that (β) holds. Suppose first that $\mathbb{P}(K)$ is transverse to $\mathrm{Gr}(2,V_0)$ at $\bigwedge^2 W_0$; then $\mathbb{P}(K)\cap \mathrm{Gr}(2,V_0)$ is not equal to the singleton $\bigwedge^2 W_0$ because $\deg \mathrm{Gr}(2,V_0)=5$ and hence $[v_0]\in \mathcal{B}(W,A)$, contradiction. If $\mathbb{P}(K)$ is not transverse to $\mathrm{Gr}(2,V_0)$ at $\bigwedge^2 W_0$ then $[v_0]\in \mathcal{B}(W,A)$ by **Claim 3.3.5** - again we get a contradiction. This proves that (3.3.17) holds. If $\dim(F_{v_0}\cap A)=1$ there is nothing to prove. If $\dim(F_{v_0}\cap A)=2$ then by **Claim 3.3.5** we get that $\mathbb{P}(K/\bigwedge^2 W_0)$ is a point not contained in $\mathrm{Gr}(2,V_0)_{W_0}$. By **Proposition 3.2.2** and (3.3.8) we get that $C_{W,A}$ is a smooth curve at $[v_0]$. Lastly suppose that $\dim(F_{v_0}\cap A)=3$. By **Claim 3.3.5** we get that $\mathbb{P}(K/\bigwedge^2 W_0)$ is a line that does not intersect $\mathrm{Gr}(2,V_0)_{W_0}$. By **Proposition 3.2.2** and (3.3.8) we get that $C_{W,A}$ is a curve with a node at $[v_0]$. This proves that if $[v_0]\notin \mathcal{B}(W,A)$ then one of Items (1), (2), (3) holds. One verifies easily that the converse holds; we leave details to the reader. □

Corollary 3.3.7. *Let $(W,A)\in\widetilde{\Sigma}(V)$. Then $C_{W,A}=\mathbb{P}(W)$ if and only if $\mathcal{B}(W,A)=\mathbb{P}(W)$. If $C_{W,A}\neq\mathbb{P}(W)$ then $\mathcal{B}(W,A)\subset \operatorname{sing} C_{W,A}$.*

Proof. If $\mathcal{B}(W,A)=\mathbb{P}(W)$ then $\dim(A\cap F_w)\geq 2$ for all $[w]\in\mathbb{P}(W)$ and hence $C_{W,A}=\mathbb{P}(W)$ by (3.1.1). If $C_{W,A}=\mathbb{P}(W)$ then $\mathcal{B}(W,A)=\mathbb{P}(W)$ by **Proposition 3.3.6**. The second statement follows at once from **Corollary 3.2.3** and **Proposition 3.3.6**. □

Given $W\in\mathrm{Gr}(3,V)$ we let

$$T_W := S_W/\bigwedge^3 W \cong \bigwedge^2 W\otimes (V/W)\cong \mathrm{Hom}(W,V/W). \tag{3.3.18}$$

(Recall (3.3.11).) Of course the second isomorphism is not canonical, it depends (up to multiplication by a scalar) on the choice of a volume form on W.

Claim 3.3.8. *Let $(W,A)\in\widetilde{\Sigma}$ and suppose that $C_{W,A}\neq\mathbb{P}(W)$. Let $[w]\in\mathbb{P}(W)$. If there exists $\alpha\in(A\cap S_W)$ such that*

(1) the equivalence class $\overline{\alpha}\in T_W$ is non-zero and

(2) $\overline{\alpha}(w)=0$ (we view $\overline{\alpha}$ as an element of $\mathrm{Hom}(W,V/W)$ thanks to (3.3.18))

then $[w]\in\operatorname{sing} C_{W,A}$.

Proof. We have $\overline{\alpha}(w)=0$ if and only if $\alpha\in S_W\cap F_w$; thus Item (2) of **Definition 3.3.3** holds and the claim follows from **Corollary 3.3.7**. □

Proof of **Proposition 3.3.1**. We may assume throughout that $C_{W,A}\neq\mathbb{P}(W)$. First we will consider

$$A\in(\mathbb{B}_{\mathcal{A}^\vee}\cup\mathbb{B}_{\mathcal{C}_2}\cup\mathbb{B}_{\mathcal{E}_2^\vee}\cup\mathbb{B}_{\mathcal{F}_1}). \tag{3.3.19}$$

By Section 2.3 of [28] we know the following:

(1) If $A\in\mathbb{B}_{\mathcal{F}_1}$ is generic then Θ_A is a line.

(2) If $A\in\mathbb{B}_{\mathcal{E}_2^\vee}$ is generic then Θ_A is a rational normal cubic and the ruled 3-fold swept out by $\mathbb{P}(W)$ for $W\in\Theta_A$ lies in a hyperplane of $\mathbb{P}(V)$.

(3) If $A\in\mathbb{B}_{\mathcal{A}^\vee}$ is generic then Θ_A is a projectively normal quintic elliptic curve and the ruled 3-fold swept out by $\mathbb{P}(W)$ for $W\in\Theta_A$ lies in a hyperplane of $\mathbb{P}(V)$.

(4) If $A\in\mathbb{B}_{\mathcal{C}_2}$ is generic then Θ_A is a projectively normal sextic elliptic curve and there exists a plane $\mathbb{P}(U)\subset\mathbb{P}(V)$ intersecting along a line each plane $\mathbb{P}(W)$ for $W\in\Theta_A$.

Suppose that (1) holds and let $W \in \Theta_A$. Let $W' \in (\Theta_A \setminus \{W\})$; then $\mathbb{P}(W \cap W')$ is a line. By **Corollary 3.3.7** $C_{W,A}$ is singular along $\mathbb{P}(W \cap W')$. Now suppose that one of Items (2), (3) or (4) holds. Let $W \in \Theta_A$ and

$$C := \bigcup_{W' \in (\Theta_A \setminus \{W\})} \mathbb{P}(W \cap W').$$

If $A \in \mathbb{B}_{\mathcal{A}^\vee}$ is generic then C is a cubic curve, this is easily checked. We claim that if $A \in (\mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee})$ is generic then C is a line. The fact is that in both cases there exists $U \in \mathrm{Gr}(3, V)$ such that $\dim(W' \cap U) = 2$ for all $W' \in \Theta_A$ and hence $C = \mathbb{P}(W \cap U)$. Existence of such a U for A generic in $\mathbb{B}_{\mathcal{C}_2}$ was stated in Item (4) above. Let's prove that such a U exists for A generic in $\mathbb{B}_{\mathcal{E}_2^\vee}$. Write $V = \mathrm{S}^2 L$ where L is a complex vector-space of dimension 3. We have embeddings

$$\begin{array}{ccccccc} \mathbb{P}(L) & \overset{k}{\hookrightarrow} & \mathrm{Gr}(3, \mathrm{S}^2 L) & & \mathbb{P}(L^\vee) & \overset{h}{\hookrightarrow} & \mathrm{Gr}(3, \mathrm{S}^2 L) \\ [l_0] & \mapsto & \{l_0 \cdot l \mid l \in L\} & & [f_0] & \mapsto & \{q \mid f_0 \in kerq\}. \end{array} \tag{3.3.20}$$

The maps k and h have the following geometric interpretation. Let $\mathcal{V}_1 \subset \mathbb{P}(\mathrm{S}^2 L)$ be the subset of tensors of rank 1 (modulo scalars) i.e. the degree-4 Veronese surface: then

$$\operatorname{im} k = \{\mathbf{T}_{[\ell_0^2]}\mathcal{V}_1 \mid [\ell_0^2] \in \mathcal{V}_1\}, \qquad \operatorname{im} h = \{\langle C \rangle \mid C \subset \mathcal{V}_1 \text{ a conic }\} \tag{3.3.21}$$

i.e. $\operatorname{im} k$ is the set of projective tangent spaces to points of $\mathcal{V}_1$ and $\operatorname{im} h$ is the set of planes spanned by conics on $\mathcal{V}_1$. Let $\mathcal{L}$ be the Plücker(ample) line-bundle on $\mathrm{Gr}(3, \mathrm{S}^2 L)$; one checks easily that

$$k^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(L)}(3), \qquad h^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(L^\vee)}(3) \tag{3.3.22}$$

and that $H^0(k^*)$, $H^0(h^*)$ are surjective. Let $R := \mathbb{P}(\ker f)$ where $[f] \in \mathbb{P}(L^\vee)$. Then $k(R) \subset \mathrm{Gr}(3, \mathrm{S}^2 L)$ is a rational normal cubic curve. Since the union of projective planes parametrized by $k(R)$ is contained in the hyperplane

$$\{[\varphi] \in \mathbb{P}(\mathrm{S}^2 L) \mid \langle \varphi, f^2 \rangle = 0\}$$

it is actually projectively equivalent to Θ_A, see Proposition 2.12 of [28]. Let

$$U' := \{[\varphi] \in \mathbb{P}(\mathrm{S}^2 L) \mid f \in \ker \varphi\}$$

Then $\dim(U' \cap W') = 2$ for all $W' \in k(R)$; since $k(R)$ is projectively equivalent to Θ_A it follows that there exists $U \in \mathrm{Gr}(3, V)$ such that $\dim(W' \cap U) = 2$ for all $W' \in \Theta_A$ as claimed. Now let's consider

$$A \in (\mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{C}_1} \cup \mathbb{B}_{\mathcal{E}_1^\vee} \cup \mathbb{B}_{\mathcal{F}_2}). \tag{3.3.23}$$

We may assume that A is generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ for $\mathcal{X} = \mathcal{A}, \ldots, \mathcal{F}_2$ where F is a basis of V given by (2.4.1). Consider first $\mathbb{B}^{\mathsf{F}}_{\mathcal{A}}$. By Table (1) we have

$$\dim(A \cap [v_0] \wedge \bigwedge^2 V_{15}) \geq 5. \tag{3.3.24}$$

We have a natural embedding $\mathrm{Gr}(2, V_{15}) \hookrightarrow \mathbb{P}([v_0] \wedge \bigwedge^2 V_{15})$ with image of codimension 3; by (3.3.24) it follows that there exists $W \in \Theta_A$ containing v_0 (actually a family of dimension at least 1). By **Corollary 3.2.3** and (3.3.24) we get that $\mathrm{mult}_{[v_0]} C_{W,A} \geq 4$. Now consider one of $\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1}$ or $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$. Then Θ_A contains $W := V_{02}$. Let $\overline{A} := A / \bigwedge^3 W$ and T_W be as in (3.3.18). We notice that the inequality which enters into the definition of $\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1}$ or $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ gives that

$$\{[w] \in \mathbb{P}(W) \mid \exists\, 0 \neq \overline{\alpha} \in (T_W \cap \overline{A}) \text{ s.t. } \overline{\alpha}(w) = 0\} \tag{3.3.25}$$

has dimension at least 1, in fact it contains a cubic curve in the case of $\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1}$ and it contains a conic in the case of $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$. This settles the case of $A \in (\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee})$. Lastly we consider $\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$. By the first inequality defining $\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$ we get that there exists $0 \neq u \in V_{23}$ such that $W := \langle v_0, v_1, u\rangle \in \Theta_A$. We claim that (3.3.25) has dimension at least 1. Let $v \in V_{23}$ be such that $\{u, v\}$ is a basis of V_{23}. Let

$$\alpha \in (\bigwedge^2 V_{01} \wedge V_{23} \oplus \bigwedge^2 V_{01} \wedge V_{45} \oplus V_{01} \wedge \bigwedge^2 V_{23}).$$

Then $\overline{\alpha}(v_0), \overline{\alpha}(v_1) \subset [\overline{v}]$ where $\overline{v} \in V/W$ is the class of v; in particular $\overline{\alpha}$ has non-trivial kernel. By the second inequality defining $\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$ we get that (3.3.25) has dimension at least 1, in fact it contains a line. This concludes the proof. □

3.4 Non-stable strata and plane sextics, II

In the present subsection we will prove the following result.

Proposition 3.4.1. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that it belongs to*

$$\mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_1} \cup \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{X}_{\mathcal{N}_3}. \tag{3.4.1}$$

Then there exists $W \in \Theta_A$ such that $C_{W,A}$ is not a curve with simple singularities; more precisely the following hold:

(1) If $A \in \mathbb{B}_{\mathcal{D}}$ or $A \in \mathbb{B}_{\mathcal{E}_1}$ then either $C_{W,A} = \mathbb{P}(W)$ or else $C_{W,A}$ has a point of multiplicity at least 4.

(2) If A is generic in $\mathbb{B}_{\mathcal{E}_2}$ or in $\mathbb{X}_{\mathcal{N}_3}$ then $C_{W,A}$ has consecutive triple points.

We will prove **Proposition 3.4.1** at the end of the subsection: first we will go through some preliminaries. We start out by giving a "classical" description of $C_{W,A}$ in a neighborhood of $[v_0]$ for $(W, A) \in \widetilde{\Sigma}$ and $[v_0] \in \mathbb{P}(W)$. For this we will suppose that there exists $V_0 \in \mathrm{Gr}(5, V)$ such that

$$v_0 \notin V_0, \qquad \bigwedge^3 V_0 \pitchfork A. \tag{3.4.2}$$

By (1.3.2) the second requirement (transversality) is equivalent to $Y_{\delta_V(A)} \neq \mathbb{P}(V^\vee)$. Let D be the direct-sum decomposition

$$V = [v_0] \oplus V_0. \tag{3.4.3}$$

Under the above hypothesis there is a "classical" description of Y_A in a neighborhood of $[v_0]$ as the discriminant hypersurface of a linear system of quadrics - see Section 1.7 of [28] - that goes as follows. We have a quadratic form $q_A = q^{\mathsf{D}}_A(0) \in \mathrm{S}^2(\bigwedge^2 V_0)^\vee$ characterized as follows:

$$\widetilde{q}_A(\alpha) = \gamma \iff (v_0 \wedge \alpha - \gamma) \in A. \tag{3.4.4}$$

Here $\widetilde{q}_A \colon \bigwedge^2 V_0 \to \bigwedge^2 V_0^\vee$ is the symmetric map associated to q_A and we make the identification

$$\begin{array}{ccc} \bigwedge^3 V_0 & \xrightarrow{\sim} & \bigwedge^2 V_0^\vee \\ \gamma & \mapsto & \alpha \mapsto \mathrm{vol}(v_0 \wedge \alpha \wedge \gamma). \end{array} \tag{3.4.5}$$

For $v \in V$ let Let $q_v \in \mathrm{S}^2(\bigwedge^2 V_0)^\vee$ be the Plücker quadratic form defined by

$$q_v(\alpha) := \mathrm{vol}(v_0 \wedge v \wedge \alpha \wedge \alpha). \tag{3.4.6}$$

Notice that (via the obvious identification) $q_v = \phi^{v_0}_{V_0}$ where $\phi^{v_0}_{V_0}$ is defined by (3.3.9). Lastly we make the identification

$$\begin{array}{ccc} V_0 & \xrightarrow{\sim} & \mathbb{P}(V) \setminus \mathbb{P}(V_0) \\ v & \mapsto & [v_0 + v]. \end{array} \tag{3.4.7}$$

(Thus $0 \in V_0$ corresponds to $[v_0]$.) By [28] we have the following local description of Y_A:

$$Y_A \cap V_0 = V(\det(q_A + q_v)). \tag{3.4.8}$$

Now suppose that $v_0 \in W$ and let $W_0 := W \cap V_0$; there is a similar description of $C_{W,A} \cap (\mathbb{P}(W) \setminus \mathbb{P}(W_0))$ which goes as follows. First notice that the restriction of (3.4.7) to W_0 may be identified with (3.2.7). Next notice that $\bigwedge^2 W_0$ is in the kernel of q_A and also in the kernel of q_w for $w \in W_0$. Let

$$\overline{q}_A, \overline{q}_w \in \mathrm{S}^2(\bigwedge^2 V_0/\bigwedge^2 W_0)^{\vee}, \qquad w \in W_0 \tag{3.4.9}$$

be the induced quadratic forms. Below is our "classical"description of $C_{W,A}$ near $[v_0]$.

Claim 3.4.2. *Keep hypotheses and notation as above - in particular assume that* (3.4.2) *holds. Then*

$$C_{W,A} \cap (\mathbb{P}(W) \setminus \mathbb{P}(W_0)) = V(\det(\overline{q}_A + \overline{q}_w)) \tag{3.4.10}$$

where $w \in W_0$ *- see* (3.2.7).

Proof. We have an isomorphism

$$\begin{array}{ccc} \ker(q_A + q_w) & \xrightarrow{\sim} & A \cap F_{v_0+w} \\ \alpha & \mapsto & (v_0 + w) \wedge \alpha \end{array} \tag{3.4.11}$$

The set-theoretic equality of the two sides of (3.4.10) follows at once from (3.1.1) and (3.4.11). In order to prove scheme-theoretic equality one may describe $C_{W,A} \cap (\mathbb{P}(W) \setminus \mathbb{P}(W_0))$ as the degeneracy locus of a family of symmetric maps parametrized by W_0 as follows. Let $U \subset V$ be complementary to W. We have a natural identification

$$(\bigwedge^2 W) \wedge U \oplus W \wedge (\bigwedge^2 U) \xrightarrow{\sim} \mathcal{E}_W. \tag{3.4.12}$$

Given the above identification we have a direct-sum decomposition into Lagrangian subspaces

$$\mathcal{E}_W = ([v_0] \wedge W_0 \wedge U \oplus [v_0] \wedge (\bigwedge^2 U)) \oplus ((\bigwedge^2 W_0) \wedge U \oplus W_0 \wedge (\bigwedge^2 U)). \tag{3.4.13}$$

(The first and second summand are the intersections of the left-hand side of (3.4.12) and F_{v_0} and $\bigwedge^3 V_0$ respectively.) Given the above decomposition the scheme $C_{W,A} \cap (\mathbb{P}(W) \setminus \mathbb{P}(W_0))$ is described as the degeneracy locus of a family of quadratic forms. One identifies the family of quadratic forms with $\{(\overline{q}_A + \overline{q}_w)\}_{w \in W_0}$ and the claim follows. □

Remark 3.4.3. Let $\mathbb{G}r(2, V_0)_{W_0} \subset \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0)$ be the projection of $\mathbb{G}r(2, V_0)$ from $\bigwedge^2 W_0$ - see (3.3.5). Let

$$Z_{W_0,A} := V(\overline{q}_A) \cap \mathbb{G}r(2, V_0)_{W_0} \subset \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0). \tag{3.4.14}$$

As w varies in W_0 the quadrics $V(\overline{q}_A + \overline{q}_w)$ vary in an open affine neighborhood of $V(\overline{q}_A)$ in $|\mathcal{I}_{Z_{W_0,A}}(2)|$ - see **Claim 3.3.2**. Thus the singularity of $C_{W,A}$ at $[v_0]$ is determined by $Z_{W_0,A}$.

Proof of **Proposition 3.4.1**. First we will prove the statement of the proposition for $A \in \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_1}$. We may suppose that $C_{W,A} \neq \mathbb{P}(W)$. We may assume that there is a basis $\mathsf{F} = \{v_0, \ldots, v_5\}$ of V such that A is generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{D}}$ or in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}$ and hence one of the following holds:

(1) $\dim A \cap ([v_0] \wedge \bigwedge^2 V_{14}) = 3$ and Θ_A is a smooth conic parametrizing planes containing $[v_0]$, see Section 2.3 of [28].

(2) $A \supset [v_0] \wedge \bigwedge^2 V_{12}$ and $\dim A \cap ([v_0] \wedge V_{12} \wedge V_{35}) = 2$.

If (1) holds let W be an arbitrary element of Θ_A, if (2) holds let $W := V_{02}$. We will prove that $C_{W,A}$ has multiplicity at least 4 at $[v_0]$. Notice that in both cases

$$\dim A \cap F_{v_0} \geq 3. \tag{3.4.15}$$

Since A is generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{D}}$ or in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}$ we may assume that (3.4.15) is an equality. Thus $\mathrm{mult}_{[v_0]} C_{W,A} \geq 2$ by **Corollary 3.2.3**: that is not good enough. We will apply **Claim 3.4.2**. First we must make sure that there exists $V_0 \in \mathrm{Gr}(5, V)$ for which (3.4.2) holds. As is easily checked V_{15} will do for A generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{D}}$ or in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}$. Next we notice that the line $\mathbb{P}(\ker \overline{q}_A)$ is contained in $\mathrm{Gr}(2, V_0)_{W_0}$ (notice that $W_0 = V_{12}$ if Item (2) holds). In fact if (1) holds the projection $\mu\colon \mathrm{Gr}(2, V_0) \dashrightarrow \mathrm{Gr}(2, V_0)_{W_0}$ maps the conic $\rho^{v_0}_{V_0}(\Theta_A)$ to $\mathbb{P}(\ker \overline{q}_A)$. If (2) holds the plane $\mathbb{P}(\rho^{v_0}_{V_0}(A \cap F_{v_0}))$ is tangent to $\mathrm{Gr}(2, V_0)$ at V_{12} and hence is mapped by μ to $\mathrm{Gr}(2, V_0)_{W_0}$; on the other hand the image by μ is exactly $\mathbb{P}(\ker \overline{q}_A)$. Since the line $\mathbb{P}(\ker \overline{q}_A)$ is contained in $\mathrm{Gr}(2, V_0)_{W_0}$ every $\overline{q}_w$ (for $w \in W_0$) vanishes on $\mathbb{P}(\ker \overline{q}_A)$ by **Claim 3.3.2**; by **Corollary 3.2.3** and **Proposition A.1.2** we get that $\mathrm{mult}_{[v_0]} C_{W,A} \geq 4$. Next we suppose that $A \in \mathbb{B}_{\mathcal{E}_2}$. Thus we may assume that A is generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$ where $\mathsf{F} = \{v_0, \ldots, v_5\}$ is a basis of V. By Proposition 2.20 of [28] we know that Θ_A is a rational normal cubic curve and that all planes parametrized by Θ_A contain $[v_0]$; as W we choose an arbitrary element of Θ_A. We will prove that $C_{W,A}$ has consecutive triple points at $[v_0]$; for the reader's convenience we notice that this holds if and only if there exists a basis $\{x, y\}$ of $W_0^\vee$ such that

$$C_{W,A} \cap W_0 = V(y^3 + c_{22}x^2y^2 + c_{13}xy^3 + c_{04}y^4 + c_{41}x^4y + c_{32}x^3y^2 + \ldots). \tag{3.4.16}$$

More precisely: the tangent cone to $C_{W,A}$ at $[v_0]$ is $V(y^3)$ and the coefficients of x^4, x^3y, x^5 (in the generator of the ideal of $C_{W,A} \cap W_0$) are zero. First we notice that (3.4.2) holds with $V_0 := V_{15}$ (if A is generic in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$) and hence we may apply **Claim 3.4.2**. By genericity of A in $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$ the inequality in the definition of $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$ is an equality; thus $\dim(\ker \overline{q}_A) = 3$. Moreover $\mathbb{P}(\ker \overline{q}_A) \cap \mathrm{Gr}(2, V_0)_{W_0}$ is a (smooth) conic C, namely the projection of $\rho^{v_0}_{V_0}(\Theta_A)$ from W_0. Let $\overline{K} := \ker \overline{q}_A$. By **Claim 3.3.2** the intersection with $\mathbb{P}(\overline{K})$ of the quadrics $V(\overline{q}_w)$ (for $w \in W_0$) equals C. Thus there exists $0 \neq w_1 \in W_0$ such that $\overline{q}_{w_1}|_{\overline{K}} = 0$. We complete $\{w_1\}$ to a basis $\{w_1, w_2\}$ of W_0; thus $V(\overline{q}_{w_2}) \cap \mathbb{P}(\overline{K}) = C$ and hence $\overline{q}_{w_2}|_{\overline{K}}$ is a non-degenerate quadratic form. In a suitable basis of $\bigwedge^2 V_0 / \bigwedge^2 W_0$ we have

$$\overline{q}_A + x\overline{q}_{w_1} + y\overline{q}_{w_2} = \begin{pmatrix} y & 0 & 0 & m_{1,4} & \cdots & m_{1,9} \\ 0 & y & 0 & m_{2,4} & \cdots & m_{2,9} \\ 0 & 0 & y & m_{3,4} & \cdots & m_{3,9} \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 + m_{4,4} & \cdots & m_{4,9} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{9,1} & m_{9,2} & m_{9,3} & m_{9,4} & \cdots & 1 + m_{9,9} \end{pmatrix} \tag{3.4.17}$$

where each $m_{i,j} \in \mathbb{C}[x, y]_1$ is homogeneous of degree 1. A straightforward computation gives that

$$\det(\overline{q}_A + x\overline{q}_{w_1} + y\overline{q}_{w_2}) = y^3 + c_{22}x^2y^2 + c_{13}xy^3 + c_{04}y^4 + c_{41}x^4y + c_{32}x^3y^2 + \ldots$$

and hence $C_{W,A}$ has consecutive triple points at $[v_0]$ - see (3.4.16). It remains to prove the statement of **Proposition 3.4.1** regarding $\mathbb{X}_{\mathcal{N}_3}$. We may assume that A is generic in $\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$ where $\mathsf{F} = \{v_0, v_1, \ldots, v_5\}$ is a basis of V. By genericity all the dimension inequalities defining $\mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$ are in fact equalities, in particular

$$\dim(A \cap F_{v_0}) = 3. \tag{3.4.18}$$

Moreover A contains

$$\begin{array}{l} v_0 \wedge v_1 \wedge (av_2 + bv_3) \\ v_0 \wedge (v_1 \wedge (cv_2 + dv_3) + v_1 \wedge v_4 + v_2 \wedge v_3) \\ v_0 \wedge (ev_1 \wedge v_4 + fv_2 \wedge v_3 + gv_1 \wedge v_5 + hv_2 \wedge v_4 + lv_3 \wedge v_4) \\ v_0 \wedge (e'v_1 \wedge v_4 + f'v_2 \wedge v_3 + g'v_1 \wedge v_5 + h'v_2 \wedge v_4 + l'v_3 \wedge v_4) + v_1 \wedge v_2 \wedge v_3. \end{array} \tag{3.4.19}$$

(We have rescaled some of the v_i's.) By genericity we also have

$$a \neq 0 \neq (ad - bc). \tag{3.4.20}$$

Define $v_2', v_4' \in V_{15}$ by

$$\begin{array}{rcl} v_2 & = & v_2' - a^{-1}bv_3, \\ v_4 & = & -cv_2' + (a^{-1}bc - d)v_3 + v_4'. \end{array}$$

Table 4: Matrix of $\overline{q}_w|_K$

	$\overline{\alpha}$	$\overline{\beta}$
$\overline{\alpha}$	0	0
$\overline{\beta}$	0	$2\,\mathrm{vol}(v_0 \wedge w \wedge w_1 \wedge w_1' \wedge w_2' \wedge u)$

Thus $\{v_0, v_1, v_2', v_3, v_4', v_5\}$ is a new basis of V. Replacing v_2 and v_4 by the above expressions we get that A contains

$$\begin{aligned} & v_0 \wedge v_1 \wedge v_2' \\ & v_0 \wedge (v_1 \wedge v_4' + v_2' \wedge v_3) \\ & v_0 \wedge (v_1 \wedge u + \omega) \\ & v_0 \wedge (v_1 \wedge x + \tau) + v_1 \wedge v_2' \wedge v_3 \end{aligned} \tag{3.4.21}$$

where $\omega, \tau \in \bigwedge^2 \langle v_2', v_3, v_4' \rangle$ and hence are decomposable. By genericity of A we have $v_2' \notin \operatorname{supp} \omega$; thus after a suitable rescaling of $v_0 \wedge (v_1 \wedge u + \omega)$ we may assume that

$$\omega = (sv_3 + v_4') \wedge (v_3 + tv_2')$$

where $s, t \in \mathbb{C}$. Let

$$w_1 := v_1, \quad w_2 := v_2' - sv_1, \quad w_1' := sv_3 + v_4', \quad w_2' := v_3 + tv_2'.$$

By genericity of A the span $\langle w_1, w_2, w_1', w_2' \rangle$ does not contain u; it follows that $\{v_0, w_1, w_2, w_1', w_2', u\}$ is yet another basis of V. Rewriting the elements of (3.4.21) in terms of the last basis we get that A contains

$$\begin{aligned} & v_0 \wedge w_1 \wedge w_2 \\ & v_0 \wedge (w_1 \wedge w_1' + w_2 \wedge w_2') \\ & v_0 \wedge (w_1 \wedge u + w_1' \wedge w_2') \\ & v_0 \wedge (w_1 \wedge \zeta + \xi) + w_1 \wedge w_2 \wedge w_2' \end{aligned} \tag{3.4.22}$$

where

$$\xi \in \bigwedge^2 \langle w_2, w_1', w_2' \rangle \tag{3.4.23}$$

(The last statement holds because $\tau \in \bigwedge^2 \langle v_2', v_3, v_4' \rangle$.) Let $W := \langle v_0, w_1, w_2 \rangle$; clearly $W \in \theta_A$. We will prove that $C_{W,A}$ has triple consecutive points at $[v_0]$. First notice that there exists $V_0 \in \mathrm{Gr}(5, V)$ such that (3.4.2) holds; in fact $V_0 := V_{15}$ will do (for generic $A \in \mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$). Thus we may appply **Claim 3.4.2**. Let $W_0 := W \cap V_0$ and $\{x, y\}$ be the basis of $W_0^{\vee}$ dual to $\{w_1, w_2\}$. By (3.4.18) we have $\dim(A \cap F_{v_0}) = 3$; thus **Corollary 3.2.3** gives that

$$C_{W,A} \cap W_0 = V(g_2 + g_3 + \ldots + g_6), \qquad g_d = \sum_{i+j=d} c_{ij} x^i y^j.$$

Let $K := \ker \overline{q}_A = \rho^{v_0}_{V_0}(A \cap F_{v_0}) / \bigwedge^2(W_0)$. Then $K = \langle \overline{\alpha}, \overline{\beta} \rangle$ where

$$\alpha := (w_1 \wedge w_1' + w_2 \wedge w_2'), \qquad \beta := (w_1 \wedge u + w_1' \wedge w_2'). \tag{3.4.24}$$

Let $w \in W_0$; the matrix of $\overline{q}_w|_K$ with respect to the basis given by (3.4.24) is given by Table (4). In particular $\overline{q}_w|_K$ is degenerate and hence $g_2 = 0$ by (3.2.10) and **Claim 3.3.2**. Let's prove that

$$g_3 = c_{03} y^3, \qquad c_{03} \neq 0. \tag{3.4.25}$$

The restriction $q_{w_1}|_K$ is zero and hence $g_3(w_1) = 0$ by **Proposition A.1.3**; thus in order to prove (3.4.25) it suffices to show that

$$g_3(x_0, y_0) \neq 0 \text{ if } y_0 \neq 0. \tag{3.4.26}$$

Table 5: Matrix of $\overline{q}_A^\vee$ restricted to $\widetilde{\overline{q}}_{w_1}(K)$

	$\widetilde{\overline{q}}_{w_1}(\overline{\alpha})$	$\widetilde{\overline{q}}_{w_1}(\overline{\beta})$
$\widetilde{\overline{q}}_{w_1}(\overline{\alpha})$	$\mathrm{vol}(v_0 \wedge (w_1 \wedge \zeta + \xi) \wedge w_1 \wedge w_2 \wedge w_2')$	$\mathrm{vol}(v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_2')$
$\widetilde{\overline{q}}_{w_1}(\overline{\beta})$	$\mathrm{vol}(v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_2')$	$\mathrm{vol}(v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_1')$

Let $w = (x_0 w_1 + y_0 w_2)$ with $y_0 \neq 0$; thus $\ker(\overline{q}_w|_K) = \langle (w_1 \wedge w_1' + w_2 \wedge w_2') \rangle$. The hypotheses of **Claim A.2.1** are satisfied by $q_* := \overline{q}_A$ and $q := \overline{q}_w$; it follows that $g_3(x_0, y_0) = 0$ if and only if

$$\overline{q}_A^\vee((x_0 w_1 + y_0 w_2) \wedge (w_1 \wedge w_1' + w_2 \wedge w_2')) = 0. \tag{3.4.27}$$

Of course here we are tacitly identifying $(\bigwedge^2 V_0 / \bigwedge^2 W_0)^\vee$ with $\mathrm{Ann}(\bigwedge^2 W_0) \subset \bigwedge^3 V_0$. In order to compute the left-hand side of (3.4.27) we notice that

$$\widetilde{\overline{q}}_A^{-1}(w_1 \wedge w_2 \wedge w_2') = -w_1 \wedge \zeta - \xi.$$

In fact the above equation follows from (3.4.4) and (3.4.22). Let

$$\widetilde{\overline{q}}_A^{-1}(w_1 \wedge w_2 \wedge w_1') = \overline{\gamma} \in \bigwedge^2 V_0 / \langle w_1 \wedge w_2, (w_1 \wedge w_2, w_1 \wedge w_1' + w_2 \wedge w_2'), (w_1 \wedge u + w_1' \wedge w_2') \rangle.$$

(Here $\gamma \in \bigwedge^2 V_0$.) Then - see (3.4.4) - we have

$$(v_0 \wedge \gamma - w_1 \wedge w_2 \wedge w_1') \in A.$$

We notice that we have

$$v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_2' = 0 \tag{3.4.28}$$

In fact the above equality holds because A is a lagrangian subspace containing the element on the fourth line of (3.4.22) and because (3.4.23) holds. From the above equations we get that

$$\overline{q}_A^\vee((x_0 w_1 + y_0 w_2) \wedge (w_1 \wedge w_1' + w_2 \wedge w_2')) = y_0^2 \,\mathrm{vol}(v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_1').$$

Since A is generic

$$v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_1' \neq 0 \tag{3.4.29}$$

and hence we get that (3.4.26) holds. We have proved (3.4.25). Next let's prove that $0 = c_{40} = c_{50}$ i.e.

$$g(xw_1, 0) \equiv 0 \pmod{x^6}. \tag{3.4.30}$$

First we apply **Proposition A.1.3** with $q_* := \overline{q}_A$ and $q := \overline{q}_{w_1}$. Let's show that $\overline{q}_A^\vee|_{\widetilde{\overline{q}}_{w_1}(K)}$ is degenerate. By definition the map $\widetilde{\overline{q}}_A$ defines an isometry between $\widetilde{\overline{q}}_A^{-1} \circ \widetilde{\overline{q}}_{w_1}(K)$ equipped with the restriction of $\overline{q}_A$ and $\widetilde{\overline{q}}_{w_1}(K)$ equipped with the restriction of $\overline{q}_A^\vee$. We have

$$\begin{aligned} \widetilde{\overline{q}}_A^{-1}(\widetilde{\overline{q}}_{w_1}(\overline{\alpha})) &= \widetilde{\overline{q}}_A^{-1}(w_1 \wedge w_2 \wedge w_2') = -w_1 \wedge \zeta - \xi, \\ \widetilde{\overline{q}}_A^{-1}(\widetilde{\overline{q}}_{w_1}(\overline{\beta})) &= \widetilde{\overline{q}}_A^{-1}(-w_1 \wedge w_2 \wedge w_1') = -\gamma. \end{aligned}$$

From this it follows that the restriction of $\overline{q}_A^\vee$ to $\widetilde{\overline{q}}_{w_1}(K)$ is given by Table (5). By (3.4.23) and (3.4.28) the entries vanish with the exception of the one on the second line and second column. Thus $\overline{q}_A^\vee|_{\widetilde{\overline{q}}_{w_1}(K)}$ is degenerate and hence $g(xw_1, 0) \equiv 0 \pmod{x^5}$ by **Proposition A.1.3**. Next we will apply **Proposition A.2.3** in order to finish proving that (3.4.30) holds. By Table (5) we have

$$\ker(\overline{q}_A^\vee|_{\widetilde{q}(K)}) \ni \widetilde{\overline{q}}_{w_1}(\overline{\alpha}) = w_1 \wedge w_2 \wedge w_2' = \widetilde{\overline{q}}_A(-w_1 \wedge \zeta - \xi).$$

Thus $v := \overline{\alpha}$ satisfies (A.2.5) (one of the hypotheses of **Proposition A.2.3**) and we may set .

$$e(\overline{q}_{w_1}; \overline{\alpha}) = -(w_1 \wedge \zeta + \xi) \tag{3.4.31}$$

Table 6: Matrix of $\overline{q}_A^\vee$ restricted to $\langle \widetilde{\overline{q}}_{w_1}(\overline{\alpha}), \widetilde{\overline{q}}_{w_2}(\overline{\alpha}) \rangle$

	$\widetilde{\overline{q}}_{w_1}(\overline{\alpha})$	$\widetilde{\overline{q}}_{w_2}(\overline{\alpha})$
$\widetilde{\overline{q}}_{w_1}(\overline{\alpha})$	$\mathrm{vol}(v_0 \wedge (w_1 \wedge \zeta + \xi) \wedge w_1 \wedge w_2 \wedge w_2')$	$\mathrm{vol}(v_0 \wedge (w_1 \wedge \zeta + \xi) \wedge w_1 \wedge w_2 \wedge w_1')$
$\widetilde{\overline{q}}_{w_2}(\overline{\alpha})$	$\mathrm{vol}(v_0 \wedge (w_1 \wedge \zeta + \xi) \wedge w_1 \wedge w_2 \wedge w_1')$	$\mathrm{vol}(v_0 \wedge \gamma \wedge w_1 \wedge w_2 \wedge w_1')$

By (3.4.23) we get that $\overline{q}_{w_1}(w_1 \wedge \zeta + \xi) = 0$ and hence (3.4.30) holds by **Proposition A.2.3**. It remains to prove that $c_{31} = 0$. Let's prove that the hypotheses of **Claim A.2.5** are satisfied by $q_* := \overline{q}_A$, $r := \overline{q}_{w_1}$ and $s := \overline{q}_{w_2}$. Item (1) holds by Table (4), moreover the kernel of $\overline{q}_{w_2}|_K$ is spanned by $\overline{\alpha}$ and hence $v := \overline{\alpha}$ in the notation of **Claim A.2.5**. Next consider Item (2): then $\widetilde{\overline{q}}_{w_1}(\overline{\alpha}) = w_1 \wedge w_2 \wedge w_2'$, $\widetilde{\overline{q}}_{w_2}(\overline{\alpha}) = -w_1 \wedge w_2 \wedge w_1'$, since they are linearly independent the first condition of that item is satisfied. Table (6) gives the restriction of $\overline{q}_A^\vee$ to $\langle \widetilde{\overline{q}}_{w_1}(\overline{\alpha}), \widetilde{\overline{q}}_{w_2}(\overline{\alpha}) \rangle$. The entry on the second line and second column is non-zero by (3.4.29), the others are zero by (3.4.23), thus the second condition of Item (2) is satisfied. Lastly we checked above that $\overline{q}_A^\vee|_{\widetilde{\overline{q}}_{w_1}(K)}$ is degenerate - see Table (5) - and hence Item (3) is satisfied. By **Claim A.2.5** we get that $c_{31} = 0$ if and only if

$$0 = \overline{q}_{w_1}(e(\overline{q}_{w_1}; \overline{\alpha})) = \overline{q}_{w_1}(w_1 \wedge \zeta + \xi).$$

(See (3.4.31) for the second equality.) The last term vanishes by (3.4.23) (as noticed above). □

We end the subsection by pointing out certain similarities between $\mathbb{B}_{\mathcal{E}_1}$, $\mathbb{B}_{\mathcal{E}_1^\vee}$ and $\mathbb{B}_{\mathcal{F}_2}$. Let F be a basis of V and $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee} \cup \mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$. Let $W \in \mathrm{Gr}(3, V)$ be defined by requiring that

$$\bigwedge^3 W = \begin{cases} [v_0] \wedge \bigwedge^2 V_{12} & \text{if } A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}, \\ \bigwedge^3 V_{02} & \text{if } A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}, \\ A \cap (\bigwedge^2 V_{01} \wedge V_{23}) & \text{if } \in \mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}. \end{cases} \tag{3.4.32}$$

Define $\widetilde{\mathcal{V}}$ as

$$\widetilde{\mathcal{V}} := \begin{cases} A \cap ([v_0] \wedge V_{12} \wedge V_{35}) & \text{if } A \in \mathbb{B}_{\mathcal{E}_1}, \\ A \cap (\bigwedge^2 V_{02} \wedge V_{34}) & \text{if } A \in \mathbb{B}_{\mathcal{E}_1^\vee}, \\ A \cap (\bigwedge^2 V_{01} \wedge V_{23} \oplus \bigwedge^2 V_{01} \wedge V_{45} \oplus V_{01} \wedge \bigwedge^2 V_{23}) & \text{if } A \in \mathbb{B}_{\mathcal{F}_2}. \end{cases} \tag{3.4.33}$$

The projection

$$\mathcal{V} := \rho_W(\widetilde{\mathcal{V}}) \subset T_W \cong \mathrm{Hom}(W, V/W) \tag{3.4.34}$$

is 2-dimensional. Let $\mathrm{Hom}(W, V/W)_r \subset \mathrm{Hom}(W, V/W)$ be the subset of maps of rank at most r. One easily checks that in each of the three cases appearing in (3.4.33) we have $\mathcal{V} \subset \mathrm{Hom}(W, V/W)_2$. The following observation is easily proved - we leave details to the reader.

Remark 3.4.4. Let A be generic in one of $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}$, $\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ or $\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}$. Let W be as in (3.4.32), $\overline{A} := A / \bigwedge^3 W$ and $\mathcal{V} \subset (\overline{A} \cap T_W)$ be given by (3.4.34). Then $\dim \mathcal{V} = 2$ and

$$(\mathcal{V} \setminus \{0\}) \subset (\mathrm{Hom}(W, V/W)_2 \setminus \mathrm{Hom}(W, V/W)_1). \tag{3.4.35}$$

By **Proposition A.3.1** $\mathcal{V}$ is equivalent modulo the natural $GL(V/W) \times GL(W)$-action on $\mathrm{Gr}(2, \mathrm{Hom}(W, V/W))$ to one of the susbpaces $\mathcal{V}_l, \mathcal{V}_c, \mathcal{V}_p$ defined by (A.3.3)-(A.3.4)-(A.3.5). Then $\mathcal{V}$ is equivalent to

$$\begin{cases} \mathcal{V}_p & \text{if } A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1}, \\ \mathcal{V}_c & \text{if } A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_1^\vee}, \\ \mathcal{V}_l & \text{if } A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{F}_2}. \end{cases} \tag{3.4.36}$$

Conversely let $A \in \mathbb{LG}(\bigwedge^3 V)$ and $W \in \Theta_A$. Let $\overline{A} := A / \bigwedge^3 W$. Suppose that there exists a 2-dimensional subspace $\mathcal{V} \subset (\overline{A} \cap T_W)$ such that (3.4.35) holds; then $A \in \mathbb{B}^*_{\mathcal{E}_1} \cup \mathbb{B}^*_{\mathcal{E}_1^\vee} \cup \mathbb{B}^*_{\mathcal{F}_2}$. More precisely $A \in \mathbb{B}^*_{\mathcal{E}_1}$ if $\mathcal{V}$ is equivalent to $\mathcal{V}_p$, $A \in \mathbb{B}^*_{\mathcal{E}_1^\vee}$ if $\mathcal{V}$ is equivalent to $\mathcal{V}_c$ and $A \in \mathbb{B}^*_{\mathcal{F}_2}$ if $\mathcal{V}$ is equivalent to $\mathcal{V}_l$.

Remark 3.4.5. Suppose that we wish to decide whether a given $A \in \mathbb{LG}(\bigwedge^3 V)$ is stable or not. **Theorem 2.1.1** provides the following algorithm:

1. Compute $\dim \Theta_A$: if $\dim \Theta_A \geq 2$ then A is not stable, if $\dim \Theta_A \leq 1$ go to Step 2 .
2. If $\dim \Theta_A = 1$ determine the irreducible components of Θ_A and hence determine whether A belongs to one of the irreducible components of Σ_∞ which appear in (2.4.10): if it does then A is not stable, if it doesn't (or $\dim \Theta_A < 1$) go to Step 3.
3. List all of the isolated elements $W \in \Theta_A$. If $\dim(A \cap S_W) \geq 4$ for one such W then A is not stable, if $\dim(A \cap S_W) \leq 3$ for all such W go to Step 4.
4. If there exists an isolated $W \in \Theta_A$ such that $\dim(A \cap S_W) = 3$ and all $\alpha \in T_W$ are degenerate (as map $W \to V/W$) then A is not stable, if there exists no such W go to Step 5.
5. If there exists an isolated $W \in \Theta_A$ such that $\dim(A \cap S_W) = 3$ and $A \in \mathbb{X}^{\mathsf{F}}_{\mathcal{N}_3}$ for a certain flag with $W = \langle v_0, v_1, av_2 + bv_3 \rangle$ then A is not stable, if there is no such W then A is stable.

4 Lagrangians with large stabilizers

4.1 Main results

In the present section we will analyze semistable lagrangians with minimal orbit and large stabilizer. Before stating the main results we will define certain elements of $\mathbb{LG}(\bigwedge^3 V)$. Let L be a three-dimensional complex vector space and k, h be given by (3.3.20). By (3.3.22) and surjectivity of $H^0(k^*)$ and $H^0(h^*)$ we get that $\operatorname{im}(k), \operatorname{im}(h)$ span 9-dimensional subspaces of $\mathbb{P}(\bigwedge^3 V)$.

Definition 4.1.1. Let $A_k(L), A_h(L) \subset \bigwedge^3 V$ be the affine cones over $\operatorname{im}(k), \operatorname{im}(h)$ respectively.

Any two planes in $\operatorname{im}(k)$ are incident and similarly for $\operatorname{im}(h)$: it follows that $A_k(L), A_h(L) \in \mathbb{LG}(\bigwedge^3 V)$. The $\mathrm{PGL}(V)$-orbit of $A_k(L)$ (or of $A_h(L)$) is independent of L: often we will denote $A_k(L), A_h(L)$ by A_k and A_h respectively. The proposition below summarizes some of the main results of the present section.

Proposition 4.1.2.

1. *There exists $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ which is stabilized by a maximal torus, and any two such lagrangians belong to the same* $\mathrm{PGL}(V)$*-orbit, which is closed in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*. Let A_{III} belong to that orbit: if $W \in \Theta_A$ then $C_{W,A}$ is a sextic of Type III-2 according to Shah's* **Theorem 1.4.2**.
2. *Let U be a four-dimensional complex vector space and $A_+(U) \in \mathbb{LG}(\bigwedge^3 V)$ be as in* (2.4.12)*. Then $A_+(U)$ is semistable with* $\mathrm{PGL}(V)$*-orbit closed in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*, and it is stabilized by* $\mathrm{PGL}(U)$ *embedded in* $\mathrm{PGL}(V)$ *via the identification $V = \bigwedge^2 U$.*
3. *Both $A_k(L)$ and $A_h(L)$ are semistable with* $\mathrm{PGL}(V)$*-orbit closed in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*, and they are stabilized by* $\mathrm{PGL}(L)$ *embedded in* $\mathrm{PGL}(V)$ *via the identification $V = \mathrm{S}^2 L$.*

We will notice that $[A_{III}] \notin \mathfrak{I}$ (this follows at once from Item (1) of **Proposition 4.1.2**) while $[A_+], [A_k], [A_h] \in \mathfrak{I}$. We will also introduce a curve $\mathfrak{X}_{\mathcal{W}} \subset \mathfrak{M}$ containing $[A_+]$ and contained in $\mathfrak{I}$ - lagrangians representing points of this curve are stabilized by $\mathrm{PSO}(4)$ suitably embedded in $\mathrm{PGL}(V)$.

4.2 A result of Luna

We start by stating an important theorem of Luna [19] that will be used throughout the rest of this work. Let G be a linearly reductive group and $\widehat{X}$ an affine variety acted on by G. Let $H < G$ be a linearly reductive subgroup and $\widehat{X}^H \subset \widehat{X}$ be the closed subset of points fixed by H. Let $N_G(H) < G$ be the normalizer of H; then $N_G(H)$ acts on $\widehat{X}^H$ and we have a natural regular map

$$\widehat{X}^H//N_G(H) := \operatorname{Spec}\Gamma(\widehat{X}^H, \mathcal{O}_{\widehat{X}^H})^{N_G(H)} \longrightarrow \operatorname{Spec}\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})^G =: \widehat{X}//G. \tag{4.2.1}$$

The following is Corollaire 1, p. 237 of [19].

Theorem 4.2.1 (Luna [19])**.** *Keep notation as above. Map* (4.2.1) *is finite. If $x \in \widehat{X}^H$ then Gx is closed if and only if $N_G(H)x$ is closed. In particular if $N_G(H)/H$ is finite then Gx is closed.*

Next suppose that $X \subset \mathbb{P}(U)$ is a projective and that G is a linearly reductive group acting on X via a homomorphism $G \to \mathrm{SL}(U)$. Let $\widehat{X} \subset U$ be the affine cone over X; applying **Theorem 4.2.1** to the induced action of G on $\widehat{X}$ one gets the following result.

Corollary 4.2.2 (Luna)**.** *Keep notation and hypotheses as above. Let $H < G$ be a linearly reductive subgroup. Let $[u] \in \mathbb{P}(\widehat{X}^H)$; then $[u]$ is G-semistable if and only if $[u]$ is $N_G(H)$-semistable, and in this case $G[u]$ is closed in X^{ss} if and only if $N_G(H)[u]$ is closed in the set of $N_G(H)$-semistable points of $\mathbb{P}(\widehat{X}^H)$. The inclusion $\mathbb{P}(\widehat{X}^H) \hookrightarrow X$ induces a finite map $\mathbb{P}(\widehat{X}^H)//N_G(H) \longrightarrow X//G$.*

4.3 Lagrangians stabilized by a maximal torus

Let

$$N := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{4.3.1}$$

The rows of N will be indexed by $0 \le i \le 5$, the columns will be indexed by $1 \le j \le 10$, i.e. $N = (n_{ij})$ where $0 \le i \le 5$ and $1 \le j \le 10$. Let $\mathsf{F} = \{v_0, \ldots, v_5\}$ be a basis of V. For $j = 1, \ldots, 10$ let $\alpha_j, \beta_j \in \bigwedge^3 V$ be the decomposable vectors given by the wedge-product of the v_i's such that $n_{ij} = 1$ and $n_{ij} = 0$ respectively (notice that on each column of N there are 3 entries equal to 1 and 3 equal to 0) in the order dictated by the ordering of the indices:

$$\alpha_1 = v_0 \wedge v_1 \wedge v_2,\ \beta_1 = v_3 \wedge v_4 \wedge v_5,\ \alpha_2 = v_0 \wedge v_1 \wedge v_3, \ldots\ldots, \beta_{10} = v_0 \wedge v_1 \wedge v_4.$$

Let $A^{\mathsf{F}}_{III} \subset \bigwedge^3 V$ be the subspace spanned by the α_j's. Let $1 \le j_0 \le 10$. By inspecting the matrix N we see that β_{j_0} is not a multiple of any of the α_j's, that it is perpendicular to each α_j with $j \ne j_0$ and that $\alpha_{j_0} \wedge \beta_{j_0} \ne 0$. It follows that A^{F}_{III} is $(,)_V$-isotropic and that $\dim A^{\mathsf{F}}_{III} = 10$ i.e. $A^{\mathsf{F}}_{III} \in \mathbb{LG}(\bigwedge^3 V)$. Let $0 \ne \omega \in \bigwedge^{10} A^{\mathsf{F}}_{III}$ and $T < GL(V)$ be the maximal torus of automorphism which are diagonal in the basis F: then

$$g(\omega) = (\det g)^5 \omega \quad \forall g \in T. \tag{4.3.2}$$

The above holds because the sum of the entries on each row of N is equal to 5. The following result will be useful in deciding whether a given $A \in \mathbb{LG}(\bigwedge^3 V)$ is in the $\mathrm{PGL}(V)$-orbit of A_{III}.

Claim 4.3.1. *Let T be a maximal torus of $\mathrm{SL}(V)$. Suppose that $A \in \mathbb{LG}(\bigwedge^3 V)$ is fixed by T and that T acts trivially on $\bigwedge^{10} A$. Then the orbit $\mathrm{PGL}(V)A$ contains A_{III}.*

Proof. Suppose that T is diagonalized in the basis $\{v_0, \ldots, v_5\}$. Since A is left invariant by T it has a basis B consisting of 10 monomials $v_i \wedge v_j \wedge v_k$ (here $0 \le i < j < k \le 5$). Let $\mathcal{T}$ be the family of "tripletons"of $\{0, 1, \ldots, 5\}$ i.e. subsets of cardinality 3. We let $\sigma\colon \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(I) := I^c := (\{0, 1, \ldots, 5\} \setminus I)$. If $a \in \{0, 1, \ldots, 5\}$ and $\mathcal{S} \subset \mathcal{T}$ we let $\mathcal{S}_a := \{I \in \mathcal{S} \mid a \in I\}$. By associating to $v_i \wedge v_j \wedge v_k$ the set $\{i, j, k\} \in \mathcal{T}$ we get an identification between the family of monomials and $\mathcal{T}$. With this identification B corresponds to a subset $\mathcal{S} \subset \mathcal{T}$ with the following properties:

(1) $\mathcal{T} = \mathcal{S} \coprod \sigma(\mathcal{S})$, and

(2) $\mathcal{S}_a$ has cardinality 5 for each $a \in \{0, 1, \ldots, 5\}$.

We claim the following:

$$\text{If } a, b \in \{0, 1, \ldots, 5\} \text{ are distinct then } |\mathcal{S}_a \cap \mathcal{S}_b| = 2. \tag{4.3.3}$$

In fact let $a, b \in \{0, 1, \ldots, 5\}$: then $|\mathcal{S}_a \cap \mathcal{S}_b| = 5 - |\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b)|$ and hence we get that

$$|\mathcal{S}_a \cap \mathcal{S}_b| = |(\mathcal{S} \setminus \mathcal{S}_a) \cap (\mathcal{S} \setminus \mathcal{S}_b)|, \qquad |\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b)| = |(\mathcal{S} \setminus \mathcal{S}_a) \cap \mathcal{S}_b|. \tag{4.3.4}$$

Now suppose that $a \ne b$. The map σ gives inclusions

$$\sigma(\mathcal{S}_a \cap \mathcal{S}_b) \subset (\mathcal{T} \setminus \mathcal{T}_a) \cap (\mathcal{T} \setminus \mathcal{T}_b), \qquad \sigma(\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b)) \subset (\mathcal{T} \setminus \mathcal{T}_a) \cap \mathcal{T}_b.$$

By (4.3.4) and Item (1) we get that

$$\begin{aligned} 2|\mathcal{S}_a \cap \mathcal{S}_b| &= |\sigma(\mathcal{S}_a \cap \mathcal{S}_b)| + |(\mathcal{S} \setminus \mathcal{S}_a) \cap (\mathcal{S} \setminus \mathcal{S}_b)| \le |(\mathcal{T} \setminus \mathcal{T}_a) \cap (\mathcal{T} \setminus \mathcal{T}_b)| = 4, \\ 2|\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b)| &= |\sigma(\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b))| + |(\mathcal{S} \setminus \mathcal{S}_a) \cap \mathcal{S}_b| \le |(\mathcal{T} \setminus \mathcal{T}_a) \cap \mathcal{T}_b| = 6. \end{aligned} \tag{4.3.5}$$

Thus $|\mathcal{S}_a \cap \mathcal{S}_b| \le 2$ and $|\mathcal{S}_a \cap (\mathcal{S} \setminus \mathcal{S}_b)| \le 3$; this proves (4.3.3). Now associate to $\mathcal{S}$ a 6×10-matrix M whose columns are the characteristic functions of the sets in $\mathcal{S}$. By (4.3.3) and a Sudoku-like argument we get that after performing a sequence of row and column permutations we may transform M into N; that proves the claim. □

Proposition 4.3.2. *A^{F}_{III} is semistable and its* $\mathrm{PGL}(V)$*-orbit is closed in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*, moreover* $Y_{A^{\mathsf{F}}_{III}} = V(X_0 \cdot X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5)$ *where* $\{X_0, \ldots, X_5\}$ *is the basis of* $V^\vee$ *dual to* F.

Proof. Let $\widehat{\mathbb{LG}}(\bigwedge^3 V) \subset \bigwedge^{10}(\bigwedge^3 V)$ be the affine cone over $\mathbb{LG}(\bigwedge^3 V)$. Let ω be a generator of $\bigwedge^{10} A^{\mathsf{F}}_{III}$; thus $\omega \in \widehat{\mathbb{LG}}(\bigwedge^3 V)$. Let $T < \mathrm{SL}(V)$ be the maximal torus of automorphisms which are diagonal in the basis F. By (4.3.2) we have $\omega \in \widehat{\mathbb{LG}}(\bigwedge^3 V)^H$. The quotient $N_{\mathrm{SL}(V)}(T)/T$ is the symmetric group $\mathfrak{S}_6$ and hence is finite. By **Theorem 4.2.1** the orbit $\mathrm{SL}(V)\omega$ is closed; thus A is semistable by the Hilbert-Mumford criterion, moreover as is well-known closedness of $\mathrm{SL}(V)\omega$ in $\widehat{\mathbb{LG}}(\bigwedge^3 V)$ implies that A is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$. Let $Y_{A^{\mathsf{F}}_{III}} = V(P)$ where $P \in \mathbb{C}[X_0, \ldots, X_5]_6$. Since A^{F}_{III} is semistable we get that $P \neq 0$ by **Corollary 2.4.6**. Since T fixes P we get that $P = cX_0 \cdot X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5$ for some $c \neq 0$. □

By **Proposition 4.3.2** it makes sense to let

$$\mathfrak{z} := [A_{III}] \in \mathfrak{M}. \tag{4.3.6}$$

Our next goal is to prove that

$$\mathfrak{z} \notin \mathfrak{I}. \tag{4.3.7}$$

By (4.3.3) the following holds: given row indices $0 \leq s < t \leq 5$ there exists exactly one set $\{s', t'\} \subset \{0, \ldots, 5\} \setminus \{s, t\}$ of two indices such that

$$v_s \wedge v_t \wedge v_{s'},\ v_s \wedge v_t \wedge v_{t'} \in A. \tag{4.3.8}$$

Thus we get the line

$$L_{s,t} := \{v_s \wedge v_t \wedge (\lambda_0 v_{s'} + \lambda_1 v_{t'}) \mid [\lambda_0, \lambda_1] \in \mathbb{P}^1\} \subset \Theta_{A^{\mathsf{F}}_{III}}. \tag{4.3.9}$$

Proposition 4.3.3. *Keeping notation as above we have*

$$\Theta_{A^{\mathsf{F}}_{III}} = \bigcup_{0 \leq s < t \leq 5} L_{s,t}. \tag{4.3.10}$$

Let $W \in \Theta_{A^{\mathsf{F}}_{III}}$ *and hence* $W = \langle v_s, v_t, (\lambda_0 v_{s'} + \lambda_1 v_{t'})\rangle$ *for a unique choice of* $0 \leq s < t \leq 5$, s', t' *as in* (4.3.8) *and* $[\lambda_0, \lambda_1] \in \mathbb{P}^1$*; then*

$$C_{W, A^{\mathsf{F}}_{III}} = 2\langle v_s, v_t\rangle + 2\langle v_s, (\lambda_0 v_{s'} + \lambda_1 v_{t'})\rangle + 2\langle v_t, (\lambda_0 v_{s'} + \lambda_1 v_{t'})\rangle. \tag{4.3.11}$$

Proof. First we will prove that $\dim \Theta_{A^{\mathsf{F}}_{III}} = 1$. By (4.3.9) we know that $\dim \Theta_{A^{\mathsf{F}}_{III}} \geq 1$. Suppose that $\dim \Theta_{A^{\mathsf{F}}_{III}} \geq 2$ and let Θ be an irreducible component of $\Theta_{A^{\mathsf{F}}_{III}}$ of dimension at least 2. Theorem 2.26 and Theorem 2.36 of [28] give the classification of couples (A, Θ) with $A \in \mathbb{LG}(\bigwedge^3 V)$ and Θ an irreducible component of Θ_A such that $\dim \Theta \geq 2$. That classification together with semistability of A^{F}_{III} gives that

$$A^{\mathsf{F}}_{III} \in (\mathbb{X}_{\mathcal{Y}} \cup \mathbb{X}_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k(L) \cup \mathrm{PGL}(V)A_h(L) \cup \mathrm{PGL}(V)A_+(U)). \tag{4.3.12}$$

(Notation as in [28].) If $A^{\mathsf{F}}_{III} \in (\mathrm{PGL}(V)A_k(L) \cup \mathrm{PGL}(V)A_h(L))$ then $Y_{A^{\mathsf{F}}_{III}}$ is a double discriminant cubic, if $A^{\mathsf{F}}_{III} \in (\mathbb{X}_{\mathcal{W}} \cup \mathrm{PGL}(V)A_+(U))$ then $Y_{A^{\mathsf{F}}_{III}}$ contains a quadric hypersurface: in both cases we contradict **Proposition 4.3.2**. This proves that $\dim \Theta_{A^{\mathsf{F}}_{III}} = 1$. Let $T < \mathrm{SL}(V)$ be the connected maximal torus of elements which are diagonal with respect to $\{v_0, \ldots, v_5\}$. By (4.3.2) T maps A^{F}_{III} to itself and hence it maps each irreducible component of $\Theta_{A^{\mathsf{F}}_{III}}$ to itself. It follows that a 0-dimensional irreducible component of $\Theta_{A^{\mathsf{F}}_{III}}$ must be of the form $v_i \wedge v_j \wedge v_k$ for $0 \leq i < j < k \leq 5$ and an irreducible 1-dimensional component of $\Theta_{A^{\mathsf{F}}_{III}}$ must be of the form (4.3.9) for some choice of pairwise distinct s, t, s', t'; it follows that s', t' satisfy (4.3.8). We have proved (4.3.10). Next we will prove the assertion about $C_{W,A}$ for $W \in \Theta_{A^{\mathsf{F}}_{III}}$. First suppose that $W = \langle v_i, v_j, v_k\rangle$. Then

$$\mathcal{B}(W, A) = \langle v_i, v_j\rangle \cup \langle v_i, v_k\rangle \cup \langle v_j, v_k\rangle. \tag{4.3.13}$$

In fact it follows from (4.3.10) that the set of $[w] \in \mathbb{P}(W)$ such that Item (1) of **Definition 3.3.3** holds is equal to the right-hand side of (4.3.13), moreover a straightforward analysis of the matrix N defining A^{F}_{III} gives that the set of $[w] \in \mathbb{P}(W)$ such that Item (2) of **Definition 3.3.3** holds is again equal to the right-hand side of (4.3.13). By **Corollary 3.3.7** we get that (4.3.11) holds if $W = \langle v_i, v_j, v_k\rangle$. Lastly suppose that $W = W_\lambda := \langle v_s, v_t, (\lambda_0 v_{s'} + \lambda_1 v_{t'})\rangle$ where $\lambda_0 \neq 0 \neq \lambda_1$. Acting by the torus T we get an isomorphism

$$C_{W_\lambda, A^{\mathsf{F}}_{III}} \xrightarrow{\sim} C_{W_{\lambda'}, A^{\mathsf{F}}_{III}} \tag{4.3.14}$$

where $\lambda' = [\lambda'_0, \lambda'_1]$ is arbitrary with $\lambda'_0 \neq 0 \neq \lambda'_1$. It follows that $C_{W_\lambda, A^{\mathsf{F}}_{III}} \neq \mathbb{P}(W_\lambda)$. In fact if we had equality then we would have $C_{W_{\lambda'}, A^{\mathsf{F}}_{III}} = \mathbb{P}(W_{\lambda'})$ whenever $\lambda'_0 \neq 0 \neq \lambda'_1$ and by continuity also for arbitrary $[\lambda'_0, \lambda'_1]$; since $W_{[1,0]} = \langle v_s, v_t, v_{s'}\rangle$ that contradicts what we have proved above. This proves that $C_{W_\lambda} \neq \mathbb{P}(W_\lambda)$. Let $T_0 < T$ be the sub-torus of g such that $g(v_{s'})/v_{s'} = g(v_{t'})/v_{t'}$. If $g \in T_0$ then $g(W_\lambda) = W_\lambda$ for every $\lambda \in \mathbb{P}^1$. Thus we have a homomorphism $\rho\colon T_0 \longrightarrow GL(W_\lambda)$. For $g \in T_0$ let

$$\overline{\rho}(g) := \rho(g)(\det g)^{-1/3} \in \mathrm{SL}(W_\lambda).$$

Write $C_{W_\lambda, A^{\mathsf{F}}_{III}} = V(P)$ where $P \in \mathrm{S}^3 W_\lambda^\vee$: by **Claim 3.2.4** we get that $\overline{\rho}(g)P = P$ for every $g \in T_0$. Since $\{\overline{\rho}(g) \mid g \in T_0\}$ is a maximal torus of $\mathrm{SL}(W_\lambda)$ it follows that (4.3.11) holds for $W = W_\lambda$. □

4.4 Lagrangians stabilized by PGL(4) or PSO(4)

Choose an isomorphism $\phi\colon \bigwedge^2 U \xrightarrow{\sim} V$. Let $A_+(U) \in \mathbb{LG}(\bigwedge^3 V)$ be defined as in (2.4.12) and similarly for $A_-(U)$: then $\mathrm{SL}(U)$ maps $A_+(U)$ to itself and it acts trivially on $\bigwedge^{10} A_+(U)$. Of course the orbits $\mathrm{PGL}(V)A_+(U)$ and $\mathrm{PGL}(V)A_-(U)$ are equal.

Proposition 4.4.1. *$A_+(U)$ is semistable and it has minimal $\mathrm{PGL}(V)$-orbit.*

Proof. The subgroup $\mathrm{SL}(U) < \mathrm{SL}(V)$ acts trivially on $\bigwedge^{10} A_+(U)$ and the index of $\mathrm{SL}(U)$ in the normalizer $N_{\mathrm{SL}(V)}(\mathrm{SL}(U))$ is 2; thus $A_+(U)$ is $\mathrm{SL}(V)$-semistable by **Corollary 4.2.2**. □

Thus $A_+(U), A_-(U)$ are semistable points with minimal orbit stabilized by SL(4). Later on we will need to have at our disposal explicit bases of $A_+(U)$ and $A_-(U)$: we define them as follows. Let $\{u_0, u_1, u_2, u_3\}$ be a basis of U and $\mathsf{F} = \{v_0, \ldots, v_5\}$ be the basis of V given by

$$v_0 = u_0 \wedge u_1,\ v_1 = u_0 \wedge u_2,\ v_2 = u_0 \wedge u_3,\ v_3 = u_1 \wedge u_2,\ v_4 = u_1 \wedge u_3,\ v_5 = u_2 \wedge u_3. \tag{4.4.1}$$

(To be precise: $v_0 = \phi(u_0 \wedge u_1)$ etc.) A straightforward computation gives that

$$i_+([\xi_0 u_0 + \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3]) = [\sum_I \alpha_I \xi^I], \quad i_-([\theta_0 u_0^\vee + \theta_1 u_1^\vee + \theta_2 u_2^\vee + \theta_3 u_3^\vee) = [\sum_I \beta_I \theta^I] \tag{4.4.2}$$

where $I = (i_0, i_1, i_2, i_3)$ runs through the set of multi-indices of length 2 and α_I, β_I are given by Table (7).

Remark 4.4.2. Let $T < \mathrm{GL}(U)$ be the maximal torus which is diagonalized in the basis $\{u_0, \ldots, u_3\}$: thus $T = \{\mathrm{diag}(t_0, \ldots, t_3) \mid t_0 t_1 t_2 t_3 \neq 0\}$. Then T acts on $A_+(U)$ and on $A_-(U)$ and is diagonalized in the basis $\{\ldots, \alpha_I, \ldots\}$ (respectively in the basis $\{\ldots, \beta_I, \ldots\}$); moreover it acts on α_I and β_I according to I or $-I$ respectively:

$$(t_0, \ldots, t_3)\alpha_I = t_0^{i_0} t_1^{i_1} t_2^{i_2} t_3^{i_3} \alpha_I, \quad (t_0, \ldots, t_3)\beta_I = t_0^{-i_0} t_1^{-i_1} t_2^{-i_2} t_3^{-i_3} \beta_I.$$

By **Remark 4.4.2** the product $(\alpha_I, \beta_J)_V$ vanishes if $I \neq J$. The products $(\alpha_I, \beta_I)_V$ are listed in Table (7). Next we will define a family of lagrangians which are stabilized by SO(4) - as usual this means that if A is such a lagrangian then there exists $\mathrm{SO}(4) < \mathrm{SL}(V)$ which acts trivially on $\bigwedge^{10} A$. The corresponding points in $\mathfrak{M}$ sweep out a curve. Let U be a complex vector-space of dimension 4 and choose an isomorphism

$$\varphi\colon V \cong \bigwedge^2 U. \tag{4.4.3}$$

Let $i_+\colon \mathbb{P}(U) \hookrightarrow \mathrm{Gr}(3, V)$ be as in (2.4.11).

Table 7: Bases of $A_+(U)$ and $A_-(U)$.

I	α_I	β_I	$(\alpha_I,\beta_I)_V$
$(2,0,0,0)$	$v_0\wedge v_1\wedge v_2$	$v_3\wedge v_4\wedge v_5$	1
$(0,2,0,0)$	$v_0\wedge v_3\wedge v_4$	$v_1\wedge v_2\wedge v_5$	1
$(0,0,2,0)$	$v_1\wedge v_3\wedge v_5$	$v_0\wedge v_2\wedge v_4$	1
$(0,0,0,2)$	$v_2\wedge v_4\wedge v_5$	$v_0\wedge v_1\wedge v_3$	1
$(1,1,0,0)$	$v_0\wedge(v_1\wedge v_4-v_2\wedge v_3)$	$v_5\wedge(v_2\wedge v_3-v_1\wedge v_4)$	2
$(1,0,1,0)$	$-v_1\wedge(v_0\wedge v_5+v_2\wedge v_3)$	$-v_4\wedge(v_0\wedge v_5+v_2\wedge v_3)$	2
$(1,0,0,1)$	$v_2\wedge(-v_0\wedge v_5+v_1\wedge v_4)$	$v_3\wedge(v_0\wedge v_5-v_1\wedge v_4)$	2
$(0,1,1,0)$	$-v_3\wedge(v_0\wedge v_5+v_1\wedge v_4)$	$v_2\wedge(v_0\wedge v_5+v_1\wedge v_4)$	2
$(0,1,0,1)$	$v_4\wedge(-v_0\wedge v_5+v_2\wedge v_3)$	$v_1\wedge(-v_0\wedge v_5+v_2\wedge v_3)$	2
$(0,0,1,1)$	$v_5\wedge(v_1\wedge v_4+v_2\wedge v_3)$	$-v_0\wedge(v_1\wedge v_4+v_2\wedge v_3)$	2

Definition 4.4.3. Keeping notation as above let $\mathbb{X}^*_{\mathcal{W}}(U)\subset\mathbb{LG}(\bigwedge^3 V)$ be the set of $A\in\mathbb{LG}(\bigwedge^3 V)$ such that $\mathbb{P}(A)$ contains $i_+(Z)$ where $Z\subset\mathbb{P}(U)$ is a smooth quadric surface (our notation is somewhat imprecise: $\mathbb{X}^*_{\mathcal{W}}(U)$ actually depends on Isomorphism (4.4.3)). Let

$$\mathbb{X}^*_{\mathcal{W}}:=\mathrm{PGL}(V)\mathbb{X}^*_{\mathcal{W}}(U).$$

Notice that $A_+(U)\in\mathbb{X}^*_{\mathcal{W}}(U)$.

Proposition 4.4.4. *Let $A\in\mathbb{X}^*_{\mathcal{W}}$. Then A is semistable and it has minimal* $\mathrm{PGL}(V)$*-orbit.*

Proof. We may assume that $A\in\mathbb{X}^*_{\mathcal{W}}(U)$ and that we have chosen Identification (4.4.3). Then $Z=V(q)$ where $q\in\mathrm{S}^2U^\vee$ is non-degenerate. Let $A_q\subset\mathrm{S}^2U$ be the annihilator of q. Let $q^\vee\in\mathrm{S}^2U$ be the dual of q (see **Section A**); thus we have the decomposition into irreducible $O(q)$-representations $\mathrm{S}^2U=A_q\oplus[q^\vee]$. We have an isomorphism

$$\begin{array}{ccc}\mathbb{P}^1 & \overset{\sim}{\longrightarrow} & \mathbb{X}^*_{\mathcal{W}}(U)\\ \mathbf{x}:=[x_0,x_1] & \mapsto & A_{\mathbf{x}}:=\langle A_q,x_0q^\vee+x_1q\rangle\end{array}\tag{4.4.4}$$

We have an embedding $\mathrm{SL}(U)<\mathrm{SL}(V)$; composing with the embedding $SO(q)<\mathrm{SL}(U)$ we get an embedding

$$SO(q)<\mathrm{SL}(V).\tag{4.4.5}$$

Since $SO(q)$ acts trivially on $\bigwedge^9 A_q$, $q^\vee$, q it acts trivially on $\bigwedge^{10}A_{\mathbf{x}}$ for every $\mathbf{x}\in\mathbb{P}^1$. The group $N_{\mathrm{SL}(V)}(SO(q))$ acts on $\mathbb{X}^*_{\mathcal{W}}(U)$. By **Corollary 4.2.2** in order to prove the proposition it suffices to show that every $A_{\mathbf{x}}$ is $N_{\mathrm{SL}(V)}(SO(q))$-semistable with closed orbit. Choose 2-dimensional vector-spaces U',U'' and an isomorphism $U\cong U'\otimes U''$ such that Z is identified with the projectivization of the subset of decomposable elements of $U'\otimes U''$. We have an isomorphism of $GL(U')\times GL(U'')$-representations

$$V=\bigwedge^2 U=\bigwedge^2(U'\otimes U'')\cong\underbrace{\mathrm{S}^2U'\otimes\bigwedge^2 U''}_{V'}\oplus\underbrace{\mathrm{S}^2U''\otimes\bigwedge^2 U'}_{V''}.$$

Composing the isogeny $\mathrm{SL}(U')\times\mathrm{SL}(U'')\longrightarrow SO(q)$ and Embedding (4.4.5) we get the isogeny $\mathrm{SL}(U')\times\mathrm{SL}(U'')\longrightarrow SO(V')\times SO(V'')$. Thus it suffices to show that each $A_{\mathbf{x}}$ is $N_{\mathrm{SL}(V)}(SO(V')\times$

$SO(V''))$-semistable with closed orbit. Let $\lambda\colon \mathbb{C}^\times \to N_{\mathrm{SL}(V)}(SO(V')\times SO(V''))$ be the 1-PS such that $\lambda(t)|_{V'} = t\,\mathrm{Id}_{V'}$, $\lambda(t)|_{V''} = t^{-1}\,\mathrm{Id}_{V''}$. The subgroup of $N_{\mathrm{SL}(V)}(SO(V')\times SO(V''))$ generated by $SO(V')\times SO(V'')$ and $\mathrm{im}\,\lambda$ is of finite index; since $SO(V')\times SO(V'')$ acts trivially on $\bigwedge^{10} A_{\mathbf{x}}$ for each $\mathbf{x}$ it follows that it suffices to prove that each $A_{\mathbf{x}}$ is λ-semistable with closed orbit. Identifying $\mathbb{X}^*_{\mathcal{W}}(U)$ with $\mathbb{P}^1$ via (4.4.4) we get that λ acts on $\mathbb{P}^1$ and on $\mathcal{O}_{\mathbb{P}^1}(1)$; let

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) = L_0\oplus L_1,\quad \dim L_i = 1,\quad \lambda(t)|_{L_i} = t^{a_i},\quad a_0+a_1 = 0 \tag{4.4.6}$$

be a diagonalization of the action of λ. Of course $\{x_0,x_1\}$ is a basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$; we claim that one may assume that $L_1 = [x_1]$. In fact we have $A_{[1,0]} = A_+(U)$ and if $\mathbf{x}\neq[1,0]$ then $A_{\mathbf{x}}$ is not projectively equivalent to $A_+(U)$ because $\dim\Theta_{A_{\mathbf{x}}} = 3$ if and only if $\mathbf{x} = [1,0]$ (this is an easy exercise); thus x_1 is an eigenvalue of $\lambda(t)$ for every $t\in\mathbb{C}^\times$ and hence we may assume that $L_1 = [x_1]$. On the other hand $A_+(U)$ is $\mathrm{SL}(V)$-semistable by **Proposition 4.4.1**. Since $A_{[1,0]} = A_+(U)$ is $\mathrm{SL}(V)$-semistable and $L_1 = [x_1]$ we get that $a_1 = 0$ and hence the λ-action on $\mathbb{P}^1$ is trivial; this proves that each $A_{\mathbf{x}}$ is λ-semistable with minimal orbit. □

By **Proposition 4.4.4** it makes sense to let

$$\mathfrak{X}_{\mathcal{W}} := \{[A]\in\mathfrak{M}\mid A\in\mathbb{X}^*_{\mathcal{W}}\},\quad \mathfrak{y} := [A_+(U)]. \tag{4.4.7}$$

Thus $\mathfrak{y}\in\mathfrak{X}_{\mathcal{W}}$.

Claim 4.4.5. *Let $A\in\mathbb{X}^*_{\mathcal{W}}$ and $W\in\Theta_A$. Then $C_{W,A}$ is in the indeterminacy locus of Map* (0.0.3)*. In particular $\mathfrak{X}_{\mathcal{W}}\subset\mathfrak{I}$.*

Proof. It suffices to show that if $A\in\mathbb{X}^*_{\mathcal{W}}(U)$ then $C_{W,A}$ is in the indeterminacy locus of Map (0.0.3) for every $W\in\Theta_A$. By definition $\mathbb{P}(A)$ contains $i_+(Z)$ where $Z\subset\mathbb{P}(U)$ is a smooth quadric. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the two families of lines on Z. The conics $i_+(\mathcal{F}_1)$ and $i_+(\mathcal{F}_2)$ span planes $\Lambda_1,\Lambda_2\subset\mathbb{P}(V)$ respectively. Let $W_1,W_2\in\mathrm{Gr}(3,V)$ be the subspaces such that $\mathbb{P}(W_i) = \Lambda_i$. Suppose that $A = A_+(U)$: as is easily checked $\mathcal{B}(W,A) = \mathbb{P}(W)$ and hence $C_{W,A} = \mathbb{P}(W)$ by **Corollary 3.3.7**. Now suppose that $A\neq A_+(U)$: then $W\in i_+(Z)\cup\{W_1,W_2\}$ (for generic $A\in\mathbb{X}^*_{\mathcal{W}}(U)$ we have $\Theta_A = i_+(Z)$). Suppose that $W\in i_+(Z)$. Then there exists a dense set of $[v]\in\mathbb{P}(W)$ for which Item (1) of **Definition 3.3.3** holds; thus $\mathcal{B}(W,A) = \mathbb{P}(W)$. By **Corollary 3.3.7** we get that $C_{W,A} = \mathbb{P}(W)$. Lastly let $i = 1,2$: applying **Proposition 3.2.2** one gets that $C_{W_i,A} = 3D$ where $D\subset\Lambda_i$ is the conic $i_+(\mathcal{F}_i)$. □

Below we will give a result for two special elements of $\mathbb{X}^*_{\mathcal{W}}(U)$ - the result will be needed in the proof of **Proposition 7.2.26**. Let $Z\subset\mathbb{P}(U)$ be the smooth quadric of **Definition 4.4.3**. Let $\mathcal{R}$ be one of the two rulings of Z by lines. We view $\mathcal{R}$ as a smooth conic in $\mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V)$: it spans a plane $\mathbb{P}(\overline{W})$ meeting the Plücker quadric hypersurface $\mathrm{Gr}(2,U)\subset\mathbb{P}(V)$ in $\mathcal{R}$. Let $p\in Z$: the unique line of $\mathcal{R}$ containing p belongs to $\mathbb{P}(i_+(p))$ and hence $\mathbb{P}(\overline{W})\cap\mathbb{P}(i_+(p))\neq\emptyset$. It follows that

$$\bigwedge^3\overline{W}\in\langle\langle i_+(Z)\rangle\rangle^\perp.$$

Here and in the following we think of $\mathrm{Gr}(3,\bigwedge^2 U) = \mathrm{im}\,i_+$ as a subset of $\mathbb{P}(\bigwedge^3 V)$ via the Plücker embedding. Byf (2.4.13) we know that $\bigwedge^3\overline{W}\notin A_+(U)$. Thus

$$A_{\mathcal{R}} := \langle\langle i_+(Z)\rangle\rangle + \bigwedge^3\overline{W} \tag{4.4.8}$$

is an element of $\mathbb{X}^*_{\mathcal{W}}(U)$. By definition we have $\overline{W}\in\Theta_{A_{\mathcal{R}}}$.

Claim 4.4.6. *Keep notation as above. Then $C_{\overline{W},A_{\mathcal{R}}} = 3\mathcal{R}$.*

Proof. Clearly $\mathcal{R}\subset\mathrm{supp}\,C_{\overline{W},A_{\mathcal{R}}}$ and hence it suffices to prove the following: if $[v]\in\mathcal{R}$ then

$$C_{\overline{W},A_{\mathcal{R}}}\cap\overline{W}_0 = V(h^3+g_4+g_5+g_6),\qquad 0\neq h\in\overline{W}_0^\vee\quad g_i\in \mathrm{S}^i\,\overline{W}_0^\vee. \tag{4.4.9}$$

(Notation as in (3.2.8).) Let $v = u \wedge u'$. We claim that

$$F_v \cap \langle\langle i_+(Z)\rangle\rangle = \langle\langle i_+(\mathbb{P}\langle u, u'\rangle)\rangle\rangle. \tag{4.4.10}$$

It is clear that the left-hand side contains the right-hand side. If the containment is strict then $\dim(F_v \cap \langle\langle i_+(Z)\rangle\rangle) \geq 4$ because the right-hand side of (4.4.10) has dimension 3: a fortiori we have $\dim(F_v \cap A_+(U)) \geq 4$. By Proposition 2.3 of [28] we get that either $Y_{A_+(U)} = \mathbb{P}(V)$ or $\operatorname{mult}_{[v]} Y_{A_+(U)} \geq 4$: that contradicts (2.4.13). This proves (4.4.10). It follows that

$$F_v \cap A_{\mathcal{R}} = \langle\langle i_+(\mathbb{P}\langle u, u'\rangle)\rangle\rangle + \bigwedge^3 \overline{W}.$$

We get (4.4.9) by applying Items (1) and (2) of **Proposition 3.2.2**. More precisely we may identify $\overline{K}$ of **Proposition 3.2.2** with $\langle\langle i_+(\mathbb{P}\langle u, u'\rangle)\rangle\rangle$ and (4.4.9) holds because the intersection of $\mathbb{P}(\overline{K})$ with $\mathrm{Gr}(2, V_0)_{\overline{W}_0}$ (notation as in **Claim 3.3.2**) is identified with $\mathcal{R}$. □

The following result shows that we will get nothing "new"if the smooth quadric Z of **Definition 4.4.3** is replaced by a singular quadric.

Proposition 4.4.7. *Let $Z \subset \mathbb{P}(U)$ be either a plane or a quadric cone. Suppose that $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ and that $\mathbb{P}(A) \supset \langle i_+(Z)\rangle$. Then A is $\mathrm{PGL}(V)$-equivalent to $A_+(U)$.*

Proof. Suppose first that Z is the plane $\mathbb{P}(U_0)$ where $U_0 \subset U$ is a subspace of codimension 1. Let $u_3 \in (U \setminus U_0)$. Let μ be the 1-PS of $\mathrm{SL}(U)$ defined by

$$\mu(t)u = tu, \quad u \in U_0, \qquad \mu(t)u_3 = t^{-3}u_3. \tag{4.4.11}$$

Let $\lambda = \bigwedge^2 \mu$ be the 1-PS of $\mathrm{SL}(V)$ corresponding to μ. There is a basis $\{\alpha_1, \ldots, \alpha_6, (\alpha_7 + \beta_7), \ldots, (\alpha_{10} + \beta_{10})\}$ of A where $\alpha_i \in \mathrm{S}^2 U$ for all i, $\{\alpha_1, \ldots, \alpha_6\}$ is a basis of $\mathrm{S}^2 U_0$ and $\beta_j \in (\mathrm{S}^2 U^\vee \cap (\mathrm{S}^2 U_0)^\perp)$ i.e. $\beta_j = x_3\phi_j$ where $x_3 \in U^\vee$ spans $\operatorname{Ann} U_0$ and $\phi_j \in U^\vee$. Let $\omega := \alpha_1 \wedge \ldots \wedge \alpha_6 \wedge (\alpha_7 + \beta_7) \wedge \ldots \wedge (\alpha_{10} + \beta_{10})$. A straightforward computation gives that

$$\lim_{t \to 0} \lambda(t)\omega = \alpha_1 \wedge \ldots \wedge \alpha_{10}. \tag{4.4.12}$$

This proves that A is $\mathrm{PGL}(V)$-equivalent to $A_+(U)$. Now suppose that Z is a quadric cone. Let $B^\vee := \{x_0, x_1, x_2, x_3\}$ be a basis of $U^\vee$ such that $Z = V(x_0x_2 + x_1^2)$. Let $B := \{u_0, u_1, u_2, u_3\}$ be the basis of U dual to B. Let μ be the 1-PS of $\mathrm{SL}(U)$ defined by

$$\mu(t)u_0 = t^{-2}u_0, \quad \mu(t)u_1 = t^{-1}u_1, \quad \mu(t)u_2 = u_2, \quad \mu(t)u_3 = t^3u_3. \tag{4.4.13}$$

Let $\lambda = \bigwedge^2 \mu$ be the 1-PS of $\mathrm{SL}(V)$ corresponding to μ. There is a basis $\{\alpha_1, \ldots, \alpha_9, (\alpha_{10} + \beta_{10})\}$ of A where $\alpha_i \in \mathrm{S}^2 U$ for all i, $\{\alpha_1, \ldots, \alpha_9\}$ is a basis of $\mathrm{S}^2 U \cap (x_0x_2 + x_1^2)^\perp$ and $\beta_{10} \in \langle(x_0x_2 + x_1^2)\rangle$. Let $\omega := \alpha_1 \wedge \ldots \wedge \alpha_9 \wedge (\alpha_{10} + \beta_{10})$. A straightforward computation gives that (4.4.12) holds in this case as well and hence A is $\mathrm{PGL}(V)$-equivalent to $A_+(U)$. □

4.5 Lagrangians stabilized by PGL(3)

For $i = 1, 2$ let $\mathcal{V}_i \subset \mathbb{P}(\mathrm{S}^2 L)$ be the closed subset of conics of rank at most i modulo scalars; thus $\mathcal{V}_1$ is a Veronese surface and $\mathcal{V}_2$ is a (discriminant) cubic hypersurface. In Section 1.5 of [28] we proved that

$$Y_{A_k(L)} = Y_{A_h(L)} = 2\mathcal{V}_2. \tag{4.5.1}$$

Proposition 4.5.1. *A_k and A_h are semistable with minimal $\mathrm{PGL}(V)$-orbits.*

Proof. Let $\widehat{\mathbb{LG}}(\bigwedge^3 V) \subset \bigwedge^{10}(\bigwedge^3 V)$ be the affine cone over $\mathbb{LG}(\bigwedge^3 V)$. Let A be one of $A_k(L)$, $A_h(L)$, and ω be a generator of $\bigwedge^{10} A$; thus $\omega \in \widehat{\mathbb{LG}}(\bigwedge^3 V)$. Let $H := \operatorname{im}(\mathrm{SL}(L) \to \mathrm{SL}(V))$. Then $\omega \in \widehat{\mathbb{LG}}(\bigwedge^3 V)^H$. We have $N_{\mathrm{SL}(V)}(H) = \operatorname{Aut}(\mathcal{V}_2)$: in fact the equality follow from (4.5.1). It follows that $N_{\mathrm{SL}(V)}(H)/H$ is trivial. By **Theorem 4.2.1** the orbit $\mathrm{SL}(V)\omega$ is closed; thus A is semistable by the Hilbert-Mumford criterion, moreover as is well-known closedness of $\mathrm{SL}(V)\omega$ in $\widehat{\mathbb{LG}}(\bigwedge^3 V)$ implies that A is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$. □

By **Proposition 4.5.1** it makes sense to let

$$\mathfrak{x} := [A_k], \qquad \mathfrak{x}^\vee := [A_h]. \tag{4.5.2}$$

We claim that

$$\mathfrak{x} \neq \mathfrak{x}^\vee, \quad \mathfrak{x}, \mathfrak{x}^\vee \in \mathfrak{I}. \tag{4.5.3}$$

First we recall [28] that

$$\Theta_{A_k(L)} = \operatorname{im}(k), \qquad \Theta_{A_h(L)} = \operatorname{im}(h). \tag{4.5.4}$$

Let $W \in \Theta_{A_k(L)}$; by (4.5.4) there exists $[l_0] \in \mathbb{P}(L)$ such that W is given by (3.3.20). Let $[l \cdot l_0] \in (\mathbb{P}(W) \setminus \{[l_0^2]\})$. Then $[l \cdot l_0] \in \mathbb{P}(W')$ where $W' := \{l \cdot l' \mid l' \in L\}$. Since $W' \neq W$ it follows that $(\mathbb{P}(W) \setminus \{[l_0^2]\}) \subset \mathcal{B}(W, A)$: by **Corollary 3.3.7** we get that

$$C_{W,A} = \mathbb{P}(W) \qquad \forall\, W \in \Theta_{A_k}. \tag{4.5.5}$$

Next let $W \in \Theta_{A_h(L)}$; by (4.5.4) there exists $f_0 \in L^\vee$ such that W is given by (3.3.20). Let $D_W := \{[l^2] \mid [l] \in \mathbb{P}(L),\ l(f_0) = 0\}$; thus $D_W \subset \mathbb{P}(W)$. Let $[l^2] \in D_W$: then $[l^2] \in h([f])$ for every $[f] \in \mathbb{P}(\operatorname{Ann}(l))$. It follows that the (smooth) conic D_W is contained in $C_{W,A}$. Applying **Proposition A.1.2** we get that

$$C_{W,A} = 3D_W \qquad \forall\, W \in \Theta_{A_h}. \tag{4.5.6}$$

Equations (4.5.5) and (4.5.6) show that $\mathfrak{x}, \mathfrak{x}^\vee \in \mathfrak{I}$ and that the orbits $\mathrm{PGL}(V)A_k$, $\mathrm{PGL}(V)A_h$ are distinct: since the orbits are minimal it follows that $\mathfrak{x} \neq \mathfrak{x}^\vee$. We have proved (4.5.3).

Table 8: Irreducible components of $\partial\mathfrak{M}$.

	$\mathfrak{B}_{\mathcal{A}}$	$\mathfrak{B}_{\mathcal{C}_1}$	$\mathfrak{B}_{\mathcal{D}}$	$\mathfrak{B}_{\mathcal{E}_1}$	$\mathfrak{B}_{\mathcal{E}_1^\vee}$	$\mathfrak{B}_{\mathcal{F}_1}$	$\mathfrak{B}_{\mathcal{F}_2}$	$\mathfrak{X}_{\mathcal{N}_3}$
dim	1	2	3	2	2	1	5	3
$C_{W,A}$ for $[A] \notin \mathfrak{I}$	II-2, II-4	II-2, II-4	II-1, II-2, II-3	II-2	II-1, II-2, II-3	II-2	?	?
$\mathrm{PGL}(V)A$ closed	III-2	III-2	equiv. to III-2	equiv. to III-2	equiv. to III-2	equiv. to III-2		
$\cdot \cap \mathfrak{I}$	$\emptyset$	$\{\mathfrak{y}\}$	$\mathfrak{X}_{\mathcal{W}}$	$\{\mathfrak{x}\}$	$\{\mathfrak{x}^\vee\}$	$\emptyset$	$\mathfrak{X}_{\mathcal{V}}$	$\mathfrak{X}_{\mathcal{W}} \cup \mathfrak{X}_{\mathcal{Z}}$

5 Description of the GIT-boundary

5.1 Main results

Below are the two main results on the GIT-boundary of $\mathfrak{M}$.

Theorem 5.1.1. *The irreducible irredundant decomposition of $\partial\mathfrak{M}$ is given by* (2.6.5), *i.e. it is*

$$\partial\mathfrak{M} = \mathfrak{B}_{\mathcal{A}} \cup \mathfrak{B}_{\mathcal{C}_1} \cup \mathfrak{B}_{\mathcal{D}} \cup \mathfrak{B}_{\mathcal{E}_1} \cup \mathfrak{B}_{\mathcal{E}_1^\vee} \cup \mathfrak{B}_{\mathcal{F}_1} \cup \mathfrak{B}_{\mathcal{F}_2} \cup \mathfrak{X}_{\mathcal{N}_3}. \tag{5.1.1}$$

The dimensions of the irreducible components are given by the entries in the first row of Table (8).

Next we will be concerned with determining $\partial\mathfrak{M} \cap \mathfrak{I}$. In **Subsection 7.2** we will define a 3-dimensional irreducible closed $\mathfrak{X}_{\mathcal{V}} \subset \mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$ and in **Subsection 7.4** we will define a 1-dimensional irreducible closed $\mathfrak{X}_{\mathcal{Z}} \subset \mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}$. We will prove (see **Remark 7.2.2** and **Proposition 7.4.6**) that

$$\mathfrak{X}_{\mathcal{W}} \subset \mathfrak{X}_{\mathcal{V}}, \qquad \mathfrak{x}, \mathfrak{x}^\vee \in \mathfrak{X}_{\mathcal{Z}}, \qquad \mathfrak{X}_{\mathcal{V}} \cap \mathfrak{X}_{\mathcal{Z}} = \{\mathfrak{y}\}. \tag{5.1.2}$$

Theorem 5.1.2. *The irreducible irredundant decomposition of $\partial\mathfrak{M} \cap \mathfrak{I}$ is given by*

$$\partial\mathfrak{M} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{V}} \cup \mathfrak{X}_{\mathcal{Z}} \tag{5.1.3}$$

The long computations that are needed in order to obtain these results are carried out in **Section 6** and **Section 7**. The first of those sections contains the analysis of all boundary components with the exception of $\mathfrak{B}_{\mathcal{F}_2}$ and $\mathfrak{X}_{\mathcal{N}_3}$, which are analyzed in **Section 7**. The reason for the distinction is that boundary components other than $\mathfrak{B}_{\mathcal{F}_2}$ and $\mathfrak{X}_{\mathcal{N}_3}$ intersect $\mathfrak{I}$ in a subset of the known subset $\mathfrak{X}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$, while in order to determine $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$ and $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}$ one needs to introduce $\mathfrak{X}_{\mathcal{V}}$ and $\mathfrak{X}_{\mathcal{Z}}$. In the present section we will state some of the intermediate results proven in **Section 6** and **Section 7**, and then we will prove that **Theorem 5.1.1** follows from those results. **Theorem 5.1.2** follows at once from the descriptions, given in **Section 6** and **Section 7**, of the intersection of each boundary component with $\mathfrak{I}$ - they are summarized in Table (8).

5.2 A GIT set-up for each standard non-stable stratum

5.2.1 Set-up

Let $\mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{D}, \mathcal{E}_1, \mathcal{E}_1^\vee, \mathcal{F}_1, \mathcal{F}_2, \mathcal{N}_3\}$ i.e. one of the subscripts appearing in (5.1.1). Let $\mathsf{F} = \{v_0, \ldots, v_5\}$ be a basis of V and $\lambda_{\mathcal{X}} \colon \mathbb{C}^\times \longrightarrow \mathrm{SL}(V)$ be the standard ordering 1-PS which is diagonal in the basis F and whose weights appear on the first column of the row of Table (1) that contains $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ (or $\mathbb{X}_{\mathcal{N}_3}$). We let $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ be the set of lagrangians $A \in \mathbb{LG}(\bigwedge^3 V)$ which are $\lambda_{\mathcal{X}}$-split, see **Section 2.6**. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$: then

$$A = A_0 + A_1 + \ldots + A_s \tag{5.2.1}$$

with $A_i \in \mathrm{Gr}(d_i, U_{e_i})$ and $A_{s-i} = A_i^\perp$ (recall that the symplectic form on $\bigwedge^3 V$ defines a perfect pairing between U_{e_i} and $U_{e_{s-i}}$). Thus we have an embedding

$$\mathbb{S}^{\mathsf{F}}_{\mathcal{X}} \hookrightarrow \mathrm{Gr}(d_0, U_{e_0}) \times \mathrm{Gr}(d_1, U_{e_1}) \times \ldots \times \mathbb{LG}(U_0) \times \mathrm{Gr}(d_1, U_{e_{[(s+2)/2]}}) \times \ldots \times \mathrm{Gr}(d_s, U_{e_s}). \tag{5.2.2}$$

Table 9: Parameter spaces for split non-stable lagrangians and the corresponding groups.

$\mathcal{X}$	$\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$	$G_{\mathcal{X}}$
$\mathcal{A}$	$\mathrm{Gr}(5, [v_0] \otimes \bigwedge^2 V_{15})$	$\mathrm{SL}(V_{15})$
$\mathcal{C}_1$	$\mathrm{Gr}(3, \bigwedge^2 V_{02} \otimes V_{35})$	$\mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$
$\mathcal{D}$	$\mathrm{Gr}(3, [v_0] \otimes \bigwedge^2 V_{14}) \times \mathbb{LG}([v_0] \otimes V_{14} \otimes [v_5] \oplus \bigwedge^3 V_{14})$	$\mathbb{C}^\times \times \mathrm{SL}(V_{14})$
$\mathcal{E}_1$	$\mathrm{Gr}(2, [v_0] \otimes V_{12} \otimes V_{35}) \times \mathbb{LG}([v_0] \otimes \bigwedge^2 V_{35} \oplus \bigwedge^2 V_{12} \otimes V_{35})$	$\mathbb{C}^\times \times \mathrm{SL}(V_{12}) \times \mathrm{SL}(V_{35})$
$\mathcal{E}_1^\vee$	$\mathrm{Gr}(2, \bigwedge^2 V_{02} \otimes V_{34}) \times \mathbb{LG}(\bigwedge^2 V_{02} \otimes [v_5] \oplus V_{02} \otimes \bigwedge^2 V_{34})$	$\mathbb{C}^\times \times \mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{34})$
$\mathcal{F}_1$	$\mathbb{LG}(V_{01} \otimes V_{23} \otimes V_{45})$	$\mathrm{SL}(V_{01}) \times \mathrm{SL}(V_{23}) \times \mathrm{SL}(V_{45})$
$\mathcal{F}_2$	$\mathbb{P}(\bigwedge^2 V_{01} \otimes V_{23}) \times \mathbb{G}r(2, \bigwedge^2 V_{01} \otimes V_{45} \oplus V_{01} \otimes \bigwedge^2 V_{23}) \times \mathbb{LG}(V_{01} \otimes V_{23} \otimes V_{45})$	$\mathbb{C}^\times \times \mathrm{SL}(V_{01}) \times \mathrm{SL}(V_{23}) \times \mathrm{SL}(V_{45})$
$\mathcal{N}_3$	$\mathbb{P}([v_0 \wedge v_1] \otimes V_{23}) \times \mathbb{P}([v_0 \wedge v_1 \wedge v_4] \oplus [v_0] \otimes \bigwedge^2 V_{23}) \times$ $\times \mathrm{Gr}(2, [v_1] \otimes \bigwedge^2 V_{23} \oplus [v_0 \wedge v_4] \otimes V_{23} \oplus [v_0 \wedge v_1 \wedge v_5]) \times$ $\times \mathbb{LG}([v_0 \wedge v_5] \otimes V_{23} \oplus [v_1 \wedge v_5] \otimes V_{23})$	$(\mathbb{C}^\times)^3 \times \mathrm{SL}(V_{23})$

with image the set of $(A_0, A_1, \ldots, A_s)$ such that for all i we have $A_{s-i} = A_i^\perp$. Notice that $U_0 = \{0\}$ (i.e. the central factor in (5.2.2) is missing) if $\mathcal{X} \in \{\mathcal{A}, \mathcal{A}^\vee, \mathcal{C}_1, \mathcal{C}_2\}$. The group $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})$ acts naturally on $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$. Table (9) gives a group $G_{\mathcal{X}}$ for each $\mathcal{X}$. Let us define a homomorphism

$$\rho_{\mathcal{X}} \colon G_{\mathcal{X}} \longrightarrow C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}}) \tag{5.2.3}$$

as follows. The group $G_{\mathcal{X}}$ is defined as a direct product of factors and hence it suffices to define a homomorphism from each factor to $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})$. Each factor of $G_{\mathcal{X}}$ is either $\mathrm{SL}(V_{ij})$ where V_{ij} is one of the isotypical summands of $\lambda_{\mathcal{X}}$ or else a torus. The restriction of $\rho_{\mathcal{X}}$ to an $\mathrm{SL}(V_{ij})$-factor is the obvious one. The restriction of $\rho_{\mathcal{X}}$ to a torus factor is as follows. Let $\mathcal{X} = \mathcal{D}$; for $s \in \mathbb{C}^\times$ we let

$$\rho_{\mathcal{D}}(s) = (s^2 \,\mathrm{Id}_{[v_0]}, s^{-1} \,\mathrm{Id}_{V_{14}}, s^2 \,\mathrm{Id}_{[v_5]}). \tag{5.2.4}$$

Let $\mathcal{X} = \mathcal{E}_1, \mathcal{E}_1^\vee$; for $s \in \mathbb{C}^\times$ we let

$$\rho_{\mathcal{E}_1}(s) = (s \,\mathrm{Id}_{[v_0]}, s^{-2} \,\mathrm{Id}_{V_{12}}, s \,\mathrm{Id}_{V_{35}}), \quad \rho_{\mathcal{E}_1^\vee}(s) = (s \,\mathrm{Id}_{V_{02}}, s^{-2} \,\mathrm{Id}_{V_{34}}, s \,\mathrm{Id}_{[v_5]}). \tag{5.2.5}$$

Let $\mathcal{X} = \mathcal{F}_2$; for $s \in \mathbb{C}^\times$ we let

$$\rho_{\mathcal{F}_2}(s) = (s \,\mathrm{Id}_{V_{01}}, s^{-2} \,\mathrm{Id}_{V_{23}}, s \,\mathrm{Id}_{V_{45}}). \tag{5.2.6}$$

Let $\mathcal{X} = \mathcal{N}_3$; for $(s_0, s_1, s_2) \in (\mathbb{C}^\times)^3$ we let

$$\rho_{\mathcal{N}_3}(s_0, s_1, s_2) = (s_0 \,\mathrm{Id}_{[v_0]},\ s_1^2 \,\mathrm{Id}_{[v_1]},\ (s_0^{-1} s_1^{-1} s_2^{-1}) \,\mathrm{Id}_{V_{23}},\ s_2^2 \,\mathrm{Id}_{[v_4]},\ s_0 \,\mathrm{Id}_{[v_5]}). \tag{5.2.7}$$

We have completed the definition of (5.2.3). Composing homomorphism $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}}) \to \mathrm{Aut}(\mathbb{S}^{\mathsf{F}}_{\mathcal{X}})$ with $\rho_{\mathcal{X}}$ we get an action of $G_{\mathcal{X}}$ on $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$. The $G_{\mathcal{X}}$-action is naturally linearized by the embedding of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ in $\mathbb{LG}(\bigwedge^3 V)$.

Claim 5.2.1. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$. Then A is $\mathrm{SL}(V)$-semistable if and only if it is $G_{\mathcal{X}}$-semistable, moreover $\mathrm{SL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ if and only if $G_{\mathcal{X}}A$ is closed in $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{X}}$. Lastly the inclusion of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ in $\mathbb{LG}(\bigwedge^3 V)$ induces* finite surjective maps

$$\mathbb{S}^{\mathsf{F}}_{\mathcal{X}} // G_{\mathcal{X}} \twoheadrightarrow \mathfrak{B}_{\mathcal{X}} \quad (\mathcal{X} \neq \mathcal{N}_3), \qquad \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3} // G_{\mathcal{N}_3} \twoheadrightarrow \mathfrak{X}_{\mathcal{N}_3}.$$

Proof. Let λ be a 1-PS which is diagonal in the basis F and whose set of weights appears in the first column of Table (1): by **Remark 2.2.3** the fixed locus $\mathbb{P}(\widehat{\mathbb{LG}}(\bigwedge^3 V)^\lambda)$ is the disjoint union of the $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ such that $\lambda_{\mathcal{X}} = \lambda$. As is easily checked the centralizer $C_{\mathrm{SL}(V)}(\lambda)$ has finite index in $N_{\mathrm{SL}(V)}(\lambda)$. By **Corollary 4.2.2** we get that inclusion induces a *finite surjective* map

$$\mathbb{S}^{\mathsf{F}}_{\mathcal{X}} // C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}}) \twoheadrightarrow \mathfrak{B}_{\mathcal{X}} \tag{5.2.8}$$

for every $\mathcal{X}$ and that if $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ then $\mathrm{SL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ if and only if $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})A$ is closed in $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{X}}$. We claim that for our purposes the action of $G_{\mathcal{X}}$ is equivalent to that of $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})$. Suppose first that $\mathcal{X} \neq \mathcal{F}_1$. Then the homomorphism

$$G_{\mathcal{X}} \longrightarrow C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})/\lambda_{\mathcal{X}} \tag{5.2.9}$$

induced by $\rho_{\mathcal{X}}$ (see (5.2.3)) is surjective with finite kernel; since $\lambda_{\mathcal{X}}$ acts trivially on $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{X}}$ we get the claim (for $\mathcal{X} \neq \mathcal{F}_1$). On the other hand if $\mathcal{X} = \mathcal{F}_1$ the subgroup

$$H_{\mathcal{F}_1} := \{(\alpha\,\mathrm{Id}_{V_{01}}, \beta\,\mathrm{Id}_{V_{23}}, \gamma\,\mathrm{Id}_{V_{45}}) \mid \alpha\beta\gamma = 1\} \tag{5.2.10}$$

of $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{F}_1})$ acts trivially on $\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$: since the restriction to $G_{\mathcal{F}_1}$ of the quotient map

$$C_{\mathrm{SL}(V)}(\lambda_{\mathcal{F}_1}) \longrightarrow C_{\mathrm{SL}(V)}(\lambda_{\mathcal{X}})/H_{\mathcal{F}_1}$$

is surjective with finite kernel the claim follows for $\mathcal{X} = \mathcal{F}_1$ as well. □

Remark 5.2.2. For each $\mathcal{X}$ we will give a list of flag conditions which are equivalent to $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ being $G_{\mathcal{X}}$-stable. In some cases, namely $\mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{E}_1, \mathcal{E}_1^{\vee}, \mathcal{F}_1\}$, we will show that the flag conditions have a nice translation into a simple geometric condition, usually of the type "a certain curve of arithmetic genus 1 associated to A is non-singular"- this it to be expected because the Baily-Borel boundary components of Type II are parametrized by the upper half-space $\mathbb{H}_1$ modulo an arithemtic group. We will not list all the closed orbits of properly $G_{\mathcal{X}}$-semistable points except for $\mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{F}_1\}$: the analysis could be carried out but is beyond what we wish to do - we beleive that it is more interesting to determine $\partial\mathfrak{M} \cap \mathfrak{I}$ in order to understand the period map $\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathbb{D}^{BB}$.

5.2.2 The Hilbert-Mumford numerical function

We will give formulae for the Hilbert-Mumford numerical function that will be handy later on. Let $\mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{D}, \mathcal{E}_1, \mathcal{E}_1^{\vee}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{N}_3\}$. The action of $G_{\mathcal{X}}$ on $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ is of the kind discussed in **Subsection 2.2**. Let $\lambda\colon \mathbb{C}^{\times} \to G_{\mathcal{X}}$ be a 1-PS of $G_{\mathcal{X}}$ and $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$: below we will make a few comments on the numerical function $\mu(A, \lambda)$. We may write

$$\bigwedge^3 \lambda = (\alpha_0, \alpha_1, \ldots, \alpha_s), \qquad \alpha_i\colon \mathbb{C}^{\times} \longrightarrow GL(U_{e_i}). \tag{5.2.1}$$

Abusing notation we will set

$$\mu(A_i, \lambda) := \mu(A_i, \alpha_i). \tag{5.2.2}$$

Definition 5.2.3. Keeping notation and hypotheses as above let $I_+(\lambda) \subset \{0, \ldots, s\}$ be the set of i such that

$$\mathrm{im}\,\alpha_i \subset \mathrm{SL}(U_{e_i}). \tag{5.2.3}$$

Let $I_-(\lambda) := \{0, \ldots, s\} \setminus I_+(\lambda)$.

Claim 5.2.4. *Keep notation and hypotheses as above. Suppose that $i \in I_+(\lambda)$. Then*

$$\mu(A_i, \lambda) = \mu(A_{s-i}, \lambda). \tag{5.2.4}$$

Proof. A straightforward computation similar to that which proves **Claim 2.2.7**. □

Claim 5.2.4 and (2.2.4) give that

$$\mu(A, \lambda) = \sum_{I_+(\lambda) \ni i < s/2} 2\mu(A_i, \lambda) + \sum_{i \in I_-(\lambda)} \mu(A_i, \lambda). \tag{5.2.5}$$

5.3 Summary of results of Sections 6 and 7

Below are the results proved in **Section 6** and **Section 7** that are needed in order to prove **Theorem 5.1.1** and **Theorem 5.1.2**. Let $\mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{D}, \mathcal{E}_1, \mathcal{E}_1^\vee, \mathcal{F}_1, \mathcal{F}_2, \mathcal{N}_3\}$ i.e. one of the subscripts appearing in (5.1.1). Then the following hold:

(1) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ is $G_{\mathcal{X}}$-stable.

(2) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ be $G_{\mathcal{X}}$-stable. The connected component of Id in $\mathrm{Stab}(A) < \mathrm{SL}(V)$ is equal to $\lambda_{\mathcal{X}}$ if $\mathcal{X} \neq \mathcal{F}_1$ and is equal to $H_{\mathcal{F}_1}$ (see (5.2.10)) if $\mathcal{X} = \mathcal{F}_1$.

(3) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ have closed $\mathrm{PGL}(V)$-orbit (in $\mathbb{LG}(\bigwedge^3 V)^{ss}$) and suppose that $[A] \notin \mathfrak{I}$. Then $C_{W,A}$ is described by the corresponding column of Table (8).

(4) $\mathfrak{B}_{\mathcal{X}} \cap \mathfrak{I}$ (or $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}$ if $\mathcal{X} = \mathcal{N}_3$) is described by the corresponding column of Table (8).

5.4 Proof of Theorem 5.1.1 assuming the results of Sections 6 and 7

5.4.1 Dimensions

The dimensions appearing in Table (8) are obtained as follows. For each $\mathcal{X}$ the generic point of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ is $G_{\mathcal{X}}$-stable (see **Subsection 5.3**). By **Claim 5.2.1** we get that

$$\dim \mathfrak{B}_{\mathcal{X}} = \dim(\mathbb{S}^{\mathsf{F}} // G_{\mathcal{X}}) = \dim \mathbb{S}^{\mathsf{F}}_{\mathcal{X}} - \dim G_{\mathcal{X}}. \tag{5.4.1}$$

The dimensions of $\mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ and $\dim G_{\mathcal{X}}$ are easily computed from Table (9): plugging the dimensions in (5.4.1) we get the dimensions appearing in Table (8).

5.4.2 No inclusion relations

We will show that no set appearing on the right-hand side of (5.1.1) is contained in another set on the right-hand side of (5.1.1). Suppose first that

$$\mathfrak{B}_{\mathcal{X}} \subset \mathfrak{B}_{\mathcal{Y}} \text{ (or } \mathfrak{B}_{\mathcal{X}} \subset \mathfrak{X}_{\mathcal{N}_3}) \text{ for } \mathcal{X} \in \{\mathcal{A}, \mathcal{C}_1, \mathcal{D}, \mathcal{E}_1, \mathcal{E}_1^\vee\} \text{ and } \mathcal{Y} \neq \mathcal{X}. \tag{5.4.2}$$

We will reach a contradiction. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{X}}$ be $G_{\mathcal{X}}$-stable. Then the orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ by **Claim 5.2.1**. By (5.4.2) it follows that there exists $A' \in \mathrm{PGL}(V)A$ which belongs to $\mathbb{S}^{\mathsf{F}}_{\mathcal{Y}}$. Since $\lambda_{\mathcal{Y}}$ acts trivially on $\bigwedge^{10} A'$ the connected component of Id in $\mathrm{Stab}(A') < \mathrm{SL}(V)$ contains $\mathrm{im}\,\lambda_{\mathcal{Y}}$: by Item (2) of **Subsection 5.3** we get that the subgroups $\mathrm{im}\,\lambda_{\mathcal{X}}, \mathrm{im}\,\lambda_{\mathcal{Y}} < \mathrm{SL}(V)$ are conjugated. Looking at Table (1) we get at once that $\{\mathcal{X}, \mathcal{Y}\} = \{\mathcal{E}_1, \mathcal{E}_1^\vee\}$ and hence $\mathfrak{B}_{\mathcal{E}_1} = \mathfrak{B}_{\mathcal{E}_1^\vee}$. That is absurd because the last row of Table (8) gives that

$$\mathfrak{B}_{\mathcal{E}_1} \cap \mathfrak{I} = \{\mathfrak{x}\} \neq \{\mathfrak{x}^\vee\} = \mathfrak{B}_{\mathcal{E}_1^\vee} \cap \mathfrak{I}. \tag{5.4.3}$$

This proves that (5.4.2) does not hold. Now consider the remaining $\mathcal{X}$ i.e. $\mathcal{X} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{N}_3\}$. Looking at the dimensions given by Table (8) we see that it remains to rule out one of the following inclusions:

$$\mathfrak{X}_{\mathcal{N}_3} \subset \mathfrak{B}_{\mathcal{D}}, \quad \mathfrak{X}_{\mathcal{N}_3} \subset \mathfrak{B}_{\mathcal{F}_2}, \quad \mathfrak{B}_{\mathcal{F}_1} \subset \mathfrak{B}_{\mathcal{X}} \ (\mathcal{X} \neq \mathcal{F}_1, \mathcal{N}_3), \quad \mathfrak{B}_{\mathcal{F}_1} \subset \mathfrak{X}_{\mathcal{N}_3}.$$

Suppose that $\mathfrak{X}_{\mathcal{N}_3} \subset \mathfrak{B}_{\mathcal{D}}$: since $\mathfrak{X}_{\mathcal{N}_3}$, $\mathfrak{B}_{\mathcal{D}}$ are closed, irreducible of the same dimension it follows that $\mathfrak{X}_{\mathcal{N}_3} = \mathfrak{B}_{\mathcal{D}}$, and we have proved above that this is impossible. Next, $\mathfrak{X}_{\mathcal{N}_3} \subset \mathfrak{B}_{\mathcal{F}_2}$ cannot hold because the last row of Table (8) gives that

$$\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{V}}, \qquad \mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I} = (\mathfrak{X}_{\mathcal{W}} \cup \mathfrak{X}_{\mathcal{Z}})$$

and $\mathfrak{X}_{\mathcal{Z}} \not\subset \mathfrak{X}_{\mathcal{V}}$ (see (5.1.2)). It remains to deal with $\mathfrak{B}_{\mathcal{F}_1}$. Suppose that $\mathfrak{B}_{\mathcal{F}_1} \subset \mathfrak{B}_{\mathcal{Y}}$ where $\mathcal{Y} \neq \mathcal{F}_1$ or $\mathfrak{B}_{\mathcal{F}_1} \subset \mathfrak{X}_{\mathcal{N}_3}$. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ be $G_{\mathcal{F}_1}$-stable. Then the orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ by **Claim 5.2.1**. Arguing as in the proof that (5.4.2) does not hold we get that $\bigwedge^{10} A$ is left invariant by a subgroup $G < \mathrm{SL}(V)$ conjugated to $\mathrm{im}\,\lambda_{\mathcal{Y}}$. By Item (2) of **Subsection 5.3** we get that $G < H_{\mathcal{F}_1}$. Going through Table (1) we see that we must have $\mathcal{Y} = \mathcal{F}_2$. It follows that the $\lambda_{\mathcal{F}_1}$-type of A is $(1,2)$ (by definition of $\mathbb{S}^{\mathsf{F}'}_{\mathcal{F}_2}$ for an arbitarry base F' of V) and not $(2,0)$ as we know it is by definition of $\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$.

6 Boundary components meeting $\mathfrak{I}$ in a subset of $\mathfrak{X}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$

6.1 $\mathfrak{B}_{\mathcal{C}_1}$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$; by definition

$$A = \bigwedge^3 V_{02} \oplus A' \oplus A'', \qquad A' \in \mathrm{Gr}(3, \bigwedge^2 V_{02} \wedge V_{35}), \quad A'' = (A')^\perp \cap (V_{02} \wedge \bigwedge^2 V_{35}). \tag{6.1.1}$$

Thus A', A'' are the summands of A which were named A_1, A_2 in **Subsection 5.2**. We choose a volume-form on V_{02} in order to have an identification $\bigwedge^2 V_{02} \wedge V_{35} \xrightarrow{\sim} \mathrm{Hom}(V_{02}, V_{35})$. Let $A' \in \mathrm{Gr}(3, \bigwedge^2 V_{02} \wedge V_{35})$. We let

$$E_{A'} := \{[\alpha] \in \mathbb{P}(A') \mid \mathrm{rk}\, \alpha \le 2\}$$

with its obvious scheme structure; thus $E_{A'}$ is either all of $\mathbb{P}(A')$ or a cubic curve. Below is the main result of the present subsection.

Proposition 6.1.1. *The following hold:*

(1) $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ is $G_{\mathcal{C}_1}$-stable if and only if $E_{A'}$ is a smooth curve.

(2) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ is $G_{\mathcal{C}_1}$-stable.

(3) If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ is $G_{\mathcal{C}_1}$-stable the connected component of Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to* $\mathrm{im}\,\lambda_{\mathcal{C}_1}$.

(4) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ have closed $\mathrm{PGL}(V)$-orbit (in $\mathbb{LG}(\bigwedge^3 V)^{ss}$), and suppose that $[A] \notin \mathfrak{I}$. Then $C_{W,A}$ is of Type II-2, II-4 or III-2.

(5) $\mathfrak{B}_{\mathcal{C}_1} \cap \mathfrak{I} = \{\mathfrak{y}\}$ where $\mathfrak{y}$ is defined by (4.4.7).

The proof of **Proposition 6.1.1** is given in **Subsubsection 6.1.5**.

6.1.1 First results

We claim that

$$\mathfrak{y} \in \mathfrak{B}_{\mathcal{C}_1}. \tag{6.1.2}$$

In fact let U be a complex vector-space of dimension 4, and choose an isomorphism $V \cong \bigwedge^2 U$; then $\mathfrak{y} = [A_+(U)]$, where $A_+(U)$ is given by (2.4.12). If $W \in \Theta_{A_+(U)}$ the affine cone over the projective tangent space to $\Theta_{A_+(U)}$ at W is contained in $A_+(U) \cap S_W$. Since $\dim \Theta_{A_+(U)} = 3$ it follows that $\dim(A_+(U) \cap S_W) \ge 4$ (in fact equality holds because otherwise $A_+(U)$ is unstable by Table (2)): this proves that (6.1.2) holds. Actually the above argument shows that for suitable U and isomorphism $V \cong \bigwedge^2 U$ we have

$$A_+(U) \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}. \tag{6.1.3}$$

(A priori this result is stronger than (6.1.2), but in fact it is equivalent by **Corollary 4.2.2**.) Next we notice that there are subschemes of $\mathbb{P}(V_{02})$ and $\mathbb{P}(V_{35}^\vee)$ which are related to $E_{A'}$. First A' defines a map $\varphi_{A'} \colon A' \otimes \mathcal{O}_{\mathbb{P}(V_{02})}(-1) \longrightarrow V_{35} \otimes \mathcal{O}_{\mathbb{P}(V_{02})}$ of locally-free sheaves. Similarly taking the transpose of elements of A' we get a map $\psi_{A'} \colon A' \otimes \mathcal{O}_{\mathbb{P}(V_{35}^\vee)}(-1) \longrightarrow V_{02} \otimes \mathcal{O}_{\mathbb{P}(V_{02})}$. Let

$$J_{V_{02}}(A') := \mathrm{div}(\det \varphi_{A'}), \qquad J_{V_{35}^\vee}(A') := \mathrm{div}(\det \psi_{A'}).$$

Thus $J_{V_{02}}(A')$ is either all of $\mathbb{P}(V_{02})$ or a cubic curve and similarly for $J_{V_{35}^\vee}(A')$. If $E_{A'}$ is smooth then it is isomorphic to $J_{V_{02}}(A')$ and to $J_{V_{35}^\vee}(A')$. By **Corollary 3.3.7** we have the following:

$$C_{V_{02},A} \text{ is equal to } \mathbb{P}(V_{02}) \text{ or to } 2J_{V_{02}}(A') \tag{6.1.4}$$

Claim 6.1.2. *Let $\overline{A} \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ and suppose that $E_{\overline{A}'}$ is a smooth curve. Then $\overline{A}$ is $G_{\mathcal{C}_1}$-stable. In particular the generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ is $G_{\mathcal{C}_1}$-stable.*

Proof. Recall that $G_{\mathcal{C}_1} = \mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$. Consider the $\mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$-equivariant rational map

$$\begin{array}{ccc} \mathrm{Gr}(3, \bigwedge^2 V_{02} \wedge V_{35}) & \overset{f}{\dashrightarrow} & |\mathcal{O}_{\mathbb{P}(V_{02})}(3)| \times |\mathcal{O}_{\mathbb{P}(V_{35}^\vee)}(3)| \\ A' & \mapsto & (J_{V_{02}}(A'), J_{V_{35}^\vee}(A')) \end{array}$$

Since $E_{\overline{A}'}$ is a smooth curve so are $J_{V_{02}}(\overline{A}')$ and $J_{V_{35}^\vee}(\overline{A}')$. Thus f is regular at $E_{\overline{A}'}$ and it maps to a stable point for the $\mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$-action on $|\mathcal{O}_{\mathbb{P}(V_{02})}(3)| \times |\mathcal{O}_{\mathbb{P}(V_{35}^\vee)}(3)|$ linearized on $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ where $\mathcal{L}_1$, $\mathcal{L}_2$ are the ample generators of $\mathrm{Pic}(|\mathcal{O}_{\mathbb{P}(V_{02})}(3)|)$ and $\mathrm{Pic}(|\mathcal{O}_{\mathbb{P}(V_{35}^\vee)}(3)|)$ respectively. It follows that $E_{\overline{A}'}$ is $\mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$-stable, say by Proposition 1.18, p. 44 of [23] applied to the complement of the indeterminacy locus of f. It is clear that for A generic $E_{A'}$ is a smooth curve and hence A is $G_{\mathcal{C}_1}$-stable. □

6.1.2 Properly semistable points of $\mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$

We will analyze the $G_{\mathcal{C}_1}$-properly semistable points of $\mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$. First we will write out the Hilbert-Mumford numerical function of $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ with respect to a 1-PS $\lambda\colon \mathbb{C}^\times \longrightarrow G_{\mathcal{C}_1} = \mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$. Let $e'_0 > \ldots > e'_j$ be the weights of the action of $\mathbb{C}^\times$ on $\bigwedge^2 V_{02} \wedge V_{35}$ defined by λ. Since $I_-(\lambda) = \emptyset$ (see **Definition 5.2.3**) Equations (5.2.5) and (2.2.9) give that

$$\mu(A, \lambda) = 2\mu'(A', \lambda) = 2\sum_{i=0}^{j} d_i^\lambda(A') e'_i. \tag{6.1.5}$$

Next we will define a closed subset $\mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ - later on we will show that it contains every minimal orbit of $G_{\mathcal{C}_1}$-properly semistable point of $\mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$. Given $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $p_i = [a_i, b_i]$ we let $A'_{\mathbf{p}} \in \mathrm{Gr}(3, \bigwedge^2 V_{02} \wedge V_{35})$ be given by

$$A'_{\mathbf{p}} := \langle a_1 v_0 \wedge v_2 \wedge v_3 + b_1 v_1 \wedge v_2 \wedge v_4, a_2 v_1 \wedge v_2 \wedge v_5 + b_2 v_0 \wedge v_1 \wedge v_3, a_3 v_0 \wedge v_1 \wedge v_4 + b_3 v_0 \wedge v_2 \wedge v_5 \rangle \tag{6.1.6}$$

We let $A''_{\mathbf{p}} := (A'_{\mathbf{p}})^\perp \cap (V_{02} \wedge \bigwedge^2 V_{35})$. Explicitly

$$A''_{\mathbf{p}} = \langle v_0 \wedge v_4 \wedge v_5, v_1 \wedge v_3 \wedge v_5, v_2 \wedge v_3 \wedge v_4, (b_1 v_1 \wedge v_4 - a_1 v_0 \wedge v_3) \wedge v_5, (b_2 v_0 \wedge v_3 + a_2 v_2 \wedge v_5) \wedge v_4, (b_3 v_2 \wedge v_5 - a_3 v_1 \wedge v_4) \wedge v_3 \rangle \tag{6.1.7}$$

We have an embedding

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \overset{\iota^{\mathsf{F}}}{\hookrightarrow} & \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1} \\ \mathbf{p} & \mapsto & A_{\mathbf{p}} := (\bigwedge^3 V_{02} \oplus A'_{\mathbf{p}} \oplus A''_{\mathbf{p}}) \end{array} \tag{6.1.8}$$

Let $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1} := \mathrm{im}(\iota^{\mathsf{F}})$. The closed subset $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ is fixed by a certain torus in $G_{\mathcal{C}_1}$ that we proceed to define. Let $T' < \mathrm{SL}(V_{02})$ and $T'' < \mathrm{SL}(V_{35})$ be the maximal tori which are diagonalized in the bases $\{v_0, v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ respectively. (We recall that $\lambda_{\mathcal{C}_1}$ is diagonal in the basis $\mathsf{F} = \{v_0, \ldots, v_5\}$.) Let $T_\star < T' \times T''$ be the torus

$$T_\star := \{(g, h) \in T' \times T'' \mid g(v_i) = s_i v_i,\ 0 \le i \le 2, \quad h(v_j) = s_{j-3}^{-1} v_j,\ 3 \le j \le 5\} \tag{6.1.9}$$

A straightforward computation gives the following result.

Claim 6.1.3. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$: then $\bigwedge^{10} A$ is fixed by $T_\star$ if and only if $A \in \mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ or*

$$A' = \langle v_0 \wedge v_1 \wedge v_5, v_0 \wedge v_2 \wedge v_4, v_1 \wedge v_2 \wedge v_3 \rangle.$$

Claim 6.1.4. *If $\mathbf{p} = ([1,0],[1,0],[1,0])$ or $\mathbf{p} = ([0,1],[0,1],[0,1])$ then $A_{\mathbf{p}} \in \mathrm{PGL}(V)A_{III}$.*

Proof. A computation gives a monomial basis of $A_{\mathbf{p}}$. Let ω be a generator of $\bigwedge^{10} A_{\mathbf{p}}$. Let $T < \mathrm{SL}(V)$ be the maximal torus diagonalized in the basis F. One checks that $g\omega = \omega$ for every $g \in T$ and hence the result follows from **Claim 4.3.1**. □

Proposition 6.1.5. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ be semistable and suppose that $E_{A'}$ is not a smooth curve. Then A is not $G_{\mathcal{C}_1}$-stable (i.e. properly semistable) and it is $\mathrm{PGL}(V)$-equivalent to an element of $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$.*

Proof. Suppose first that A' contains a non-zero decomposable element. Then there exist a subspace $U \subset V_{02}$ of dimension 2 and $0 \neq z_0 \in V_{35}$ such that $\bigwedge^2 U \wedge [z_0] \subset A'$. Choose direct-sum decompositions

$$V_{02} = [u_0] \oplus U, \qquad V_{35} = [z_0] \oplus Z. \tag{6.1.10}$$

Let λ be the 1-PS of $G_{\mathcal{C}_1}$ defined by

$$\lambda(t)u_0 = t^{-2}u_0, \quad \lambda(t)|_U = t\,\mathrm{Id}_U, \quad \lambda(t)z_0 = t^2 z_0, \quad \lambda(t)|_Z = t^{-1}\,\mathrm{Id}_Z\,. \tag{6.1.11}$$

The isotypical summands of the action of λ on $\bigwedge^2 V_{02} \wedge V_{35}$ are the following:

$$\begin{array}{ccc} \bigwedge^2 U \wedge [z_0] & ([u_0] \wedge U \wedge [z_0] \oplus \bigwedge^2 U \wedge Z) & [u_0] \wedge U \wedge Z \\ t^4 & t & t^{-2} \end{array} \tag{6.1.12}$$

The λ-type of A' is $(1, d_1', d_2')$ with $d_1' + d_2' = 2$. Thus $\mu(A', \lambda) = 6 - 3d_2'$. By (6.1.5) we get that A' is not $G_{\mathcal{C}_1}$-stable and that $d_2' = 2$ (because by hypothesis A is semistable). Moreover **Claim 2.2.4** gives that A is $G_{\mathcal{C}_1}$-equivalent to

$$A_0 = \bigwedge^3 V_{02} \oplus (\bigwedge^2 U \wedge [z_0] \oplus H) \oplus (\bigwedge^2 U \wedge [z_0] \oplus H)^{\perp} \cap (V_{02} \wedge \bigwedge^2 V_{35}), \quad H \in \mathrm{Gr}(2, [u_0] \wedge U \wedge Z).$$

The intersection $\mathrm{Gr}(3, [u_0] \oplus U \oplus Z) \cap \mathbb{P}([u_0] \wedge U \wedge Z)$ is a quadric hypersurface: it follows that the intersection $\mathbb{P}(H) \cap \mathrm{Gr}(3, [u_0] \oplus U \oplus Z)$ is one of the following:

(1) a set with exactly two elements,

(2) a set with exactly one element,

(3) a line.

Suppose that (1) holds: there exist bases $\{u_1, u_2\}$, $\{z_1, z_2\}$ of U and Z respectively such that $H = \langle u_0 \wedge u_1 \wedge z_1, u_0 \wedge u_2 \wedge z_2 \rangle$. A straightforward computation gives that A is $A_{III}^{\mathsf{F}'}$ for some basis F' of V - see **Claim 4.3.1**. By **Claim 6.1.4** we get that A is $\mathrm{PGL}(V)$-equivalent to $A_{\mathbf{p}}$ for $\mathbf{p}$ equal to $([1,0],[1,0],[1,0])$ or $([0,1],[0,1],[0,1])$. If (2) or (3) above hold then A_0 is in the closure of the set of A's for which Item (1) holds and hence it belongs to the orbit $\mathrm{SL}(V)A_{III}^{\mathsf{F}}$ by **Proposition 4.3.2**. This settles the case of A' containing a non-zero decomposable element. Now assume that $E_{A'}$ is not a smooth curve but it does not contain non-zero decomposable elements. Then there exists $[\alpha] \in E_{A'}$ such that

$$\dim T_{[\alpha]} E_{A'} = 2. \tag{6.1.13}$$

In what follows we will identify $\bigwedge^2 V_{02} \wedge V_{35}$ with $\mathrm{Hom}(V_{02}, V_{35})$. By hypothesis $\mathrm{rk}\,\alpha = 2$; let $[u_0] = \ker \alpha$. Equation (6.1.13) is equivalent to $\beta(u_0) \in \mathrm{im}\,\alpha$ for all $\beta \in A'$. Let $Z := \mathrm{im}\,\alpha$; by hypothesis $\dim Z = 2$. Choose direct-sum decompositions as in (6.1.10). Let λ be the 1-PS of $G_{\mathcal{C}_1}$ defined by (6.1.11) and λ^{-1} its inverse: $\lambda^{-1}(t) := \lambda(t^{-1})$. Replacing each weight appearing in (6.1.12) by its opposite we get the isotypical decomposition of the representation of λ^{-1} on $\bigwedge^2 V_{02} \wedge V_{35}$. Notice that $\alpha \in [u_0] \wedge U \wedge Z$ and that A' is contained in the second term of the λ^{-1}-weight filtration of $\bigwedge^2 V_{02} \wedge V_{35}$. It follows that the λ^{-1}-type of A' is $(d_0', 3 - d_0', 0)$ where $d_0' \geq 1$ and hence $\mu(A', \lambda^{-1}) = 3d_0' - 3 \geq 0$. By (6.1.5) we get that A is not $G_{\mathcal{C}_1}$-stable and that its λ^{-1}-type is $(1,2,0)$ (because it is semistable by hypothesis). Moreover **Claim 2.2.4** gives that if A is $G_{\mathcal{C}_1}$-equivalent to

$$A_0 = \bigwedge^3 V_{02} \oplus A_0' \oplus (A_0')^{\perp} \cap V_{02} \wedge \bigwedge^2 V_{35} \text{ where } A_0' \text{ is } \lambda^{-1}\text{-split of type } (1,2,0).$$

Let α_0 be a generator of $A_0' \cap [u_0] \wedge U \wedge Z$ and $\{\beta_0, \gamma_0\}$ be a basis of $A_0' \cap ([u_0] \wedge U \wedge [z_0] \oplus \bigwedge^2 U \wedge Z)$; a straightforward computation gives that $\det(x\alpha_0 + y\beta_0 + w\gamma_0) = x\phi(y, w)$ where $\phi \in \mathbb{C}[y, w]_2$. Suppose first that the zero-locus $V(\phi)$ is either all of $\mathbb{C}^2$ or the union of two distinct lines. Let

(y_1, w_1) and (y_2, w_2) be linearly independent solutions of $\phi(y, w) = 0$. We let $\delta_i := y_i\beta_0 + w_i\gamma_0$ for $i = 1, 2$. We may choose bases $\{u_1, u_2\}$, $\{z_1, z_2\}$ of U and Z respectively such that

$$\alpha_0 = u_0 \wedge u_2 \wedge z_1 + u_0 \wedge u_1 \wedge z_2, \quad \delta_1 = u_1 \wedge u_2 \wedge z_1 + au_0 \wedge u_1 \wedge z_0, \quad \delta_2 = u_1 \wedge u_2 \wedge z_2 + bu_0 \wedge u_2 \wedge z_0. \tag{6.1.14}$$

It follows at once that there exists $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $A_{\mathbf{p}}$ is $\mathrm{SL}(V)$-equivalent to A. Lastly suppose that the zero-locus $V(\phi)$ is a single line (with multiplicity 2). Arguing as above we get a basis of A'_0 given by

$$u_0 \wedge u_2 \wedge z_1 + u_0 \wedge u_1 \wedge z_2, \quad u_1 \wedge u_2 \wedge z_1 + au_0 \wedge u_1 \wedge z_0, \quad u_1 \wedge u_2 \wedge z_2 + bu_0 \wedge u_2 \wedge z_0 + cu_0 \wedge u_1 \wedge z_0.$$

Let $g \in GL(V)$ be defined by $g(u_i) = v_{2-i}$, $g(z_0) = v_5$, $g(z_1) = v_3$ and $g(z_2) = v_4$. Consider the torus $g^{-1}T_\star g$ where $T_\star$ is defined by (6.1.9); applying it to A'_0 we get as limit a subspace generated by $\alpha_0, \delta_1, \delta_2$ given by (6.1.14) and hence we are done again. $\square$

Next we notice that $T' \times T''$ maps $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ to itself and hence it acts on $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$.

Corollary 6.1.6. *The inclusion $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1} \hookrightarrow \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ induces a finite map*

$$\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}//T' \times T'' \longrightarrow \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}//\,\mathrm{SL}(V_{02} \times \mathrm{SL}(V_{35}). \tag{6.1.15}$$

with image the equivalence classes of $G_{\mathcal{C}_1}$-properly semistable points.

Proof. The product $T' \times T''$ is of finite index in the normalizer of $T_\star$ in $\mathrm{SL}(V_{02}) \times \mathrm{SL}(V_{35})$, hence the corollary follows from **Claim 6.1.3** and **Corollary 4.2.2**. $\square$

We define an action of T' on $(\mathbf{P}^1)^3$ as follows. Let $g \in T'$ be given by $g(v_i) = s_i v_i$ for $0 \le i \le 2$, and $([a_1, b_1], [a_2, b_2], [a_3, b_3])$: then

$$g([a_1, b_1], [a_2, b_2], [a_3, b_3]) = [s_1^{-1}a_1, s_0^{-1}b_1], [s_0^{-1}a_2, s_2^{-1}b_2], [s_2^{-1}a_3, s_1^{-1}b_3]) \tag{6.1.16}$$

A straightforward computation gives that (6.1.8) induces an isomorphism

$$(\mathbf{P}^1)^3//T' \cong \mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}//T' \times T''. \tag{6.1.17}$$

(Recall that $T_\star$ acts trivially on $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$.) The quotient $(\mathbf{P}^1)^3//T'$ is isomorphic to $\mathbb{P}^1$ via the map

$$\begin{array}{ccc} (\mathbf{P}^1)^3 & \longrightarrow & \mathbb{P}^1 \\ ([a_1, b_1], [a_2, b_2], [a_3, b_3]) & \mapsto & [a_1a_2a_3, b_1b_2b_3] \end{array} \tag{6.1.18}$$

Before stating the next result we notice that if $\mathbf{p} \in (\mathbb{P}^1)^3$ and $\{f, g, h\}$ is the basis of $A'_{\mathbf{p}}$ given by the elements on the right-hand side of (6.1.6) then

$$E_{A'_{\mathbf{p}}} = V(\det(xf + yg + zh)) = V((a_1a_2a_3 + b_1b_2b_3)xyz). \tag{6.1.19}$$

Corollary 6.1.7. *If $\mathbf{q} = ([1, 1], [1, -1], [1, 1])$ then $A_{\mathbf{q}} \in \mathrm{PGL}(V)A_+$.*

Proof. By (6.1.3) we know that $A_+(U) \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ for some choice of 4-dimensional vector-space U and isomorphism $V \cong \bigwedge^2 U$. We claim that $E_{A'_+(U)} = \mathbb{P}(A'_+(U))$; in fact one may easily give an isomorphism $V_{35} \cong V_{02}^\vee$ such that $A'_+(U) \subset \mathrm{Hom}(V_{02}, V_{35})$ consists of the subspace of skew-symmetric maps. By **Proposition 6.1.5** it follows that there exists $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $A_{\mathbf{p}} \in \mathrm{PGL}(V)A_+(U)$. Let $\mathbf{p} = ([a_1, b_1], [a_2, b_2], [a_3, b_3])$; since $E_{A'_+(U)} = \mathbb{P}(A'_+(U))$ Equation (6.1.19) gives that

$$a_1a_2a_3 + b_1b_2b_3 = 0.$$

Since $A_+(U)$ is $\mathrm{PGL}(V)$-semistable the point $\mathbf{p}$ is T'-semistable by **Corollary 4.2.2**, and hence $a_1a_2a_3 \neq 0 \neq b_1b_2b_3$ because (6.1.18) is the T'-quotient map. Thus we may assume that $1 = a_1 = a_2 = a_3$ and hence $b_1b_2b_3 = -1$. As is easily checked it follows that there exists $g \in T'$ such that $T'\mathbf{p} = \mathbf{q}$. $\square$

6.1.3 Semistable lagrangians A with $\dim \Theta_A \geq 2$ or $C_{W,A} = \mathbb{P}(W)$.

We will prove results that will be used several times in order to describe $C_{W,A}$.

Lemma 6.1.8. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ and suppose that $\dim \Theta_A \geq 2$. Then A is $\mathrm{PGL}(V)$-equivalent to an element of*

$$\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k \cup \mathrm{PGL}(V)A_h. \tag{6.1.20}$$

On the other hand if A belongs to (6.1.20) then $\dim \Theta_A \geq 2$.

Proof. Suppose that $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ and that $\dim \Theta_A \geq 2$. By Theorem 2.26 and Theorem 2.36 of [28] it follows that either A itself belongs to (6.1.20) or else there exist an isomorphism $V \cong \bigwedge^2 U$ and a singular quadric $Z \subset \mathbb{P}(U)$ such that $\mathbb{P}(A) \supset \langle i_+(Z) \rangle$. By **Proposition 4.4.7** we get that A is $\mathrm{PGL}(V)$-equivalent to an element of (6.1.20). Now suppose that A belongs to (6.1.20). If $A \in \mathbb{X}^*_{\mathcal{W}}$ then Θ_A contains $i_+(Z)$ where $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$ (notation as in **Definition 4.4.3**), if $A \in (\mathrm{PGL}(V)A_k \cup \mathrm{PGL}(V)A_h)$ then Θ_A contains $k(\mathbb{P}(L))$ or $h(\mathbb{P}(L^\vee))$ i.e. a Veronese surfaces (of degree 9): in both cases we get that $\dim \Theta_A \geq 2$. □

Proposition 6.1.9. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ and suppose that there exists $W \in \Theta_A$ such that $C_{W,A} = \mathbb{P}(W)$. Then A is $\mathrm{PGL}(V)$-equivalent to an element of $\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k$.*

Proof. By **Corollary 3.3.7** we have $\mathcal{B}(W, A) = \mathbb{P}(W)$ i.e. one of the following holds:

(a) For generic $[w] \in \mathbb{P}(W)$ there exists $W' \in (\Theta_A \setminus \{W\})$ with $[w] \in W'$.

(b) For all $[w] \in \mathbb{P}(W)$ there exists $0 \neq \overline{\alpha} \in T_W$ such that $\overline{\alpha}(w) = 0$. (Recall (3.3.18).)

Suppose that (a) holds. It follows that $\dim \Theta_A \geq 2$. By **Lemma 6.1.8** we get that A is $\mathrm{PGL}(V)$-equivalent to an element of $\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k \cup \mathrm{PGL}(V)A_h$. On the other hand if $W \in \Theta_{A_h}$ then $C_{W,A_h} \neq \mathbb{P}(W)$ (it is a triple conic) and hence A is not $\mathrm{PGL}(V)$-equivalent to A_h. Now suppose that (b) holds. We may suppose that (a) does not hold. Then necessarily $\dim(A \cap S_W) \geq 4$. By Table (1) it follows that A is $\mathrm{PGL}(V)$-equivalent to an element $A_0 \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ such that $E_{A'_0} = \mathbb{P}(V_{02})$. By **Proposition 6.1.5** it follows that A_0 is $G_{\mathcal{C}_1}$-equivalent to an element $A_{\mathbf{p}} \in \mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ such that $E_{A'_{\mathbf{p}}} = \mathbb{P}(V_{02})$. Now look at (6.1.19): by (6.1.18) and **Corollary 6.1.7** we get that $A_{\mathbf{p}}$ is $G_{\mathcal{C}_1}$-equivalent to A_+, and since $\mathrm{PGL}(V)A_+ \subset \mathbb{X}^*_{\mathcal{W}}$ we are done. □

Corollary 6.1.10. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$. Suppose that $\dim \Theta_A \leq 1$ and A has minimal $\mathrm{PGL}(V)$-orbit. Let $W \in \Theta_A$: then $C_{W,A} \neq \mathbb{P}(W)$.*

Proof. Suppose that $C_{W,A} = \mathbb{P}(W)$. By **Proposition 6.1.9** we get that A is $\mathrm{PGL}(V)$-equivalent to an element $A_0 \in (\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k)$. By **Proposition 4.4.4** and **Proposition 4.5.1** A_0 has minimal $\mathrm{PGL}(V)$-orbit: by our hypothesis $\mathrm{PGL}(V)A = \mathrm{PGL}(V)A_0$ i.e. we may assume that $A_0 = A$: that is a contradiction because by **Lemma 6.1.8** we know that $\dim \Theta_A \geq 2$ for all $A \in (\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k)$. □

6.1.4 Analysis of Θ_A and $C_{W,A}$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ and A'' be as in (6.1.1); then

$$\Theta_A \supset \{V_{02}\} \coprod \Theta_{A''}. \tag{6.1.21}$$

Now suppose that $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$: we will describe curves in $\Theta_{A_{\mathbf{p}}}$ which are not contained in the right-hand side of (6.1.21). Let $C_{\mathbf{p},i} \subset \mathrm{Gr}(3, V)$ for $i = 0, 1, 2$ be the conics given by

$$\begin{aligned} C_{\mathbf{p},0} &:= \{\langle v_0, (\lambda v_1 - b_3\mu v_5), (\lambda v_2 + a_3\mu v_4)\rangle \mid [\lambda, \mu] \in \mathbb{P}^1\} \\ C_{\mathbf{p},1} &:= \{\langle v_1, (\lambda v_0 + a_2\mu v_5), (\lambda v_2 + b_2\mu v_3)\rangle \mid [\lambda, \mu] \in \mathbb{P}^1\} \\ C_{\mathbf{p},2} &:= \{\langle v_2, (\lambda v_0 + b_1\mu v_4), (\lambda v_1 - a_1\mu v_3)\rangle \mid [\lambda, \mu] \in \mathbb{P}^1\}. \end{aligned} \tag{6.1.22}$$

A straightforward computation (use (6.1.7)) shows that $C_{\mathbf{p},i} \subset \Theta_{A_{\mathbf{p}}}$ for $i = 0, 1, 2$.

Proposition 6.1.11. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ be semistable (and hence by* **Proposition 6.1.5** *either $E_{A'}$ is smooth or else there exist $g \in \mathrm{PGL}(V)$ and $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $gA = A_{\mathbf{p}}$) with minimal orbit, not equal to that of A_{III} nor to that of A_+.*

(1) If $E_{A'}$ is a smooth curve then $\Theta_{A''}$ is a smooth curve and moreover (6.1.21) *is an equality.*

(2) Suppose that $gA = A_{\mathbf{p}}$ where $g \in \mathrm{PGL}(V)$ and $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$g\Theta_A = \{V_{02}\} \cup \Theta_{A''} \cup C_{\mathbf{p},0} \cup C_{\mathbf{p},1} \cup C_{\mathbf{p},2}.$$

(3) $\dim \Theta_A = 1$.

Proof. Let's show that

$$E_{A'} \neq \mathbb{P}(A'). \tag{6.1.23}$$

In fact suppose that $E_{A'} = \mathbb{P}(A')$. By **Proposition 6.1.5** there exist $g \in \mathrm{PGL}(V)$ and $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $gA = A_{\mathbf{p}}$. By (6.1.19) we get that $A'_{\mathbf{p}}$ is T'-equivalent to $([1,1],[1,-1],[1,1])$. By hypothesis $A_{\mathbf{p}}$ has minimal orbit: it follows that $\mathbf{p} \in T'([1,1],[1,-1],[1,1])$ and by **Corollary 6.1.7** that contradicts the hypothesis that $gA \neq gA_+$. We have proved (6.1.23). Let $W \in (\Theta_A \setminus \{V_{02}\})$. Let $0 \neq \omega \in \bigwedge^3 W$; then

$$\omega = \alpha + \beta + \gamma, \qquad \alpha \in \bigwedge^3 V_{02}, \quad \beta \in A', \quad \gamma \in A'', \quad \beta + \gamma \neq 0. \tag{6.1.24}$$

Since $V_{02} \in \Theta_A$ we know that $\dim W \cap V_{02} > 0$. Let $\xi \in W \cap V_{02}$; multiplying both sides of the equality of (6.1.24) by ξ we get that $0 = \xi \wedge \beta = \xi \wedge \gamma$. It follows that if $\dim W \cap V_{02} = 2$ then $\gamma = 0$ and β is non-zero decomposable. Thus $[\beta] \in E_{A'}$: by (6.1.23) it follows that $E_{A'}$ is singular at $[\beta]$. By **Proposition 6.1.5** it follows that the orbt $\mathrm{PGL}(V)A$ intersects $\mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ and hence we might as well assume that $A \in \mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$. In the proof of **Proposition 6.1.5** we showed that if there exists $[\beta] \in E_{A'}$ with β decomposable then the T'-orbit of A' contains $A'_{\mathbf{p}}$ where $\mathbf{p}$ is either $([1,0],[1,0],[1,0])$ or $([0,1],[0,1],[0,1])$; by **Claim 6.1.4** it follows that $\mathrm{PGL}(V)A$ contains A_{III}, that contradicts our hypothesis. This proves that if $W \in (\Theta_A \setminus \{V_{02}\})$ then $\dim W \cap V_{02} = 1$. We claim that either $W \in \Theta_{A''}$ or else $W \cap V_{35} = \{0\}$. In fact if $W \cap V_{35} \neq \{0\}$ let $0 \neq \eta \in W \cap V_{35}$; then $0 = \eta \wedge \alpha = \eta \wedge \beta = \eta \wedge \gamma$. Thus $\alpha = 0$ and β is decomposable (it is a multiple of $\xi \wedge \eta$ where $0 \neq \xi \in W \cap V_{02}$), if $\beta \neq 0$ we get a contradiction as above, if $\beta = 0$ then $W \in \Theta_{A''}$. Thus from now on we may assume that $W \cap V_{35} = \{0\}$. It follows that there exist a basis $\{\xi_0, \xi_1, \xi_2\}$ of V_{02} and linearly independent $\eta_1, \eta_2 \in V_{35}$ such that

$$W = \langle \xi_0, \xi_1 + \eta_1, \xi_2 + \eta_2 \rangle.$$

Thus $\omega := \xi_0 \wedge (\xi_1 + \eta_1) \wedge (\xi_2 + \eta_2) \in A$. Decomposing ω according to the direct-sum decomposition $\bigwedge^3 V = \bigoplus_i \bigwedge^{3-i} V_{02} \wedge \bigwedge^i V_{35}$ we get that

$$\xi_0 \wedge (\xi_1 \wedge \eta_2 - \xi_2 \wedge \eta_1) \in A', \qquad \xi_0 \wedge \eta_1 \wedge \eta_2 \in A''.$$

In particular $[\xi_0 \wedge (\xi_1 \wedge \eta_2 - \xi_2 \wedge \eta_1)] \in E_{A'}$. Since $\xi_0 \wedge \eta_1 \wedge \eta_2 \in A''$ we have $A' \subset (\xi_0 \wedge \eta_1 \wedge \eta_2)^{\perp}$; it follows that $[\xi_0 \wedge (\xi_1 \wedge \eta_2 - \xi_2 \wedge \eta_1)]$ is a singular point of $E_{A'}$ (recall that $E_{A'}$ is a curve by (6.1.23)). This proves Item (1). Next let $A = A_{\mathbf{p}}$. Let $W \in (\Theta_A \setminus \{V_{02}\} \setminus \Theta_{A''})$; the argument above shows that $W \in (C_{\mathbf{p},0} \cup C_{\mathbf{p},1} \cup C_{\mathbf{p},2})$. This proves Item (2). Let's prove Item (3). By Items (1) and (2) it suffices to show that $\dim \Theta_{A''} = 1$. We have

$$\Theta_{A''} = \mathbb{P}(A'') \cap (\mathbb{P}(V_{02}) \times \mathbb{P}(\bigwedge^2 V_{35})) \subset \mathbb{P}(V_{02} \wedge \bigwedge^2 V_{35}) \tag{6.1.25}$$

and hence the expected dimension of $\Theta_{A''}$ is 1. Suppose that $W \in \Theta_{A''}$ and $\dim T_W \Theta_{A''} > 1$. Let $W = \langle [\xi_0], U \rangle$ where $\xi_0 \in V_{02}$ and $U \in \mathrm{Gr}(2, V_{35})$. Since $A' = (A'')^{\perp}$ we get that for every $\alpha \in A'$ we have $\alpha(\xi_0) \subset U$ (we view α as an element of $\mathrm{Hom}(V_{02}, V_{35})$). Since $\dim T_W \Theta_{A''} > 1$ we have

$$\dim(A'' \cap ([\xi_0] \wedge \bigwedge^2 V_{35} + V_{02} \wedge \bigwedge^2 U)) \geq 3. \tag{6.1.26}$$

Let $Z \subset V_{02}$ be a subspace complementary to $[\xi_0]$. Then

$$([\xi_0] \wedge \bigwedge^2 V_{35} + V_{02} \wedge \bigwedge^2 U)^{\perp} = [\xi_0] \wedge Z \wedge U.$$

By (6.1.26) we get that $0 \neq \alpha_0 \in (A' \cap [\xi_0] \wedge Z \wedge U)$ (recall that $A' = (A'')^{\perp}$). Then $[\alpha_0] \in E_{A'}$ and $E_{A'}$ is singular at $[\alpha_0]$ because $\alpha(\xi_0) \subset U$ for every $\alpha \in A'$. Moreover we get that $[\xi_0] \wedge \bigwedge^2 U = \bigwedge^3 W$ i.e. W is determined by α_0. This proves that if $E_{A'}$ is a smooth curve then $\Theta_{A''}$ is a smooth (irreducible) curve of genus 1 and that if $A = A_{\mathbf{p}}$ is as in Item (2) then there are exactly 3 singular points of $\Theta_{A''}$ (they are in one-to-one correspondence with the singular points of $E_{A'}$) and hence $\dim \Theta_{A''} = 1$. It follows that in both cases $\dim \Theta_A = 1$. □

Corollary 6.1.12. *Let $A_{\mathbf{p}}$ be as in Item (2) of* **Proposition 6.1.11**. *Then*

$$\Theta_{A''_{\mathbf{p}}} = \{\langle v_3, xv_1+yv_2, a_3yv_4-b_3xv_5 | [x,y]\in\mathbb{P}^1\rangle\} \cup \{\langle v_4, xv_0+yv_2, b_2yv_3+a_2xv_5 | [x,y]\in\mathbb{P}^1\rangle\} \cup \{\langle v_5, xv_0+yv_1, a_1yv_3-b_1xv_4\rangle | [x,y]\in\mathbb{P}^1\}. \tag{6.1.27}$$

Proof. A computation gives that $\Theta_{A''_{\mathbf{p}}}$ contains the three conics appearing in the right-hand side of (6.1.27). By **Proposition 6.1.11** we know that $\Theta_{A''_{\mathbf{p}}}$ is a curve of degree 6: the corollary follows. □

Corollary 6.1.13. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ be semistable with minimal orbit. Suppose that $\mathrm{PGL}(V)A$ does not contain A_+. Then one of the following holds:*

(1) $E_{A'}$ is a smooth curve and $C_{V_{02},A}$ is a semistable sextic curve of Type II-4.

(2) $E_{A'}$ is a triangle (the union of 3 non concurrent lines) and $C_{V_{02},A}$ is a semistable sextic curve of Type III-2.

Proof. By **Claim 5.2.1** we know that A is $\mathrm{PGL}(V)$-semistable with minimal orbit. Suppose first that $\mathrm{PGL}(V)A$ contains A_{III}: then Item (2) holds by (4.3.11) and (6.1.4). Next suppose that $\mathrm{PGL}(V)A$ does not contain A_{III}. By **Proposition 6.1.11** we have $\dim \Theta_A = 1$ and hence $C_{V_{02},A} \neq \mathbb{P}(V_{02})$ by **Corollary 6.1.10**. We have proved that $C_{V_{02},A} \neq \mathbb{P}(V_{02})$: by (6.1.4) we get that $C_{V_{02},A} = 2J_{V_{02}}(A')$ and that $\dim J_{V_{02}}(A') = 1$. Suppose that $E_{A'}$ is a smooth curve: it follows that $J_{V_{02}}(A') \cong E_{A'}$ and hence Item (1) holds. Now suppose that $E_{A'}$ is not a smooth curve: by **Proposition 6.1.5** we may assume that $A = A_{\mathbf{p}}$ and hence Item (2) holds by (6.1.19). □

Proposition 6.1.14. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$ and suppose that $E_{A'}$ is a smooth curve. Let $W \in \Theta_{A''}$: then $C_{W,A}$ is a semistable sextic curve of Type II-2.*

Proof. By **Claim 6.1.2** and **Claim 5.2.1** we know that A is $\mathrm{PGL}(V)$-semistable with minimal orbit. By **Proposition 6.1.11** we have $\dim \Theta_A = 1$ and hence we get that $C_{W,A} \neq \mathbb{P}(W)$ by **Corollary 6.1.10**. Let $\{\xi_0, \xi_1, \xi_2\}$ be a basis of W with $\xi_0 \in V_{02}$ and $\xi_1, \xi_2 \in V_{35}$. Let $\{X_0, X_1, X_2\}$ be the dual basis of $W^{\vee}$; then $C_{W,A} = V(P)$ where $0 \neq P \in \mathbb{C}[X_0, X_1, X_2]_6$. Let $t \in \mathbb{C}^{\times}$: then $\mathrm{diag}(t, t, t, t^{-1}, t^{-1}, t^{-1}) \in \mathrm{SL}(V)$ (the basis is F) acts trivially on $\bigwedge^{10} A$ and moreover it sends W to itself. By **Claim 3.2.4** we get that $\mathrm{diag}(s^2, s^{-1}, s^{-1}) \in \mathrm{SL}(W)$ acts trivially on P: by **Remark 1.4.3** we get that $P = X_0^2 F(X_1, X_2)$. It remains to prove that F has no multiple factors. Let $Z \subset \mathbb{P}(V_{35}^{\vee})$ be the image of the intersection map

$$\begin{array}{ccc} \Theta_{A''} & \overset{\tau}{\longrightarrow} & \mathbb{P}(V_{35}^{\vee}) \\ W' & \mapsto & \mathbb{P}(W' \cap V_{35}). \end{array}$$

By **Proposition 6.1.11** we get that Z is a smooth cubic. Let $L = W \cap V_{35} = V(X_0)$; then $L \in Z$. We have a regular map $f_0 \colon (Z \setminus \{L\}) \to \mathbb{P}(L)$ given by intersection with L: since Z is smooth it extends to a regular map $f \colon Z \to \mathbb{P}(L)$. Let $[\eta_1], \ldots, [\eta_4] \in L$ be the branch points of f. We claim that

$$\mathrm{mult}_{[\eta_i]} C_{W,A} \geq 3 \tag{6.1.28}$$

and hence the (X_1, X_2)-coordinates of $[\eta_1], \ldots, [\eta_4]$ are zeroes of F; since $\deg F = 4$ it will follow that F has no multiple factors. First notice that if $[\eta] \in \mathbb{P}(V_{35})$ then $\dim(F_\eta \cap A) \geq 3$: in fact $\operatorname{cod}(F_\eta \cap V_{02} \wedge \bigwedge^2 V_{35}, V_{02} \wedge \bigwedge^2 V_{35}) = 3$ and hence $\dim(F_\eta \cap A'') \geq 3$ because $\dim A'' = 6$. Now let $i = 1, \ldots, 4$. If $\dim(F_{\eta_i} \cap A) > 3$ then (6.1.28) holds by **Corollary 3.2.3**. Thus we may suppose that $\dim(F_{\eta_i} \cap A) = 3$ (in fact one can show that $\dim(F_\eta \cap A) = 3$ for all $[\eta] \in V_{35}$). We will apply **Proposition 3.2.2** in order to compute the term g_2 of the Taylor expansion (3.2.8) of $C_{W,A}$ near $[\eta_i]$. Let $\overline{K}$ be as in **Proposition 3.2.2**; the projection $\widetilde{\mu}$ of (3.3.4) realizes $\mathbb{P}(\overline{K})$ as a 1-dimensional linear subspace of $\mathbb{P}(\bigwedge^2 V_0 / \bigwedge^2 W_0)$ which intersects $\operatorname{Gr}(2, V_0)_{W_0}$ in one point with multiplicity 2. By (3.2.10) and (3.3.8) we get that $g_2 = 0$ and hence (6.1.28) holds. □

Proposition 6.1.15. *Let $A'_{\mathbf{p}} \in \mathbb{M}^{\mathsf{F}}_{\mathcal{C}_1}$ be $T' \times T''$-semistable with minimal orbit. Suppose that $A_{\mathbf{p}} \notin \mathrm{PGL}(V)A_+$. Let $W \in \Theta_{A_{\mathbf{p}}}$: then $C_{W,A}$ is a semistable sextic curve of Type III-2.*

Proof. If $A_{\mathbf{p}} \in \mathrm{PGL}(V)A_{III}$ then $C_{W,A}$ is a semistable sextic curve of Type III-2 by **Proposition 4.3.3**. Thus we may assume that $A_{\mathbf{p}} \notin \mathrm{PGL}(V)A_{III}$. By **Proposition 6.1.11** we know that $\dim \Theta_A = 1$ and by **Theorem 4.2.1** $A_{\mathbf{p}}$ is $\mathrm{PGL}(V)$-semistable with minimal orbit: it follows from **Corollary 6.1.10** that $C_{W,A} \neq \mathbb{P}(W)$. Thus $C_{W,A} = V(P)$ where $0 \neq P \in \mathrm{S}^6 W^\vee$. Looking at the explicit description of $C_{\mathbf{P},i}$ and $\Theta_{A''}$ provided by (6.1.22) and **Corollary 6.1.12** we get that there is a 2-dimensional torus $T_{\mathbf{p}} < T_\star$ which sends W to itself. Applying **Claim 3.2.4** one gets that P is fixed by a maximal torus in $\mathrm{SL}(W)$ and hence $C_{W,A}$ is of Type III-2 by **Remark 1.4.3**. □

6.1.5 Wrapping it up

We will prove **Proposition 6.1.1**. Item (1) and Item (2) are gotten by putting together the statements of **Claim 6.1.2** and **Proposition 6.1.5**. Let's prove Item (3). Since A is $G_{\mathcal{C}_1}$-stable the stabilizer of A in $G_{\mathcal{C}_1}$ is a finite group. Thus it suffices to show that if $g \in \mathrm{Stab}(A)$ then g belongs to the centralizer $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{C}_1})$ of $\lambda_{\mathcal{C}_1}$ in $\mathrm{SL}(V)$. $E_{A'}$ is a smooth curve because A is $G_{\mathcal{C}_1}$-stable. By **Proposition 6.1.11** we get that $\Theta_A = \{V_{02}\} \cup \Theta_{A''}$, moreover $\Theta_{A''}$ is a smooth curve. It follows that V_{35} is the unique 3-dimensional vector subspace of V intersecting every $W \in \Theta_{A''}$ in a subspace of dimension 2. From these facts we get that if $g \in \mathrm{Stab}(A)$ then $g(V_{02}) = V_{02}$ and $g(V_{35}) = V_{35}$ i.e. $g \in C_{\mathrm{SL}(V)}(\lambda_{\mathcal{C}_1})$. We have proved Item (3). Lastly let's prove Items (4) and (5). First we notice that $\mathfrak{y} \in \mathfrak{B}_{\mathcal{C}_1}$ by (6.1.2) and $\mathfrak{y} \in \mathfrak{I}$ by **Claim 4.4.5**: thus $\{\mathfrak{y}\} \subset \mathfrak{B}_{\mathcal{C}_1} \cap \mathfrak{I}$. Now suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{C}_1}$, that the orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ and not equal to that of A_+. By **Claim 6.1.2** and **Proposition 6.1.5** either $E_{A'}$ is smooth or else we may assume that $A = A_{\mathbf{p}}$ where $\mathbf{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By (4.3.11) we may assume from now on that $\mathrm{SL}(V)A \neq \mathrm{SL}(V)A_{III}$. Suppose that $E_{A'}$ is smooth: by **Proposition 6.1.11** either $W = V_{02}$ or $W \in \Theta_{A''}$. If $W = V_{02}$ then $C_{W,A}$ is a sextic curve of Type II-4 by **Corollary 6.1.13** and if $W \in \Theta_{A''}$ then $C_{W,A}$ is a sextic curve of Type II-2 by **Proposition 6.1.14**. Suppose that $A = A_{\mathbf{p}}$ (and $A_+ \notin \mathrm{PGL}(V)A$): if $W \in \Theta_A$ then $C_{W,A}$ is of Type III-2 by **Proposition 6.1.15**.

6.2 $\mathfrak{B}_{\mathcal{A}}$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$; by definition

$$A = A' \oplus A'', \qquad A' \in \operatorname{Gr}(5, [v_0] \wedge \bigwedge^2 V_{15}), \quad A'' = (A')^\perp \cap (\bigwedge^3 V_{15}). \tag{6.2.1}$$

In other words A', A'' are the summands denoted A_0, A_1 in **Subsection 5.2**. Notice that $\Theta_{A'}$ and $\Theta_{A''}$ both have expected dimension 1. The following is the main result of the present subsection.

Proposition 6.2.1. *The following hold:*

(1) $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ is $G_{\mathcal{A}}$-stable if and only if $\Theta_{A'}$ is a smooth curve.

(2) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ is $G_{\mathcal{A}}$-stable.

(3) *If* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ *is* $G_{\mathcal{A}}$*-stable the connected component of* Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to* $\mathrm{im}\,\lambda_{\mathcal{A}}$.

(4) *Let* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ *have closed* $\mathrm{PGL}(V)$*-orbit (in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*). Then* $C_{W,A}$ *is of Type II-2, II-4 or III-2. In particular* $\mathfrak{B}_{\mathcal{A}} \cap \mathfrak{I} = \emptyset$.

The proof of **Proposition 6.2.1** will be given in **Subsubsection 6.2.3**.

6.2.1 The GIT analysis

Let λ be a 1-PS of $G_{\mathcal{A}}$. By definition $G_{\mathcal{A}}$ is identified with $\mathrm{SL}(V_{15})$: it follows that $I_{-}(\lambda) = \emptyset$, see **Definition 5.2.3**. The 1-PS λ defines an action of $\mathbb{C}^{\times}$ on $[v_0] \wedge \bigwedge^2 V_{15}$: let $e'_0 > \ldots > e'_j$ be the weights of the action. Now let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$: by (5.2.5) and (2.2.9) we have

$$\mu(A,\lambda) = 2\mu(A',\lambda) = 2\sum_{i=0}^{j} d'_i(A')e'_i. \tag{6.2.2}$$

Next we notice that $A^{\mathsf{F}}_{III} \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$, see (4.3.1).

Proposition 6.2.2. *Suppose that* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ *is semistable and that* $\Theta_{A'}$ *is not a smooth curve. Then* A *is not* $G_{\mathcal{A}}$*-stable and it is* $G_{\mathcal{A}}$*-equivalent to* A^{F}_{III}.

Proof. Every irreducible component of $\Theta_{A'}$ has dimension at least 1: it follows that $\Theta_{A'}$ contains a point W whose tangent space has dimension greater than 1. Let $\overline{W} := W \cap V_{15}$ (thus $\dim \overline{W} = 2$) and choose a direct-sum decomposition $V_{15} = \overline{W} \oplus U$. Let λ be the 1-PS of $G_{\mathcal{A}}$ such that

$$\lambda(t)|_{\overline{W}} = t^3\,\mathrm{Id}_{\overline{W}}, \qquad \lambda(t)|_U = t^{-2}\,\mathrm{Id}_U\,. \tag{6.2.3}$$

The λ-type of A' is $(1, d'_1(A'), 4 - d'_1(A'))$ and hence $\mu(A',\lambda) = 5d'_1(A') - 10$. Since the tangent space to $\Theta_{A'}$ at W has dimension greater than 1 we have $d'_1(A') = \dim(A' \cap \overline{W} \wedge U) \geq 2$ and thus $\mu(A',\lambda) \geq 0$. By (6.2.2) and semistability of A it follows that $\mu(A',\lambda) = 0$ i.e. $d'_1(A') = 2$. By **Claim 2.2.4** we get that A is $G_{\mathcal{A}}$-equivalent to $A_0 = A'_0 \oplus A''_0$ where $A'_0 \in \mathrm{Gr}(5, [v_0] \wedge \bigwedge^2 V_{15})$ and $A''_0 \in \mathrm{Gr}(5, \bigwedge^3 V_{15})$ are λ-split of types $(1,2,2)$ and $(1,4,0)$ respectively. There exists a basis $\{u_1, u_2, u_3, w_1, w_2\}$ of V_{15} such that $u_i \in U$, $w_j \in \overline{W}$ and $A'_0 \cap \bigwedge^2 U = \langle u_1 \wedge u_2, u_1 \wedge u_3\rangle$. Let $U_{23} := \langle u_2, u_3\rangle$. We let λ_0 be the 1-PS of $G_{\mathcal{A}}$ defined by

$$\lambda_0(t)u_1 = t^2 u_1, \quad \lambda_0(t)|_{U_{23}} = \mathrm{Id}_{U_{23}}, \quad \lambda_0(t)|_{\overline{W}} = t^{-1}\,\mathrm{Id}_{\overline{W}}\,.$$

The λ_0-type of A'_0 is $(2, d'_1(A'_0), 0, d'_3(A'_0), 1)$ and $d'_1(A'_0) + d'_3(A'_0) = 2$; it follows that $\mu(A'_0,\lambda_0) = d'_1(A'_0) - d'_3(A'_0) + 2 \geq 0$. By (6.2.2) and semistability of A we get that $d'_1(A') = 0$ and $d'_3(A') = 2$. By **Claim 2.2.4** we get that A_0 is $G_{\mathcal{A}}$-equivalent to $A_{00} = A'_{00} \oplus A''_{00}$ where A'_{00} is λ_0-split of type $(2,0,0,2,1)$. In particular we have $\dim(A'_{00} \cap (U_{23} \wedge \overline{W})) = 2$. The Grassmannian $\mathrm{Gr}(2, U_{23} \oplus \overline{W})$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2(U_{23} \oplus \overline{W}))$: it follows that the intersection $R := \mathbb{P}(A'_{00} \cap (U_{23} \wedge \overline{W})) \cap \mathrm{Gr}(2, U_{23} \oplus \overline{W})$ is one of the following:

(1) a set with exactly two elements,

(2) a set with exactly one element,

(3) a line.

Suppose that (1) holds: then there exist bases $\{u'_2, u'_3\}$, $\{w'_1, w'_2\}$ of U_{23} and $\overline{W}$ respectively such that $R = \{u'_2 \wedge w'_1, u'_3 \wedge w'_2\}$. Let $\mathsf{F}' := \{u_1, u'_2, u'_3, w'_1, w'_2\}$; as is easily checked $A_{00} = A^{\mathsf{F}'}_{III}$. Now suppose that (2) or (3) holds: such an A_{00} is in the closure of the set of A_{00}'s for which Item (1) holds, since they are in the orbit $\mathrm{SL}(V)A^{\mathsf{F}'}_{III}$ we get that A_{00} itself belongs to that orbit by **Proposition 4.3.2**. □

Proposition 6.2.3. *Suppose that* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ *and that* $\Theta_{A'}$ *is a smooth curve. Then* A *is* $G_{\mathcal{A}}$*-stable. Moreover the generic* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ *is* $G_{\mathcal{A}}$*-stable.*

Proof. Let $\mathrm{Gr}(5,\bigwedge^2 V_{15})^0 \subset \mathrm{Gr}(5,\bigwedge^2 V_{15})$ be the open dense subset of B' such that $\Theta_{[v_0]\wedge B'}$ is a smooth curve. The j-invariant provides a regular $\mathrm{SL}(V_{15})$-invariant map $j\colon \mathrm{Gr}(5,\bigwedge^2 V_{15})^0 \to \mathbb{A}^1$. Let $p \in (\mathbb{A}^1 \setminus j(A'))$ and $D \subset \mathrm{Gr}(5,\bigwedge^2 V_{15})$ be the closure of $j^{-1}(p)$. Then D is $\mathrm{SL}(V_{15})$-invariant and does not contain A'; it follows that A' is $\mathrm{SL}(V_{15})$-semistable. Now suppose that A' is not stable. Then there exists a minimal orbit $\mathrm{SL}(V_{15})A'_0$ contained in $\overline{\mathrm{SL}(V_{15}A')} \cap \mathrm{Gr}(5,\bigwedge^2 V_{15})^{ss}$ and $\mathrm{SL}(V_{15})A'_0 \neq \mathrm{SL}(V_{15})A'$. In particular $\dim \mathrm{SL}(V_{15})A'_0 < \dim \mathrm{SL}(V_{15})A'$; it follows that $A'_0 \notin \mathrm{Gr}(5,\bigwedge^2 V_{15})^0$. By **Proposition 6.2.2** we get that $A'_0 = A'_{III}$ and hence $\Theta_{A'_0}$ is a curve whose singularities are nodes - in fact a cycle of 5 lines; by monodromy considerations that contradicts the hypothesis that A'_0 is in the closure of $\mathrm{SL}(V_{15})A'$. □

The result below follows at once from **Proposition 6.2.3**.

Corollary 6.2.4. *The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ is $G_{\mathcal{A}}$-stable.*

6.2.2 Analysis of Θ_A and $C_{W,A}$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$: we have an embedding

$$\begin{array}{ccc} \Theta_{A'} & \overset{\iota}{\hookrightarrow} & \mathrm{Gr}(2,V_{15}) \\ W & \mapsto & W \cap V_{15} \end{array} \tag{6.2.4}$$

We will often identify $\Theta_{A'}$ with its image via ι.

Proposition 6.2.5. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$. Then $\Theta_{A'}$ is a smooth curve if and only if $\Theta_{A''}$ is a smooth curve. If this is the case then $\Theta_{A'} \cong \Theta_{A''}$ and $\Theta_A = \Theta_{A'} \coprod \Theta_{A''}$.*

Proof. Suppose that $\Theta_{A'}$ is a smooth curve. Let's prove the following:

$$\text{if } W_1 \in \Theta_{A'} \text{ and } W_2 \in \Theta_{A''} \text{ then } \dim(W_1 \cap W_2) = 1. \tag{6.2.5}$$

We know that $\dim(W_1 \cap W_2) \geq 1$; the point is to show that we can not have strict inequality. Suppose that $\dim(W_1 \cap W_2) = 2$. Let $U := W_1 \cap V_{15}$; thus $U = W_1 \cap W_2 = \rho^{v_0}_{V_{15}}(W_1)$ where $\rho^{v_0}_{V_{15}}$ is given by (3.3.12) (with V_0 replaced by V_{15}). Choose bases $\{u_1,u_2\}$, $\{u_1,u_2,v\}$ of U and W_2 respectively. Since $A' = (A'')^{\perp}$ we get that $A' \subset (u_1 \wedge u_2 \wedge v)^{\perp}$. Since the projective tangent space to $\mathrm{Gr}(2,V_{15})$ at U is contained in $\mathbb{P}((u_1 \wedge u_2 \wedge v)^{\perp})$ it follows that the tangent space to $\iota(\Theta_{A'}) = \mathbb{P}(\rho^{v_0}_{V_{15}}(A')) \cap \mathrm{Gr}(2,V_{15})$ at U has dimension at least 2: that contradicts the hypothesis that $\Theta_{A'}$ is a smooth curve. This proves (6.2.5). Let's define a morphism

$$\varphi\colon \operatorname{Pic}^{-3}\Theta_{A'} \longrightarrow \Theta_{A''}. \tag{6.2.6}$$

Let $\mathcal{E}$ be the restriction to $\Theta_{A'}$ of the tautological rank-2 vector-bundle on $\mathrm{Gr}(2,V_{15})$. Let

$$\epsilon\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}(V_{15}) \tag{6.2.7}$$

be the natural morphism and $R_{\mathcal{E}} := \operatorname{im}\epsilon$. We notice that ϵ is injective: in fact if $U_1, U_2 \in \rho(\Theta_{A'})$ are distinct then $U_1 \cap U_1 = \{0\}$ because $\Theta_{A'}$ does not contain lines. Clearly $\deg\mathcal{E} = -5$. We claim that $\mathcal{E}$ is stable. In fact $\mathcal{E}^{\vee}$ is globally generated and hence if it is not stable then $\mathcal{E} \cong L_1 \oplus L_2$ where $\deg L_1 = 3$ and $\deg L_2 = 2$; that contradicts injectivity of ϵ. Let $L \in \operatorname{Pic}^{-3}\Theta_{A'}$; since $\mathcal{E}$ is stable $\dim \operatorname{Hom}(L,\mathcal{E}) = 1$. Let $\tau \in \operatorname{Hom}(L,\mathcal{E})$ be non-zero; then τ does not vanish anywhere and hence $\epsilon(\operatorname{im}\tau)$ is a cubic curve (recall that ϵ is injective) spanning a plane $\mathbb{P}(W)$ such that $W \cap U \neq \{0\}$ for every $U \in \Theta_{A'}$. Since $\Theta_{A'}$ spans $\mathbb{P}(A')$ and $A'' = (A')^{\perp}$ it follows that $W \in \Theta_{A''}$. We define the morphism φ of (6.2.6) by setting $\varphi([L]) := W$. The morphism φ is injective because ϵ is injective. Using (6.2.5) one proves easily that φ is surjective. Thus $\Theta_{A''}$ has the expected dimension 1 and hence it is an irreducible curve of arithmetic genus 1: it follows that φ is an isomorphism. We have proved that if $\Theta_{A'}$ is a smooth curve then $\Theta_{A''}$ is isomorphic to $\Theta_{A'}$, in particular it is a smooth curve. By duality it follows that if $\Theta_{A''}$ is a smooth curve then $\Theta_{A'} \cong \Theta_{A''}$, in particular it is a smooth curve. Now assume that $\Theta_{A''}$ is a smooth curve: we must prove that $\Theta_A = \Theta_{A'} \coprod \Theta_{A''}$. Suppose that $\alpha \in A$ is non-zero decomposable and that $\operatorname{supp}(\alpha) \notin (\Theta_{A'} \coprod \Theta_{A''})$. Then there exist linearly independent $u_1,u_2,v \in V_{15}$ such that $\alpha = v_0 \wedge u_1 \wedge u_2 + u_1 \wedge u_2 \wedge v$. Thus $v_0 \wedge u_1 \wedge u_2 \in A'$ and $u_1 \wedge u_2 \wedge v \in A''$ and hence $\langle v_0,u_1,u_2\rangle \in \Theta_{A'}$, $\langle u_1,u_2 \wedge v\rangle \in \Theta_{A'}$; that contradicts (6.2.5). □

Proposition 6.2.6. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$. Suppose that $\Theta_{A'}$ is a smooth curve. If $W \in \Theta_{A'}$ or $W \in \Theta_{A''}$ then $C_{W,A}$ is a sextic curve of Type II-2 or II-4 respectively.*

Proof. By **Proposition 6.2.5** we have $\dim \Theta_A = 1$. By **Proposition 6.2.3** we know that A is $G_{\mathcal{A}}$-stable and hence A is $\mathrm{PGL}(V)$-semistable with closed orbit by **Claim 5.2.1**. Let $W \in \Theta_A$: since $\dim \Theta_A < 2$ it follows from **Corollary 6.1.10** that $C_{W,A} \neq \mathbb{P}(W)$. Let $W \in \Theta_{A'}$. Let $\{v_0, u_1, u_2\}$ be a basis of W where $u_1, u_2 \in V_{15}$, and $\{X_0, X_1, X_2\}$ be the dual basis of $W^{\vee}$. For $t \in \mathbb{C}^{\times}$ let $g(t) := \mathrm{diag}(t^5, t^{-1}, \ldots, t^{-1}) \in \mathrm{SL}(V)$. Then $g(t)$ acts trivially on $\bigwedge^{10} A$ and it maps W to itself. Applying **Claim 3.2.4** we get that $C_{W,A} = V(P)$ where $P = X_0^2 F(X_1, X_4)$ - and we know that $F \neq 0$. It remains to prove that F does not have multiple factors. Let's examine $C_{W,A}$ in a neighborhood of $[v_0]$. We identify $U := W \cap V_{15}$ with an open affine neighborhood of $[v_0]$ in $\mathbb{P}(W)$ via (3.2.7). We have $C_{W,A} \cap U = V(g_4)$ where $g_4 = F/X_0^4$. Let $Z_{U,A} \subset \mathbb{P}(\bigwedge^2 V_{15} / \bigwedge^2 U)$ be the projection of $\iota(\Theta_{A'})$ from $\bigwedge^2 U$ - notation as in **Remark 3.4.3**. By (3.2.10) the set of zeroes (up to scalars) of g_4 is in one-to-one correspondence with the set of singular quadrics in $\mathbb{P}(\rho^{v_0}_{V_{15}}(A') / \bigwedge^2 U)$ containing $Z_{U,A}$. Since $Z_{U,A}$ is a linearly normal quartic elliptic curve in the 3-dimensional projective space $\mathbb{P}(\rho^{v_0}_{V_{15}}(A') / \bigwedge^2 U)$ there are exactly 4 singular quadrics containing it; thus F does not have multiple factors. Now let $W \in \Theta_{A''}$. If $W' \in \Theta_{A'}$ then $\dim W' \cap W = 1$ by (6.2.5). As W' varies in $\Theta_{A'}$ the intersection $W' \cap W$ describes a curve $E_W \subset \mathbb{P}(W)$ (recall that ϵ is injective). One checks easily that $E_W = \epsilon(\mathbb{P}(L))$ where $L \hookrightarrow \mathcal{E}$ is a sub-line-bundle of degree -3 (a sub-line-bundle of $\mathcal{E}$ of degree less than -3 will give a non-planar curve in $\mathbb{P}(V_{15})$); it follows that E_W is a smooth cubic curve in $\mathbb{P}(W)$. By **Corollary 3.3.7** we get that $C_{W,A} = 2E_W$ (recall that $C_{W,A} \neq \mathbb{P}(W)$) and hence $C_{W,A}$ is of Type II-4. □

6.2.3 Wrapping it up

We will prove **Proposition 6.2.1**. Item (1) and Item (2) are gotten by putting together the statements of **Proposition 6.2.2**, **Proposition 6.2.3** and **Corollary 6.2.4**. Let's prove Item (3). Since A is $G_{\mathcal{A}}$-stable the stabilizer of A in $G_{\mathcal{A}}$ is a finite group. Thus it suffices to show that if $g \in \mathrm{Stab}(A)$ then g belongs to the centralizer $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{A}})$ of $\lambda_{\mathcal{A}}$ in $\mathrm{SL}(V)$. By Item (1) and $G_{\mathcal{A}}$-stability of A we know that $\Theta_{A'}$ is a smooth curve. By **Proposition 6.2.5** we get that $\Theta_A = \Theta_{A'} \cup \Theta_{A''}$ and $\Theta_{A''}$ is a smooth elliptic curve of degree 5. It follows that $[v_0]$ is the unique 1-dimensional vector subspace of V contained in every $W \in \Theta_{A'}$ and V_{15} is the unique 5-dimensional vector subspace of V containing every $W \in \Theta_{A''}$ (and there is no 1-dimensional subspace of V contained in every $W \in \Theta_{A''}$ and no proper subspace of V containing all $W \in \Theta_{A'}$). From these facts we get that if $g \in \mathrm{Stab}(A)$ then $g([v_0]) = [v_0]$ and $g(V_{15}) = V_{15}$ i.e. $g \in C_{\mathrm{SL}(V)}(\lambda_{\mathcal{A}})$. We have proved Item (3). Lastly we prove Item (4). Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{A}}$ be $G_{\mathcal{A}}$-semistable with minimal orbit. Suppose that $\Theta_{A'}$ is a smooth curve: then $C_{W,A}$ is of Type II-2 or II-4 by **Proposition 6.2.5** and **Proposition 6.2.6**. Suppose that $\Theta_{A'}$ is not a smooth curve: then $A \in \mathrm{PGL}(V) A_{III}$ by **Proposition 6.2.2** and hence $C_{W,A}$ is of Type III-2 by (4.3.11).

6.3 $\mathfrak{B}_{\mathcal{D}}$

Below is the main result of the present subsection.

Proposition 6.3.1. *The following hold:*

(1) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ is $G_{\mathcal{D}}$-stable.

(2) If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ is $G_{\mathcal{D}}$-stable the connected component of Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to* $\mathrm{im}\, \lambda_{\mathcal{D}}$.

(3) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ have closed $\mathrm{PGL}(V)$*-orbit (in $\mathbb{LG}(\bigwedge^3 V)^{ss}$), and suppose that $[A] \notin \mathfrak{I}$. Then $C_{W,A}$ is of Type II-1, II-2, II-3 or* $\mathrm{PGL}(V)$*-equivalent to Type III-2.*

(4) $\mathfrak{B}_{\mathcal{D}} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{W}}$, where $\mathfrak{X}_{\mathcal{W}}$ is as in (4.4.7).

The proof of **Proposition 6.3.1** will be given in **Subsubsection 6.3.4**.

6.3.1 Quadrics associated to $A \in \mathbb{S}_{\mathcal{D}}^{\mathsf{F}}$

Let $A \in \mathbb{S}_{\mathcal{D}}^{\mathsf{F}}$; by definition $A = A' \oplus A'' \oplus A'''$ where

$$A' \in \mathrm{Gr}(3, [v_0] \wedge \bigwedge^2 V_{14}), \quad A'' \in \mathbb{LG}([v_0] \wedge V_{14} \wedge [v_5] \oplus \bigwedge^3 V_{14}), \quad A''' = (A')^{\perp} \cap (\bigwedge^2 V_{14} \wedge [v_5]). \tag{6.3.1}$$

In other words A', A'', A''' are the summands named A_0, A_1, A_2 in **Subsection 5.2**. We define closed subsets $Q_{A'}, Q_{A''}, Q_{A'''} \subset \mathbb{P}(V_{14})$ as follows:

$$\begin{aligned} Q_{A'} &:= \{[\xi] \in \mathbb{P}(V_{14}) \mid \dim(A' \cap F_\xi) > 0\}, \\ Q_{A''} &:= \{[\xi] \in \mathbb{P}(V_{14}) \mid \dim(A'' \cap F_\xi) > 0\}, \\ Q_{A'''} &:= \{[\xi] \in \mathbb{P}(V_{14}) \mid \dim(A''' \cap F_\xi) > 0\}. \end{aligned}$$

Thus $Q_{A'}$ is swept out by the lines $\mathbb{P}(W \cap V_{14})$ for W varying in $\Theta_{A'}$ and similarly for $Q_{A'''}$. In particular each of $Q_{A'}$, $Q_{A'''}$ is either a quadric or $\mathbb{P}(V_{14})$, moreover $Q_{A'''} = Q_{A'}$ because $A''' = (A')^{\perp}$. Similarly $Q_{A''}$ is either a quadric or $\mathbb{P}(V_{14})$. Suppose that $A'' \cap \bigwedge^3 V_{14} = \{0\}$; a simpler description of $Q_{A''}$ goes as follows. We have an isomorphism $\bigwedge^3 V_{14} \cong ([v_0] \wedge V_{14} \wedge [v_5])^{\vee}$ given by wedge-product followed by vol and A'' is the graph of a map $q_{A''} \colon [v_0] \wedge V_{14} \wedge [v_5] \to \bigwedge^3 V_{14}$ which is symmetric because A'' is lagrangian. As is easily checked $Q_{A''} = V(q_{A''})$. The intersection $Y_A \cap \mathbb{P}(V_{14})$ is supported on $Q_{A'} \cup Q_{A''}$ and it has multiplicity at least 2 along $Q_{A'}$: it follows that either $\mathbb{P}(V_{14}) \subset Y_A$ or $Y_A \cap \mathbb{P}(V_{14}) = 2Q_{A'} + Q_{A''}$. In the following subsubsection we will compare $G_{\mathcal{D}}$-(semi)stability of A with geometric properties of $Q_{A'}$ and $Q_{A''}$: for example we will show that if $Q_{A'} \cap Q_{A''}$ is a smooth curve (the generic case) then A is $G_{\mathcal{D}}$-stable. In the present subsubsection we will go through basic results about $Q_{A''}$ and the computation of $Q_{A'}$ for one explicit A'.

Proposition 6.3.2. *Let A'' be as in* (6.3.1) *and $[\xi_0] \in Q_{A''}$. Then $\dim T_{[\xi_0]} Q_{A''} = 3$ (i.e. either $Q_{A''}$ is a quadric singular at $[\xi_0]$ or it is equal to $\mathbb{P}(V_{14})$) if and only if one of the following holds:*

(a) $A'' \cap F_{\xi_0} \cap ([v_0] \wedge V_{14} \wedge [v_5]) \neq \{0\}$.

(b) $A'' \cap F_{\xi_0} \cap \bigwedge^3 V_{14} \neq \{0\}$.

On the other hand suppose that

$$A'' \cap F_{\xi_0} = \langle v_0 \wedge \xi_0 \wedge v_5 + \alpha \rangle, \qquad 0 \neq \alpha \in \bigwedge^3 V_{14}.$$

Then the embedded **projective tangent space** *of $Q_{A''}$ at $[\xi_0]$ is*

$$\mathbf{T}_{[\xi_0]} Q_{A''} = \mathbb{P}(\operatorname{supp} \alpha).$$

Proof. In order to simplify notation we let $S := ([v_0] \wedge V_{14} \wedge [v_5] \oplus \bigwedge^3 V_{14})$. Let $B \in \mathbb{LG}(S)$ be transversal both to A'' and F_{ξ_0}. The symplectic form on S defines an isomorphism $B \cong (A'')^{\vee}$. Choose a subspace $U \subset V_{14}$ complementary to $[\xi_0]$. We have an isomorphism

$$\begin{array}{ccc} U & \xrightarrow{\sim} & \mathbb{P}(V_{14}) \setminus \mathbb{P}(U) \\ \xi & \mapsto & [\xi_0 + \xi] \end{array}$$

onto a neighborhood of $[\xi_0]$. There is an open $U_0 \subset U$ containing 0 such that $F_{\xi_0+\xi}$ is transverse to B for all $\xi \in U_0$. Let $\xi \in U_0$: then $F_{\xi_0+\xi}$ is the graph of a linear map $\psi(\xi) \colon A'' \to B = (A'')^{\vee}$. Since $F_{\xi_0+\xi}$ is lagrangian the map $\psi(\xi)$ is symmetric. Clearly we have

$$Q_{A''} \cap U_0 = V(\det \psi), \qquad \ker \psi(0) = A'' \cap F_{\xi_0}. \tag{6.3.2}$$

Now suppose that $\dim(A'' \cap F_{\xi_0}) \geq 2$. Then $\psi(0)$ has corank at least 2 and hence $\dim T_{[\xi_0]} Q_{A''} = 3$. On the other hand one checks at once that Item (b) holds. Thus from now on we may suppose that $\dim(A'' \cap F_{\xi_0}) = 1$. Let

$$A'' \cap F_{\xi_0} = \langle \xi_0 \wedge (x v_0 \wedge v_5 + \alpha_0) \rangle, \qquad \alpha_0 \in \bigwedge^2 V_{14}.$$

Given $\tau \in U_0 = T_{[\xi_0]}\mathbb{P}(V_{14})$ we have

$$\tau \in T_{[\xi_0]}Q_{A''} \iff \frac{d\psi}{d\tau}(\xi_0 \wedge (xv_0 \wedge v_5 + \alpha_0)) = 0.$$

(Here we view $\frac{d\psi}{d\tau}$ as a quadratic form on A''.) Equation (2.26) of [26] (warning: the v_0 of [26] is our ξ_0!) gives that

$$\frac{d\psi}{d\tau}(\xi_0 \wedge (xv_0 \wedge v_5 + \alpha_0)) = \mathrm{vol}(\tau \wedge \xi_0 \wedge (xv_0 \wedge v_5 + \alpha_0) \wedge (xv_0 \wedge v_5 + \alpha_0)) = 2x\,\mathrm{vol}(\tau \wedge \xi_0 \wedge v_0 \wedge v_5 \wedge \alpha_0).$$

(Notice that α_0 is decomposable and hence $\alpha_0 \wedge \alpha_0 = 0$.) The proposition follows. □

In **Subsubsection 6.3.3** we will need the following explicit computation. Let $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ be a basis of V_{14} and $\{T_0, T_1, T_2, T_3\}$ be the dual basis of $V_{14}^\vee$. Let

$$A' = [v_0] \wedge \langle \eta_0 \wedge \eta_1 + \eta_2 \wedge \eta_3, \eta_0 \wedge \eta_2 - \eta_1 \wedge \eta_3, \eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2 \rangle \in \mathrm{Gr}(3, [v_0] \wedge \bigwedge^2 V_{14}). \quad (6.3.3)$$

A straightforward computation gives that

$$Q_{A'} = V(T_0^2 + T_1^2 + T_2^2 + T_3^2). \quad (6.3.4)$$

Notice that

$$A''' = (A')^\perp = [v_0] \wedge \langle \eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3, \eta_0 \wedge \eta_2 + \eta_1 \wedge \eta_3, \eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2 \rangle \in \mathrm{Gr}(3, [v_0] \wedge \bigwedge^2 V_{14}). \quad (6.3.5)$$

6.3.2 The GIT analysis

Let λ be a 1-PS of $G_{\mathcal{D}}$. We claim that $I_-(\lambda) = \emptyset$, see **Definition 5.2.3**. In fact $G_{\mathcal{D}} = \mathbb{C}^\times \times \mathrm{SL}(V_{14})$ and hence it suffices to check that (5.2.3) holds for λ with image in the $\mathbb{C}^\times$-factor: now look at (5.2.4). The 1-PS λ defines actions of $\mathbb{C}^\times$ on $[v_0] \wedge \bigwedge^2 V_{14}$ and $([v_0] \wedge V_{14} \wedge [v_5] \oplus \bigwedge^3 V_{14})$: we let $e'_0 > \ldots > e'_{j(0)}$ and $e''_0 > \ldots > e''_{j(1)}$ be the corresponding weights. Now let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$. By (5.2.5) and (2.2.9) we have

$$\mu(A, \lambda) = 2\mu(A', \lambda) + \mu(A'', \lambda) = 2\sum_{i=0}^{j(0)} e'_i d_i^\lambda(A') + \sum_{i=0}^{j(1)} e''_i d_i^\lambda(A''). \quad (6.3.6)$$

Proposition 6.3.3. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$. Then A is not $G_{\mathcal{D}}$-stable if and only if one of the following holds:*

(1) $\dim(A'' \cap [v_0] \wedge V_{14} \wedge [v_5]) \geq 2$.

(2) $\dim(A'' \cap \bigwedge^3 V_{14}) \geq 2$.

(3) There exists a basis $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ of V_{14} such that one of the following holds:

  (3a) $A' \ni v_0 \wedge \xi_0 \wedge \xi_1$ *and* $A'' \supset \langle v_0 \wedge \xi_0 \wedge v_5, \xi_0 \wedge \xi_1 \wedge \xi_2 \rangle$.

  (3b) $A' \supset \langle v_0 \wedge \xi_0 \wedge \xi_1, v_0 \wedge \xi_0 \wedge \xi_2 \rangle$.

  (3c) $A' \supset \langle v_0 \wedge \xi_0 \wedge \xi_1, v_0 \wedge (\xi_0 \wedge \xi_3 + \xi_1 \wedge \xi_2) \rangle$ *and there exists* $0 \neq (av_0 \wedge \xi_0 \wedge v_5 + b\xi_0 \wedge \xi_1 \wedge \xi_2) \in A''$.

Proof. Let $\lambda_0 \colon \mathbb{C}^\times \to G_{\mathcal{D}}$ be the 1-PS of $G_{\mathcal{D}}$ mapping identically to the $\mathbb{C}^\times$-factor and trivially to the $\mathrm{SL}(V_{14})$-factor. We let $\lambda_0^{-1}(t) := \lambda_0(t^{-1})$ be the inverse. We notice that λ_0 acts trivially on $[v_0] \wedge \bigwedge^2 V_{14}$ and the weight-decomposition of the λ_0-action on $([v_0] \wedge V_{14} \wedge [v_5]) \oplus \bigwedge^3 V_{14}$ is the following:

$$\underbrace{[v_0] \wedge V_{14} \wedge [v_5]}_{t^3} \oplus \underbrace{\bigwedge^3 V_{14}}_{t^{-3}}. \quad (6.3.7)$$

Let

$$\mathsf{B} = \{\xi_0, \xi_1, \xi_2, \xi_3\} \quad (6.3.8)$$

be a basis of V_{14}. Let $\lambda_1\colon \mathbb{C}^\times \to \mathrm{SL}(V_{14})$ be defined by

$$\lambda_1(t)\xi_0 = t\xi_0, \quad \lambda_1(t)\xi_1 = \xi_1, \quad \lambda_1(t)\xi_2 = \xi_2, \quad \lambda_1(t)\xi_3 = t^{-1}\xi_3. \tag{6.3.9}$$

We view λ_1 as a 1-PS of $G_{\mathcal{D}}$. The weight-decomposition of the λ_1-action on $[v_0] \wedge \bigwedge^2 V_{14}$ is the following:

$$\underbrace{[v_0 \wedge \xi_0] \wedge \langle \xi_1, \xi_2 \rangle}_{t} \oplus \underbrace{\langle v_0 \wedge \xi_0 \wedge \xi_3, v_0 \wedge \xi_1 \wedge \xi_2 \rangle}_{1} \oplus \underbrace{[v_0 \wedge \xi_3] \wedge \langle \xi_1, \xi_2 \rangle}_{t^{-1}}. \tag{6.3.10}$$

The weight-decomposition of the λ_1-action on $([v_0] \wedge V_{14} \wedge [v_5]) \oplus \bigwedge^3 V_{14}$ is the following:

$$\underbrace{\langle v_0 \wedge \xi_0 \wedge v_5, \xi_0 \wedge \xi_1 \wedge \xi_2 \rangle}_{t} \oplus \underbrace{\langle v_0 \wedge \xi_1 \wedge v_5, v_0 \wedge \xi_2 \wedge v_5, \xi_0 \wedge \xi_1 \wedge \xi_3, \xi_0 \wedge \xi_2 \wedge \xi_3 \rangle}_{1} \oplus \underbrace{\langle v_0 \wedge \xi_3 \wedge v_5, \xi_1 \wedge \xi_2 \wedge \xi_3 \rangle}_{t^{-1}}. \tag{6.3.11}$$

A straightforward computation gives the following:

($1'$) If A satisfies Item (1) then $\mu(A, \lambda_0) \geq 0$.

($2'$) If A satisfies Item (2) then $\mu(A, \lambda_0^{-1}) \geq 0$.

($3a'$) If A satisfies Item (3a) then $d^{\lambda_1}(A') \succeq (1,0,2)$ and $d^{\lambda_1}(A'') \succeq (2,2,0)$ thus $\mu(A, \lambda_1) \geq 0$.

($3b'$) If A satisfies Item (3b) then $d^{\lambda_1}(A') \succeq (2,0,1)$ and $d^{\lambda_1}(A'') \succeq (0,2,2)$ thus $\mu(A, \lambda_1) \geq 0$.

($3c'$) If A satisfies Item (3c) then $d^{\lambda_1}(A') \succeq (1,1,1)$ and $d^{\lambda_1}(A'') \succeq (1,2,1)$ thus $\mu(A, \lambda_1) \geq 0$.

(The relation $\succeq$ is defined as in **Definition 2.4.1**.) This proves that if one of Items (1)-(3c) holds then A is not $G_{\mathcal{D}}$-stable. We will prove the converse by applying the Cone Decomposition Algorithm of **Subsection 2.3**. We choose the maximal torus $T < G_{\mathcal{D}}$ to be $T = \mathbb{C}^\times \times \{\mathrm{diag}(t_0, t_1, t_2, t_3) \mid t_0 \cdot \ldots \cdot t_4 = 1\}$ where the matrices are diagonal with respect to the basis B. We let $C \subset \check{X}(T)_{\mathbb{R}}$ be the standard cone. Thus

$$\check{X}(T)_{\mathbb{R}} := \{(n, r_0, \ldots, r_3) \in \mathbb{R}^5 \mid r_0 + \ldots + r_3 = 0\}, \qquad C := \{(n, r_0, \ldots, r_3) \in \mathbb{R}^5 \mid r_0 \geq r_1 \geq \ldots \geq r_3\}.$$

Let

$$x_i := r_{i-1} - r_i, \qquad i = 1, 2, 3.$$

In the new coordinates (n, x_1, x_2, x_3) we have

$$C := \{(n, x_1, \ldots, x_3) \mid x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}.$$

The linear functions $r_0, \ldots, r_3$ (restricted to $\check{X}(T)_{\mathbb{R}}$) are expressed as follows in terms of the coordinates $x_1, \ldots, x_3$:

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ -1/4 & 1/2 & 1/4 \\ -1/4 & -1/2 & 1/4 \\ -1/4 & -1/2 & -3/4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6.3.12}$$

A hyperplane $H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and ony if it is the kernel of one of the following linear functions:

$$x_1,\ x_2,\ x_3,\ x_1 - x_3,\ x_1 + x_3 \pm 12n,\ \ x_1 - x_3 \pm 12n,\ x_1 + 2x_2 + x_3 \pm 12n. \tag{6.3.13}$$

Thus the hypotheses of **Proposition 2.3.4** are satisfied. An easy computation gives that the ordering rays in the (n, x_1, x_2, x_3)-coordinates are generated by the vectors

$$(\pm 1, 0, 0, 0), \quad (0, 1, 0, 1), \quad (0, 1, 0, 0), \quad (0, 0, 0, 1), \quad (0, 0, 1, 0).$$

Switching to (n, r_0, r_1, r_2, r_3)-coordinates via (6.3.12) we get the 1-PS's

$$\lambda_0^{\pm 1}, \quad \lambda_1, \quad (1, \mathrm{diag}(t^3, t^{-1}, t^{-1}, t^{-1})), \quad (1, \mathrm{diag}(t, t, t, t^{-3})), \quad (1, \mathrm{diag}(t, t, t^{-1}, t^{-1})).$$

A straightforward case-by-case analysis gives that if $\mu(A, \lambda) \geq 0$ for one of the last three 1-PS's then one of Items (1)-(3c) holds. $\square$

Corollary 6.3.4. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ and let A', A'' be as in (6.3.1). Suppose that $A'' \cap \bigwedge^3 V_{14} = \{0\}$. Then A is $G_{\mathcal{D}}$-stable if and only if $Q_{A'} \cap Q_{A''}$ is a smooth curve. In particular the generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ is $G_{\mathcal{D}}$-stable.*

Proof. Let $[\xi_0] \in Q_{A''}$: then $\dim(A'' \cap F_{\xi_0}) = 1$ because $A'' \cap \bigwedge^3 V_{14} = \{0\}$. Let

$$A'' \cap F_{\xi_0} = [v_0 \wedge \xi_0 \wedge v_5 + \alpha], \qquad \alpha \in \bigwedge^3 V_{14}.$$

By **Proposition 6.3.2** the projective tangent space to $Q_{A''}$ at $[\xi_0]$ is equal to $\mathbb{P}(\operatorname{supp} \alpha)$. Now assume that $\dim(A'' \cap [v_0] \wedge V_{14} \wedge [v_5]) \geq 2$. Then on one hand A is not $G_{\mathcal{D}}$-stable by **Proposition 6.3.3**, on the other hand $Q_{A''}$ is either $\mathbb{P}(V_{14})$ or a quadric whose singular locus has dimension at least 1 and hence $Q_{A'} \cap Q_{A''}$ is not a smooth curve. Thus from now on we may assume that $\dim(A'' \cap [v_0] \wedge V_{14} \wedge [v_5]) \leq 1$. Notice that since $A'' \cap \bigwedge^3 V_{14} = \{0\}$ we get that neither (1), (2) or (3a) of **Proposition 6.3.3** holds. Next notice that (3b) of **Proposition 6.3.3** holds if and only if $\Theta_{A'}$ is not a smooth conic i.e. $Q_{A'}$ is either all of $\mathbb{P}(V_{14})$ or a quadric of rank at most 2: it follows that we may suppose that $Q_{A'}$ is a smooth quadric. With these hypotheses $Q_{A'} \cap Q_{A''}$ is not transverse at $[\xi_0]$ if and only if there exists a basis $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ of V_{14} such that (3c) of **Proposition 6.3.3** holds. □

Proposition 6.3.5. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ and suppose that A is $G_{\mathcal{D}}$-semistable. Suppose in addition that one of Items (1), (2), (3a), (3b) of* **Proposition 6.3.3** *holds. Then A is* $\mathrm{PGL}(V)$*-equivalent to A_{III}.*

Proof. Suppose that Item (1) or (2) holds. Taking $\lim_{t\to 0} \lambda_0(t)A$ (respectively $\lim_{t\to 0} \lambda_0^{-1}(t)A$) and applying **Claim 2.2.4** we get that A is $G_{\mathcal{D}}$-equivalent to

$$A_0 = A' \oplus B \oplus C \oplus A''', \qquad B \in \mathrm{Gr}(2, [v_0] \wedge V_{14} \wedge [v_5]), \quad C = B^{\perp} \cap \bigwedge V_{14}.$$

It follows easily that A_0 satisfies Item (3*a*) in the statement of **Proposition 6.3.3**. Thus we may assume from the start that one of Items (3*a*), (3*b*) holds. Suppose that Item (3*a*) holds. As shown in the proof of **Proposition 6.3.3** it follows that $d^{\lambda_1}(A') \succeq (1,0,2)$ and $d^{\lambda_1}(A'') \succeq (2,2,0)$. Taking $\lim_{t\to 0} \lambda_1(t)A$ we get that A is $G_{\mathcal{D}}$-equivalent to a λ_1-split $A_0 \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ with

$$A'_0 = \langle v_0 \wedge \xi_0 \wedge \xi_1, v_0 \wedge \xi_1 \wedge \xi_3, v_0 \wedge \xi_2 \wedge \xi_3 \rangle, \quad A''_0 \supset \langle v_0 \wedge \xi_0 \wedge v_5, \xi_0 \wedge \xi_1 \wedge \xi_2 \rangle.$$

Let λ_2 be the 1-PS of $\mathrm{SL}(V_{14})$ defined by

$$\lambda_2(t)\xi_1 = t\xi_1, \quad \lambda_2(t)\xi_3 = t\xi_3, \quad \lambda_2(t)\xi_0 = t^{-1}\xi_0, \quad \lambda_2(t)\xi_2 = t^{-1}\xi_2.$$

One checks easily that $\mu(A_0, \lambda_2) = 0$ and that $A_{00} = \lim_{t\to 0} \lambda_2(t)A_0$ has a monomial basis. By **Claim 4.3.1** we get that $A_{00} \in \mathrm{PGL}(V)A_{III}$ and hence A_{00} is $G_{\mathcal{D}}$-equivalent to A_{III}. It follows by duality that if Item (3*b*) holds then A is $G_{\mathcal{D}}$-equivalent to A_{III}. □

Corollary 6.3.6. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be semistable and suppose that A is not* $\mathrm{PGL}(V)$*-equivalent to A_{III}. Then $Q_{A'}$ is a smooth quadric.*

Proof. Suppose that $Q_{A'}$ is not a smooth quadric: then Item (3*b*) of **Proposition 6.3.3** holds and hence we get a contradiction by **Proposition 6.3.5**. □

Remark 6.3.7. Let

$$A := \langle v_0 \wedge \xi_0 \wedge \xi_1, v_0 \wedge \xi_0 \wedge \xi_3, v_0 \wedge \xi_2 \wedge \xi_3, v_0 \wedge \xi_1 \wedge v_5, v_0 \wedge \xi_2 \wedge v_5, \xi_0 \wedge \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_2 \wedge \xi_3, \xi_0 \wedge \xi_2 \wedge v_5, \xi_0 \wedge \xi_3 \wedge v_5, \xi_1 \wedge \xi_3 \wedge v_5 \rangle. \tag{6.3.14}$$

Then $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$. Applying **Claim 4.3.1** we get that the left-hand side belongs to $\mathrm{PGL}(V)A_{III}$. Thus $\mathrm{PGL}(V)A_{III} \cap \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ is not empty.

Let B be the basis of V_{14} appearing in the proof of **Proposition 6.3.3** - see (6.3.8). Let λ_1 be the 1-PS of $G_{\mathcal{D}}$ defined by (6.3.9). Let $\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{D}}$ be the affine cone over $\mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$; then $G_{\mathcal{D}}$ acts on $\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{D}}$. The fixed locus $(\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{D}})^{\lambda_1}$ is the set of A which are mapped to themselves by $\wedge^3\lambda_1(t)$ and such that $\wedge^3\lambda_1(t)$ acts trivially on $\bigwedge^{10} A$.

Definition 6.3.8. Let $\mathbb{M}^{\mathsf{B}}_{\mathcal{D}} \subset \mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{D}})^{\lambda_1})$ be the set of A such that $\wedge^3\lambda_1(t)$ acts trivially on $\bigwedge^3 A'$, $\bigwedge^4 A''$, and $\bigwedge^3 A'''$ (as usual A', A'', A''' are as in (6.3.1)).

Remark 6.3.9. Let's adopt the notation introduced in the proof of **Proposition 6.3.3**. Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$; then $A \in \mathbb{M}^{\mathsf{B}}_{\mathcal{D}}$ if and only if A', A'' are λ_1-split of types $d^{\lambda_1}(A') = (1,1,1)$ and $d^{\lambda_1}(A'') = (1,2,1)$. Moreover $\mathbb{M}^{\mathsf{B}}_{\mathcal{D}}$ is an irreducible component of $\mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{D}})^{\lambda_1})$.

Proposition 6.3.10. *Suppose that A is properly $G_{\mathcal{D}}$-semistable i.e. $G_{\mathcal{D}}$-semistable but not $G_{\mathcal{D}}$-stable. Then there exists $A_0 \in \mathbb{M}^{\mathsf{B}}_{\mathcal{D}}$ which is $G_{\mathcal{D}}$-equivalent to A.*

Proof. By **Proposition 6.3.3** one of Items (1), (2), ($3a$), ($3b$) or ($3c$) of that proposition holds. We will adopt the notation introduced in the proof of **Proposition 6.3.3**. If Item ($3c$) holds then by **Remark 6.3.9** there exists $A_0 \in \mathbb{M}^{\mathsf{B}}_{\mathcal{D}}$ which is $G_{\mathcal{D}}$-equivalent to A. Now suppose that Item (1) or (2) holds. Taking $\lim_{t\to 0}\lambda_0(t)A$ (respectively $\lim_{t\to 0}\lambda_0^{-1}(t)A$) and applying **Claim 2.2.4** we get that A is $G_{\mathcal{D}}$-equivalent to

$$A_0 = A' \oplus B \oplus C \oplus A''', \qquad B \in \mathrm{Gr}(2, [v_0] \wedge V_{14} \wedge [v_5]), \quad C = B^{\perp} \cap \bigwedge V_{14}.$$

It follows easily that A_0 satisfies Item ($3a$) in the statement of **Proposition 6.3.3**. Thus we may assume from the start that one of Items ($3a$), ($3b$) holds. Suppose that Item ($3a$) holds. As shown in the proof of **Proposition 6.3.3** it follows that $d^{\lambda_1}(A') \succeq (1,0,2)$ and $d^{\lambda_1}(A'') \succeq (2,2,0)$. Taking $\lim_{t\to 0}\lambda_1(t)A$ and applying **Claim 2.2.4** we get that A is $G_{\mathcal{D}}$-equivalent to a λ_1-split $A_0 \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ with

$$A_0' = \langle v_0\wedge\xi_0\wedge\xi_1, v_0\wedge\xi_1\wedge\xi_3, v_0\wedge\xi_2\wedge\xi_3\rangle, \quad A'' \supset \langle v_0\wedge\xi_0\wedge v_5, \xi_0\wedge\xi_1\wedge\xi_2\rangle.$$

Let λ_2 be the 1-PS of $\mathrm{SL}(V_{14})$ defined by

$$\lambda_2(t)\xi_1 = t\xi_1, \quad \lambda_2(t)\xi_3 = t\xi_3, \quad \lambda_2(t)\xi_0 = t^{-1}\xi_0, \quad \lambda_2(t)\xi_2 = t^{-1}\xi_2.$$

One checks easily that $\mu(A_0,\lambda_2) = 0$ and that $A_{00} = \lim_{t\to 0}\lambda_2(t)A_0$ has a monomial basis. By **Claim 4.3.1** we get that $A_{00} \in \mathrm{PGL}(V)A_{III}$ and hence A_{00} is $G_{\mathcal{D}}$-equivalent to an element of $\mathbb{M}^{\mathsf{F}}_{\mathcal{D}}$ by (6.3.14). This proves that if Item ($3a$) holds then A is $G_{\mathcal{D}}$-equivalent to an element of $\mathbb{M}^{\mathsf{F}}_{\mathcal{D}}$. It follows by duality that if Item ($3b$) holds then A is $G_{\mathcal{D}}$-equivalent to an element of $\mathbb{M}^{\mathsf{F}}_{\mathcal{D}}$. □

6.3.3 Analysis of Θ_A and $C_{W,A}$

Proposition 6.3.11. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be $G_{\mathcal{D}}$-semistable and suppose that it is not* $\mathrm{PGL}(V)$*-equivalent to A_{III}. Let $W \in \Theta_A$. Then one of the following holds:*

(1) $\dim(W\cap V_{14}) = 1$ *and* $W = \langle \eta_0, v_0+\eta_2, \eta_1+v_5\rangle$ *where* $\eta_0,\eta_1,\eta_2 \in V_{14}$. *Moreover we may assume that one of the following holds:*

- *(1a)* $v_0\wedge\eta_0\wedge v_5 \in A''$ *and* $\eta_1 = 0$ *or* $\eta_2 = 0$.
- *(1b)* $\mathbf{T}_{[\eta_0]}Q_{A'} \subset \mathbf{T}_{[\eta_0]}Q_{A''}$ *and A is not $G_{\mathcal{D}}$-stable.*

(2) $\dim(W\cap V_{14}) = 2$ *and*

- *(2a)* $W \in (\Theta_{A'} \cup \Theta_{A'''})$ *or*
- *(2b)* $W = \langle v_0+\eta_2, \eta_0, \eta_1\rangle$ *where* $\eta_0,\eta_1,\eta_2 \in V_{14}$ *are linearly independent.*
- *(2c)* $W = \langle v_5+\eta_2, \eta_0, \eta_1\rangle$ *where* $\eta_0,\eta_1,\eta_2 \in V_{14}$ *are linearly independent.*

If either one of (2b), (2c) holds then A is not $G_{\mathcal{D}}$-stable.

(3) $W \subset V_{14}$.

Proof. First notice that $Q_{A'}$ is a smooth quadric by **Corollary 6.3.6**. Clearly $\dim(W \cap V_{14}) \geq 1$. We proceed to a case-by-case analysis according to the dimension of $W \cap V_{14}$.

$\boxed{\dim(W \cap V_{14}) = 1}$ Then W is necessarily as in Item (1). It remains to show that we may assume that (1a) or (1b) holds. We have

$$A \ni \eta_0 \wedge (v_0 + \eta_2) \wedge (\eta_1 + v_5) = -v_0 \wedge \eta_0 \wedge \eta_1 - (v_0 \wedge \eta_0 \wedge v_5 + \eta_0 \wedge \eta_1 \wedge \eta_2) + \eta_0 \wedge \eta_2 \wedge v_5.$$

It follows that

$$v_0 \wedge \eta_0 \wedge \eta_1 \in A', \quad (v_0 \wedge \eta_0 \wedge v_5 + \eta_0 \wedge \eta_1 \wedge \eta_2) \in A'', \quad \eta_0 \wedge \eta_2 \wedge v_5 \in A'''. \tag{6.3.15}$$

If one (at least) among $\eta_0 \wedge \eta_1$, $\eta_0 \wedge \eta_2$ vanishes then we may rename η_1, η_2 so that (1a) holds. Thus we may assume that $\eta_0 \wedge \eta_1 \neq 0 \neq \eta_0 \wedge \eta_2$. By (6.3.15) we get that the lines $\mathbb{P}\langle \eta_0, \eta_1 \rangle$ and $\mathbb{P}\langle \eta_0, \eta_2 \rangle$ are lines on the smooth quadric $Q_{A'}$ belonging to different rulings: it follows that $\mathbf{T}_{[\eta_0]} Q_{A'} = \mathbb{P}(\langle \eta_0, \eta_1, \eta_2 \rangle)$. On the other hand $\mathbb{P}(\langle \eta_0, \eta_1, \eta_2 \rangle) \subset \mathbf{T}_{[\eta_0]} Q_{A''}$ by (6.3.15) and **Proposition 6.3.2**. This proves that $\mathbf{T}_{[\eta_0]} Q_{A'} \subset \mathbf{T}_{[\eta_0]} Q_{A''}$, moreover we get that Item (3c) of **Proposition 6.3.3** holds with $\xi_i = \eta_i$ for $i = 0, 1, 2$ and ξ_3 such that $\mathbf{T}_{[\eta_1]} Q_{A'} = \mathbb{P}(\langle \eta_0, \eta_1, \xi_3 \rangle)$: it follows that A is not $G_{\mathcal{D}}$-stable. Thus Item (1b) holds.

$\boxed{\dim(W \cap V_{14}) = 2}$ Let $\{\eta_0, \eta_1\}$ be a basis of $W \cap V_{14}$. Let $0 \neq \alpha \in \bigwedge^3 W$: then $\alpha = \alpha' + \alpha'' + \alpha'''$ where $\alpha' \in A'$ etc. Multiplying α by η_0 or η_1 we get that

$$\alpha = x v_0 \wedge \eta_0 \wedge \eta_1 + \eta_0 \wedge \eta_1 \wedge \eta_2 + y \eta_0 \wedge \eta_1 \wedge v_5, \qquad x, y \in \mathbb{C}, \quad \eta_2 \in V_{14}.$$

Since $Q_{A'}$ is a smooth quadric it follows that one at least among x, y vanishes. On the other hand x, y do not both vanish because $W \not\subset V_{14}$. If $\eta_0 \wedge \eta_1 \wedge \eta_2 = 0$ then $W \in (\Theta_{A'} \cup \Theta_{A'''})$ i.e. Item(2a) holds. Assume that $\eta_0 \wedge \eta_1 \wedge \eta_2 \neq 0$: rescaling the η_i's we get that W is as in Item (2b) if $x \neq 0$, as in Item (2c) if $y \neq 0$. It remains to prove that if Item (2b) or (2c) holds then A is not $G_{\mathcal{D}}$-stable. By symmetry it suffices to assume that (2b) holds. Thus $v_0 \wedge \eta_0 \wedge \eta_1 \in A'$ and $\eta_0 \wedge \eta_1 \wedge \eta_2 \in A''$. In particular the smooth quadric $Q_{A'}$ contains the line $L := \mathbb{P}\langle \eta_0, \eta_1 \rangle$. Let $P := \mathbb{P}\langle \eta_0, \eta_1, \eta_2 \rangle$. Since $Q_{A'}$ is a smooth quadric $P \cap Q_{A'} = L + L'$ where L' is line distinct from L. We may choose a basis of $\langle \eta_0, \eta_1 \rangle$ and rename its elements η_0, η_1 so that $L \cap L' = [\eta_0]$. Then $\mathbf{T}_{[\eta_0]} = P = \mathbb{P}(\langle \eta_0, \eta_1, \eta_2 \rangle)$; it follows that Item (3c) of **Proposition 6.3.3** holds with ξ_i replaced by η_i for $i = 0, 1, 2$ and a suitable ξ_3 (up to a scalar ξ_3 is determined by requiring that $\mathbf{T}_{[\eta_1]} = P = \mathbb{P}(\langle \eta_0, \eta_1, \xi_3 \rangle)$). Thus A is not $G_{\mathcal{D}}$-stable.

$\boxed{\dim(W \cap V_{14}) = 3}$ Then Item (3) holds. □

Corollary 6.3.12. *Let $A \in \mathbb{S}_{\mathcal{D}}^{\mathsf{F}}$ be $G_{\mathcal{D}}$-stable. Then $\Theta_A = \Theta_{A'} \cup \Theta_{A'''} \cup Z_A$ where Z_A is a finite set. Moreover each of $\Theta_{A'}$, $\Theta_{A'''}$ is a smooth conic.*

Proof. Each of $\Theta_{A'}$, $\Theta_{A'''}$ is a smooth conic by **Corollary 6.3.6**. Let $W \in \Theta_A$ and suppose that $W \notin (\Theta_{A'} \cup \Theta_{A'''})$. Then either Item (1a) or Item (3) of **Proposition 6.3.11** holds. Suppose that Item (1a) holds. By Item (1) of **Proposition 6.3.3** we get that $[\eta_0] \in \mathbb{P}(V_{14})$ is unique. If $0 = \eta_1 = \eta_2$ there are no other choices involved and hence W is uniquely determined. Next suppose that one of $\eta_0 \wedge \eta_1$ or $\eta_0 \wedge \eta_2$ is non-zero (if they both vanish we may rename η_1, η_2 so that $0 = \eta_1 = \eta_2$). Since $Q_{A'} = Q_{A'''}$ is a smooth quadric (by **Corollary 6.3.6**) we get that either $\eta_2 = 0$ and $\langle \eta_0, \eta_1 \rangle$ is the unique line of $Q_{A'}$ through $[\eta_0]$ or else $\eta_1 = 0$ and $\langle \eta_0, \eta_2 \rangle$ is the unique line of $Q_{A'''}$ through $[\eta_0]$. This shows that there is at most a finite set of choices for W such that Item (1a) of **Proposition 6.3.11** holds. By Item (2) of **Proposition 6.3.3** there is at most one choice for W such that Item (3) of **Proposition 6.3.11** holds. □

Definition 6.3.13. Suppose that Item (3) of **Proposition 6.3.11** holds. Let

$$C'_W := \{[\eta] \in \mathbb{P}(W) \mid \dim(A' \cap F_\eta) > 0\}, \quad C''_W := \{[\eta] \in \mathbb{P}(W) \mid \dim(A'' \cap F_\eta) > 1\}.$$

Remark 6.3.14. Suppose that Item (3) of **Proposition 6.3.11** holds. Then

$$C'_W = \mathbb{P}(W) \cap Q_{A'} = \mathbb{P}(W) \cap Q_{A'''} = \{[\eta] \in \mathbb{P}(W) \mid \dim(A''' \cap F_\eta) > 0\}. \tag{6.3.16}$$

(Recall that $Q_{A'}$ is a smooth quadric - see the proof of **Proposition 6.3.11**.) It follows that either $C_{W,A} = 2C'_W + C''_W$ or $C_{W,A} = \mathbb{P}(W)$.

We continue to assume that Item (3) of **Proposition 6.3.11** holds. Let $W = \langle \eta_0, \eta_1, \eta_2 \rangle$. By hypothesis A is not $\mathrm{PGL}(V)$-equivalent to A_{III}: thus **Proposition 6.3.5** gives that

$$A'' = \langle \eta_0 \wedge \eta_1 \wedge \eta_2, v_0 \wedge \eta_0 \wedge v_5 + \alpha_0, v_0 \wedge \eta_1 \wedge v_5 + \alpha_1, v_0 \wedge \eta_2 \wedge v_5 + \alpha_2, \rangle, \quad \alpha_i \in \bigwedge^3 V_{14}. \tag{6.3.17}$$

The condition that A'' be lagrangian translates into

$$\eta_i \wedge \alpha_j = \eta_j \wedge \alpha_i, \qquad 0 \le i, j \le 2. \tag{6.3.18}$$

It follows that

$$C''_W = \left\{ \left[\sum_{i=0}^{2} X_i \eta_i \right] \mid \sum_{0 \le i,j \le 2} \eta_i \wedge \alpha_j X_i X_j = 0 \right\}. \tag{6.3.19}$$

Lemma 6.3.15. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be $G_{\mathcal{D}}$-stable. Suppose that $W \in \Theta_A$ and that $W \subset V_{14}$. Then $C_{W,A} = 2C'_W + C''_W$ and C'_W is a smooth conic intersecting transversely C''_W.*

Proof. First we claim that $C_{W,A} \ne \mathbb{P}(W)$. In fact A has minimal $\mathrm{PGL}(V)$-orbit by **Claim 5.2.1**, moreover it follows from **Proposition 6.3.11** that $\dim \Theta_A = 1$. Thus $C_{W,A} \ne \mathbb{P}(W)$ by **Corollary 6.1.10**. By **Remark 6.3.14** we get that $C_{W,A} = 2C'_W + C''_W$. Suppose that C'_W is a singular conic. Then Item (3c) of **Proposition 6.3.3** is satisfied with $a = 0$ and $W = \langle \xi_0, \xi_1, \xi_2 \rangle$: by **Proposition 6.3.3** that contradicts the hypothesis that A is $G_{\mathcal{D}}$-stable. This proves that C'_W is a smooth conic. Now suppose that there is a point $p \in C'_W \cap C''_W$ such that $\mathbf{T}_p C'_W \subset \mathbf{T}_p C''_W$. We may choose a basis $\{\eta_0, \eta_1, \eta_2\}$ of W such that $p = [\eta_0]$ and $\mathbf{T}_p C'_W = \mathbb{P}\langle \eta_0, \eta_1 \rangle$. We let $\alpha_0, \alpha_1, \alpha_2$ be as in (6.3.17). The explicit equation (6.3.19) gives that $\eta_0 \wedge \alpha_0 = 0$ and allows us to compute $\mathbf{T}_p C''_W$: it follows that $\eta_0 \wedge \alpha_1 = 0$. By (6.3.18) we get that $\eta_1 \wedge \alpha_0 = 0$; thus $\alpha_0 = \eta_0 \wedge \eta_1 \wedge \eta$. Since $\mathbf{T}_p C'_W \subset \mathbf{T}_p Q_{A'}$ and $\mathbb{P}(W)$ is not tangent to $Q_{A'}$ we may extend $\{\eta_0, \eta_1\}$ to a basis $\{\eta_0, \eta_1, \eta_3, \eta_4\}$ (notice that η_2 does not belong to the chosen basis)) so that $v_0 \wedge \eta_0 \wedge \eta_3 \in A'$ (i.e. $\mathbb{P}\langle \eta_0, \eta_3 \rangle$ is a line of the ruling of $Q_{A'}$ corresponding to A') and $v_0 \wedge (\eta_0 \wedge \eta_4 + \eta_3 \wedge \eta_1) \in A'$. Suppose first that $\eta_0 \wedge \eta_1 \wedge \eta$ and $\eta_0 \wedge \eta_1 \wedge \eta_2$ are linearly dependent. Then $v_0 \wedge \eta_0 \wedge v_5 \in A''$ and hence A is not $G_{\mathcal{D}}$-stable by Item (3c) of **Proposition 6.3.3**, that is a contradiction. Next suppose that $\eta_0 \wedge \eta_1 \wedge \eta$ and $\eta_0 \wedge \eta_1 \wedge \eta_2$ are linearly independent: then there exist $x, y \in \mathbb{C}$ such that $x\eta_0 \wedge \eta_1 \wedge \eta + y\eta_0 \wedge \eta_1 \wedge \eta_2 = -\eta_0 \wedge \eta_1 \wedge \eta_3$. It follows that $(xv_0 \wedge \eta_0 \wedge v_5 + \eta_0 \wedge \eta_1 \wedge \eta_3) \in A''$. Set $\xi_0 = \eta_0$, $\xi_1 = \eta_3$, $\xi_2 = \eta_1$ and $\xi_3 = \eta_4$; then A satisfies Item (3c) of **Proposition 6.3.3** and hence A is not $G_{\mathcal{D}}$-stable, that is a contradiction. □

Lemma 6.3.16. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be $G_{\mathcal{D}}$-stable. Suppose that $W \in \Theta_A$ and that Item (1) of* **Proposition 6.3.11** *holds. Then $C_{W,A}$ is a semistable sextic of Type II-1.*

Proof. By **Proposition 6.3.11** there exists $0 \ne \eta_0 \in V_{14}$ such that $W = \langle v_0, \eta_0, v_5 \rangle$. Arguing as in the proof of **Lemma 6.3.15** we get that $C_{W,A} \ne \mathbb{P}(W)$: thus $C_{W,A} = V(P)$ where $0 \ne P \in \mathrm{S}^6 W^\vee$. Let $\lambda_{\mathcal{D}}$ be the 1 PS of $\mathrm{SL}(V)$ defined in **Subsection 5.2** i.e. $\lambda_{\mathcal{D}}(t) = \mathrm{diag}(t, 1, 1, 1, 1, t^{-1})$ in the basis F. Then $\lambda_{\mathcal{D}}(t) W = W$ for all $t \in \mathbb{C}^\times$. Now apply **Claim 3.2.4** to $\lambda_{\mathcal{D}}(t)$ and P: by **Remark 1.4.3** we get that P is given by (1.4.2) i.e. $C_{W,A}$ is the "union"of 3 conics tangent at $[v_0]$ and $[v_5]$ (because $P \ne 0$). It remains to prove that the 3 conics are distinct. The proof is achieved by a brutal computation. By **Corollary 6.3.6** we know that $Q_{A'}$ is a smooth quadric, moreover $[\eta_0] \notin Q_{A'}$ because if $[\eta_0] \in Q_{A'}$ then Item (3c) of **Proposition 6.3.3** holds and hence A is not $G_{\mathcal{D}}$-stable. Since $[\eta_0]$ is outside the smooth quadric $Q_{A'}$ we may complete η_0 to a basis $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ of V_{14} such that A' is given by (6.3.3). Then A' and A''' are transverse to $\langle \eta_1, \eta_2, \eta_3 \rangle$: thus there are linear maps $f, g \colon \bigwedge^2 \langle \eta_1, \eta_2, \eta_3 \rangle \to \langle \eta_1, \eta_2, \eta_3 \rangle$ such that

$$A' = \{v_0 \wedge (\eta_0 \wedge f(\beta') + \beta') \mid \beta' \in \bigwedge^2 \langle \eta_1, \eta_2, \eta_3 \rangle\}, \quad A''' = \{[v_5] \wedge (\eta_0 \wedge g(\beta''') + \beta''') \mid \beta''' \in \bigwedge^2 \langle \eta_1, \eta_2, \eta_3 \rangle\}.$$

Choose the basis $\mathcal{B}=\{\eta_1,\eta_2,\eta_3\}$ of $\langle\eta_1,\eta_2,\eta_3\rangle$ and let $\mathcal{B}^\vee=\{\eta_2\wedge\eta_3,\eta_3\wedge\eta_1,\eta_1\wedge\eta_2\}$ be the dual basis of $\bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle$: the matrices associated to f and g are the unit matrix 1_3 and -1_3 respectively: in particular we have $g=-f$. By **Proposition 6.3.3** we have $A\cap[v_0]\wedge V_{14}\wedge[v_5]=[v_0\wedge\eta_0\wedge v_5]$: it follows that there exists a linear map $h\colon \bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle\to\langle\eta_1,\eta_2,\eta_3\rangle$ such that

$$A''=[v_0\wedge\eta_0\wedge v_5]\oplus\{(v_0\wedge h(\beta'')\wedge v_5+\eta_0\wedge\beta'')\mid \beta''\in\bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle\}.$$

By definition $[xv_0+\eta_0+yv_5]\in C_{W,A}$ if and only if $\dim(A\cap F_{xv_0+\eta_0+yv_5})\geq 2$ i.e. there exists

$$(0,0,0)\neq(\beta',\beta'',\beta''')\in\bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle\times\bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle\times\bigwedge^2\langle\eta_1,\eta_2,\eta_3\rangle$$

such that

$$0=(xv_0+\eta_0+yv_5)\wedge(v_0\wedge(\eta_0\wedge f(\beta')+\beta')+(v_0\wedge h(\beta'')\wedge v_5+\eta_0\wedge\beta'')+v_5\wedge(\eta_0\wedge g(\beta''')+\beta''')). \quad (6.3.20)$$

We may write out the right-hand side as the sum of 3 elements respectively in $[v_0]\wedge\bigwedge^3 V_{14}$, $[v_5]\wedge\bigwedge^3 V_{14}$ and $[v_0]\wedge\bigwedge^2 V_{14}\wedge[v_5]$: we get that

$$0=\beta'-x\beta''=\beta'''-y\beta''=xg(\beta''')-yf(\beta')-h(\beta'')=x\beta''-y\beta'. \quad (6.3.21)$$

Thus (recall that $g=-f$)

$$[xv_0+\eta_0+yv_5]\in C_{W,A}\text{ if and only if }(h+2xyf)\text{ is singular.} \quad (6.3.22)$$

To finish the proof we distinguish between the two cases:

(a) $A''\cap\bigwedge^3 V_{14}=\{0\}$ or

(b) $A''\cap\bigwedge^3 V_{14}\neq\{0\}$.

Item (a) holds Then $Q_{A''}$ is a quadric with $\operatorname{sing} Q_{A''}=\{[\eta_0]\}$ and $Q_{A'}\cap Q_{A''}$ is a smooth curve of genus 1 (by **Proposition 6.3.3** it cannot have singular points). Let $Q_{A'}=V(q_{A'})$ and $Q_{A''}=V(q_{A''})$. Since $Q_{A'}\cap Q_{A''}$ is smooth there are exactly 4 singular quadrics in the pencil spanned by $Q_{A'}$ and $Q_{A''}$: since $Q_{A'}$ is smooth and $Q_{A''}$ is singular it follows that

$$|\{r\neq 0\mid\det(q_{A'}+rq_{A''})=0\}|=3. \quad (6.3.23)$$

Now let $M(q_{A'})$ and $M(q_{A''})$ be the symmetric matrices associated to $q_{A'}$ and $q_{A''}$ by the choice of the basis $\{\eta_0,\eta_1,\eta_2,\eta_3\}$ of V_{14} and the dual basis $\{\eta_1\wedge\eta_2\wedge\eta_3,\eta_0\wedge\eta_2\wedge\eta_3,\eta_0\wedge\eta_3\wedge\eta_1,\eta_0\wedge\eta_1\wedge\eta_2\}$ of $\bigwedge^3 V_{14}$. Then $M(q_{A''})$ has first row and first column equal to zero. Let N be the 3×3-matrix obtained by deleting first row and first column of $M(q_{A''})$: thus N is the matrix $M^{\mathcal{B}}_{\mathcal{B}^\vee}(h^{-1})$ associated to h^{-1} by the choice of bases $\mathcal{B}$, $\mathcal{B}^\vee$ given above. By (6.3.4) we know that $M(q_{A'})$ is the unit matrix: thus (6.3.23) gives that N has exactly 3 distinct (non-zero) eigenvalues and hence so does $M^{\mathcal{B}^\vee}_{\mathcal{B}}(h)$. Since $M^{\mathcal{B}^\vee}_{\mathcal{B}}(f)=1_3$ we get that $(h+2sf)$ is singular for exactly 3 distinct non-zero values of s, say s_1,s_2,s_3. Now look at (1.4.2): we get that $a_i/b_i=-s_i$ and hence the 3 conics are indeed distinct.

Item (b) holds Then $\dim(A''\cap\bigwedge^3 V_{14})=1$ by **Proposition 6.3.3**. By an orthogonal change of basis in $\langle\eta_1,\eta_2,\eta_3\rangle$ we may assume that $A''\cap\bigwedge^3 V_{14}=[\eta_0\wedge\eta_1\wedge\eta_2]$ and moreover (6.3.4) continues to hold (recall that C'_{W_0} is smooth by **Lemma 6.3.15**). Thus A'' is given by (6.3.17) with $\alpha_0=0$. Let $W_0:=\langle\eta_0,\eta_1,\eta_2\rangle$: then $W\in\Theta_A$ and we have the conics $C'_{W_0},C''_{W_0}\subset\mathbb{P}(W_0)$, see **Definition 6.3.13**. By (6.3.19) we know that C''_{W_0} is singular at $[\eta_0]$ (recall (6.3.18)); in order to be coherent with our current use of coordinates (see (6.3.4)) we replace the X_i's in (6.3.19) by T_i's. Let $C'_{W_0}=V(c'_{W_0})$ and $C''_{W_0}=V(c''_{W_0})$: by **Lemma 6.3.15** we have

$$|\{r\neq 0\mid\det(c'_{W_0}+rc''_{W_0})=0\}|=2. \quad (6.3.24)$$

The matrix $M_{\mathcal{B}}^{\mathcal{B}^\vee}(h)$ has third row and third column equal to zero: let P be the 2×2-matrix obtained by deleting third row and third column, it is invertible because $\dim(A'' \cap \bigwedge^3 V_{14}) = 1$. Let R be the 3×3-matrix with vanishing first row and first column and with P^{-1} in the remaining space. Then R is the symmetric matrix giving c''_{W_0}: since (6.3.4) continues to hold (6.3.24) gives that P^{-1} has exactly 2 distinct eigenvalues. Thus P has exactly 2 distinct eigenvalues as well: it follows that $(h + 2sf)$ is singular for exactly 2 distinct non-zero values of s, say s_1, s_2. Moreover h is singular because Item (b) holds. Now look at (1.4.2): we get that $a_i/b_i = -s_i$ for $i = 1, 2$ and $a_3 = 0$, thus the 3 conics are indeed distinct. □

Proposition 6.3.17. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be $G_{\mathcal{D}}$-stable. Let $W \in \Theta_A$. Then*

(i) If Item (1) of **Proposition 6.3.11** *holds then $C_{W,A}$ is a semistable sextic of Type II-1.*

(ii) If Item (2) of **Proposition 6.3.11** *holds then $C_{W,A}$ is a semistable sextic of Type II-2.*

(iii) If Item (3) of **Proposition 6.3.11** *holds then $C_{W,A}$ is a semistable sextic of Type II-3.*

In particular $[A] \notin \mathfrak{I}$.

Proof. Item (i) is the content of **Lemma 6.3.16** and Item (iii) is the content of **Lemma 6.3.15**. Thus it remains to prove Item (ii). First we claim that $C_{W,A} \neq \mathbb{P}(W)$. In fact A has minimal $\mathrm{PGL}(V)$-orbit by **Claim 5.2.1**, moreover it follows from **Proposition 6.3.11** that $\dim \Theta_A = 1$. Thus $C_{W,A} \neq \mathbb{P}(W)$ by **Corollary 6.1.10**. Since A is $G_{\mathcal{D}}$-stable we have $W \in (\Theta_{A'} \cup \Theta_{A'''})$. We will give the proof for $W \in \Theta_{A'}$ (if $W \in \Theta_{A'''}$ the proof is analogous). There exist $\eta_1, \eta_2 \in V_{14}$ such that $W = \langle v_0, \eta_1, \eta_2 \rangle$. Let $\{X_0, X_1, X_2\}$ be the dual basis of $W^\vee$ and $0 \neq P \in \mathbb{C}[X_0, X_1, X_2]_6$ be the homogeneous . The 1 PS $\lambda_{\mathcal{D}}$ defined in **Subsection 5.2** maps W to itself: by applying **Claim 3.2.4** to $\lambda_{\mathcal{D}}(t)$ and P we get that $P = X_0^2 F$ where $0 \neq F \in \mathbb{C}[X_1, X_2]_4$. It remains to prove that F has no multiple roots. Let $L := \mathbb{P}(W \cap V_{14})$. The line L is contained in $Q_{A'}$ (by definition). By **Corollary 6.3.6** we know that $Q_{A'}$ is a smooth quadric and hence there is a projection $\pi\colon Q_{A'} \to L$. The line L has equation $X_0 = 0$ in $\mathbb{P}(W)$ and the roots of F give 4 points $p_1, p_2, p_3, p_4 \in L$: we must show that the p_i's are distinct. In order to describe the p_i's we distinguish between the two cases:

(a) There is no $W_0 \in \Theta_A$ contained in V_{14}.

(b) There exist $W_0 \in \Theta_A$ contained in V_{14}.

Item (a) holds Then $E := Q_{A'} \cap Q_{A''}$ is a smooth elliptic curve by **Proposition 6.3.3**. Restricting the projection π to E we get a degree-2 map $f\colon E \to L$. Since E is smooth of genus 1 there are 4 (distinct) ramification points $q_1, \ldots, q_4$ of f: we will show that $\{p_1, \ldots, p_4\} = \{\pi(q_1), \ldots, \pi(q_4)\}$. Let $[\eta_2] \in E$ be a ramification point of f and let $\pi([\eta_2]) = [\eta_0]$. We must prove that

$$\mathbb{P}(\langle v_0, \eta_0 \rangle) \subset C_{W,A}. \tag{6.3.25}$$

By hypothesis the line $\mathbb{P}(\langle \eta_0, \eta_2 \rangle)$ is contained in $Q_{A'}$ and it belongs to the ruling parametrized by A''' i.e. $\eta_0 \wedge \eta_2 \wedge v_5 \in A'''$. We may extend $\{\eta_0, \eta_2\}$ to a basis $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ (we may need to rescale η_0) of V_{14} so that

$$A' = \langle v_0 \wedge \eta_0 \wedge \eta_1, v_0 \wedge (\eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2), v_0 \wedge \eta_2 \wedge \eta_3 \rangle. \tag{6.3.26}$$

Since $[\eta_2]$ is a ramification point of f the line $\mathbb{P}(\langle \eta_0, \eta_2 \rangle)$ is tangent to $Q_{A''}$ at $[\eta_2]$: by **Proposition 6.3.2** it follows that there exists $\gamma \in \langle \eta_1, \eta_3 \rangle$ such that

$$(v_0 \wedge \eta_2 \wedge v_5 + \eta_0 \wedge \eta_2 \wedge \gamma) \in A''. \tag{6.3.27}$$

Thus $\gamma = s\eta_1 + t\eta_3$. A straightforward computation gives that

$$(-sv_0 \wedge (\eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2) + tv_0 \wedge \eta_2 \wedge \eta_3 + x^2 \eta_0 \wedge \eta_2 \wedge v_5 + x(v_0 \wedge \eta_2 \wedge v_5 + s\eta_0 \wedge \eta_2 \wedge \eta_1 + t\eta_0 \wedge \eta_2 \wedge \eta_3)) \in A \cap F_{v_0 + x\eta_0}. \tag{6.3.28}$$

Since $(v_0 \wedge \eta_0 \wedge \eta_1) \in A \cap F_{v_0+x\eta_0}$ it follows that $\dim(A \cap F_{v_0+x\eta_0}) \geq 2$. This shows that (6.3.25) holds and hence that $C_{W,A}$ is a semistable sextic of Type II-2.

Item (b) holds Let $C'_{W_0}, C''_{W_0} \subset \mathbb{P}(W_0)$ be as in **Definition 6.3.13**: by **Lemma 6.3.15** we know that $C'_{W_0} \cap C''_{W_0}$ consists of 4 distinct points, say $q_1, \ldots, q_4$: moreover no two of the points $q_1, \ldots, q_4$ belong to the same line on $Q_{A'}$ because C'_{W_0} is a smooth conic (see **Lemma 6.3.15**). Let's show that $\{p_1, \ldots, p_4\} = \{\pi(q_1), \ldots, \pi(q_4)\}$. Let $q_i = [\eta_2]$. By hypothesis $[\eta_2] \in Q_{A'}$: it follows that we may complete η_2 to a basis $\{\eta_0, \ldots, \eta_3\}$ of V_{14} so that $\eta_0 \wedge \eta_2 \wedge v_5 \in A'''$ and (6.3.26) holds. By definition of the q_i's there exists $0 \neq \eta_2 \wedge \beta \in A'' \cap \bigwedge^3 V_{14}$ and moreover $\dim(A'' \cap F_{\eta_2}) \geq 2$: since A is $G_{\mathcal{D}}$-stable $\dim(A'' \cap \bigwedge^3 V_{14}) = 1$ and hence there exists $(v_0 \wedge \eta_2 \wedge v_5 + \eta_2 \wedge \delta) \in A''$. Moreover $\eta_2 \wedge \delta \neq 0$ because otherwise $[\eta_2]$ is a singular point of C''_{W_0} (see (6.3.19)) and that would contradict **Lemma 6.3.15**. Now notice that $\eta_0 \wedge \eta_2 \wedge \beta \neq 0$ because by **Lemma 6.3.15** we know that C'_{W_0} is smooth. Thus there exists $x \in \mathbb{C}$ such that $\eta_0 \wedge (\eta_2 \wedge \delta + x\eta_2 \wedge \beta) = 0$ and hence (6.3.27) holds for a suitable $\gamma \in \langle \eta_1, \eta_3 \rangle$. It follows that (6.3.28) holds in this case as well and we are done again. □

Proposition 6.3.18. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$. Suppose that A is properly $G_{\mathcal{D}}$-semistable with minimal* $\mathrm{PGL}(V)$*-orbit (equivalently minimal $G_{\mathcal{D}}$-orbit by* **Claim 5.2.1***) and that $[A] \notin \mathfrak{X}_{\mathcal{W}}$. If $W \in \Theta_A$ then $C_{W,A}$ is a* $\mathrm{PGL}(W)$*-semistable sextic curve* $\mathrm{PGL}(W)$*-equivalent to a sextic of Type III-2, in particular $[A] \notin \mathfrak{I}$.*

Proof. First we notice that $C_{W,A} \neq \mathbb{P}(W)$. In fact suppose the contrary. By **Corollary 6.1.10** we get that A is $\mathrm{PGL}(V)$-equivalent to a lagrangian in $(\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k)$. Since A has minimal $\mathrm{PGL}(V)$-orbit it follows that $A \in (\mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k)$. Since $[A] \notin \mathfrak{X}_{\mathcal{W}}$ we must have $A \in \mathrm{PGL}(V)A_k$. As is easily checked $\Theta_{A_k} = k(\mathbb{P}(L))$ and hence Θ_{A_k} is a Veronese surface of degree 9: thus Θ_{A_k} does not contain any conic and therefore $A_k \notin \mathbb{B}_{\mathcal{D}}$, that is a contradiction. This proves that $C_{W,A} \neq \mathbb{P}(W)$. Next we may suppose that $A \notin \mathrm{PGL}(V)A_{III}$ because in that case $C_{W,A}$ is a sextic of Type III-2 by **Proposition 4.3.3**: thus $Q_{A'}$ is a smooth quadric by **Corollary 6.3.6**. Let $\lambda_{\mathcal{D}}, \lambda_1$ be the 1-PS's of $\mathrm{SL}(V)$ defined in **Subsection 5.2** and (6.3.9) respectively: notice that they commute and hence they define a homomorphism

$$\begin{array}{ccc} (\mathbb{C}^\times)^2 & \xrightarrow{\rho} & \mathrm{SL}(V) \\ (s,t) & \mapsto & \lambda_{\mathcal{D}}(s) \cdot \lambda_1(t) \end{array}$$

Both $\lambda_{\mathcal{D}}$ and λ_1 act trivially on $\bigwedge^{10} A$: thus $\rho(s,t)$ acts on Θ_A and hence we get an action of $(\mathbb{C}^\times)^2$ on Θ_A. Suppose first that W is fixed by $\rho(s,t)$ for every $(s,t) \in (\mathbb{C}^\times)^2$: we will prove that $C_{W,A}$ is a sextic of Type III-2. Let $\{\xi_0, \ldots, \xi_3\}$ be the basis of V_{14} appearing in the definition of λ_1, see (6.3.9). We claim that W is one of the following:

$$\langle v_0, \xi_0, a_1\xi_1 + a_2\xi_2 \rangle, \langle v_0, a_1\xi_1 + a_2\xi_2, \xi_3 \rangle, \langle \xi_0, a_1\xi_1 + a_2\xi_2, v_5 \rangle, \langle a_1\xi_1 + a_2\xi_2, \xi_3, v_5 \rangle, \langle \xi_0, \xi_1, \xi_2 \rangle, \langle \xi_1, \xi_2, \xi_3 \rangle.$$

In fact this is a simple consequence of **Proposition 6.3.11**: one invokes the hypothesis that $Q_{A'}$ is smooth (recall that a polynomial defining $Q_{A'}$ is left invariant by $\lambda_{\mathcal{D}}$) in order to exclude the cases $W = \langle v_0, \xi_1, \xi_2 \rangle$ or $W = \langle \xi_1, \xi_2, v_5 \rangle$. In each of the cases above the image of $(\mathbb{C}^\times)^2 \to GL(W)$ is a 2-dimensional torus. Let $C_{W,A} = V(P)$, thus $P \neq 0$: applying **Claim 3.2.4** we get that P is left invariant by a maximal torus of $\mathrm{SL}(W)$ and hence $C_{W,A}$ is a sextic of Type III-2 by **Remark 1.4.3**. Now let $W \in \Theta_A$ be arbitrary. Then the closure of $\{\rho(s,t)W\}$ contains a $W_0 \in \Theta_A$ which is fixed by $\rho(s,t)$ for every $(s,t) \in (\mathbb{C}^\times)^2$. It follows that $C_{W,A}$ is $\mathrm{PGL}(W)$-equivalent to $C_{W_0,A}$: we have proved that $C_{W_0,A}$ is a sextic of Type III-2 and hence we are done. □

6.3.4 Wrapping it up

We will prove **Proposition 6.3.1**. Item (1) is the content of **Corollary 6.3.4**. Let's prove Item (2). By **Corollary 6.3.12** we have $\Theta_A = \Theta_{A'} \cup \Theta_{A'''} \cup Z_A$ where $\Theta_{A'}$, $\Theta_{A'''}$ are smooth conics, Z_A is a finite set, every $W \in \Theta_{A'}$ contains $[v_0]$ and every $W \in \Theta_{A'''}$ contains $[v_5]$. It follows that $[v_0]$ is the unique 1-dimensional vector subspace of V contained in every $W \in \Theta_{A'}$ and $[v_5]$ is the unique 1-dimensional vector subspace of V contained in every $W \in \Theta_{A'''}$. From these facts we get that if $g \in \mathrm{Stab}(A)$ then g preserves the set $\{[v_0],[v_5]\}$ and maps V_{14} to itself. Thus the the connected component of Id in $\mathrm{Stab}(A)$ belongs to the centralizer $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{D}})$. Since A is $G_{\mathcal{D}}$-stable the stabilizer of A in $G_{\mathcal{D}}$ is a finite group and hence Item (2) follows. Lastly let's prove Items (3) and (4). Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ be $G_{\mathcal{D}}$-stable with minimal orbit: then $C_{W,A}$ is of Type II-1, II-2 or II-3 by **Proposition 6.3.17**. Next suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{D}}$ is properly $G_{\mathcal{D}}$-semistable with minimal orbit and $[A] \notin \mathfrak{X}_{\mathcal{W}}$: then $C_{W,A}$ is $\mathrm{PGL}(W)$-semistable and $\mathrm{PGL}(W)$-equivalent to a sextic of Type III-2 by **Proposition 6.3.18**. It remains to prove that

$$\mathfrak{X}_{\mathcal{W}} \subset \mathfrak{B}_{\mathcal{D}}. \tag{6.3.29}$$

In fact let U be a 4-dimensional vector-space and $\varphi\colon V \cong \bigwedge^2 U$ be an isomorphism as in (4.4.3). It suffices to prove that $\mathbb{X}^*_{\mathcal{W}}(U) \subset \mathbb{B}^*_{\mathcal{D}}$. Let $A \in \mathbb{X}^*_{\mathcal{W}}(U)$. By **Definition 4.4.3** there exists a smooth quadric $Z \subset \mathbb{P}(U)$ such that $A \supset i_+(Z)$. Let $L \subset Z$ be a line. Then $i_+(L)$ is a smooth conic contained in Θ_A; we claim that the intersection of $\mathrm{Gr}(3,V)$ and the linear span $\langle i_+(L)\rangle \subset \mathbb{P}(\bigwedge^3 V)$ is equal to $i_+(L)$. In fact if it is not then the plane $\langle i_+(L)\rangle$ is contained in Θ_A (because $\mathrm{Gr}(3,V)$ is cut out by quadrics) and hence $A \in \mathbb{X}^*_{\mathcal{F}_1,+}$; thus A is unstable and that contradicts **Proposition 4.4.4**. Since the intersection of $\mathrm{Gr}(3,V)$ and the linear span $\langle i_+(L)\rangle$ is equal to the smooth conic $i_+(L)$ it follows by [28] that $A \in \mathbb{B}^*_{\mathcal{D}}$. This proves (6.3.29).

6.4 $\mathfrak{B}_{\mathcal{E}_1}$

The isotypical decomposition of $\bigwedge^3 \lambda_{\mathcal{E}_1}$ with decreasing weights is

$$\bigwedge^3 V = \bigwedge^3 V_{02} \oplus [v_0] \wedge V_{12} \wedge V_{35} \oplus \left([v_0] \wedge \bigwedge^2 V_{35} \oplus \bigwedge^2 V_{12} \wedge V_{35}\right) \oplus V_{12} \wedge \bigwedge^2 V_{35} \oplus \bigwedge^3 V_{35}. \tag{6.4.1}$$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$. By definition $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ where

$$A_0 = \bigwedge^3 V_{02}, \quad A_1 \in \mathrm{Gr}(2,[v_0]\wedge V_{12}\wedge V_{35}), \quad A_2 \in \mathbb{LG}([v_0]\wedge \bigwedge^2 V_{35} \oplus \bigwedge^2 V_{12}\wedge V_{35}), \quad A_3 = A_1^{\perp} \cap (V_{12}\wedge \bigwedge^2 V_{35}).$$

We will associate to A two closed subsets of $\mathbb{P}(\bigwedge^2 V_{35})$ that will be conics for A generic. First we notice that $\mathbb{P}(V_{12} \wedge \bigwedge^2 V_{35}) \cap \mathbb{G}(3,V)$ is isomorphic to $\mathbb{P}(V_{12}) \times \mathbb{P}(V_{35})$ embedded by the Segre map. Since $\mathbb{P}(A_3)$ has codimension 2 in $\mathbb{P}(V_{12} \wedge \bigwedge^2 V_{35})$ it follows that Θ_{A_3} has dimension at least 1 and that generically it is a twisted rational cubic curve. The projection $\mathbb{P}(V_{12}) \times \mathbb{P}(\bigwedge^2 V_{35}) \to \mathbb{P}(\bigwedge^2 V_{35})$ defines a regular map $\pi\colon \Theta_{A_3} \to \mathbb{P}(\bigwedge^2 V_{35})$. Let $D_{A_3} := \operatorname{im} \pi$. If Θ_{A_3} is a twisted rational cubic curve then D_{A_3} is a smooth conic. On the other hand let

$$D_{A_2} := \{[\gamma] \in \mathbb{P}(\bigwedge^2 V_{35}) \mid A_2 \cap ([v_0 \wedge \gamma] \oplus \bigwedge^2 V_{12} \wedge \langle \operatorname{supp} \gamma\rangle) \neq \{0\}\}. \tag{6.4.2}$$

Then D_{A_2} is a lagrangian degeneracy locus and either it is a conic or all of $\mathbb{P}(\bigwedge^2 V_{35})$.

Remark 6.4.1. If $A_2 \cap \bigwedge^2 V_{12} \wedge V_{35} = \{0\}$ we may describe D_{A_2} as follows. By our assumption A_2 is the graph of a linear map $[v_0] \wedge \bigwedge^2 V_{35} \longrightarrow \bigwedge^2 V_{12} \wedge V_{35}$ which is symmetric because A_2 is lagrangian: let q_{A_2} be the associated quadratic form. Then $D_{A_2} = V(q_{A_2})$.

If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ is generic then D_{A_2}, D_{A_3} are conics intersecting transversely. Below is the main result of the present subsection.

Proposition 6.4.2. *The following hold:*

1. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$. Then A is $G_{\mathcal{E}_1}$-stable if and only if D_{A_3} is a a smooth conic and D_{A_2} is a conic intersecting D_{A_3} transversely.*
2. *The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ is $G_{\mathcal{E}_1}$-stable.*
3. *If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ is $G_{\mathcal{E}_1}$-stable the connected component of* Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to* $\operatorname{im} \lambda_{\mathcal{E}_1}$.

(4) *Let* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ *have closed* $\mathrm{PGL}(V)$*-orbit (in* $\mathbb{LG}(\bigwedge^3 V)^{ss}$*), and suppose that* $[A] \notin \mathfrak{I}$*. Then* $C_{W,A}$ *is of Type II-2 or* $\mathrm{PGL}(V)$*-equivalent to Type III-2.*

(5) $\mathfrak{B}_{\mathcal{E}_1} \cap \mathfrak{I} = \{\mathfrak{x}\}$ *where* $\mathfrak{x} \in \mathfrak{M}$ *is as in* (4.5.2) *.*

The proof of **Proposition 6.4.2** is given in **Subsubsection 6.4.3**.

6.4.1 The GIT analysis

Let λ be a 1-PS of $G_{\mathcal{E}_1}$. Since $G_{\mathcal{E}_1} = \mathbb{C}^\times \times \mathrm{SL}(V_{12}) \times \mathrm{SL}(V_{35})$ there exist bases $\{\xi_1, \xi_2\}$, $\{\beta_1, \beta_2, \beta_3\}$ of V_{12} and V_{35} respectively such that

$$\lambda(t) = (t^m, \mathrm{diag}(t^r, t^{-r}), \mathrm{diag}(t^{s_1}, t^{s_2}, t^{s_3})). \tag{6.4.3}$$

and

$$m, s_1, s_2, s_3 \in \mathbb{Z}, \quad r \in \mathbb{N}, \quad s_1 \geq s_2 \geq s_3, \quad (m, r, s_1, s_2, s_3) \neq (0,0,0,0,0), \quad \sum s_i = 0. \tag{6.4.4}$$

We recall that the action of the $\mathbb{C}^\times$-factor on V is given by (5.2.5). We write below the action of $\bigwedge^3 \lambda$ on the second and third summands of (6.4.1):

$$[v_0] \wedge V_{12} \wedge V_{35} = \underbrace{[v_0 \wedge \xi_1 \wedge \beta_1]}_{t^{r+s_1}} + \underbrace{[v_0 \wedge \xi_1 \wedge \beta_2]}_{t^{r+s_2}} + \underbrace{[v_0 \wedge \xi_1 \wedge \beta_3]}_{t^{r+s_3}} + \underbrace{[v_0 \wedge \xi_2 \wedge \beta_1]}_{t^{-r+s_1}} + \underbrace{[v_0 \wedge \xi_2 \wedge \beta_2]}_{t^{-r+s_2}} + \underbrace{[v_0 \wedge \xi_2 \wedge \beta_3]}_{t^{-r+s_3}}. \tag{6.4.5}$$

$$[v_0] \wedge \bigwedge^2 V_{35} \oplus \bigwedge^2 V_{12} \wedge V_{35} = \underbrace{[v_0 \wedge \beta_1 \wedge \beta_2]}_{t^{3m-s_3}} + \underbrace{[v_0 \wedge \beta_1 \wedge \beta_3]}_{t^{3m-s_2}} + \underbrace{[v_0 \wedge \beta_2 \wedge \beta_3]}_{t^{3m-s_1}} + \underbrace{[\xi_1 \wedge \xi_2 \wedge \beta_1]}_{t^{s_1-3m}} + \underbrace{[\xi_1 \wedge \xi_2 \wedge \beta_2]}_{t^{s_2-3m}} + \underbrace{[\xi_1 \wedge \xi_2 \wedge \beta_3]}_{t^{s_3-3m}}. \tag{6.4.6}$$

In particular $I_-(\lambda) \subset \{0, 4\}$, see **Definition 5.2.3**. We let $e^1_0 > \ldots > e^1_{j(1)}$ and $e^2_0 > \ldots > e^2_{j(2)}$ be the weights (in decreasing order) of the action of $\bigwedge^3 \lambda$ on the second and third summands of (6.4.1). By (5.2.5) and (2.2.9) we have

$$\mu(A, \lambda) = -3m + 2\mu(A_1, \lambda) + \mu(A_2, \lambda) = -3m + 2\sum_{i=0}^{j(1)} d^\lambda_i(A_1) e^1_i + \sum_{i=0}^{j(2)} d^\lambda_i(A_2) e^2_i. \tag{6.4.7}$$

Proposition 6.4.3. $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ *is not* $G_{\mathcal{E}_1}$*-stable if and only if one of the following holds:*

(1) *There exists a non-zero decomposable element of* A_1*.*

(2) $\dim(A_2 \cap \bigwedge^2 V_{12} \wedge V_{35}) \geq 1$.

(3) *There exist bases* $\{\xi_1, \xi_2\}$ *of* V_{12}*,* $\{\beta_1, \beta_2, \beta_3\}$ *of* V_{35} *and* $x, y \in \mathbb{C}$ *not both zero such that*

$$\langle \xi_1 \wedge \beta_1 \wedge \beta_2, (x\xi_1 \wedge \beta_1 \wedge \beta_3 + y\xi_2 \wedge \beta_1 \wedge \beta_2) \rangle \subset A_3 \tag{6.4.8}$$

and

$$\dim(A_2 \cap \langle v_0 \wedge \beta_1 \wedge \beta_2, \xi_1 \wedge \xi_2 \wedge \beta_1 \rangle) \geq 1. \tag{6.4.9}$$

Proof. We will use the data displayed in Tables (25), (26) and (27). The first two tables give for each of a series of 1-PS's of $G_{\mathcal{E}_1}$ the weights of the action on the second and third summands of (6.4.1). Each such 1-PS is diagonalized as in (6.4.3) and we denote it by the corresponding string of weights (m, r, s_1, s_2, s_3). One computes the numerical function $\mu(A, \lambda)$ of such a 1-PS by plugging the data in Formula (6.4.7): the results are listed in Table (27). The 1-PS's will be obtained by applying the Cone Decomposition Algorithm of **Subsection 2.3**: below we will give the details. First let's prove that if one of Items (1), (2), (3) above holds then A is not $G_{\mathcal{E}_1}$-stable. Suppose that Item (1) holds. There exist bases $\{\xi_1, \xi_2\}$ of V_{12} and $\{\beta_1, \beta_2, \beta_3\}$ of V_{35} such that $v_0 \wedge \xi_1 \wedge \beta_1 \in A_1$. Let λ be the 1-PS which is diagonal in the basis $\{v_0, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3\}$ and which is denoted by $(0,1,0,0,0)$: then $\mu(A, \lambda) \geq 0$ (see Tables (25) and (27)) and hence A is not $G_{\mathcal{E}_1}$-stable. Next suppose that Item (2) holds. There exist bases $\{\xi_1, \xi_2\}$ of V_{12} and $\{\beta_1, \beta_2, \beta_3\}$ of V_{35} such that $\xi_1 \wedge \xi_2 \wedge \beta_1 \in A_2$. Let λ be the 1-PS which is diagonal in the basis $\{v_0, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3\}$ and which is denoted by $(-1,0,0,0,0)$: then $\mu(A, \lambda) \geq 0$ (see Tables (25) and (27)) and hence A is not $G_{\mathcal{E}_1}$-stable. Before dealing with Item (3) we notice that the equality $A_1 = A_3^\perp \cap (\bigwedge^2 V_{12} \wedge V_{35})$ gives the following

Remark 6.4.4. (6.4.8) holds (for some $x, y \in \mathbb{C}$ not both zero) if and only if there exist $w_1, w_2, z \in \mathbb{C}$ not all zero such that

$$[v_0] \wedge [w_1\xi_1 \wedge \beta_1 + w_2\xi_1 \wedge \beta_2 + z\xi_2 \wedge \beta_1] \subset A_1 \subset [v_0] \wedge ([\xi_1] \wedge V_{35} \oplus \langle \xi_2 \wedge \beta_1, \xi_2 \wedge \beta_2 \rangle). \quad (6.4.10)$$

Now suppose that Item (3) holds. Thus we have the bases $\{\xi_1, \xi_2\}$ of V_{12} and $\{\beta_1, \beta_2, \beta_3\}$ of V_{35} which appear in the statement of Item (3). Let λ be the 1-PS of $G_{\mathcal{E}_1}$ which corresponds to $(0, 3, 6, 0, -6)$ (with respect to the given basis of V). By **Remark 6.4.4** we know that (6.4.10) holds, and of course (6.4.9) holds: it follows that $\mu(A, \lambda) \geq 0$ (see Tables (25) and (27)) and hence A is not $G_{\mathcal{E}_1}$-stable. It remains to prove the converse i.e. that if A is not $G_{\mathcal{E}_1}$-stable then one of Items (1), (2), (3) holds. We will apply the Cone Decomposition Algorithm of **Subsection 2.3**. We choose the maximal torus $T < G_{\mathcal{E}_1}$ to be

$$T = \{(u, \mathrm{diag}(t, t^{-1}), \mathrm{diag}(t_1, t_2, t_3)) \mid u, t, t_i \in \mathbb{C}^\times,\ t_1 \cdot t_2 \cdot t_3 = 1\}. \quad (6.4.11)$$

(The maps are diagonal with respect to the bases $\{\xi_1, \xi_2\}$ and $\{\beta_1, \beta_2, \beta_3\}$.) Thus

$$\check{X}(T)_{\mathbb{R}} := \{(m, r, s_1, s_2, s_3) \in \mathbb{R}^5 \mid s_1 + s_2 + s_3 = 0\}$$

We let $C \subset \check{X}(T)_{\mathbb{R}}$ be the standard cone:

$$C := \{(m, r, s_1, s_2, s_3) \in \mathbb{R}^5 \mid r \geq s_1 \geq s_2 \geq s_3\}.$$

$H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and only if is equal to the kernel of one the following linear functions:

$$s_i - s_j,\ r,\ 2r - s_i + s_j,\ s_i + 6m,\ s_i - 3m.$$

In particular the hypotheses of **Proposition 2.3.4** are satisfied. One computes the ordering rays by passing to coordinates (m, r, x_1, x_2) where

$$x_i := s_i - s_{i+1}, \qquad i = 1, 2. \quad (6.4.12)$$

In the above coordinates

$$C = \{(n, r, x_1, x_2) \mid r \geq 0, \quad x_1 \geq 0,\ x_2 \geq 0\}.$$

The linear functions s_1, s_2, s_3 on W are expressed as follows in terms of the coordinates x_1, x_2:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \\ -1/3 & -2/3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (6.4.13)$$

It follows that $H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and only if, in the (m, r, x_1, x_2)-coordinates, it is equal to the kernel of one of the following linear functions:

$$x_1,\ x_2,\ r,\ 2r - x_1,\ 2r - x_2,\ 2r - x_1 - x_2,\ 2x_1 + x_2 + 18m,\ x_1 - x_2 - 18m,\ x_1 + 2x_2 - 18m,\ 2x_1 + x_2 - 9m,\ x_1 - x_2 + 9m,\ x_1 + 2x_2 + 9m.$$

An easy computation gives the ordering rays in the (m, r, x_1, x_2)-coordinates. Switching back to (m, r, s_1, s_2, s_3)-coordinates we get the following generators for ordering rays. First we get the vectors

$$(\pm 1, 0, 0, 0, 0), \quad (0, 1, 0, 0, 0), \qquad (m, r, 6, 0, -6)\ \ (m, r) \in \{(0, 0), (0, 3), (1, 3), (2, 3)\}. \quad (6.4.14)$$

Secondly we get the vectors

$$(m, r, 12, -6, -6), \quad m = r = 0 \text{ or } m \in \{1, 4, -2\} \text{ and } r \in \{0, 9\}. \quad (6.4.15)$$

and lastly the vectors

$$(m, r, 6, 6, -12), \quad m = r = 0 \text{ or } m \in \{-1, -4, 2\} \text{ and } r \in \{0, 9\}. \quad (6.4.16)$$

Thus we get exactly the 1-PS's that appear in Tables (25), (26) and (27). As is easily checked the following hold:

(a) Let λ be the 1-PS indicized by $(0,1,0,0,0)$ and suppose that $\mu(A,\lambda)\ge 0$. Then Item (1) of **Proposition 6.4.3** holds.

(b) Let λ be the 1-PS indicized by $(-1,0,0,0,0)$ and suppose that $\mu(A,\lambda)\ge 0$. Then Item (2) of **Proposition 6.4.3** holds.

(c) Let λ be the 1-PS indicized by $(0,3,6,0,-6)$ and suppose that $\mu(A,\lambda)\ge 0$. Suppose in addition that neither Item (1) nor Item (2) of **Proposition 6.4.3** holds: then Item (3) of **Proposition 6.4.3** holds (use **Remark 6.4.4**).

In order to finish the proof it suffices to show that if $\mu(A,\lambda)\ge 0$ for one of the remaining ordering 1-PS's (i.e. different from those appearing in Items (a), (b) and (c) above) then one of Items (1), (2) or (3) holds. This consists of a series of routine checks. We summarize the main points. First consider the 1-PS λ indicized by $(1,0,0,0,0)$ and suppose that $\mu(A,\lambda)\ge 0$. By Table (25) we get that

$$\dim(A_2\cap[v_0]\wedge\bigwedge^2 V_{35})\ge 2. \tag{6.4.17}$$

Let's show that if (6.4.17) holds there exist bases $\{\xi_1,\xi_2\}$ of V_{12}, $\{\beta_1,\beta_2,\beta_3\}$ of V_{35} and $w_1,w_2,z\in\mathbb{C}$ not all zero such that (6.4.9) and (6.4.10) hold. It will be convenient to identify $[v_0]\wedge V_{12}\wedge V_{35}$ with $\mathrm{Hom}(V_{12},V_{35})$ via the perfect pairing $V_{12}\times V_{12}\to\bigwedge^2 V_{12}$ given by wedge product. First one shows that there exist

$$0\ne\alpha_1\in A_1,\quad 0\ne v_0\wedge\theta\in A_2,\ \theta\in\bigwedge^2 V_{35}$$

such that the following holds. Let $f_1\colon V_{12}\to V_{35}$ be the map associated to α_1: then $\mathrm{im}\, f_1\subset\mathrm{supp}\,\theta$. Now complete α_1 to basis $\{\alpha_1,\alpha_2\}$ of A_1 and let $f_2\colon V_{12}\to V_{35}$ be the map associated to α_2. Since $\dim f_2^{-1}(\mathrm{supp}\,\theta)\ge 1$ there exists a basis $\{\xi_1,\xi_2\}$ of V_{12} such that $f_2(\xi_1)\in\mathrm{supp}\,\theta$. Let $0\ne\beta_1$ such that $f_1(\xi_1)\in[\beta_1]$: thus $\beta_1\in\mathrm{supp}\,\theta$. Now complete β_1 to basis $\{\beta_1,\beta_2,\beta_3\}$ of V_{35} such that $\mathrm{supp}\,\theta=\langle\beta_1,\beta_2\rangle$. Then (6.4.9) and (6.4.10) hold: by **Remark 6.4.4** we get that Item (3) of **Proposition 6.4.3** holds. Next one examines the other ordering 1-PS's, i.e. those indicized by $(m,r,6,0-6)$, $(m,r,6,0-6)$, $(m,r,12,-6,-6)$ and $(m,r,6,6,-12)$. Suppose that λ is one such 1-PS and that $\mu(A,\lambda)\ge 0$. We may assume that neither Item (1) nor Item (2) nor Item (3) of **Proposition 6.4.3** holds (with respect to arbitrary bases $\{\xi_1,\xi_2\}$ of V_{12}, $\{\beta_1,\beta_2,\beta_3\}$ of V_{35}): then one must check that $\mu(A,\lambda)<0$. This is time-consuming but straightforward. □

Corollary 6.4.5. *$A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ is $G_{\mathcal{E}_1}$-stable if and only if D_{A_3} is a smooth conic (equivalently Θ_{A_3} is a smooth curve) and D_{A_2} is a conic intersecting D_{A_3} transversely.*

Proof. First notice the following:

(A) The equality $A_3=A_1^{\perp}\cap(V_{12}\wedge\bigwedge^2 V_{35})$ gives: Item (1) of **Proposition 6.4.3** holds if and only if the intersection

$$\mathbb{P}(A_3)\cap\left(\mathbb{P}(V_{12})\times\mathbb{P}(\bigwedge^2 V_{35})\right) \tag{6.4.18}$$

in $\mathbb{P}(V_{12}\wedge\bigwedge^2 V_{35})$ is **not** transverse.

(B) D_{A_2} is a double line or all of $\mathbb{P}(\bigwedge^2 V_{35})$ if and only if either Item (2) of **Proposition 6.4.3** holds or (6.4.17) holds.

Let's prove that if D_{A_3} is not a smooth conic or if D_{A_3} is a smooth conic but D_{A_2} is not a conic intersecting D_{A_3} transversely then A is not $G_{\mathcal{E}_1}$-stable. If D_{A_3} is not a smooth conic then Θ_{A_3} is not a smooth curve i.e. the intersection (6.4.18) is not transverse. By Item (A) above it follows that Item (1) of **Proposition 6.4.3** holds and thus A is not $G_{\mathcal{E}_1}$-stable by **Proposition 6.4.3**. Now let's assume that D_{A_3} is a smooth conic but D_{A_2} is not a conic intersecting D_{A_3} transversely. In order to prove that A is not $G_{\mathcal{E}_1}$-stable we need first to write out the tangent space to D_{A_3} at a point $[\theta]$ (here $0\ne\theta\in\bigwedge^2 V_{35}$). Since $[\theta]\in D_{A_3}$ there exists $0\ne\xi_1\in V_{12}$ such that $[\xi_1\wedge\theta]$ belongs to (6.4.18). By the assumption that D_{A_3} is a smooth conic we get that the intersection (6.4.18) is transverse at $[\xi_1 \wedge \theta]$ (as intersection in $\mathbb{P}(V_{12} \wedge \bigwedge^2 V_{35})$). Let $M\colon A_3 \to \bigwedge^2 V_{12} \wedge \bigwedge^2 V_{35}$ be multiplication by ξ_1. Then $\ker M = [\xi_1 \wedge \theta]$ because the intersection (6.4.18) is transverse at $[\xi_1 \wedge \theta]$. Thus M is surjective and hence

$$M^{-1}(\bigwedge^2 V_{12} \wedge [\theta]) = \langle \xi_1 \wedge \theta, \xi_1 \wedge \gamma + \xi_2 \wedge \theta \rangle, \quad \gamma \in \bigwedge^2 V_{35}. \tag{6.4.19}$$

Moreover γ, θ are linearly independent because the intersection (6.4.18) is transverse at $[\xi_1 \wedge \theta]$; thus there exists $0 \neq \beta_1 \in V_{35}$ such that $\operatorname{supp} \gamma \cap \operatorname{supp} \theta = [\beta_1]$. The projective tangent space to D_{A_3} at $[\theta]$ is given by

$$\mathbf{T}_{[\theta]} D_{A_3} = \mathbb{P}\langle \operatorname{Ann} \beta_1 \rangle. \tag{6.4.20}$$

Here we make the identification $\mathbb{P}(\bigwedge^2 V_{35}^{\vee}) = \mathbb{P}(V_{35})$. We may complete β_1 to a basis $\{\beta_1, \beta_2, \beta_3\}$ of V_{35} such that $\theta = \beta_1 \wedge \beta_2$ and $\gamma = \beta_1 \wedge \beta_3$. Thus (6.4.19) gives that

$$A_3 \supset \langle \xi_1 \wedge \beta_1 \wedge \beta_2, \xi_1 \wedge \beta_1 \wedge \beta_3 + \xi_2 \wedge \beta_1 \wedge \beta_2 \rangle. \tag{6.4.21}$$

Now suppose that $[\theta] = [\beta_1 \wedge \beta_2] \in D_{A_2}$ i.e.

$$(v_0 \wedge \beta_1 \wedge \beta_2 + \xi_1 \wedge \xi_2 \wedge \beta) \in A_2, \qquad \beta \in \langle \beta_1, \beta_2 \rangle \tag{6.4.22}$$

and that either D_{A_2} is all of $\mathbb{P}(\bigwedge^2 V_{35})$ or a conic which does not intersect D_{A_3} transversely at $[\theta]$. If D_{A_2} is all of $\mathbb{P}(\bigwedge^2 V_{35})$ or a a double line then by Item (B) above we get that Item (2) of **Proposition 6.4.3** holds, thus A is not $G_{\mathcal{E}_1}$-stable by **Proposition 6.4.3**. Next we assume that D_{A_2} is a conic of rank at least 2. By Item (B) above it follows that Item (2) of **Proposition 6.4.3** does not hold. Thus D_{A_2} is described as in **Remark 6.4.1**: it follows that $\mathbf{T}_{[\beta_1 \wedge \beta_2]} D_{A_2} = \mathbb{P}\langle \operatorname{Ann} \beta \rangle$. Since D_{A_2} and the smooth conic D_{A_3} do not intersect transversely at $[\beta_1 \wedge \beta_2]$ we get that $\beta \in [\beta_1]$. By (6.4.21) and (6.4.22) we get that Item (3) of **Proposition 6.4.3** holds and hence A is not $G_{\mathcal{E}_1}$-stable. We have proved that if D_{A_3} is not a smooth conic or if D_{A_3} is a smooth conic but D_{A_2} is not a conic intersecting D_{A_3} transversely then A is not $G_{\mathcal{E}_1}$-stable. Now suppose that A is not $G_{\mathcal{E}_1}$-stable and hence one of Items (1), (2), (3) of **Proposition 6.4.3** holds. If Item (1) holds then by Item (A) above we get that D_{A_3} is not a smooth conic. If Item (2) holds then by Item (B) above D_{A_2} is all of $\mathbb{P}(\bigwedge^2 V_{35})$ or else a double line (and hence it cannot intersect transversely a conic). Lastly suppose that Item (3) holds. We may assume that neither Item (1) nor Item (2) hold: thus (6.4.8) and (6.4.9) give (after a rescaling of β_3) that

$$\xi_1 \wedge \beta_1 \wedge \beta_2, (\xi_1 \wedge \beta_1 \wedge \beta_3 + \xi_2 \wedge \beta_1 \wedge \beta_2) \in A_3, \quad (v_0 \wedge \beta_1 \wedge \beta_2 + z\xi_1 \wedge \xi_2 \wedge \beta_1) \in A_2. \tag{6.4.23}$$

Since Item (1) of **Proposition 6.4.3** does not hold the conic D_{A_3} is smooth. By (6.4.23) we have that $[\beta_1 \wedge \beta_2] \in D_{A_3} \cap D_{A_2}$ and the analysis carried out above shows that the intersection is not transverse at $[\beta_1 \wedge \beta_2]$. □

Let $\mathsf{B} = \{v_0, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3\}$ be a basis of V with $\{\xi_1, \xi_2\}$ a basis of V_{12} and $\{\beta_1, \beta_2, \beta_3\}$ a basis of V_{35}. Let λ_1 be the 1-PS of $G_{\mathcal{E}_1}$ indicized by $(0, 3, 6, 0, -6)$ (given the choice of the basis B) i.e. the 1-PS that intervenes in the proof that if Item (3) of **Proposition 6.4.3** holds for A then A is not $G_{\mathcal{E}_1}$-stable. Let $\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1}$ be the affine cone over $\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$; then $G_{\mathcal{E}_1}$ acts on $\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1}$. The fixed locus $(\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1})^{\lambda_1}$ is the set of A which are mapped to themselves by $\wedge^3 \lambda_1(t)$ and such that $\wedge^3 \lambda_1(t)$ acts trivially on $\bigwedge^{10} A$.

Definition 6.4.6. Let $\mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1} \subset \mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1})^{\lambda_1})$ be the set of A such that $\wedge^3 \lambda_1(t)$ acts trivially on $\bigwedge^2 A_1$, $\bigwedge^3 A_2$ and $\bigwedge^4 A_3$.

Remark 6.4.7. Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$; then $A \in \mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1}$ if and only if it is λ_1-split of types $d^{\lambda_1}(A_1) = (0, 1, 1, 0)$ and $d^{\lambda_1}(A_2) = (1, 1, 1)$. Moreover $\mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1}$ is an irreducible component of $\mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1})^{\lambda_1})$.

Proposition 6.4.8. *Suppose that A is properly $G_{\mathcal{E}_1}$-semistable. Then there exists a semistable $A_0 \in \mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1}$ which is $G_{\mathcal{E}_1}$-equivalent to A.*

Proof. One of Items (1), (2), (3) of **Proposition 6.4.3** holds. Suppose that Item (3) holds. We showed in the proof of **Proposition 6.4.3** that there exists a semistable $A_0 \in \mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1}$ which is $G_{\mathcal{E}_1}$-equivalent to A, namely the limit $\lim_{t\to 0} \lambda_1(t)A$. We will finish the proof by showing that if one of Items (1), (2) of **Proposition 6.4.3** holds then there exists $A_0 \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ which is $G_{\mathcal{E}_1}$-equivalent to A and for which Item (3) of **Proposition 6.4.3** holds. Suppose that Item (2) holds. We will refer to the notation introduced in the proof that if Item (2) of **Proposition 6.4.3** holds then A is not $G_{\mathcal{E}_1}$-stable. Let λ_2 be the 1-PS of $G_{\mathcal{E}_1}$ indicized by $(-1,0,0,0,0)$. We showed in the proof of **Proposition 6.4.3** that $\mu(A,\lambda_2)=0$. Thus $\lim_{t\to 0}\lambda_2(t)A$ is a semistable lagrangian A' which is $G_{\mathcal{E}_1}$-equivalent to A and which is λ_2-split with $d_0(A'_2)=1$ (and hence $d_1(A'_2)=2$). It follows that $\dim(A'_2 \cap [v_0]\wedge \bigwedge^2 V_{35})=2$: as shown in the proof of **Proposition 6.4.3** (see the text right below (6.4.17)) it follows that Item (3) holds for A'. This proves the result if Item (2) holds. Lastly suppose that Item (1) of **Proposition 6.4.3** holds. Let $\lambda=(0,1,0,0,0)$. As shown in the proof of **Proposition 6.4.3** we have $\mu(A,\lambda)\ge 0$. Since A is $G_{\mathcal{E}_1}$-semistable $\mu(A,\lambda)=0$ and hence A is $G_{\mathcal{E}_1}$-equivalent to a λ-split A' with type $d^{\lambda}(A_1)=(1,1)$. It follows that there exist bases $\{\xi_1,\xi_2\}$ of V_{12} and $\{\beta_1,\beta_2,\beta_3\}$ of V_{35} such that either $A'_1=\langle v_0\wedge\xi_1\wedge\beta_1, v_0\wedge\xi_2\wedge\beta_2\rangle$ or $A'_1=\langle v_0\wedge\xi_1\wedge\beta_1, v_0\wedge\xi_2\wedge\beta_1\rangle$. Suppose that the latter holds. Let λ' be the 1-PS of $\mathrm{SL}(V_{35})$ defined by $\lambda'(t)=\mathrm{diag}(t,1,t^{-1})$ (the basis is $\{\beta_1,\beta_2,\beta_3\}$): then $\mu(A',\lambda')>0$, that is a contradiction. Thus $A'_1=\langle v_0\wedge\xi_1\wedge\beta_1, v_0\wedge\xi_2\wedge\beta_2\rangle$. Let λ'' be the 1-PS of $\mathrm{SL}(V_{35})$ defined by $\lambda''(t)=\mathrm{diag}(t,t,t^{-2})$: then $\mu(A',\lambda'')\ge 0$ and hence it is zero by semistability of A'. Let $A'':=\lim_{t\to 0}\lambda''(t)A'$. As is easily checked $A''_2 \ni \xi_1\wedge\xi_2\wedge\beta_3$ and hence A'' satisfies Item (2) of **Proposition 6.4.3**. □

6.4.2 Analysis of Θ_A and $C_{W,A}$

Proposition 6.4.9. *Let $A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ be $G_{\mathcal{E}_1}$-stable and $W\in\Theta_A$. Then one of the following holds:*

(a) $W=V_{02}$.

(b) $W\in\Theta_{A_3}$.

(c) $W=\langle v_0,\beta_1,\beta_2\rangle$ where $\beta_1,\beta_2\in V_{35}$.

Proof. Let $W\in\Theta_A$. We distinguish between the three cases:

(I) $W\supset V_{12}$.

(II) $\dim(W\cap V_{12})=1$.

(III) $W\cap V_{12}=\{0\}$.

One checks easily that if (I) holds then $W=V_{02}$ and that if (II) holds then $W\in\Theta_{A_3}$. Suppose that (III) holds. Since $V_{02}\in\Theta_A$ we have $W\cap V_{02}\ne\{0\}$: it follows that W is not contained in V_{15} and hence $\dim(W\cap V_{15})=2$. Thus there exist linearly independent $\beta_1,\beta_2\in V_{35}$ and $\xi_1,\xi_2,\xi\in V_{12}$ such that

$$W=\langle v_0+\xi,\xi_1-\beta_1,\xi_2-\beta_2\rangle.$$

Thus

$$A\ni(v_0+c\xi_1)\wedge(\xi_1-\beta_1)\wedge(\xi_2-\beta_2)=v_0\wedge\xi_1\wedge\xi_2+v_0\wedge(-\xi_1\wedge\beta_2+\xi_2\wedge\beta_1)+(v_0\wedge\beta_1\wedge\beta_2-\xi\wedge\xi_1\wedge\beta_2+\xi\wedge\xi_2\wedge\beta_1)+\xi\wedge\beta_1\wedge\beta_2. \tag{6.4.24}$$

The addends of (6.4.24) belong to different summands of the isotypycal decomposition of $\bigwedge^3\lambda_{\mathcal{E}_1}$ - see (6.4.1) - hence each addend belongs to A. One checks easily that unless $0=\xi_1=\xi_2=\xi$ one of Items (1) or (3) of **Proposition 6.4.3** holds and hence A is not $G_{\mathcal{E}_1}$-stable, that is a contradiction. Thus $0=\xi_1=\xi_2=\xi$. □

Corollary 6.4.10. *Let $A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ be $G_{\mathcal{E}_1}$-stable. Then $\Theta_A=\{V_{02}\}\cup\Theta_{A_3}\cup Z_A$ where Z_A is a finite set.*

Proof. It suffices to prove that there is at most one $W \in \Theta_A$ such that Item (c) of **Proposition 6.4.9** holds. Let $W = \langle v_0, \beta_1, \beta_2\rangle$. By Item (2) of **Proposition 6.4.3** we may describe D_{A_2} as in **Remark 6.4.1**; it follows that $[\beta_1 \wedge \beta_2]$ is a singular point of the conic D_{A_2}. On the other hand D_{A_2} is a conic with at most one singular point by **Corollary 6.4.5**: thus there is at most one choice for $\langle \beta_1, \beta_2\rangle$ and hence for W as well. □

Proposition 6.4.11. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ be $G_{\mathcal{E}_1}$-stable and $W \in \Theta_A$. Then $C_{W,A}$ is a sextic curve of Type II-2.*

Proof. The orbit $\mathrm{PGL}(V)A$ is minimal because A is $G_{\mathcal{E}_1}$-stable (see **Claim 5.2.1**) and $\dim \Theta_A = 1$ by **Proposition 6.4.9**: thus $C_{W,A} \neq \mathbb{P}(W)$ by **Corollary 6.1.10**. One of Items (a), (b), (c) of **Proposition 6.4.9** holds. Let $\{X_0, X_1, X_2\}$ be a basis of $W^\vee$ such that

(a') $[X_0] = \mathrm{Ann}\langle v_1, v_2\rangle$ and $[v_0] = \mathrm{Ann}\langle X_1, X_2\rangle$ if (a) holds.

(b') $[X_0] = \mathrm{Ann}(W \cap V_{35})$ and $W \cap V_{12} = \mathrm{Ann}\langle X_1, X_2\rangle$ if (b) holds.

(c') $[X_0] = \mathrm{Ann}(W \cap V_{35})$ and $[v_0] = \mathrm{Ann}\langle X_1, X_2\rangle$ if (c) holds.

The 1-PS $\lambda_{\mathcal{E}_1}$ maps W to itself. Now we look at the action of $\lambda_{\mathcal{E}_1}$ on W: by **Claim 3.2.4** and **Remark 1.4.3** we get that

$$C_{W,A} = X_0^2 F(X_1, X_2), \qquad 0 \neq F \in \mathbb{C}[X_1, X_2]_4. \tag{6.4.25}$$

It remains to prove that F does not have multiple roots. We will carry out a case-by-case analysis.

$\boxed{W = V_{02}}$ Let $0 \neq \xi \in V_{12}$. Let

$$\rho\colon A \cap F_{(v_0-\xi)} \longrightarrow V_{12} \wedge \bigwedge^2 V_{35} \tag{6.4.26}$$

be the projection determined by Decomposition (6.4.1). Let's prove that

$$\ker \rho = \bigwedge^3 V_{02}, \quad \dim(\operatorname{im} \rho) \leq 1. \tag{6.4.27}$$

Let $\alpha \in (A \cap F_{(v_0-\xi)})$ and write $\alpha = \sum_{i=0}^3 \alpha_i$ where α_i belongs to the $(i+1)$-th summand of Decomposition (6.4.1) (we start from the left of course). Then $v_0 \wedge \alpha = \xi \wedge \alpha$. Decomposing $v_0 \wedge \alpha$ and $\xi \wedge \alpha$ according to Decomposition (6.4.1) we get that $\xi \wedge \alpha_3 = 0$, in particular α_3 is decomposable i.e. $[\alpha_3] \in \Theta_{A_3}$. By **Corollary 6.4.5** we know that Θ_{A_3} is a smooth curve: it follows that the projection $\Theta_{A_3} \to \mathbb{P}(V_{12})$ is an isomorphism. This proves that $\dim \operatorname{im} \rho \leq 1$. Next suppose that $\alpha_3 = 0$. From $0 = v_0 \wedge \alpha_3 = \xi \wedge \alpha_2$ we get that $\alpha_2 = 0$ (recall that $A \cap (\bigwedge^2 V_{12} \wedge V_{35}) = \{0\}$ by $G_{\mathcal{E}_1}$-stability of A). We also have $\xi \wedge \alpha_1 = 0$: since A is $G_{\mathcal{E}_1}$-stable A_1 contains no non-zero decomposables and thus $\alpha_1 = 0$. This finishes the proof of (6.4.27). Now suppose that $[v_0 - \xi] \in C_{W,A}$. By (6.4.27) we have $\dim((A \cap F_{(v_0-\xi)}) = 2$. We claim that $[v_0 - \xi] \notin \mathcal{B}(W, A)$. First there is no $W' \in (\Theta_A \setminus \{W\})$ containing $[v_0 - \xi]$ by **Proposition 6.4.9**. Secondly suppose that $\alpha \in (A \cap F_{(v_0-\xi)})$ and $\alpha_3 = \rho(\alpha) \neq 0$: if $\xi' \in V_{12}$ is not a multiple of ξ then $0 \neq v_0 \wedge \xi' \wedge \alpha_3 = v_0 \wedge \xi' \wedge \alpha$, this proves that $A \cap F_{(v_0-\xi)} \cap S_W = \bigwedge^3 W$ and hence we get that $[v_0 - \xi] \notin \mathcal{B}(W, A)$. By **Proposition 3.3.6** it follows that F has no multiple roots.

$\boxed{W \in \Theta_{A_3}}$ Let $W \cap V_{12} = [\xi]$ and $\neq \beta \in W \cap V_{35}$. Let

$$\pi\colon A \cap F_{(\xi+\beta)} \longrightarrow \bigwedge^3 V_{02} \tag{6.4.28}$$

be the projection determined by Decomposition (6.4.1). Arguing as in the previous case one checks that $\ker(\pi) = [\xi \wedge \beta_1 \wedge \beta_2]$ where $\xi \in V_{12}$, $\beta_1, \beta_2 \in V_{35}$ and $\langle \xi, \beta_1, \beta_2\rangle$ is the unique element of Θ_{A_3} mapped to $[\xi]$ by the projection $\Theta_{A_3} \to \mathbb{P}(V_{12})$. Suppose that $[\xi + \beta] \in C_{W,A}$: it follows that $\dim(A \cap F_{(\xi+\beta)}) = 2$. A straightforward computation shows that $[\xi + \beta] \notin \mathcal{B}(W, A)$ and hence $C_{W,A}$ is smooth at $[\xi + \beta]$. This proves that F has no multiple factors.

$\boxed{W = \langle v_0, U\rangle \text{ where } U \in \mathrm{Gr}(2, V_{35})}$ Let

$$T := \{[\beta] \in \mathbb{P}(U) \mid \mathrm{mult}_{[\beta]} C_{W,A} \geq 3\}.$$

By (6.4.25) it suffices to prove that T has cardinality at least 4. Let $[\beta] \in \mathbb{P}(V_{35})$: as is easily checked $\dim(F_\beta \cap A_3) = 2$ and moreover

$$|\mathbb{P}(F_\beta \cap A_3) \cap \mathrm{Gr}(3, V)| = \begin{cases} 2 & \text{if } [\beta] \notin D^\vee_{A_3}, \\ 1 & \text{if } [\beta] \in D^\vee_{A_3}. \end{cases}$$

(We have the identification $\mathbb{P}(\bigwedge^2 V_{35}^\vee) = \mathbb{P}(V_{35})$.) Since A is $G_{\mathcal{E}_1}$-stable we have $\bigwedge^2 U \notin D_{A_3}$ and hence $|\mathbb{P}(U) \cap D^\vee_{A_3}| = 2$. Applying **Proposition 3.2.2** we get that

$$\mathbb{P}(U) \cap D^\vee_{A_3} \subset T. \tag{6.4.29}$$

Next we examine A_2. By hypothesis $D_{A_2} = L_1 \cup L_2$ where $L_1, L_2 \subset \mathbb{P}(\bigwedge^2 V_{35})$ are distinct lines intersecting in $\bigwedge^2 U$. It follows that there exist bases $\{\xi_1, \xi_2\}$ of V_{12} and $\{\beta_1, \beta_2, \beta_3\}$ of V_{35} such that

$$A_2 \supset \langle v_0 \wedge \beta_1 \wedge \beta_3 + \xi_1 \wedge \xi_2 \wedge \beta_1, v_0 \wedge \beta_2 \wedge \beta_3 + \xi_1 \wedge \xi_2 \wedge \beta_2\rangle.$$

Thus $\dim(F_{\beta_i} \cap A) \geq 4$ for $i = 1, 2$: by **Corollary 3.2.3** we get that

$$[\beta_1], [\beta_2] \in T. \tag{6.4.30}$$

We have $[\beta_1], [\beta_2] \notin D^\vee_{A_3}$ because D_{A_2} is transverse to D_{A_3} (see **Corollary 6.4.5**). Thus (6.4.29) and (6.4.30) give that T has cardinality at least 4. □

Proposition 6.4.12. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ be properly $G_{\mathcal{E}_1}$-semistable with minimal orbit. Then either $[A] = \mathfrak{x}$ (here $\mathfrak{x} \in \mathfrak{M}$ is as in* (4.5.2)*) or else the following holds: if $W \in \Theta_A$ then $C_{W,A}$ is a semistable sextic curve* $\mathrm{PGL}(W)$*-equivalent to a sextic of Type III-2.*

Proof. By **Claim 5.2.1** A is $\mathrm{PGL}(V)$-semistable with minimal orbit. We claim that $A \notin \mathbb{X}^*_{\mathcal{W}}$. In fact suppose that $A \in \mathbb{X}^*_{\mathcal{W}}$. Since A is the limit of A' generic in $\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ we get that Θ_A contains a curve of degree 3 (with respect to the Plücker embedding) namely the limit of $\Theta_{A'_3}$. On the other hand if $A \in \mathbb{X}^*_{\mathcal{W}}$ then any curve in Θ_A has even degree, that is a contradiction. Now suppose that $[A] \neq \mathfrak{x}$: by **Proposition 6.1.9** we get that $C_{W,A} \neq \mathbb{P}(W)$. Since A is not $G_{\mathcal{E}_1}$-stable we may assume that $A \in M^{\mathsf{B}}_{\mathcal{E}_1}$ by **Proposition 6.4.8**. It follows that $\bigwedge^{10} A$ is fixed by the 1-PS of $\mathrm{SL}(V)$ given by $(m, r, s_1, s_2, s_3) = (0, 1, 2, 0, -2)$ - see (6.4.3). On the other hand $\bigwedge^{10} A$ is fixed by $\lambda_{\mathcal{E}_1}$ because $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$. Thus $\bigwedge^{10} A$ is fixed by the torus

$$T := \{\mathrm{diag}(s^4, st, st^{-1}, s^{-2}t^2, s^{-2}, s^{-2}t^{-2}) \mid (s,t) \in \mathbb{C}^\times \times \mathbb{C}^\times\}.$$

(The basis of V is $\mathsf{B} = \{v_0, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3\}$.) Now suppose that $W \in \Theta_A$ is fixed by T: then W is spanned by vectors of B. Let $C_{W,A} = V(P)$ where $0 \neq P \in \mathrm{S}^6 W^\vee$. Applying **Claim 3.2.4** we get that P is fixed by a maximal torus of $\mathrm{SL}(W)$: it follows that $C_{W,A}$ is of Type III-2. Next assume that W is not fixed by T: then we may find a 1-PS $\lambda\colon \mathbb{C}^\times \to T$ such that $\lim_{t\to 0} \lambda(t)W$ exists and is equal to $W_0 \in \Theta_A$ fixed by T: it follows that $C_{W,A}$ is a semistable sextic $\mathrm{PGL}(W)$-equivalent to a sextic of Type III-2. □

6.4.3 Wrapping it up

We will prove **Proposition 6.4.2**. Item (1) is the content of **Corollary 6.4.5**. We have noticed that if $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$ is generic then D_{A_2}, D_{A_3} are conics intersecting transversely: together with Item (1) that gives Item (2). Let's prove Item (3). By **Corollary 6.4.10** we have $\Theta_A = \{V_{02}\} \cup \Theta_{A_3} \cup Z_A$ where Z_A is finite. By **Corollary 6.4.5** we know that Θ_{A_3} is a rational normal twisted curve parametrizing subspaces $W \subset V_{15}$. By the classification of [28] (see Table 2) the following holds: V_{35} is the unique 3-dimensional vector-subspace of V intersecting every $W \in \Theta_{A_3}$ in a subspace of dimension 2. In addition **Proposition 6.4.11** gives that $C_{V_{02},A}$ is a sextic of Type II-2 with isolated singular point in $[v_0]$ (for the last statement go to the proof of **Proposition 6.4.11**). Now let $g \in \mathrm{Stab}(A)$ belong to the connected component of Id. The facts quoted above about Θ_A and $C_{V_{02},A}$ give that $g([v_0]) = [v_0]$, $g(V_{12}) = V_{12}$ and $g(V_{35}) = V_{35}$. Thus g belongs to the centralizer $C_{\mathrm{SL}(V)}(\lambda_{\mathcal{E}_1})$. Since A is $G_{\mathcal{E}_1}$-stable the stabilizer of A in $G_{\mathcal{E}_1}$ is a finite group and Item (3) follows. Lastly let us prove Items (4) and (5). First we will show that

$$\mathfrak{x} \in \mathfrak{B}_{\mathcal{E}_1}. \tag{6.4.31}$$

By definition it suffices to show that $A_k(L) \in \mathbb{B}^*_{\mathcal{E}_1}$, where $A_k(L)$ is given by **Definition 4.1.1**. Let $W \in \Theta_{A_k(L)}$. There exists a basis $\{X, Y, Z\}$ of L such that $W = \langle X^2, XY, XZ\rangle$. Let $\mathsf{F} = \{v_0, \ldots, v_5\}$ be the basis of $\mathrm{S}^2 L$ defined by $\mathsf{F} := \{X^2, XY, XZ, Y^2, YZ, Z^2\}$. A straightforward computation shows that $v_0 \wedge (v_1 \wedge v_4 - v_2 \wedge v_3), v_0 \wedge (v_1 \wedge v_5 - v_2 \wedge v_4) \in A$. Since $v_0 \wedge v_1 \wedge v_2 \in A$ it follows that $A_k(L) \in \mathbb{B}^*_{\mathcal{E}_1}$. This proves (6.4.31). Items (4) and (5) follow at once from (6.4.31), **Proposition 6.4.11** and **Proposition 6.4.12**.

6.5 $\mathfrak{B}_{\mathcal{E}_1^\vee}$

The isotypical decomposition of $\bigwedge^3 \lambda_{\mathcal{E}_1^\vee}$ with decreasing weights is

$$\bigwedge^3 V = \bigwedge^3 V_{02} \oplus \bigwedge^2 V_{02} \wedge V_{34} \oplus \left(V_{02} \wedge \bigwedge^2 V_{34} \oplus \bigwedge^2 V_{02} \wedge [v_5] \right) \oplus V_{02} \wedge V_{34} \wedge [v_5] \oplus \bigwedge^3 V_{35}. \tag{6.5.1}$$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$; by definition $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ where

$$A_0 = \bigwedge^3 V_{02}, \quad A_1 \in \mathrm{Gr}(2, \bigwedge^2 V_{02} \wedge V_{34}), \quad A_2 \in \mathbb{LG}(V_{02} \wedge \bigwedge^2 V_{34} \oplus \bigwedge^2 V_{02} \wedge [v_5]), \quad A_3 = A_1^{\perp} \cap (V_{02} \wedge V_{34} \wedge [v_5]).$$

We associate to the generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ two closed subsets of $\mathbb{P}(V_{02})$ (generically conics) as follows. First we notice that $\mathbb{P}(V_{02} \wedge V_{34} \wedge [v_5]) \cap \mathbb{G}(3, V)$ is isomorphic to $\mathbb{P}(V_{02}) \times \mathbb{P}(V_{34})$ embedded by the Segre map. Since $\mathbb{P}(A_3)$ has codimension 2 in $\mathbb{P}(V_{02} \wedge V_{34} \wedge [v_5])$ it follows that Θ_{A_3} has dimension at least 1 and that generically it is a twisted rational cubic curve. The projection $\mathbb{P}(V_{02}) \times \mathbb{P}(V_{34}) \to \mathbb{P}(V_{02})$ defines a regular map $\pi \colon \Theta_{A_3} \to \mathbb{P}(V_{02})$. Let $C_{A_3} := \operatorname{im} \pi$. If Θ_{A_3} is a twisted rational cubic curve then C_{A_3} is a smooth conic. On the other hand let

$$C_{A_2} := \{[\beta] \in \mathbb{P}(V_{02}) \mid A_2 \cap ([\beta] \wedge \bigwedge^2 V_{34} \oplus [\beta] \wedge V_{02} \wedge [v_5]) \neq \{0\}\}. \tag{6.5.2}$$

Then C_{A_2} is a lagrangian degeneracy locus and either it is a conic or all of $\mathbb{P}(V_{02})$. If $A_2 \cap \bigwedge^2 V_{02} \wedge [v_5] = \{0\}$ we may describe C_{A_2} as follows. By our assumption A_2 is the graph of a linear map $V_{02} \wedge \bigwedge^2 V_{34} \longrightarrow \bigwedge^2 V_{02} \wedge [v_5]$ which is symmetric because A_2 is lagrangian: let q_{A_2} be the associated quadratic form. Then $C_{A_2} = V(q_{A_2})$. If A is generic in $\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ then C_{A_2}, C_{A_3} are conics intersecting transversely. Below is the main result of the present subsection.

Proposition 6.5.1. *The following hold:*

(1) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$. Then A is $G_{\mathcal{E}_1^\vee}$-stable if and only if C_{A_3} is a smooth conic and C_{A_2} is a conic intersecting D_{A_3} transversely.

(2) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is $G_{\mathcal{E}_1^\vee}$-stable.

(3) If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is $G_{\mathcal{E}_1^\vee}$-stable the connected component of Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to* $\operatorname{im} \lambda_{\mathcal{E}_1^\vee}$.

(4) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ have closed $\mathrm{PGL}(V)$*-orbit (in $\mathbb{LG}(\bigwedge^3 V)^{ss}$), and suppose that $[A] \notin \mathfrak{I}$. Then $C_{W,A}$ is of Type II-1, II-2, II-3 or* $\mathrm{PGL}(V)$*-equivalent to Type III-2.*

(5) $\mathfrak{B}_{\mathcal{E}_1^\vee} \cap \mathfrak{I} = \{\mathfrak{x}^\vee\}$.

The proof of **Proposition 6.5.1** is in **Subsubsection 6.5.3**.

6.5.1 The GIT analysis

Proposition 6.5.2. *$A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is not $G_{\mathcal{E}_1^\vee}$-stable if and only if one of the following holds:*

(1) There exists a non-zero decomposable element of A_1.

(2) $\dim(A_2 \cap \bigwedge^2 V_{02} \wedge [v_5]) \geq 1$.

(3) There exist bases $\{\beta_1, \beta_2, \beta_3\}$ of V_{02}, $\{\xi_1, \xi_2\}$ of V_{34} and $x, y \in \mathbb{C}$ not both zero such that

$$\langle \beta_1 \wedge \xi_1 \wedge v_5, (x\beta_1 \wedge \xi_2 + y\beta_2 \wedge \xi_1) \wedge v_5 \rangle \subset A_3$$

and

$$\dim(A_2 \cap \langle \xi_1 \wedge \xi_2 \wedge \beta_1, \beta_1 \wedge \beta_2 \wedge v_5 \rangle) \geq 1.$$

Proof. A is not $G_{\mathcal{E}_1^\vee}$-stable if and only if $\delta_V(A)$ is not $G_{\mathcal{E}_1}$-stable - see (1.3.1) for the definition of δ_V. Now $\delta_V(A) \in \mathbb{S}^{\mathsf{G}}_{\mathcal{E}_1}$ where $\mathsf{G} = \{v_5^\vee, v_4^\vee, \dots, v_0^\vee\}$. The proposition follows at once from **Proposition 6.4.3**. □

By copying the proof of **Corollary 6.4.5** one gets the following result.

Corollary 6.5.3. *$A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is $G_{\mathcal{E}_1^\vee}$-stable if and only if C_{A_3} is a a smooth conic (equivalently Θ_{A_3} is a smooth curve) and C_{A_2} is a conic intersecting C_{A_3} transversely.*

Let $\{\beta_1, \beta_2, \beta_3\}$ be a basis of V_{02} and $\{\xi_1, \xi_2\}$ be a basis of V_{34}. Let $\lambda_1^\vee$ be the 1-PS of $\mathrm{SL}(V)$ defined by $\lambda_1^\vee(t) := \mathrm{diag}(t^2, 1, t^{-2}, t, t^{-1}, 1)$ where we mean diagonal with respect to the basis $\{\beta_1, \beta_2, \beta_3, \xi_1, \xi_2, v_5\}$. The group $G_{\mathcal{E}_1^\vee}$ acts on the affine cone $\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ over $\mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1}$. The fixed locus $(\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1^\vee})^{\lambda_1}$ is the set of A which are mapped to themselves by $\wedge^3\lambda_1(t)$ and such that $\wedge^3\lambda_1(t)$ acts trivially on $\bigwedge^{10} A$.

Definition 6.5.4. Let $\mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1^\vee} \subset \mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1^\vee})^{\lambda_1})$ be the set of A such that $\wedge^3\lambda_1(t)$ acts trivially on $\bigwedge^2 A_1$, $\bigwedge^3 A_2$ and $\bigwedge^4 A_3$.

Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$; then $A \in \mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1^\vee}$ if and only if it is $\lambda_1^\vee$-split of types $d^{\lambda_1^\vee}(A_1) = (0, 1, 1, 0)$ and $d^{\lambda_1^\vee}(A_2) = (1, 1, 1)$. Moreover $\mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1^\vee}$ is an irreducible component of $\mathbb{P}((\widehat{\mathbb{S}}^{\mathsf{F}}_{\mathcal{E}_1^\vee})^{\lambda_1})$. By copying the proof of **Proposition 6.4.8** one gets the following result.

Proposition 6.5.5. *Suppose that A is properly $G_{\mathcal{E}_1^\vee}$-semistable. Then there exists a semistable $A_0 \in \mathbb{M}^{\mathsf{B}}_{\mathcal{E}_1^\vee}$ with minimal orbit which is $G_{\mathcal{E}_1^\vee}$-equivalent to A.*

6.5.2 Analysis of Θ_A and $C_{W,A}$

Proposition 6.5.6. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ be $G_{\mathcal{E}_1^\vee}$-stable and $W \in \Theta_A$. Then one of the following holds:*

(a) $W = V_{02}$.

(b) $W \in \Theta_{A_3}$.

(c) $W = \langle \beta, \xi_1, \xi_2 \rangle$ *where* $\beta \in V_{02}$ *and* $\xi_1, \xi_2 \in V_{34}$.

Proof. Follows from the equality $\delta_V(\Theta_A) = \Theta_{\delta_V(A)}$ and **Proposition 6.4.9**. □

Proposition 6.5.7. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ be $G_{\mathcal{E}_1^\vee}$-stable. Let $W \in \Theta_A$ and hence one of Items (a), (b), (c) of* **Proposition 6.5.6** *holds. Then $C_{W,A}$ is a sextic curve of*

(1) Type II-3 if Item (a) holds.

(2) Type II-1 if Item (b) holds.

(3) Type II-2 if Item (c) holds.

Proof. The orbit $\mathrm{PGL}(V)A$ is minimal because A is $G_{\mathcal{E}_1^\vee}$-stable (see **Claim 5.2.1**) and $\dim \Theta_A = 1$ by **Proposition 6.5.6**: thus $C_{W,A} \neq \mathbb{P}(W)$ by **Corollary 6.1.10**. Let us carry out a case-by-case analysis.

$\boxed{W = V_{02}}$ We have $C_{A_2}, C_{A_3} \subset C_{V_{02},A}$. Moreover $\dim(A_3 \cap F_\beta) \geq 2$ for all $[\beta] \in \mathbb{P}(V_{02})$: thus $\mathrm{mult}_{[\beta]} C_{V_{02},A} \geq 2$ for all $[\beta] \in C_{A_3}$. Since C_{A_2} and C_{A_3} are conics and $C_{V_{02},A}$ is a sextic it follows that $C_{V_{02},A} = C_{A_2} + 2C_{A_3}$: by **Corollary 6.5.3** the conics C_{A_2}, C_{A_3} are transverse and hence $C_{V_{02},A}$ is of Type II-3.

$\boxed{W \in \Theta_{A_3}}$ Thus $W = \langle \beta, v_5, \xi \rangle$ where $\beta \in V_{02}$ and $\xi \in V_{34}$. Notice that $\lambda_{\mathcal{E}_1^\vee}(t)$ maps W to itself for every $t \in \mathbb{C}^\times$. Let $\{x, y, z\}$ be the basis of $W^\vee$ dual to $\{\beta, v_5, \xi\}$: applying **Claim 3.2.4** we get that

$$C_{W,A} = V((xy + a_1 z^2)(xy + a_2 z^2)(xy + a_3 z^2)). \tag{6.5.3}$$

It remains to prove that a_1, a_2, a_3 are pairwise distinct. It suffices to show that

$$\mathrm{mult}_{[x\beta + yv_5 + \xi]} C_{W,A} \leq 1 \text{ if } y \neq 0. \tag{6.5.4}$$

The key step is the proof that

$$\dim(A \cap F_{(x\beta + yv_5 + \xi)}) \leq 2, \quad y \neq 0. \tag{6.5.5}$$

Let $\alpha \in A \cap F_{(x\beta + yv_5 + \xi)}$. Write $\alpha = \sum_{i=0}^{3} \alpha_i$ where α_i belongs to the $(i+1)$-th (starting from the left) summand of (6.5.1). We set $\alpha_2 = \alpha_2' + \alpha_2''$ where $\alpha_2' \in V_{02} \wedge \bigwedge^2 V_{34}$, $\alpha_2'' \in \bigwedge^2 V_{02} \wedge [v_5]$. We have $(x\beta + yv_5 + \xi) \wedge \alpha = 0$. Now decompose $(x\beta + yv_5 + \xi) \wedge \alpha$ according to the direct-sum decomposition of $\bigwedge^4 V$ determined by $V = V_{02} \oplus V_{34} \oplus [v_5]$: we get that

$$0 = yv_5 \wedge \alpha_2' + \xi \wedge \alpha_3 = yv_5 \wedge \alpha_1 + \xi \wedge \alpha_2'' + x\beta \wedge \alpha_3 = x\beta \wedge \alpha_2' + \xi \wedge \alpha_1 = x\beta \wedge \alpha_2'' + yv_5 \wedge \alpha_0 = x\beta \wedge \alpha_1 + \xi \wedge \alpha_0. \tag{6.5.6}$$

Now suppose that $y \neq 0$: then

$$\begin{array}{ccc} A \cap F_{(x\beta + yv_5 + \xi)} & \xrightarrow{\rho} & V_{02} \wedge V_{34} \wedge [v_5] \\ \alpha & \mapsto & \alpha_3 \end{array} \tag{6.5.7}$$

is injective. This follows at once from (6.5.6). Now we prove (6.5.5) arguing by contradiction. Suppose that (6.5.5) does not hold. Since the map ρ of (6.5.7) is injective it follows that $\dim(\mathrm{im}\,\rho) \geq 3$. Now consider the intersection of $\mathbb{P}(\mathrm{im}\,\rho)$ and $\mathbb{P}(V_{02}) \times \mathbb{P}(V_{34}) \times \{[v_5]\}$: it contains $[\beta \wedge \xi \wedge v_5]$ and the expected dimension is zero. Since the Segre 3-fold $\mathbb{P}(V_{02}) \times \mathbb{P}(V_{34})$ has degree 3 it follows that one of the following holds:

(I) $\mathbb{P}(\mathrm{im}\,\rho)$ contains $[\beta' \wedge \xi' \wedge v_5] \neq [\beta \wedge \xi \wedge v_5]$.

(II) $\mathbb{P}(\mathrm{im}\,\rho)$ contains a tangent vector to $\mathbb{P}(V_{02}) \times \mathbb{P}(V_{34}) \times \{[v_5]\}$ at $[\beta \wedge \xi \wedge v_5]$ i.e. there exists $\alpha \in A \cap F_{(x\beta + yv_5 + \xi)}$ such that $\alpha_3 = (\beta \wedge \xi' + \beta' \wedge \xi) \wedge v_5$.

Suppose that (I) holds. We let $\beta_3 := \beta$, $\xi_2 := \xi$, $\beta_1 := \beta'$ and $\xi_1 := \xi'$. By hypothesis there exists $\alpha \in A \cap F_{(x\beta + yv_5 + \xi)}$ such that $\alpha_3 = \beta_1 \wedge \xi_1 \wedge v_5$. The first equality of (6.5.6) gives that $\alpha_2' = y^{-1}\beta_1 \wedge \xi_1 \wedge \xi_2$. The third equality of (6.5.6) gives that $\alpha_1 = -xy^{-1}\beta_1 \wedge \beta_3 \wedge \xi_1 + \gamma \wedge \xi_2$ for some $\gamma \in \bigwedge^2 V_{02}$. Since $\beta_1 \wedge \xi_1 \wedge v_5 \in A_3$ and $A_1 \bot A_3$ we get that $\gamma \wedge \beta_1 = 0$. Thus $\gamma = \beta_1 \wedge \theta$ for some $\theta \in V_{02}$ and $\alpha_1 = -xy^{-1}\beta_1 \wedge \beta_3 \wedge \xi_1 + \beta_1 \wedge \theta \wedge \xi_2$. Since A_1 contains no non-zero decomposable element we get that $\{\beta_1, \beta_3, \theta\}$ is a basis of V_{02}: we let $\beta_2 := \theta$. The second equality of (6.5.6) gives that $\alpha_2'' = y\beta_1 \wedge \beta_2 \wedge v_5$. Summarizing:

$$\alpha_1 = -xy^{-1}\beta_1 \wedge \beta_3 \wedge \xi_1 + \beta_1 \wedge \beta_2 \wedge \xi_2, \quad \alpha_2 = y^{-1}\beta_1 \wedge \xi_1 \wedge \xi_2 + y\beta_1 \wedge \beta_2 \wedge v_5. \tag{6.5.8}$$

The equality $A_3 = A_1^\bot \cap (V_{02} \wedge V_{34} \wedge [v_5])$ together with the first equality of (6.5.8) gives that there exist $s, t \in \mathbb{C}$ not both zero such that $(s\beta_1 \wedge \xi_2 + t\beta_2 \wedge \xi_1) \wedge v_5 \in A_3$. By hypothesis $\beta_1 \wedge \xi_1 \wedge v_5 = \alpha_3 \in A_3$. Thus Item (3) of **Proposition 6.5.2** holds and hence A is not $G_{\mathcal{E}_1^\vee}$-stable;

that is a contradiction. Next suppose that (II) holds. Let $\beta_1 := \beta$, $\xi_1 := \xi$, $\beta_2 := \beta'$ and $\xi_2 := \xi'$. Thus

$$\beta_1 \wedge \xi_1 \wedge v_5, (\beta_1 \wedge \xi_2 + \beta_2 \wedge \xi_1) \wedge v_5 \in A_3$$

and there exists $\alpha \in A \cap F_{(x\beta+yv_5+\xi)}$ such that $\alpha_3 = (\beta_1 \wedge \xi_2 + \beta_2 \wedge \xi_1) \wedge v_5$. Since Θ_{A_3} is a smooth curve β_1, β_2 are linearly independent and $\{\xi_1, \xi_2\}$ is a basis of V_{34}. On the other hand an argument similar to that of the previous case gives that $\alpha_2 = -y^{-1}\beta_1 \wedge \xi_1 \wedge \xi_2 - x\beta_1 \wedge \beta_2 \wedge v_5$. Thus A is not $G_{\mathcal{E}_1^\vee}$-stable by **Proposition 6.5.2**; that is a contradiction. We have proved (6.5.5). Next assume that $[x\beta + yv_5 + \xi] \in C_{W,A}$ and $y \neq 0$. Thus $\dim(A \cap F_{(x\beta+yv_5+\xi)}) = 2$. One shows that $[x\beta + yv_5 + \xi] \notin \mathcal{B}(W, A)$. The computations are similar to those which prove (6.5.5): we leave details to the reader. This finishes the proof that if $W \in \Theta_{A_3}$ then $C_{W,A}$ is a semistable sextic of Type II-1.

$\boxed{W = \langle \beta, \xi_1, \xi_2 \rangle \text{ where } \beta \in V_{02},\ \xi_1, \xi_2 \in V_{34}}$ *Mutatis mutandis* the proof is that (given in **Proposition 6.4.11**) that if Item (a) of **Proposition 6.4.9** holds then $C_{W,A}$ is of Type II-2. Let $\{X_0, X_1, X_2\}$ be the basis of $W^\vee$ dual to $\{\beta, \xi_1, \xi_2\}$: applying **Claim 3.2.4** one gets that

$$C_{W,A} = V(X_0^2 F(X_1, X_2)), \qquad 0 \neq F \in \mathbb{C}[X_1, X_2]_4.$$

It remains to prove that F does not have multiple roots. Let $0 \neq \xi \in V_{34}$ and $\pi\colon A \cap F_{(\beta-\xi)} \to V_{02} \wedge V_{34} \wedge [v_5]$ be the projection. Arguing as in the proof of **Proposition 6.4.11** one shows that the image is either $\{0\}$ or it belongs to Θ_{A_3}, and it has dimension at most 1. Moreover the kernel is spanned by $\beta \wedge \xi_1 \wedge \xi_2$. Now suppose that $[\beta - \xi] \in C_{W,A}$: then it follows that $\dim(A \cap F_{(\beta-\xi)}) = 2$. Moreover one checks easily that $[\beta - \xi] \notin \mathcal{B}(W, A)$. By **Proposition 3.3.6** it follows that $C_{W,A}$ is smooth at $[\beta - \xi]$: thus F does not have multiple roots. □

Arguing as in the proof of **Proposition 6.4.12** one gets the following result.

Proposition 6.5.8. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ be properly $G_{\mathcal{E}_1^\vee}$-semistable with minimal orbit. Then either $[A] = \mathfrak{x}^\vee$ or else the following holds: if $W \in \Theta_A$ then $C_{W,A}$ is a semistable sextic curve* $\mathrm{PGL}(W)$*-equivalent to a sextic of Type III-2.*

6.5.3 Wrapping it up

We will prove **Proposition 6.5.1**. Item (1) is the content of **Corollary 6.5.3**. We have noticed that if $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is generic then C_{A_2}, C_{A_3} are conics intersecting transversely: together with Item (1) that gives Item (2). Item (3) follows from Item (3) of **Proposition 6.4.2** because if $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{E}_1^\vee}$ is $G_{\mathcal{E}_1^\vee}$-stable then $\delta_V(A)$ belongs to $\mathbb{S}^{\mathsf{F}'}_{\mathcal{E}_1}$ for a suitable basis F' of $V^\vee$ and is $G_{\mathcal{E}_1}$-stable. In order to prove Items (4) and (5) we notice that $\delta(\mathfrak{B}_{\mathcal{E}_1}) = \mathfrak{B}_{\mathcal{E}_1^\vee}$ and hence $\mathfrak{x}^\vee \in \mathfrak{B}_{\mathcal{E}_1^\vee}$ by (6.4.31). Since $\mathfrak{x}^\vee \in \mathfrak{B}_{\mathcal{E}_1^\vee}$ Items (4) and (5) follow from **Proposition 6.5.7** and **Proposition 6.5.8**.

6.6 $\mathfrak{B}_{\mathcal{F}_1}$

Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$. Then

$$A = \bigwedge^2 V_{01} \wedge V_{23} \oplus A_2 \oplus V_{01} \wedge \bigwedge^2 V_{45} \oplus \bigwedge^2 V_{23} \wedge V_{45}, \qquad A_2 \in \mathbb{LG}(V_{01} \wedge V_{23} \wedge V_{45}). \qquad (6.6.1)$$

Below is the main result of the present subsection.

Proposition 6.6.1. *The following hold:*

(1) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$. Then A is $G_{\mathcal{F}_1}$-stable if and only if A_2 contains no non-zero decomposable element.

(2) The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ is $G_{\mathcal{F}_1}$-stable.

(3) If $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ is $G_{\mathcal{F}_1}$-stable the connected component of Id *in* $\mathrm{Stab}(A) < \mathrm{SL}(V)$ *is equal to $H_{\mathcal{F}_1}$ (see* (5.2.10)*).*

(4) Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ have closed $\mathrm{PGL}(V)$*-orbit (in $\mathbb{LG}(\bigwedge^3 V)^{ss}$). Then $C_{W,A}$ is of Type II-2 or III-2. In particular $\mathfrak{B}_{\mathcal{F}_1} \cap \mathfrak{I} = \emptyset$.*

The proof of **Proposition 6.6.1** is in **Subsubsection 6.6.3**.

6.6.1 The GIT analysis

Let λ be a 1-PS of $G_{\mathcal{F}_1}$. Since $G_{\mathcal{F}_1} = \mathrm{SL}(V_{01})\times\mathrm{SL}(V_{23})\times\mathrm{SL}(V_{45})$ we have $I_-(\lambda) = \emptyset$, see **Definition 5.2.3**. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$: by (5.2.5) we have

$$\mu(A,\lambda) = \mu(A_2,\lambda). \tag{6.6.2}$$

Let $\{\xi_0,\xi_1\}$, $\{\xi_2,\xi_3\}$, $\{\xi_4,\xi_5\}$ be bases of V_{01}, V_{23} and V_{45} respectively such that

$$\lambda(t) := \mathrm{diag}(t^{r_1},t^{-r_1},t^{r_2},t^{-r_2},t^{r_3},t^{-r_3}), \qquad r_1\geq 0,\ r_2\geq 0,\ r_3\geq 0. \tag{6.6.3}$$

We denote λ by (r_1,r_2,r_3): thus (r_1,r_2,r_3) belongs to the first quadrant of $\mathbb{R}^3$. Below are the weights of the action of $\bigwedge^3\lambda(t)$ on $V_{01}\wedge V_{23}\wedge V_{45}$:

$$\begin{matrix} [\xi_0\wedge\xi_2\wedge\xi_4] & [\xi_0\wedge\xi_2\wedge\xi_5] & [\xi_0\wedge\xi_3\wedge\xi_4] & [\xi_1\wedge\xi_2\wedge\xi_4] & [\xi_0\wedge\xi_3\wedge\xi_5] & [\xi_1\wedge\xi_2\wedge\xi_5] & [\xi_1\wedge\xi_3\wedge\xi_4] & [\xi_1\wedge\xi_3\wedge\xi_5] \\ r_1+r_2+r_3 & r_1+r_2-r_3 & r_1-r_2+r_3 & -r_1+r_2+r_3 & r_1-r_2-r_3 & -r_1+r_2-r_3 & -r_1-r_2+r_3 & -r_1-r_2-r_3 \end{matrix} \tag{6.6.4}$$

Proposition 6.6.2. *$A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ is $G_{\mathcal{F}_1}$-stable if and only if A_2 contains no non-zero decomposable element.*

Proof. Suppose that A_2 contains a non-zero decomposable element α. Since we have an isomorphism

$$\begin{matrix} \mathbb{P}(V_{01})\times\mathbb{P}(V_{23})\times\mathbb{P}(V_{45}) & \hookrightarrow & \mathbb{P}(V_{01}\wedge V_{23}\wedge V_{45})\cap\mathrm{Gr}(3,V) \\ ([u],[v],[w]) & \mapsto & [u\wedge v\wedge w] \end{matrix} \tag{6.6.5}$$

there exists bases $\{\xi_0,\xi_1\},\{\xi_2,\xi_3\},\{\xi_4,\xi_5\}$ as above such that $\alpha = \xi_0\wedge\xi_2\wedge\xi_4$. Let λ_1 be the 1-PS of $G_{\mathcal{F}_1}$ denoted $(1,1,1)$ i.e. $\lambda_1(t) := \mathrm{diag}(t,t^{-1},t,t^{-1},t,t^{-1})$. Then $\mu(A_2,\lambda_1)\geq 0$: by (6.6.2) we get that A is not $G_{\mathcal{F}_1}$-stable. We prove the converse by running the Cone Decomposition algorithm. We choose the maximal torus $T < G_{\mathcal{F}_1}$ to be

$$T = \{\mathrm{diag}(s_1,s_1^{-1},s_2,s_2^{-1},s_3,s_3^{-1})) \mid s_i\in\mathbb{C}^\times\}. \tag{6.6.6}$$

(The maps are diagonal with respect to the basis $\{\xi_0,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5\}$.) Thus

$$\check{X}(T)_{\mathbb{R}} = \{(r_1,r_2,r_3)\in\mathbb{R}^3\} \tag{6.6.7}$$

where the r_i's are those appearing in (6.6.3) and $C = \{(r_1,r_2,r_3)\in\mathbb{R}^3 \mid r_i\geq 0\}$. Let $H\subset\check{X}(T)_{\mathbb{R}}$ be a hyperplane: by (6.6.4) H is an ordering hyperplane if and only if it is the kernel of one of the following following linear functions on $\check{X}(T)_{\mathbb{R}}$:

$$r_i, \quad r_i - r_j, \quad r_i - r_j - r_k\ (j\neq k).$$

A quick computation gives that the ordering rays are those spanned by

$$(1,0,0), \quad (1,1,0), \quad (2,1,1), \quad (1,1,1)$$

and their permutations. Computing $\mu(A_2,\lambda)$ and imposing $\mu(A_2,\lambda)\geq 0$ we get that in each case A_2 contains a non-zero decomposable element. □

6.6.2 Analysis of Θ_A and $C_{W,A}$

Proposition 6.6.3. *Let $A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ be $G_{\mathcal{F}_1}$-stable. Then*

$$\Theta_A = \{W\in\mathrm{Gr}(3,V) \mid V_{01}\subset W\subset V_{03}\}\cup\{W\in\mathrm{Gr}(3,V) \mid V_{23}\subset W\subset V_{25}\}\cup\{W\in\mathrm{Gr}(3,V) \mid V_{45}\subset W\subset(V_{45}\oplus V_{01})\}. \tag{6.6.8}$$

Let $W\in\Theta_A$: then $C_{W,A}$ is a semistable sextic curve of Type II-2.

Proof. The right-hand side of (6.6.8) is contained in Θ_A by (6.6.1). Now suppose that $W_0 \in \Theta_A$. Since A is lagrangian

$$W_0 \text{ has non-trivial intersection with every } W \text{ belonging to the right-hand side of (6.6.8).} \quad (6.6.9)$$

Suppose that W_0 contains one of V_{01}, V_{23} or V_{45}: it follows from (6.6.9) that W_0 must belong to the right-hand side of (6.6.8). Now suppose that W_0 does not contain V_{01} nor V_{23} nor V_{45}. It follows from (6.6.9) that W_0 has non-trivial intersection with two at least among V_{01}, V_{23} and V_{45}. That easily leads to a contradiction because by **Proposition 6.6.2** we know that A_2 contains no non-zero decomposable elements. We have proved (6.6.8). Now suppose that $W \in \Theta_A$ i.e. W belongs to the right-hand side of (6.6.8): we will prove that $C_{W,A}$ is a semistable sextic curve of Type II-2. By (6.6.8) we have $\dim \Theta_A = 1$: by **Corollary 6.1.10** it follows that $C_{W,A} \neq \mathbb{P}(W)$. From now on we will assume that $V_{01} \subset W \subset V_{03}$, if W belongs to one of the other two subsets on the right-hand side of (6.6.8) the proof is analogous. Let ξ be a generator of $W \cap V_{23}$: thus $W = \langle \xi, v_0, v_1 \rangle$. Let $\{X_0, X_1, X_2\}$ be the basis of $W^\vee$ dual to $\{\xi, v_0, v_1\}$. Then $\lambda_{\mathcal{F}_1}(t)$ maps W to itself for every $t \in \mathbb{C}^\times$: applying **Claim 3.2.4** we get that

$$C_{W,A} = V(X_0^2 P), \qquad 0 \neq P \in \mathbb{C}[X_1, X_2]_4. \qquad (6.6.10)$$

It remains to prove that P has no multiple factors. Let $0 \neq u \in V_{01}$. We claim that

$$\dim(A \cap F_{(\xi - u)}) \leq 2. \qquad (6.6.11)$$

In fact assume that $\alpha \in A \cap F_{(\xi-u)}$. Thus $(\xi - u) \wedge \alpha = 0$. Write $\alpha = \alpha_0 + \alpha_2 + \alpha_3' + \alpha_3''$ where $\alpha_0 \in \bigwedge^2 V_{01} \wedge V_{23}$, $\alpha_2 \in V_{01} \wedge V_{23} \wedge V_{45}$, $\alpha_3' \in V_{01} \wedge \bigwedge^2 V_{45}$ and $\alpha_3'' \in \bigwedge^2 V_{23} \wedge V_{45}$. The equality $(\xi - u) \wedge \alpha = 0$ is equivalent to the following equalities:

$$0 = \xi \wedge \alpha_0 = u \wedge \alpha_2 = \xi \wedge \alpha_3' = u \wedge \alpha_3', \qquad \xi \wedge \alpha_2 = u \wedge \alpha_3''. \qquad (6.6.12)$$

In particular $\alpha_0 \in \bigwedge^2 V_{01} \wedge [\xi]$. One also gets easily that the projection

$$\pi \colon A \cap F_{(\xi-u)} \longrightarrow V_{01} \wedge V_{23} \wedge V_{45}$$

has 1-dimensional kernel namely $\bigwedge^2 V_{01} \wedge [\xi]$. On the other hand

$$\operatorname{im} \pi \subset \{u \wedge \theta \mid \theta \in V_{23} \wedge V_{45}\}. \qquad (6.6.13)$$

A subspace of the right-hand side of (6.6.13) of dimension at least 2 contains non-zero decomposable elements: since A_2 does not contain non-zero decomposables it follows that $\dim(\operatorname{im} \pi) \leq 1$. This proves (6.6.11). Next assume that $[\xi - u] \in C_{W,A}$: by (6.6.11) we get that $\dim(A \cap F_{(\xi-u)}) = 2$. As is easily checked $\mathcal{B}(W, A) = \emptyset$. This proves that $C_{W,A}$ is smooth at $[\xi - u]$: it follows that the polynomial P of (6.6.10) does not have multiple roots. □

Before stating the next result we notice that $\mathrm{PGL}(V) A_{III} \cap \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1} \neq \emptyset$.

Proposition 6.6.4. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_1}$ be properly $G_{\mathcal{F}_1}$-semistable: then $A \in \mathrm{PGL}(V) A_{III}$. In particular $C_{W,A}$ is a semistable sextic curve of Type III-2.*

Proof. By **Proposition 6.6.2** A_2 contains a non-zero decomposable element, say $\xi_0 \wedge \xi_2 \wedge \xi_4$. Proceeding as in the proof of **Proposition 6.6.2** we define a 1-PS λ_1 such that $\mu(A, \lambda_1) = 0$. Considering the action of λ_1 on $V_{01} \wedge V_{23} \wedge V_{45}$ we get that $A' := \lim_{t \to 0} \lambda_1(t) A$ has a monomial basis. Thus either A' is not $G_{\mathcal{F}_1}$-semistable or else it belongs to $\mathrm{PGL}(V) A_{III}$ by **Claim 4.3.1** - one checks that in fact the latter holds. □

6.6.3 Wrapping it up

We will prove **Proposition 6.6.1**. Item (1) is the content of **Proposition 6.6.2**. The generic $A_2 \in \mathbb{LG}(V_{01} \wedge V_{23} \wedge V_{45})$ contains no non-zero decomposable element because the dimension of the right-hand side of (6.6.5) is equal to 3, thus Item (2) follows from Item (1). Let's prove Item (3). Let $g \in \mathrm{Stab}(A)$ belong to the connected component of Id. **Proposition 6.6.3** gives that $g(V_{01}) = V_{01}$, $g(V_{23}) = V_{23}$ and $g(V_{45}) = V_{45}$ i.e. $g \in C_{\mathrm{SL}(V)}(\lambda_{\mathcal{F}_1})$. Since A is $G_{\mathcal{F}_1}$-stable the stabilizer of A in $G_{\mathcal{F}_1}$ is finite: it follows that $g \in H_{\mathcal{F}_1}$. Lastly Item (4) follows from **Proposition 6.6.3** and **Proposition 6.6.4**.

7 The remaining boundary components

7.1 $\mathfrak{B}_{\mathcal{F}_2}$

The isotypical decomposition of $\bigwedge^3 \lambda_{\mathcal{F}_2}$ is the following:

$$\bigwedge^2 V_{01}\wedge V_{23}\oplus\big(\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23}\big)\oplus V_{01}\wedge V_{23}\wedge V_{45}\oplus\big(V_{01}\wedge\bigwedge^2 V_{45}\oplus\bigwedge^2 V_{23}\wedge V_{45}\big)\oplus V_{23}\wedge\bigwedge^2 V_{45}. \tag{7.1.1}$$

Let $A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$: then $A=A_0+\ldots+A_4$ where

$$A_0\in\mathbb{P}(\bigwedge^2 V_{01}\wedge V_{23}),\ A_1\in\mathrm{Gr}(2,(\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23}))\ A_2\in\mathbb{LG}(V_{01}\wedge V_{23}\wedge V_{45})\ A_3\in\mathrm{Gr}(2,(V_{01}\wedge\bigwedge^2 V_{45}\oplus\bigwedge^2 V_{23}\wedge V_{45}))\ A_4\in\mathbb{P}(V_{23}\wedge\bigwedge^2 V_{45}). \tag{7.1.2}$$

and $A_{4-i}\bot A_i$. Let λ be a 1-PS of $G_{\mathcal{F}_2}$. There exist bases $\{\xi_0,\xi_1\}$, $\{\xi_2,\xi_3\}$, $\{\xi_4,\xi_5\}$ of V_{01}, V_{23}, V_{45} respectively such that

$$\lambda(t)=(t^m,(\mathrm{diag}(t^{r_1},t^{-r_1}),\mathrm{diag}(t^{r_2},t^{-r_2}),\mathrm{diag}(t^{r_3},t^{-r_3}))),\qquad r_1\geq 0,\ r_2\geq 0,\ r_3\geq 0. \tag{7.1.3}$$

We denote such a 1-PS by (m,r_1,r_2,r_3). Below are the weights of the action of $\bigwedge^3\lambda(t)$ on the first two summands of (7.1.1):

$$\bigwedge^2 V_{01}\wedge V_{23} \;=\; \underset{r_2}{[\xi_0\wedge\xi_1\wedge\xi_2]}\;\oplus\;\underset{-r_2}{[\xi_0\wedge\xi_1\wedge\xi_3]} \tag{7.1.4}$$

$$\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23} \;=\; \underset{r_1-3m}{[\xi_0\wedge\xi_2\wedge\xi_3]}\;\oplus\;\underset{-r_1-3m}{[\xi_1\wedge\xi_2\wedge\xi_3]}\;\oplus\;\underset{r_3+3m}{[\xi_0\wedge\xi_1\wedge\xi_4]}\;\oplus\;\underset{-r_3+3m}{[\xi_0\wedge\xi_1\wedge\xi_5]} \tag{7.1.5}$$

The weights of the action of $\bigwedge^3\lambda(t)$ on $V_{01}\wedge V_{23}\wedge V_{45}$ are given by (6.6.4). In particular we get that $I_-(\lambda)=\emptyset$: by (5.2.5) and (2.2.9) we have

$$\mu(A,\lambda)=2\mu(A_0,\lambda)+2\mu(A_1,\lambda)+\mu(A_2,\lambda).$$

Proposition 7.1.1. *$A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is not $G_{\mathcal{F}_2}$-stable if and only if one of the following holds:*

(1) $\dim A_1\cap(V_{01}\wedge\bigwedge^2 V_{23})\geq 1$ *or* $\dim A_1\cap(\bigwedge^2 V_{01}\wedge V_{45})\geq 1$.

(2) There exist $0\neq\beta\in V_{23}$ and $0\neq\theta\in V_{01}\wedge V_{45}$ such that $v_0\wedge v_1\wedge\beta\in A_0$ and $\beta\wedge\theta\in A_2$.

(3) There exist $0\neq\alpha\in V_{01}$, $0\neq\beta\in V_{23}$, $0\neq\gamma\in V_{45}$ such that $(\alpha\wedge v_2\wedge v_3+v_0\wedge v_1\wedge\gamma)\in A_1$ and $\alpha\wedge\beta\wedge\gamma\in A_2$.

(4) There exists $0\neq\alpha\in V_{01}$ such that $\dim A_2\cap([\alpha]\wedge V_{23}\wedge V_{45})\geq 2$, or there exists $0\neq\gamma\in V_{45}$ such that $\dim A_2\cap(V_{01}\wedge V_{23}\wedge[\gamma])\geq 2$.

Proof. We begin by considering the duality operator. If A is not $G_{\mathcal{F}_2}$-stable then so is $\delta_V(A)$ where δ_V is defined by (1.3.1). More precisely let $\{\xi_0,\xi_1,\ldots,\xi_5\}$ be a basis of V as above and $\{\xi_0^\vee,\xi_1^\vee,\ldots,\xi_5^\vee\}$ be the dual basis of $V^\vee$. Let $\phi\colon V^\vee\xrightarrow{\sim}V$ be the isomorphism such that $\phi(\xi_i^\vee)=\xi_{5-i}$. Let $A\in\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$: then

$$B:=\bigwedge^3\phi(\delta_V(A))\in\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}. \tag{7.1.6}$$

Now suppose that λ_1 is the 1-PS of $G_{\mathcal{F}_2}$ denoted by (m,r_1,r_2,r_3) and let λ_2 be the 1-PS of $G_{\mathcal{F}_2}$ denoted by $(-m,r_3,r_2,r_1)$. An easy computation shows that $\mu(A,\lambda_1)=\mu(B,\lambda_2)$; in particular if $\mu(A,\lambda_1)\geq 0$ then $\mu(B,\lambda_2)\geq 0$. Thus non-stable elements of $\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ come in dual pairs. One can easily check that if A satisfies one of Items (1) - (4) above then B satisfies the same Item. Now let's prove that if one of Items (1) - (4) holds then A is not $G_{\mathcal{F}_2}$-stable. We will freely use the data listed in Tables (28) and (29). Suppose that Item (1) holds. Let $\{\xi_0,\xi_1,\ldots,\xi_5\}$ be a basis of V as above and $\lambda_1^\pm$ be the 1-PS of $G_{\mathcal{F}_2}$ which is diagonal in the chosen basis and is indicized by $(\pm 1,0,0,0)$ - see (7.1.3). Explicitly

$$\lambda_1^+(s)=\mathrm{diag}(s,s,s^{-2},s^{-2},s,s),\quad \lambda_1^-(s)=\mathrm{diag}(s^{-1},s^{-1},s^2,s^2,s^{-1},s^{-1}). \tag{7.1.7}$$

If $\dim(A_1 \cap V_{01} \wedge \bigwedge^2 V_{23}) \geq 1$ then $\mu(A, \lambda_1^+) \geq 0$ (see (28)), if $\dim(A_1 \cap \bigwedge^2 V_{01} \wedge V_{45}) \geq 1$ then $\mu(A, \lambda_1^-) \geq 0$: in both cases it follows that A is not $G_{\mathcal{F}_2}$-stable. Next suppose that Item (2) holds. Let $\xi_2 := \beta$ and extend ξ_2 to a basis $\{\xi_0, \ldots, \xi_5\}$ of V as above. Let λ_2 be the 1-PS's of $G_{\mathcal{F}_2}$ which is diagonal in the chosen basis and is indicized by $(0, 0, 1, 0)$. Explicitly

$$\lambda_2(s) = \mathrm{diag}(1, 1, s, s^{-1}, 1, 1). \tag{7.1.8}$$

Then $\mu(A, \lambda_2) \geq 0$ - see Tables (28) and (29). Now suppose that Item (3) holds. Let $\xi_0 := \alpha$, $\xi_2 := \beta$ and $\xi_4 := \gamma$. Extend $\{\xi_0, \xi_2, \xi_4\}$ to a basis $\{\xi_0, \ldots, \xi_5\}$ as above: we require that $\xi_0 \wedge \xi_1 = v_0 \wedge v_1$ and $\xi_2 \wedge \xi_3 = v_2 \wedge v_3$. Let λ_3 be the 1-PS's of $G_{\mathcal{F}_2}$ which is diagonal in the chosen basis and is indicized by $(0, 3, 0, 3)$. Explicitly

$$\lambda_3(s) = \mathrm{diag}(s^3, s^{-3}, 1, 1, s^3, s^{-3}). \tag{7.1.9}$$

Then $\mu(A, \lambda_3) \geq 0$ - see Tables (28) and (29). Now suppose that Item (4) holds. We may assume that Item (1) does not hold. Thus there exists an isomorphism $\varphi \colon V_{01} \xrightarrow{\sim} V_{45}$ such that

$$A_1 = \{v_0 \wedge v_1 \wedge \varphi(\alpha) + \alpha \wedge v_2 \wedge v_3 \mid \alpha \in V_{01}\}. \tag{7.1.10}$$

Assume first that there exists $0 \neq \alpha \in V_{01}$ such that $\dim(A_2 \cap [\alpha] \wedge V_{23} \wedge V_{45}) \geq 2$. Let $\xi_0 := \alpha$ and $\xi_4 := \varphi(\alpha)$. We extend $\{\xi_0, \xi_4\}$ to a basis $\{\xi_0, \ldots, \xi_5\}$ as above: we require that $\xi_0 \wedge \xi_1 = v_0 \wedge v_1$ and $\xi_2 \wedge \xi_3 = v_2 \wedge v_3$. Let λ_4^+ be the 1-PS's of $G_{\mathcal{F}_2}$ which is diagonal in the chosen basis and is indicized by $(1, 6, 0, 0)$. Then $\mu(A, \lambda_4^+) \geq 0$ - see Tables (28) and (29). Now assume that there exists $0 \neq \gamma \in V_{45}$ such that $\dim(A_2 \cap V_{01} \wedge V_{23} \wedge [\gamma]) \geq 2$. Let B be given by (7.1.6): then $\dim(B_2 \cap [\alpha] \wedge V_{23} \wedge V_{45}) \geq 2$ for a certain $0 \neq \alpha \in V_{01}$ and hence A is not $G_{\mathcal{F}_2}$-stable. More precisely let λ_4^- be the 1-PS's of $G_{\mathcal{F}_2}$ indicized by $(-1, 0, 0, 6)$: then $\mu(A, \lambda_4^-) \geq 0$. The 1-PS's $\lambda_4^{\pm}$ are given explicitly by

$$\lambda_4^+(s) = \mathrm{diag}(s^7, s^{-5}, s^{-2}, s^{-2}, s, s), \quad \lambda_4^-(s) = \mathrm{diag}(s^{-1}, s^{-1}, s^2, s^2, s^5, s^{-7}). \tag{7.1.11}$$

It remains to prove that if $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is not $G_{\mathcal{F}_2}$-stable then one of Items (1) - (4) holds. We will run the Cone Decomposition algorithm. We choose the maximal torus $T < G_{\mathcal{F}_2}$ to be

$$T = \{(u, \mathrm{diag}(s_1, s_1^{-1}), \mathrm{diag}(s_2, s_2^{-1}), \mathrm{diag}(s_3, s_3^{-1}))) \mid u, s_i \in \mathbb{C}^\times\}. \tag{7.1.12}$$

(The maps are diagonal with respect to the bases $\{\xi_0, \xi_1\}$, $\{\xi_2, \xi_3\}$, $\{\xi_4, \xi_5\}$.) Thus

$$\check{X}(T)_{\mathbb{R}} = \{(m, r_1, r_2, r_3) \mid m, r_i \in \mathbb{R}\}, \quad C = \{(m, r_1, r_2, r_3) \mid r_i \geq 0\}$$

with notation as in (7.1.3). Looking at (6.6.4), (7.1.4) and (7.1.5) we get that $H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and only if it is the kernel of one of the following linear functions:

$$r_i, \quad r_i - r_j, \quad r_i - r_j - r_k \ (j \neq k), \quad r_1 - r_3 + 6m, \quad r_1 - r_3 - 6m, \quad r_1 + r_3 + 6m, \quad r_1 + r_3 - 6m.$$

In particular the hypotheses of **Proposition 2.3.4** are satisfied. It follows that the ordering rays are generated by vectors (m, r_1, r_2, r_3) such that $m \in \{0, \pm 1\}$ and

$$(r_1,r_2,r_3) \in \{(0,0,0),\ (0,1,0),\ (6,0,0),\ (0,0,6),\ (6,6,0),\ (0,6,6),\ (3,0,3),\ (3,3,3),\ (3,6,3),\ (12,6,6),\ (6,6,12),\ (4,2,2),\ (2,2,4)\}.$$

Actually the ordering 1-PS with $m = 0$ are $(0, 0, 1, 0)$, $(0, 3, 0, 3)$, $(0, 3, 3, 3)$ and $(0, 3, 6, 3)$ while all combinations of $m = \pm 1$ and the (r_1, r_2, r_3) listed above occur. By the self-duality of $\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ that we discussed above it suffices to prove that if $\mu(A, \lambda) \geq 0$ for an ordering 1-PS λ with $m \in \{0, 1\}$ then A satisfies one of Items (1)-(4). In other words it suffices to check that if none of Items (1)-(4) is satisfied then $\mu(A, \lambda) < 0$ for all ordering 1-PS λ with $m \in \{0, 1\}$. One gets the above statement for all ordering 1-PS, with the exception of the one indicized by $(0, 0, 1, 0)$, by consulting the last column of Tables (28) and of Table (29). It remains to exclude the existence of A such that $d^\lambda(A_0) = 0$

and $d^{\lambda}(A_2) \geq 3$ for λ indicized by $(0,0,1,0)$. By hypothesis Item (1) is not satisfied: it follows that the subset of $\mathbb{P}(V_{01}) \times \mathbb{P}(V_{45})$ defined by

$$\{([\alpha],[\gamma]) \mid (\alpha \wedge v_2 \wedge v_3 + v_0 \wedge v_1 \wedge \gamma) \in A_1\} \tag{7.1.13}$$

is a curve (a conic if we embed $\mathbb{P}(V_{01}) \times \mathbb{P}(V_{45})$ via the Segre map). On the other hand the subset of $\mathbb{P}(V_{01}) \times \mathbb{P}(V_{45})$ defined by

$$\{([\alpha],[\gamma] \mid (\alpha \wedge \xi_2 \wedge \gamma) \in A_2\} \tag{7.1.14}$$

is a curve or all of $\mathbb{P}(V_{01}) \times \mathbb{P}(V_{45})$ (look at the second row of Table (28) and recall that $d^{\lambda}(A_2) \geq 3$). Thus there is point $([\alpha],[\beta])$ of intersection between (7.1.13) and (7.1.14), i.e. A satisfies Item (3) (with $\beta = \xi_2$), and that is a contradiction. □

Corollary 7.1.2. *The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is $G_{\mathcal{F}_2}$-stable.*

Proof. It suffices to show that the generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ satisfies none of Items (1)-(4) of **Proposition 7.1.1**. A dimension count shows that the set of A's satisfying Item (1) or (2) has codimension (at least) 1, and the set of A's satisfying Item (3) or (4) has codimension (at least) 2. □

Proposition 7.1.3. *Let $\lambda_1^{\pm}$, $\lambda_2^{\pm}$, λ_3 and λ_4 be the 1-PS's of $G_{\mathcal{F}_2}$ defined by* (7.1.7), (7.1.8), (7.1.9) *and* (7.1.11) *respectively. Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is properly $G_{\mathcal{F}_2}$-semistable. Then A is $G_{\mathcal{F}_2}$-equivalent to $A' \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ satisfying one of the following conditions:*

(1') A' is $\lambda_1^{\pm}$-split and $d^{\lambda_1^{\pm}}(A'_1) = (1,1)$.

(2') A' is λ_2-split, $d^{\lambda_2}(A'_0) = (1,0)$ and $d^{\lambda_1^{\pm}}(A'_2) = (1,3)$ (non-reduced type).

(3') A' is λ_3-split, $d^{\lambda_3}(A'_0) = (1,0)$, $d^{\lambda_3}(A'_1) = (1,1)$ and $d^{\lambda_3}(A'_2) = (1,2,1)$ (non-reduced type).

(4') A' is $\lambda_4^{\pm}$-split, $d^{\lambda_4^{\pm}}(A'_1) = (1,1)$ and $d^{\lambda_4^{\pm}}(A'_2) = (2,2)$ (non-reduced type).

Proof. Follows from the proof of **Proposition 7.1.1** together with the observation that the types indicated above are those for which the numerical function $\mu(A,\cdot)$ is equal to 0 (i.e. not > 0). □

The proof of the above proposition gives also the following observation.

Remark 7.1.4. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ be $G_{\mathcal{F}_2}$-semistable. If Item (1) of **Proposition 7.1.1** holds then either $\dim A_1 \cap (V_{01} \wedge \bigwedge^2 V_{23}) = 1$ or $\dim A_1 \cap (\bigwedge^2 V_{01} \wedge V_{45}) = 1$. If Item (2) of **Proposition 7.1.1** holds then θ is unique up to rescaling.

Below we will prove a result on $C_{W,A}$ for certain semistable $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ (in **Subsection 7.2** we will examine $C_{W,A}$ for arbitrary semistable $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ with minimal orbit). Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$; there exists $\beta_0 \in V_{23}$ well-defined up to rescaling such that

$$A_0 = [v_0 \wedge v_1 \wedge \beta_0], \quad A_4 = [\beta_0 \wedge v_4 \wedge v_5]. \tag{7.1.15}$$

We set

$$W_\infty := \langle v_0, v_1, \beta_0 \rangle, \qquad W_0 := \langle v_4, v_5, \beta_0 \rangle. \tag{7.1.16}$$

Proposition 7.1.5. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ be $G_{\mathcal{F}_2}$-semistable with closed orbit and suppose that Item (1) of* **Proposition 7.1.1** *holds. Let $W \in \Theta_A$. Then $C_{W,A}$ is a semistable sextic curve of Type II-2 or of Type III-2.*

Proof. By **Proposition 7.1.3** we know that A is $G_{\mathcal{F}_2}$-equivalent to A' which is $\lambda_1^{\pm}$-split with $d^{\lambda_1^{\pm}}(A') = (1,1)$. Since A has closed orbit we may assume that $A' = A$. Let $\{\xi_0,\ldots,\xi_5\}$ be the basis of V introduced in the proof of **Proposition 7.1.1**. If A is λ_1^{+}-split we get that there exists $0 \neq \gamma \in V_{45}$ such A contains $\xi_0 \wedge \xi_1 \wedge \gamma$, if A is λ_1^{-}-split there exists $0 \neq \alpha \in V_{01}$ such A contains $\alpha \wedge \xi_4 \wedge \xi_5$. Let β_0 be as in (7.1.15): then A contains $\xi_0 \wedge \xi_1 \wedge \beta_0$ and $\beta_0 \wedge \xi_4 \wedge \xi_5$. It follows that $A \in \mathbb{B}^*_{\mathcal{F}_1}$: thus the proposition follows from **Proposition 6.6.3** and **Proposition 6.6.4**. □

Corollary 7.1.6. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ be $G_{\mathcal{F}_2}$-semistable and suppose that Item (1) of* **Proposition 7.1.1** *holds. Let $W \in \Theta_A$. Then $C_{W,A}$ is a semistable sextic curve and the period map* (0.0.3) *is regular at $C_{W,A}$.*

Proof. By contradiction. Suppose that $C_{W,A}$ is either $\mathbb{P}(W)$ or a sextic curve in the indeterminacy locus of the period map (0.0.3). Let $A' \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ be $G_{\mathcal{F}_2}$-semistable with closed orbit and $G_{\mathcal{F}_2}$-equivalent to A: thus A' belongs to the closure of $G_{\mathcal{F}_2}A$. It follows that there exists $W' \in \Theta_{A'}$ such that $C_{W',A'}$ is either $\mathbb{P}(W')$ or a sextic curve in the indeterminacy locus of the period map (0.0.3) (for $W = W'$): that contradicts **Proposition 7.1.5**. □

7.2 $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$

7.2.1 Set-up and statement of the main results

Let U be a complex vector-space of dimension 4 and choose an isomorphism

$$\psi\colon \bigwedge^2 U \xrightarrow{\sim} V. \tag{7.2.1}$$

Let $\{u_0, u_1, u_2, u_3\}$ be a basis of U and F the basis of V given by

$$v_0 = u_0 \wedge u_1,\ v_1 = u_0 \wedge u_2,\ v_2 = u_0 \wedge u_3,\ v_3 = u_1 \wedge u_2,\ v_4 = u_1 \wedge u_3,\ v_5 = u_2 \wedge u_3. \tag{7.2.2}$$

Consider the action of $\mathbb{C}^\times$ on $\mathbb{P}(U)$ defined by $g(t) := \operatorname{diag}(t, 1, 1, t^{-1})$ in the basis $\{u_0, u_1, u_2, u_3\}$: then

$$\bigwedge^2 g(t) = \lambda_{\mathcal{F}_2}(t). \tag{7.2.3}$$

Let $D \subset \mathbb{P}(U)$ be the smooth conic

$$D := \{[\lambda^2 u_0 + \lambda\mu u_1 + \mu^2 u_3] \mid [\lambda, \mu] \in \mathbb{P}^1\}. \tag{7.2.4}$$

(no misprint, the vectors are u_0, u_1 and u_3) and $i_+\colon \mathbb{P}(U) \hookrightarrow \operatorname{Gr}(3, V)$ be the map of (2.4.11): then $i_+(D)$ is an irreducible curve (of Type **Q** according to the classification of [28]) parametrizing pairwise incident projective planes. Next let

$$\mathbb{W}^\psi := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \Theta_A \supset i_+(D)\}. \tag{7.2.5}$$

Let $t \in \mathbb{C}^\times$: then D is sent to itself by $g(t)$ and hence $\lambda_{\mathcal{F}_2}(t)$ defines a projectivity of $\mathbb{P}(V)$ mapping $i_+(D)$ to itself. It follows that $\bigwedge^{10} \lambda_{\mathcal{F}_2}$ defines an action of $\mathbb{C}^\times$ on the affine cone over $\mathbb{W}^\psi$. Let

$$\mathbb{W}^\psi_{\text{fix}} := \{A \in \mathbb{W}^\psi \mid \bigwedge^{10} A \text{ is in the fixed locus of } \bigwedge^{10} \lambda_{\mathcal{F}_2}(t) \text{ for all } t \in \mathbb{C}^\times\}. \tag{7.2.6}$$

We claim that

$$\mathbb{W}^\psi_{\text{fix}} \subset \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}. \tag{7.2.7}$$

In fact let $A \in \mathbb{W}^\psi_{\text{fix}}$. Then $\bigwedge^{10} A$ is fixed by $\bigwedge^{10} \lambda_{\mathcal{F}_2}(t)$ for every $t \in \mathbb{C}^\times$: it follows that A is $\lambda_{\mathcal{F}_2}(t)$-split, say $A = (A_0 \bigoplus \ldots \bigoplus A_4)$, of reduced type $(2,0)$, $(1,2)$ or $(0,4)$. Now notice that

$$i_+([1,0,0,0]) = \langle v_0, v_1, v_2\rangle, \quad i_+([0,0,0,1]) = \langle v_4, v_5, v_2\rangle$$

and hence $v_0 \wedge v_1 \wedge v_2 \in A_0$ and $v_2 \wedge v_4 \wedge v_5 \in A_4$. Thus $\dim A_0 \geq 1$ and $\dim A_4 \geq 1$. It follows that A is of reduced type $(1,2)$ i.e. it belongs to $\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$. We have proved (7.2.7). Let $A \in \mathbb{W}^\psi_{\text{fix}}$: then

$$W_\infty = i_+([1,0,0,0]) = \langle v_0, v_1, v_2\rangle, \quad W_0 = i_+([0,0,0,1]) = \langle v_4, v_5, v_2\rangle. \tag{7.2.8}$$

In particular, letting β_0 be is as in (7.1.16), we may set

$$\beta_0 = v_2. \tag{7.2.9}$$

Let $\{X_0, X_1, X_2\}$ be the basis of $W_\infty^\vee$ dual to the basis $\{v_0, v_1, v_2\}$. Write $C_{W_\infty,A} = V(P_\infty)$ where $P_\infty \in \mathbb{C}[X_0, X_1, X_2]_6$. Since $\lambda_{\mathcal{F}_2}$ acts trivially on $\bigwedge^{10} A$ and it maps W_∞ to itself we may apply **Claim 3.2.4**: it follows that P_∞ is fixed by every element of $\{\mathrm{diag}(t, t, t^{-2})\}$. Thus

$$C_{W_\infty,A} = V(F_\infty X_2^2), \qquad F_\infty \in \mathbb{C}[X_0, X_1]_4. \tag{7.2.10}$$

Next we notice the following. Let

$$\Lambda := \mathbb{P}(\psi(\bigwedge^2 \langle u_0, u_1, u_3\rangle)) = \mathbb{P}(\langle v_0, v_2, v_4\rangle) \subset \mathbb{P}(V).$$

Given $p \in D$ let $W(p) = i_+(p)$. The projective plane Λ intersects $\mathbb{P}(W(p))$ in the line $L_{W(p)} \subset \mathbb{P}(W(p))$ parametrizing lines contained in $\mathbb{P}\langle u_0, u_1, u_3\rangle$ and containing p: each such line, with the exception of the line tangent to D, is parametrized by the intersection (in $\mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V)$) $\mathbb{P}(W(p)) \cap \mathbb{P}(W(q))$ for a suitable $q \in (D\backslash\{p\})$. By **Corollary 3.3.7** it follows that $C_{W(p),A}$ is singular along $L_{W(p)}$ (or $C_{W(p),A} = \mathbb{P}(W)$). Now we consider $W_\infty = W([1, 0, 0, 0])$: then $L_{W_\infty} = V(X_1)$ and recalling (7.2.10) we get that

$$C_{W_\infty,A} = V((a_2X_0^2 + a_3X_0X_1 + a_4X_1^2)X_1^2X_2^2). \tag{7.2.11}$$

We let

$$\mathbb{X}^\psi := \{A \in \mathbb{W}^\psi_{\mathrm{fix}} \mid C_{W_\infty,A} = V((a_3X_0X_1 + a_4X_1^2)X_1^2X_2^2)\}. \tag{7.2.12}$$

Thus $A \in \mathbb{X}^\psi$ if and only if $C_{W_\infty,A}$ is not a semistable sextic in the regular locus of the period map (0.0.3).

Definition 7.2.1. $\mathfrak{X}_{\mathcal{V}} \subset \mathfrak{M}$ is the set of points represented by semistable lagrangians $A \in \mathbb{X}^\psi$ (of course $\mathfrak{X}_{\mathcal{V}}$ is independent of ψ).

By definition we have $\mathfrak{X}_{\mathcal{V}} \subset \mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$.

Remark 7.2.2. The smooth quadric $Z \subset \mathbb{P}(U)$ given by

$$Z := \{[\eta_0u_0 + \eta_1u_1 + \eta_2u_2 + \eta_3u_3] \mid \eta_0\eta_3 - \eta_1^2 = 0\}$$

is left invariant by $g(t)$ for every $t \in \mathbb{C}^\times$ and contains D. It follows that if $A \in \mathbb{X}^*_{\mathcal{W}}(U)$ then there exists $g \in \mathrm{PGL}(V)$ such that $gA \in \mathbb{W}^\psi_{\mathrm{fix}}$ and hence $gA \in \mathbb{X}^\psi$. This proves that $\mathfrak{X}_{\mathcal{W}} \subset \mathfrak{X}_{\mathcal{V}}$.

Below is the main result of the present subsection - it is obtained by putting together **Proposition 7.2.6** and **Subsubsection 7.2.7**.

Proposition 7.2.3. *$\mathfrak{X}_{\mathcal{V}}$ is irreducible of dimension 3, and it is equal to $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$.*

7.2.2 The 3-fold swept out by the projective planes parametrized by $i_+(D)$

We will examine the curve $\Theta := i_+(D)$ and the variety

$$R_\Theta := \bigcup_{W\in\Theta} \mathbb{P}(W). \tag{7.2.13}$$

Let $\{W_1, -Z_2, W_3, Z_3, W_2, Z_1\}$ be the basis of $V^\vee$ dual to the basis F of (7.2.2): thus

$$v = W_1v_0 - Z_2v_1 + W_3v_2 + Z_3v_3 + W_2v_4 + Z_1v_5. \tag{7.2.14}$$

Let W, Z be the column vectors with entries W_1, W_2, W_3 and Z_1, Z_2, Z_3 respectively. Let

$$B := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The Plücker quadratic relation is $W^t \cdot Z = 0$ and we have

$$R_\Theta = V(W^t \cdot Z) \cap V(Z^t \cdot B \cdot Z).$$

Thus

$$|\mathcal{I}_{R_\Theta}(2)| = \mathbb{P}(\langle Q_0, Q_\infty\rangle), \quad Q_0 := V(W^t \cdot Z),\ Q_\infty := V(Z^t \cdot B \cdot Z). \tag{7.2.15}$$

Remark 7.2.4. It follows from (7.2.15) that there is a unique singular quadric containing R_Θ, namely Q_∞.

We will describe $\mathrm{Aut}(R_\Theta) < \mathrm{PGL}(V)$. Let $g \in \mathrm{Aut}(R_\Theta)$. Then $g(Q_\infty) = Q_\infty$ because of **Remark 7.2.4**. It follows that $g(V(Z_1, Z_2, Z_3)) = V(Z_1, Z_2, Z_3)$ and hence

$$f^* \begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} L & M \\ 0_3 & N \end{pmatrix} \cdot \begin{pmatrix} W \\ Z \end{pmatrix} \tag{7.2.16}$$

where L, M, N are 3×3 matrices, 0_3 is the 3×3 zero matrix. Equation (7.2.15) gives that

$$N^t \cdot B \cdot N = \mu B, \quad L^t \cdot N = \nu 1_3, \quad M^t \cdot N = \tau B + P, \qquad \mu, \nu, \tau \in \mathbb{C}, \quad P^t = -P. \tag{7.2.17}$$

The intersection $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ acts on $\mathbb{W}^\psi_{\mathrm{fix}}$. It follows from (7.2.17) that the elements of $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ are represented by matrices

$$\begin{pmatrix} a^{-2} & 0 & 0 & 0 & m_1 & 0 \\ 0 & b^{-2} & 0 & m_2 & 0 & 0 \\ 0 & 0 & a^{-1}b^{-1} & 0 & 0 & m_3 \\ 0 & 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & ab \end{pmatrix}, \qquad a^2 m_1 + b^2 m_2 + ab m_3 = 0. \tag{7.2.18}$$

In particular

$$\dim \mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2} = 3. \tag{7.2.19}$$

Claim 7.2.5. *Let $Q, Q' \in |\mathcal{I}_{R_\Theta}(2)|$ be smooth quadrics and $h \in \mathrm{Aut}(\Theta)$. There exists $g \in \mathrm{Aut}(R_\Theta)$ such that $g(Q) = Q'$ and the automorphism $\overline{g} \in \mathrm{Aut}(\Theta)$ induced by g is equal to h.*

Proof. Let $Q_s := V(W^t \cdot Z + s Z^t \cdot B \cdot Z)$ - the notation is consistent with (7.2.15). Thus $Q_s \in |\mathcal{I}_{R_\Theta}(2)|$ is a smooth quadric and conversely every smooth quadric in $|\mathcal{I}_{R_\Theta}(2)|$ is equal to Q_s for some $s \in \mathbb{C}$. Let $g_s \in \mathrm{PGL}(V)$ be such that

$$g_s^* W_1 = W_1 + 2sZ_2, \quad g_s^* W_2 = W_2, \quad g_s^* W_3 = W_3 - 2sZ_3, \qquad g_s^* Z_i = Z_i.$$

Then $g_s \in \mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ (it corresponds to $a = b = 1$, $m_1 = 2s$, $m_2 = 0$ and $m_3 = -2s$ in (7.2.18)) and $g_s^*(Q_0) = (Q_s)$. To finish the proof it suffices to notice that every $\varphi \in \mathrm{Aut}(D)$ extends to an automorphism of $\mathbb{P}(U)$ and hence it induces a projectivity of $\mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V)$ sending R_Θ to itself. □

7.2.3 Explicit description of $\mathbb{W}^\psi_{\mathrm{fix}}$.

First we explain Table (10). Let $\langle\langle i_+(D) \rangle\rangle \subset A_+(U)$ be the span of the affine cone over $i_+(D)$. Going through Table (7) one gets that a basis of $\langle\langle i_+(D) \rangle\rangle$ is given by the first five entries of Table (10). It follows by a straightforward computation that the elements of Table (10) form a basis of $i_+(D)^\perp$. Notice that each such element spans a subspace invariant under the action of $\lambda_{\mathcal{F}_2}(t)$ for $t \in \mathbb{C}^\times$: the corresponding character of $\mathbb{C}^\times$ is contained in the third column of Table (10). Let $P_D \subset A_+(U)$ be the subspace spanned by the elements of Table (15) which belong to lines 6 through 10 and $Q_D \subset A_-(U)$ be the subspace spanned by the elements of Table (15) which belong to lines 11 through 15. Both P_D and Q_D are isotropic for $(,)_V$ and the symplectic form identifies one with the dual of the other; thus the restriction of $(,)_V$ to $P_D \oplus Q_D$ is a symplectic form. It follows that a lagrangian $A \in \mathbb{LG}(\bigwedge^3 V)$ contains $i_+(D)$ if and only if it is equal to $\langle\langle i_+(D) \rangle\rangle \oplus R$ where $R \in \mathbb{LG}(P_D \oplus Q_D)$. Let $P^0_D \subset P_D$ and $Q^0_D \subset Q_D$ be the subspaces of elements which are invariant for $\lambda_{\mathcal{F}_2}$ i.e. the spaces spanned by the elements on rows 7 through 9 and 12 through 14 of Table (10) respectively. The symplectic form $(,)_V$ identifies P^0_D with the dual of Q^0_D and the restriction of $(,)_V$ to $P^0_D \oplus Q^0_D$ is a symplectic form: we let $\mathbb{LG}(P^0_D \oplus Q^0_D)$ be the corresponding symplectic grassmannian. Let $\mathbf{c} = [c_0, c_1] \in \mathbb{P}^1$ and $\mathbf{L} \in \mathbb{LG}(P^0_D \oplus Q^0_D)$; we let

$$R_{\mathbf{c}} := \langle c_0 \alpha_{(1,0,1,0)} + c_1 \beta_{(0,0,1,1)}, c_0 \alpha_{(0,0,1,1)} + c_1 \beta_{(1,0,1,0)} \rangle. \tag{7.2.20}$$

Table 10: Bases of $\langle\langle i_+(D)\rangle\rangle$ and of $\langle\langle i_+(D)\rangle\rangle^\perp$.

α-β notation	explicit expression	action of $\lambda_{\mathcal{F}_2}(t)$
$\alpha_{(2,0,0,0)}$	$v_0 \wedge v_1 \wedge v_2$	t^2
$\alpha_{(1,1,0,0)}$	$v_0 \wedge (v_1 \wedge v_4 - v_2 \wedge v_3)$	t
$\alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)}$	$v_0 \wedge v_2 \wedge v_5 + v_0 \wedge v_3 \wedge v_4 - v_1 \wedge v_2 \wedge v_4$	1
$\alpha_{(0,1,0,1)}$	$v_0 \wedge v_4 \wedge v_5 + v_2 \wedge v_3 \wedge v_4$	t^{-1}
$\alpha_{(0,0,0,2)}$	$v_2 \wedge v_4 \wedge v_5$	t^{-2}
$\alpha_{(1,0,1,0)}$	$v_0 \wedge v_1 \wedge v_5 - v_1 \wedge v_2 \wedge v_3$	t
$\alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}$	$-v_0 \wedge v_2 \wedge v_5 + v_0 \wedge v_3 \wedge v_4 + v_1 \wedge v_2 \wedge v_4$	1
$\alpha_{0,1,1,0)}$	$v_0 \wedge v_3 \wedge v_5 + v_1 \wedge v_3 \wedge v_4$	1
$\alpha_{(0,0,2,0)}$	$v_1 \wedge v_3 \wedge v_5$	1
$\alpha_{(0,0,1,1)}$	$v_1 \wedge v_4 \wedge v_5 + v_2 \wedge v_3 \wedge v_5$	t^{-1}
$\beta_{(0,0,1,1)}$	$-v_0 \wedge v_1 \wedge v_4 - v_0 \wedge v_2 \wedge v_3$	t
$2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)}$	$v_0 \wedge v_3 \wedge v_5 + 2v_1 \wedge v_2 \wedge v_5 - v_1 \wedge v_3 \wedge v_4$	1
$\beta_{(0,1,1,0)}$	$-v_0 \wedge v_2 \wedge v_5 - v_1 \wedge v_2 \wedge v_4$	1
$\beta_{(0,0,2,0)}$	$4v_0 \wedge v_2 \wedge v_4$	1
$\beta_{(1,0,1,0)}$	$v_0 \wedge v_4 \wedge v_5 - v_2 \wedge v_3 \wedge v_4$	t^{-1}

$$A_{\mathbf{c},\mathbf{L}} := \langle\langle i_+(D)\rangle\rangle \oplus R_{\mathbf{c}} \oplus \mathbf{L}. \tag{7.2.21}$$

Looking at the action of $\lambda_{\mathcal{F}_2}(t)$ on the given bases of P_D and Q_D one gets that

$$\mathbb{W}^{\psi}_{\text{fix}} = \{A_{\mathbf{c},\mathbf{L}} \mid (\mathbf{c},\mathbf{L}) \in \mathbb{P}^1 \times \mathbb{LG}(P^0_D \oplus Q^0_D)\}. \tag{7.2.22}$$

7.2.4 $\mathfrak{X}_{\mathcal{V}}$ is irreducible of dimension 3

The main result of the present subsubsection is the following.

Proposition 7.2.6. *$\mathfrak{X}_{\mathcal{V}}$ is irreducible of dimension* 3*.*

The proof will be given at the end of the subsubsection. Let $\mathcal{U} \subset \mathbb{LG}(P^0_D \oplus Q^0_D)$ be the dense open subset of $\mathbf{L}$ such that $\mathbf{L} \cap Q^0_D = \{0\}$. Let

$$\begin{aligned}\mathbf{L}_M := \langle &\alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)} + m_{11}(2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)}) + 2m_{12}\beta_{(0,1,1,0)} + 4m_{13}\beta_{(0,0,2,0)},\\ &\alpha_{(0,1,1,0)} + m_{12}(2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)}) + m_{22}\beta_{(0,1,1,0)} + 2m_{23}\beta_{(0,0,2,0)},\\ &\alpha_{(0,0,2,0)} + m_{13}(2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)}) + m_{23}\beta_{(0,1,1,0)} + 2m_{33}\beta_{(0,0,2,0)}\rangle\end{aligned} \tag{7.2.23}$$

where m_{ij} are arbitrary complex numbers - here M is the symmetric 3×3-matrix with entries the given m_{ij}'s. A straightforward computation (use the last column of Table (7)) gives that $\mathbf{L}_M \in \mathcal{U}$ and that conversely every $\mathbf{L} \in \mathcal{U}$ is equal to $\mathbf{L}_M$ for a unique M. Next recall that $A_{\mathbf{c},\mathbf{L}} \in \mathbb{W}^{\psi}_{\text{fix}}$ is sent to itself by the 1-PS $\lambda_{\mathcal{F}_2}$ and hence $A_{\mathbf{c},\mathbf{L}}$ decomposes as the direct sum of its weight subspaces: we let $A_{\mathbf{c},\mathbf{L}}(i) \subset A_{\mathbf{c},\mathbf{L}}$ be the weight-i subspace (thus $A_{\mathbf{c},\mathbf{L}}(i)$ is $A_{\mathbf{c},\mathbf{L},2-i}$ in the old notation). Tables (11), (12) and (13) give bases of $A_{\mathbf{c},\mathbf{L}_M}(i)$ for $i = 0, \pm 1$. A few explanations regarding notation: we denote $v_i \wedge v_j \wedge v_k$ by (ijk), we let ℓ_j be the j-th element of the basis of $\mathbf{L}_M$ given by (7.2.23). In order to determine whether $A_{\mathbf{c},\mathbf{L}} \in \mathbb{W}^{\psi}_{\text{fix}}$ belongs to $\mathbb{X}^{\psi}$ we will analyze $C_{W_\infty, A_{\mathbf{c},\mathbf{L}}}$ in a neighborhood of $[v_0 + v_2]$. The first step is the computation of $F_{v_0+v_2} \cap A_{\mathbf{c},\mathbf{L}}$. Notice that

$$(F_{v_0+v_2} \cap A_{\mathbf{c},\mathbf{L}_M}) \supset \langle \alpha_{(2,0,0,0)}, \alpha_{(1,1,0,0)} + \alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)} + \alpha_{(0,1,0,1)} + \alpha_{(0,0,0,2)}\rangle. \tag{7.2.24}$$

(Of course (7.2.24) holds also if $\mathbf{L}_M$ is replaced by an arbitrary element of $\mathbb{LG}(P^0_D \oplus Q^0_D)$.)

Table 11: Basis of $A_{\mathbf{c},\mathbf{L}_M}(0)$.

(024)	(025)	(034)	(035)	(124)	(125)	(134)	(135)	element of basis
0	1	1	0	-1	0	0	0	$\alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)}$
$16m_{13}$	$-2m_{12}-1$	1	m_{11}	$-2m_{12}+1$	$2m_{11}$	$-m_{11}$	0	ℓ_1
$8m_{23}$	$-m_{22}$	0	$m_{12}+1$	$-m_{22}$	$2m_{12}$	$-m_{12}+1$	0	ℓ_2
$8m_{33}$	$-m_{23}$	0	m_{13}	$-m_{23}$	$2m_{13}$	$-m_{13}$	1	ℓ_3

Table 12: Basis of $A_{\mathbf{c},\mathbf{L}}(1)$.

(014)	(015)	(023)	(123)	element of basis
1	0	-1	0	$\alpha_{(1,1,0,0)}$
$-c_1$	c_0	$-c_1$	$-c_0$	$c_0\alpha_{(1,0,1,0)} + c_1\beta_{(0,0,1,1)}$

Lemma 7.2.7. *Keep notation as above. If $c_0 m_{11} \neq 0$ then right-hand side and left-hand side of* (7.2.24) *are equal. On the other hand*

$$F_{v_0+v_2} \cap A_{[0,1],\mathbf{L}_M} \supset \langle \alpha_{(2,0,0,0)}, \alpha_{(1,1,0,0)} + \alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)} + \alpha_{(0,1,0,1)} + \alpha_{(0,0,0,2)}, \\ \alpha_{(1,1,0,0)} + \beta_{(1,1,0,0)}, \alpha_{(0,1,0,1)} + 2\alpha_{(0,0,0,2)} + \beta_{(1,0,1,0)} \rangle. \quad (7.2.25)$$

Proof. By (7.2.24) the first two elements spanning the right-hand side of (7.2.25) are contained in $A_{[0,1],\mathbf{L}_M}$. On the other hand the third and fourth element are contained in $A_{[0,1],\mathbf{L}_M}$ because $\beta_{(1,1,0,0)}, \beta_{(1,0,1,0)} \in A_{[0,1],\mathbf{L}_M}$. Thus the right-hand side of (7.2.25) is contained in $A_{[0,1],\mathbf{L}_M}$. Looking at Table (10) we get that the right-hand side of (7.2.25) is contained in $F_{v_0+v_2}$ as well: this proves that (7.2.25) holds. Now suppose that $c_0 m_{11} \neq 0$. Let $\gamma \in A_{\mathbf{c},\mathbf{L}_M}$. Write $\gamma = \sum_i \gamma(i)$ where $\gamma(i) \in A_{\mathbf{c},\mathbf{L}_M}(i)$, i.e. $\lambda_{\mathcal{F}_2}(t)\gamma(i) = t^i\gamma(i)$. Then $\gamma \in F_{v_0+v_2}$ if and only if $(v_0+v_2)\wedge\gamma = 0$. Now $v_0 \in A_{\mathbf{c},\mathbf{L}_M}(1)$ and $v_2 \in A_{\mathbf{c},\mathbf{L}_M}(0)$: it follows that $\gamma \in F_{v_0+v_2} \cap A_{\mathbf{c},\mathbf{L}_M}$ if and only if

$$0=v_2\wedge\gamma(-2)=v_0\wedge\gamma(-2)+v_2\wedge\gamma(-1)=v_0\wedge\gamma(-1)+v_2\wedge\gamma(0)=v_0\wedge\gamma(0)+v_2\wedge\gamma(1)=v_0\wedge\gamma(1)+v_0\wedge\gamma(2)=v_2\wedge\gamma(2). \quad (7.2.26)$$

Now let $\gamma \in F_{v_0+v_2} \cap A_{\mathbf{c},\mathbf{L}_M}$: we will show that γ belongs to the right-hand side of (7.2.24). Subtracting from γ a suitable multiple of $\alpha_{(2,0,0,0)}$ we might assume that $\gamma(2) = 0$. By (7.2.26) we get that $v_0 \wedge \gamma(1) = 0$; since $c_0 \neq 0$ it follows that $\gamma(1) \in \langle \alpha_{(1,1,0,0)} \rangle$ - see Table (12). Subtracting a suitable multiple of the second element appearing in the right-hand side of (7.2.24) we may assume that $\gamma(1) = 0$: we must prove that $\gamma = 0$. By (7.2.26) we get that $v_0 \wedge \gamma(0) = 0$; a straightforward computation - see Table (11) - gives that $\gamma(0) = 0$ (recall that by hypothesis $m_{11} \neq 0$). By (7.2.26) we get that $v_0 \wedge \gamma(-1) = 0$, this implies that $\gamma(-1) = 0$ - see Table (13). By (7.2.26) we get that $v_0 \wedge \gamma(-2) = 0$ and hence $\gamma(-2) = 0$ because $\gamma(-2) \in \langle v_2 \wedge v_4 \wedge v_5 \rangle$. This proves that $\gamma = 0$. □

Proposition 7.2.8. *Let $c_1 \in \mathbb{C}$. Then $A_{[1,c_1],\mathbf{L}} \notin \mathbb{X}^\psi$ for generic $\mathbf{L} \in \mathbb{LG}(P_D^0 \oplus Q_D^0)$.*

Proof. We will analyze $C_{W_\infty, A_{[1,c_1],\mathbf{L}_M}}$ in a neighborhood of $[v_0+v_2]$. Let

$$V_0 := \langle v_0, v_1, v_3, v_4, v_5 \rangle.$$

Table 13: Basis of $A_{\mathbf{c},\mathbf{L}}(-1)$.

(045)	(145)	(234)	(235)	element of basis
1	0	1	0	$\alpha_{(0,1,0,1)}$
c_1	c_0	$-c_1$	c_0	$c_0\alpha_{(0,0,1,1)} + c_1\beta_{(1,0,1,0)}$

(No typo: we omit v_2 !) Going through Tables (11), (12) and (13) one gets that

$$\bigwedge^3 V_0 \cap A_{[1,c_1],\mathbf{L}_M} = \{0\} \text{ if } \det \begin{pmatrix} 2m_{13} & 2m_{12} & m_{11} \\ m_{23} & 2m_{22} & m_{12} \\ m_{33} & 2m_{23} & m_{13} \end{pmatrix} \neq 0. \tag{7.2.27}$$

The determinant appearing in (7.2.27) is not identically zero: we assume that M is such that the determinant does not vanish. We will also assume that $m_{11} \neq 0$ and hence the right-hand side and left-hand side of (7.2.24) are equal. The lagrangians $\bigwedge^3 V_0$ and $A_{[1,c_1],\mathbf{L}_M}$ are transverse because On the other hand we have a direct-sum decomposition $V = [v_0 + v_2] \oplus V_0$. Thus **Claim 3.4.2** applies. We adopt the notation of that claim: of course in the present context v_0 is $(v_0 + v_2)$ and $W_0 = W_\infty \cap V_0 = \langle v_0, v_1 \rangle$. **Claim 3.4.2** states that

$$C_{W_\infty, A_{[1,c_1],\mathbf{L}_M}} \cap (\mathbb{P}(W_\infty \setminus \mathbb{P}(W_0)) = V(\det(\overline{q}_{A_{[1,c_1],\mathbf{L}_M}} + z_0 \overline{q}_{v_0} + z_1 \overline{q}_{v_1}). \tag{7.2.28}$$

(Beware that the point with affine coordinates (z_0, z_1) is $(1 + z_0)v_0 + z_1 v_1 + v_2$.) Here $q_{A_{[1,c_1],\mathbf{L}_M}}$ is as in (3.4.4) and $\overline{q}_{A_{[1,c_1],\mathbf{L}_M}}$, $\overline{q}_{v_0}$, $\overline{q}_{v_1}$ are the quadratic forms on $\bigwedge^2 V_0 / \bigwedge^2 W_0$ given by (3.4.9). The kernel of $\overline{q}_{A_{[1,c_1],\mathbf{L}_M}}$ is as follows. First notice that

$$-(\alpha_{(2,0,0,0)} + \alpha_{(1,1,0,0)} + \alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)} + \alpha_{(0,1,0,1)} + \alpha_{(0,0,0,2)}) = (v_0 + v_2) \wedge (v_1 + v_3 - v_5) \wedge (v_0 - v_4).$$

By **Lemma 7.2.7** it follows that

$$\ker \overline{q}_{A_{[1,c_1],\mathbf{L}_M}} = \langle e_1 \rangle, \qquad e_1 := (v_1 + v_3 - v_5) \wedge (v_0 - v_4). \tag{7.2.29}$$

(The notation is somewhat sloppy: we mean that the kernel is generated by the image of e_1 in $\bigwedge^2 V_0 / \bigwedge^2 W_0$.) Since e_1 is a decomposable tensor we have $\overline{q}_{v_1}(e_1) = 0$ and hence by **Proposition A.1.2** we have

$$\det(\overline{q}_{A_{[1,c_1],\mathbf{L}_M}} + z_1 \overline{q}_{v_1}) = b_2 z_1^2 + b_3 z_1^3 + \ldots + b_6 z_1^6.$$

(Of course this agrees with (7.2.11).) We will show that $b_2 \neq 0$ for M generic and that will prove the proposition. We will apply **Proposition A.1.3** as reformulated in **Remark A.1.4**. In the case at hand $q_* = \overline{q}_{A_{[1,c_1],\mathbf{L}_M}}$ and $q = \overline{q}_{v_1}$. It follows that e_2 is such that

$$(v_0 + v_2) \wedge e_2 - v_1 \wedge (v_1 + v_3 - v_5) \wedge (v_0 - v_4) \in A_{[1,c_1],\mathbf{L}_M}.$$

(Once again notation is potentially confusing: $e_2 \in \bigwedge^2 V_0 / \bigwedge^2 W_0$ and is determined modulo $\langle e_1 \rangle$, we think of e_2 as an element of $\bigwedge^2 V_0$ determined modulo $\langle v_0 \wedge v_1, e_1 \rangle$.) By **Remark A.1.4** we get that $b_2 = 0$ if and only if

$$(v_0 + v_2) \wedge e_2 \wedge v_1 \wedge (v_1 + v_3 - v_5) \wedge (v_0 - v_4) = 0. \tag{7.2.30}$$

One computes e_2 by using Table (14). We explain Table (14). Let $\pi \colon \bigwedge^3 V \to \bigwedge^3 V_0$ be the projection determined by the direct-sum decomposition $\bigwedge^3 V = F_{v_0+v_2} \oplus \bigwedge^3 V_0$. Then $\pi(A_{[1,c_1],\mathbf{L}_M}) = \langle v_0 \wedge v_1, e_1 \rangle^\perp$, in particular $\pi(A_{[1,c_1],\mathbf{L}_M})$ is contained in the subspace generated by $v_i \wedge v_j \wedge v_k$ where $i < j < k$, $i, j, k \in \{0, 1, 3, 4, 5\}$ and $(i, j, k) \neq (3, 4, 5)$. Table (14) gives $\pi(\gamma)$ as linear combination of the $v_i \wedge v_j \wedge v_k$'s listed above for a collection of $\gamma \in A_{[1,c_1],\mathbf{L}_M}$ giving a basis of a subspace complementary to $F_{v_0+v_2} \cap A_{[1,c_1],\mathbf{L}_M}$. (The elements ℓ_1, ℓ_2, ℓ_3 are as in Table (11).) It follows from Table (14) that

$$e_2 = (c_1 + m_{22} - m_{11}^{-1} m_{12}(2m_{12} - 1))\alpha_{(1,1,0,0)} + (c_1 - m_{11}^{-1} m_{12})\alpha_{(0,1,0,1)} + (2c_1 - m_{11}^{-1} m_{12})\alpha_{(0,0,0,2)} + \\ + (\alpha_{(1,0,1,0)} + c_1 \beta_{(0,0,1,1)}) + (\alpha_{(0,0,1,1)} + c_1 \beta_{(1,0,1,0)}) - m_{11}^{-1} m_{12} \ell_1 + \ell_2. \tag{7.2.31}$$

Computing we get that (7.2.30) holds (assuming that $m_{11} \neq 0$ and the determinant appearing in (7.2.27) does not vanish) if and only if

$$2m_{12}^2 - m_{11} m_{22} - 2m_{11} c_1 = 0. \tag{7.2.32}$$

This proves that for generic M we have $A_{[1,c_1],\mathbf{L}_M} \notin \mathbb{X}^\psi$. □

Table 14: $\pi(A_{[1,c_1],\mathbf{L}_M}(0))$.

(013)	(014)	(015)	(034)	(035)	(045)	(134)	(135)	(145)	γ
0	1	0	0	0	0	0	0	0	$\alpha_{(1,1,0,0)}$
0	0	0	-1	0	1	0	0	0	$\alpha_{(0,1,0,1)}$
0	0	0	0	0	-1	0	0	0	$\alpha_{(0,0,0,2)}$
-1	$-c_1$	1	0	0	0	0	0	0	$\alpha_{(1,0,1,0)} + c_1\beta_{(0,0,1,1)}$
0	0	0	c_1	-1	c_1	0	1	0	$\alpha_{(0,0,1,1)} + c_1\beta_{(1,0,1,0)}$
0	$1-2m_{12}$	$2m_{11}$	1	m_{11}	0	$-m_{11}$	0	0	ℓ_1
0	$-m_{22}$	$2m_{12}$	0	$m_{12}+1$	0	$1-m_{12}$	0	0	ℓ_2
0	$-m_{23}$	$2m_{13}$	0	m_{13}	0	$-m_{13}$	1	0	ℓ_3

Corollary 7.2.9. *Keep notation as above. Then*

$$\mathbb{X}^{\psi} = \{A_{[0,1],\mathbf{L}} \mid \mathbf{L} \in \mathbb{LG}(P_D^0 \oplus Q_D^0)\} \cup \mathbb{X}_{\mathcal{V}}^{\psi} \tag{7.2.33}$$

where $\mathbb{X}_{\mathcal{V}}^{\psi}$ *is an irreducible divisor in* $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L}|$ *where* $\mathcal{L}$ *is the ample generator of the Picard group of* $\mathbb{LG}(P_D^0 \oplus Q_D^0)$ *(i.e. the Plücker line-bundle).*

Proof. One gets right away that $\mathbb{X}^{\psi}$ is the zero-locus of a section σ of $\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{L}$ - see (3.2.21) and (3.2.26). Moreover σ is not identically zero by **Proposition 7.2.8** and hence $\mathbb{X}^{\psi}$ is a divisor in $|\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{L}|$. By **Lemma 7.2.7** and **Corollary 3.2.3** the "vertical" divisor $\mathbb{V} \subset \mathbb{P}^1 \times \mathbb{LG}(P_D^0 \oplus Q_D^0)$ given by $c_0 = 0$ is an irreducible component of $\mathbb{X}^{\psi}$. Thus $\mathbb{X}^{\psi} = \mathbb{V} \cup \mathbb{X}_{\mathcal{V}}^{\psi}$ where $\mathbb{X}_{\mathcal{V}}^{\psi} \in |\mathcal{O}_{\mathbb{P}^1}(d) \boxtimes \mathcal{L}|$ with $d \leq 1$. Looking at (7.2.32) we get that in fact $d = 1$ and $\mathbb{X}_{\mathcal{V}}^{\psi}$ is irreducible. □

Remark 7.2.10. Let p_{ijk} for $1 \leq i < j < k \leq 6$ be homogeneous coordinates on $\mathbb{P}(\bigwedge^3 (P_D^0 \oplus Q_D^0))$ associated to the basis of $(P_D^0 \oplus Q_D^0)$ given by

$$\alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)},\ \alpha_{(0,1,1,0)},\ \alpha_{(0,0,2,0)},\ 2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)},\ \beta_{(0,1,1,0)},\ \beta_{(0,0,2,0)}.$$

Then **Corollary 7.2.9** and (7.2.32) give that $\mathbb{X}_{\mathcal{V}}^{\psi} \subset \mathbb{P}^1 \times \mathbb{LG}(P_D^0 \oplus Q_D^0))$ has equation

$$c_0 p_{345} - 2c_1 p_{234} = 0. \tag{7.2.34}$$

The following result shows that only the second component of (7.2.33) will contribute to $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$.

Proposition 7.2.11. *If* $\mathbf{L} \in \mathbb{LG}(P_D^0 \oplus Q_D^0)$ *then* $A_{[0,1],\mathbf{L}}$ *is unstable. On the other hand the generic* $A_{[c_0,c_1],\mathbf{L}} \in \mathbb{X}_{\mathcal{V}}^{\psi}$ *is* $G_{\mathcal{F}_2}$*-stable.*

Proof. We have $v_0 \wedge v_1 \wedge v_4, v_0 \wedge v_2 \wedge v_3 \in A_{[0,1],\mathbf{L}}(1)$ - see Table (12). Thus Item (1) of **Proposition 7.1.1** holds with $A = A_{[0,1],\mathbf{L}}$. On the other hand $C_{W_\infty, A_{[0,1],\mathbf{L}}}$ is not a sextic curve in the regular locus of the period map (0.0.3): by **Corollary 7.1.6** we get that $A_{[0,1],\mathbf{L}}$ is $G_{\mathcal{F}_2}$-unstable and hence unstable. Next we will prove that the generic $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}_{\mathcal{V}}^{\psi}$ is $G_{\mathcal{F}_2}$-stable. By **Proposition 7.1.1** it suffices to check that if $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}_{\mathcal{V}}^{\psi}$ is generic then none of Items (1) - (4) of **Proposition 7.1.1** holds. First Item (1) never holds (because $c_0 = 1$!). Item (2) holds if and only if $F_{v_2} \cap A_{[1,c_1],\mathbf{L}_M}(0) \neq \{0\}$; looking at Table (11) we get that Item (2) holds if and only if

$$0 = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & m_{11} & -m_{11} & 0 \\ 0 & m_{12}+1 & -m_{12}+1 & 0 \\ 0 & m_{13} & -m_{13} & 1 \end{pmatrix} = 2m_{11}.$$

On the other hand if M is generic and (7.2.32) holds then $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}_{\mathcal{V}}^{\psi}$: it follows that if $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}_{\mathcal{V}}^{\psi}$ is generic then Item (2) does not hold. Next we will show that if $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}_{\mathcal{V}}^{\psi}$ is generic then $A_{[1,c_1],\mathbf{L}_M}(0)$ contains no non-zero decomposable tensor: that will prove that neither Item (3) nor Item (4) holds. First notice that if $A \in \mathbb{W}^{\psi}_{\text{fix}}$ is generic then $\Theta_A = i_+(D)$: it follows that $A(0)$ contains no non-zero decomposable tensor. On the other hand Table (11) gives that the condition "$A_{\mathbf{c},\mathbf{L}_M}(0)$ contains a non-zero decomposable tensor "is independent of $\mathbf{c}$. It follows that if M is generic then for every choice of $\mathbf{c} \in \mathbb{P}^1$ we have that $A_{\mathbf{c},\mathbf{L}_M}(0)$ contains no non-zero decomposable tensors: choosing $c_0 = 1$ and c_1 such that (7.2.32) holds we get $A_{[1,c_1],\mathbf{L}_M} \in \mathbb{X}^{\psi}_{\mathcal{V}}$ such that $A_{[1,c_1],\mathbf{L}_M}(0)$ contains no non-zero decomposable tensors. □

Proof of **Proposition 7.2.6**. By **Proposition 7.2.11** every point of $\mathfrak{X}_{\mathcal{V}}$ is represented by a (semistable) point of $\mathbb{X}^{\psi}_{\mathcal{V}}$. Thus we have a surjection

$$\mathbb{X}^{\psi,ss}_{\mathcal{V}} \twoheadrightarrow \mathfrak{X}_{\mathcal{V}} \tag{7.2.35}$$

and hence $\mathfrak{X}_{\mathcal{V}}$ is irreducible because $\mathbb{X}^{\psi}_{\mathcal{V}}$ is irreducible. It remains to prove that $\dim \mathfrak{X}_{\mathcal{V}} = 3$. Let $\widetilde{\mathfrak{X}}_{\mathcal{V}} \subset (\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}//G_{\mathcal{F}_2})$ be the image of $\mathbb{X}^{\psi,ss}_{\mathcal{V}}$ under the quotient map $(\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2})^{ss} \twoheadrightarrow (\mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}//G_{\mathcal{F}_2})$. We have a natural factorization of Map (7.2.35):

$$\mathbb{X}^{\psi,ss}_{\mathcal{V}} \overset{\pi}{\twoheadrightarrow} \widetilde{\mathfrak{X}}_{\mathcal{V}} \overset{\varphi}{\twoheadrightarrow} \mathfrak{X}_{\mathcal{V}}.$$

The map φ is finite by **Claim 5.2.1**, and $\dim \mathbb{X}^{\psi}_{\mathcal{V}} = 6$ by **Corollary 7.2.9**; it follows that it suffices to show that the generic fiber of π has dimension 3. The open set $\mathbb{X}^{\psi,s}_{\mathcal{V}}$ parametrizing $G_{\mathcal{F}_2}$-stable A's is dense by **Proposition 7.2.11**. Let $A \in \mathbb{X}^{\psi,s}_{\mathcal{V}}$. By $G_{\mathcal{F}_2}$-stability we have

$$\pi^{-1}(\pi(A)) = \{A' \in \mathbb{X}^{\psi,s}_{\mathcal{V}} \mid A' = gA, \quad g \in G_{\mathcal{F}_2}\}. \tag{7.2.36}$$

We will show that the right-hand side has dimension 3. Let $\Theta = i_+(D)$ and let R_Θ be as in (7.2.13). The group $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ acts on $\mathbb{X}^{\psi,s}_{\mathcal{V}}$ with finite stabilizers: by (7.2.19) we get that the right-hand side of (7.2.36) has dimension at least 3. On the other hand $\dim \Theta_A = 1$ for $A \in \mathbb{X}^{\psi,s}_{\mathcal{V}}$. In fact suppose the contrary: by **Lemma 6.1.8** either $A \in \mathbb{X}^*_{\mathcal{W}}$ or it is in the $\mathrm{PGL}(V)$-orbit of A_k or A_h. By **Lemma 7.2.15** we get that $A \in \mathbb{X}^*_{\mathcal{W}}$ and hence A is properly $G_{\mathcal{F}_2}$-semistable, that is a contradiction. Let $A \in \mathbb{X}^{\psi,s}_{\mathcal{V}}$: since $\dim \Theta_A = 1$ the right-hand side of (7.2.36) is a union of sets isomorphic to the $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$-orbit of A and hence it has dimension 3. □

7.2.5 Points of $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$ are represented by lagrangians in $\mathbb{W}^{\psi}_{\text{fix}}$

Below is the main result of the present subsubsection.

Proposition 7.2.12. *Let* F *be a basis of* V *and* ψ *be as in* (7.2.1). *Suppose that* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ *is semistable with minimal orbit and that* $[A] \in \mathfrak{I}$. *Then there exist* $g \in \mathrm{PGL}(V)$ *such that* $gA \in \mathbb{W}^{\psi}_{\text{fix}}$.

The proof of **Proposition 7.2.12** is given at the end of the subsubsection.

Lemma 7.2.13. *Suppose that* $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ *is semistable with minimal orbit and that* $[A] \in \mathfrak{I}$. *There exists* $\overline{W} \in \Theta_A$ *such that* $C_{\overline{W},A}$ *is either* $\mathbb{P}(\overline{W})$ *or a sextic curve in the indeterminacy locus of Map* (0.0.3) *and*

$$\overline{W} \in \{W_0, W_\infty, \langle\alpha,\beta,\gamma\rangle\}, \quad \textit{where } \alpha \in V_{01}, \beta \in V_{23}, \gamma \in V_{45}. \tag{7.2.37}$$

Proof. By hypothesis there exists $W \in \Theta_A$ such that $C_{W,A}$ is either $\mathbb{P}(W)$ or a sextic curve in the indeterminacy locus of Map (0.0.3). Taking $\lim_{t\to 0} \lambda_{\mathcal{F}_2}(t)W$ we get that there exists $\overline{W} \in \Theta_A$ such that $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map (0.0.3) and $\overline{W}$ is fixed by $\lambda_{\mathcal{F}_2}(t)$ for all $t \in \mathbb{C}^\times$. Thus $\overline{W}$ is the direct sum of 3 irreducible summands for the representation $\lambda_{\mathcal{F}_2}\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ i.e. one of

$$W_\infty,\ W_0,\ V_{01} \oplus [\gamma],\ V_{23} \oplus [\alpha],\ V_{23} \oplus [\gamma],\ V_{45} \oplus [\alpha],\ \langle\alpha,\beta,\gamma\rangle, \qquad \alpha \in V_{01},\ \beta \in V_{23},\ \gamma \in V_{45}.$$

Suppose that $\overline{W}$ does not belong to the set appearing on the right-hand side of (7.2.37). Then Item (1) of **Proposition 7.1.1** holds (if $\overline{W} = V_{23} \oplus [\gamma]$ then $v_2 \wedge v_3 \wedge \gamma \in A_3$, since $A_1 \bot A_3$ it follows that $\dim A_1 \cap (V_{01} \wedge \bigwedge^2 V_{23}) \geq 1$) and hence $[A] \notin \mathfrak{I}$ by **Proposition 7.1.5**, that is a contradiction. □

Proposition 7.2.14. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is semistable with minimal orbit and that $[A] \in \mathfrak{I}$. Then* $\dim \Theta_A \geq 1$.

Proof. By contradiction. Suppose that $\dim \Theta_A = 0$. In particular

$$\text{if } W_1 \neq W_2 \in \Theta_A \text{ then } \dim(W_1 \cap W_2) = 1. \tag{7.2.38}$$

Moreover $C_{W,A}$ is a sextic curve for every $W \in \Theta_A$ by **Corollary 6.1.10**. By **Lemma 7.2.13** there exists $\overline{W} \in \Theta_A$ such that (7.2.37) holds and $C_{\overline{W},A}$ is a sextic curve in the indeterminacy locus of Map (0.0.3). Notice that

$$\dim S_{\overline{W}} \leq 3. \tag{7.2.39}$$

In fact suppose that (7.2.39) does not hold. Then $A \in \mathbb{B}_{\mathcal{C}_1}$: by **Proposition 6.1.1** we get that $A \in \mathrm{PGL}(V)A_+$, that is a contradiction because $\dim \Theta_{A_+} = 3$. Let $\{w_0, w_1, w_2\}$ be the basis of $\overline{W}$ appearing in (7.1.16) or in (7.2.37): thus $w_0 = v_0$ if $\overline{W} = W_\infty$, $w_0 = \alpha$ if $\overline{W} = \langle \alpha, \beta, \gamma \rangle$ and $w_0 = v_4$ if $\overline{W} = W_0$ etc. Let $\{X_0, X_1, X_2\}$ be the basis of $\overline{W}^\vee$ dual to $\{w_0, w_1, w_2\}$. The 1-PS $\lambda_{\mathcal{F}_2}$ acts trivially on $\bigwedge^{10} A$; applying **Claim 3.2.4** we get that $C_{\overline{W},A} = V(P)$ where

$$P = \begin{cases} F_4 X_2^2, \quad F_4 \in \mathbb{C}[X_0, X_1]_4 & \text{if } \overline{W} = W_\infty \text{ or } \overline{W} = W_0, \\ (b_1 X_0 X_2 + a_1 X_1^2)(b_2 X_0 X_2 + a_2 X_1^2)(b_3 X_0 X_2 + a_3 X_1^2) & \text{if } \overline{W} = \langle \alpha, \beta, \gamma \rangle. \end{cases}$$

Since $C_{\overline{W},A}$ is a sextic curve in the indeterminacy locus of Map (0.0.3) one gets that one of the following holds:

(1) $C_{\overline{W},A} = V((bX_0X_2 + aX_1^2)^3)$.

(2a) $C_{\overline{W},A} = V(X_0^2 X_2^2 (bX_0X_2 + X_1^2))$.

(2b) $C_{\overline{W},A} = V(L \cdot M^3 \cdot X_2^2)$ where $L, M \in \mathbb{C}[X_0, X_1]_1$.

(3) $C_{\overline{W},A} = V(X_1^4 (bX_0X_2 + aX_1^2))$.

Let $Z \subset \mathbb{P}(\overline{W})$ be the union of 1-dimensional components of $\operatorname{sing} C_{\overline{W},A}$: in all of the above cases Z is non-empty. By **Proposition 3.3.6** we get that $Z \subset \mathcal{B}(\overline{W}, A)$. Let $[v] \in Z$ be generic: there does not exist $W \in \Theta_A$ containing $[v]$ and different from $\overline{W}$ because $\dim \Theta_A = 0$. It follows that $\dim(A \cap F_v \cap S_{\overline{W}}) \geq 2$. Since $[v]$ moves on a curve it follows that $\dim S_{\overline{W}} \geq 3$ (recall that (7.2.38) holds): by (7.2.39) we get that

$$\dim S_{\overline{W}} = 3. \tag{7.2.40}$$

Let $V = \overline{W} \oplus U$ where U is $\lambda_{\mathcal{F}_2}$-invariant and let $\mathcal{V} := S_{\overline{W}} \cap (\bigwedge^2 \overline{W} \wedge U)$. By (7.2.40) we have $\dim \mathcal{V} = 2$. View $\mathcal{V}$ as a subspace of $\mathrm{Hom}(\overline{W}, U)$ by choosing a volume form on $\overline{W}$: every $\phi \in \mathcal{V}$ has rank 2 (by (7.2.38), (7.2.40) and the fact that Z is not empty). Now suppose that (1) above holds. Since Z is a smooth conic we get that $A \in \mathbb{B}_{\mathcal{E}_1^\vee}$ by **Remark 3.4.4**. By **Proposition 6.5.1** we get that $A \in \mathrm{PGL}(V)A_h$: that is a contradiction because $\dim \Theta_{A_h} = 2$. Now suppose that (2a) or (2b) above holds: then Z is the union of two lines and that contradicts **Proposition A.3.1**. Lastly suppose that (3) above holds. Then $K(\mathcal{V})$ (notation as in (A.3.6)) is the line $V(X_1)$. By **Proposition A.3.1** we get that $\mathcal{V}$ is $\mathrm{GL}(\overline{W}) \times \mathrm{GL}(U)$-equivalent to $\mathcal{V}_l$. Thus there exists a basis $\{u_0, u_1, u_2\}$ of U such that

$$\mathcal{V} = \langle w_0 \wedge w_1 \wedge u_0 + w_0 \wedge w_2 \wedge u_1,\ w_0 \wedge w_2 \wedge u_2 + w_1 \wedge w_2 \wedge u_0 \rangle. \tag{7.2.41}$$

Up to scalars there is a unique non-zero element of $\mathcal{V}$ mapping w_0 to 0 and similarly there is a unique (up to scalars) non-zero element of $\mathcal{V}$ mapping w_2 to 0: since $\mathcal{V}$, $[w_0]$ and $[w_2]$ are $\lambda_{\mathcal{F}_2}$-invariant it follows that the two elements of $\mathcal{V}$ appearing in (7.2.41) generate $\lambda_{\mathcal{F}_2}$-invariant subspaces. Since each w_i generates a $\lambda_{\mathcal{F}_2}$-invariant subspace it follows that each u_j generates a $\lambda_{\mathcal{F}_2}$-invariant subspace. Now suppose that $\overline{W} = \langle \alpha, \beta, \gamma \rangle$. Considering the possible weights of the u_j's we get that $u_0 \in V_{23}$, $u_1 \in V_{01}$ and $u_2 \in V_{45}$. Thus we have

$$\mathcal{V} = \langle \alpha \wedge \beta \wedge u_0 + \alpha \wedge \gamma \wedge u_1,\ \alpha \wedge \gamma \wedge u_2 + \beta \wedge \gamma \wedge u_0 \rangle, \quad u_0 \in V_{23},\ u_1 \in V_{01},\ u_2 \in V_{45}.$$

It follows that Item (3) of **Proposition 7.1.1** holds and hence $\mu(A,\lambda_3) \ge 0$ where λ_3 is given by (7.1.9). Since the $G_{\mathcal{F}_2}$-orbit of A is closed in $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{F}_2}$ we may assume that λ_3 acts trivially on $\bigwedge^{10} A$. By **Claim 3.2.4** we get that P is left invariant by $\mathrm{diag}(s^3 t, s^{-3}t, t^{-2})$ for $s,t \in \mathbb{C}^\times$: it follows that $P = aX_0^2X_1^2X_2^2$, that is a contradiction. Now suppose that $\overline{W} = W_\infty$. We may (and will) choose $v_2 := w_2 = \beta_0$. Considering the possible weights of the u_j's we get that $u_0 \in V_{45}$, $u_1 \in V_{23}$ and $u_2 \in V_{45}$. Thus we may assume that $v_3 = u_1$, $v_4 = u_0$ and $v_5 = u_2$. It follows that

$$\mathcal{V} = \langle v_0 \wedge v_1 \wedge v_4 + v_0 \wedge v_2 \wedge v_3,\ v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4 \rangle.$$

Thus $(v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4) \in A \cap S_{\overline{W}}$. Now $A \cap S_{\overline{W}}$ contains a 3-dimensional subspace R dictated by the condition $A \in \mathbb{B}_{\mathcal{F}_2}$ - see Table (1) - and $(v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4) \notin R$. Thus $\dim(A \cap S_{\overline{W}}) \ge 4$ and that contradicts (7.2.40). It remains to deal with the case $\overline{W} = W_0$: it is similar to the case $\overline{W} = W_\infty$. □

Lemma 7.2.15. *$\mathfrak{B}_{\mathcal{F}_2}$ does not contain $\mathfrak{x}$ nor $\mathfrak{x}^\vee$.*

Proof. Suppose the contrary. Then $A_k(L) \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ or $A_h(L) \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$, in particular $\lambda_{\mathcal{F}_2}(t)$ acts trivially on $\bigwedge^{10} A_k(L)$ (respectively $\bigwedge^{10} A_h(L)$). The stabilizer of $\bigwedge^{10} A_k(L)$ (respectively $\bigwedge^{10} A_h(L)$) is the image of the homomorphism $\rho\colon \mathrm{SL}(L) \to \mathrm{SL}(\mathrm{S}^2 L)$ (we have chosen an isomorphism $V = \mathrm{S}^2 L$): since $\{\lambda_{\mathcal{F}_2}(t) \mid t \in \mathbb{C}^\times\}$ is not in the image of ρ we get a contradiction. □

Proposition 7.2.16. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{F}_2}$ is semistable with minimal orbit and that $[A] \in \mathfrak{I}$. Then Θ_A contains $i_+(D)$ for some choice of Isomorphism* (7.2.1).

Proof. Suppose first that $\dim \Theta_A \ge 2$. By **Lemma 6.1.8** we have $A \in \mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k \cup \mathrm{PGL}(V)A_h$. By **Lemma 7.2.15** we get that $[A] \in \mathbb{X}^*_{\mathcal{W}}$ and hence Θ_A contains $i_+(D)$ for some choice of isomorphism (7.2.1) - see **Remark 7.2.2**. Now suppose that $\dim \Theta_A \le 1$. By **Proposition 7.2.14** we have $\dim \Theta_A = 1$. Let Θ be a 1-dimensional irreducible component of Θ_A. By Theorem 3.9 of [28] the curve Θ belongs to one of the Types

$$\mathcal{F}_1, \mathcal{D}, \mathcal{E}_2, \mathcal{E}_2^\vee, \mathbf{Q}, \mathcal{A}, \mathcal{A}^\vee, \mathcal{C}_2, \mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{T}^\vee$$

defined in [28]. Moreover if Θ is of Type $\mathcal{X}$ then $A \in \mathbb{B}_{\mathcal{X}}$ - see Claim 3.22 of [28]. Thus if Θ has calligraphic Type then $A \in \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_2}$; by (2.6.4) we get that $[A] \in \mathfrak{B}_{\mathcal{A}} \cup \mathfrak{B}_{\mathcal{C}_1} \cup \mathfrak{B}_{\mathcal{D}} \cup \mathfrak{B}_{\mathcal{E}_1} \cup \mathbb{B}_{\mathcal{E}_1^\vee}$ and hence $[A] \in \mathfrak{B}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$ by **Proposition 6.1.1**, **Proposition 6.2.1**, **Proposition 6.3.1**, **Proposition 6.4.2** and **Proposition 6.5.1**. It follows that $\dim \Theta_A \ge 2$, that is a contradiction. Thus we may assume that Θ is of Type **Q**, **R**, **S**, **T** or $\mathbf{T}^\vee$. Now notice that if $t \in \mathbb{C}^\times$ then $\lambda_{\mathcal{F}_2}(t)$ acts on Θ i.e. $\lambda_{\mathcal{F}_2}(t)|_\Theta$ is an automorphism of Θ. Suppose that $\lambda_{\mathcal{F}_2}(t)|_\Theta$ is the identity for each $t \in \mathbb{C}^\times$: looking at the action of $\lambda_{\mathcal{F}_2}(t)$ on V we get that Θ is a line and hence $A \in \mathbb{B}_{\mathcal{F}_1}$. By **Proposition 6.6.1** we have $\mathfrak{B}_{\mathcal{F}_1} \cap \mathfrak{I} = \emptyset$ and hence we get a contradiction. It follows that if $t \in \mathbb{C}^\times$ is generic then $\lambda_{\mathcal{F}_2}(t)|_\Theta$ is not the identity - in particular there exist points in Θ with dense orbit and hence Θ has geometric genus 0. We claim that there does not exist a Θ of Type **R**, **S**, **T** or $\mathbf{T}^\vee$ such that $\lambda_{\mathcal{F}_2}(t)(\Theta) = \Theta$ for $t \in \mathbb{C}^\times$. In fact suppose that Θ has type **R**. Then we may assume that $\Theta = i_+(C)$ where $C \subset \mathbb{P}(U)$ is a rational normal cubic curve and each $\lambda_{\mathcal{F}_2}(t)$ is induced by a projectivity of $\mathbb{P}(U)$: as is easily checked that is impossible. On the other hand Θ cannot be of Type **S**, **T** or $\mathbf{T}^\vee$ because there is no 1-PS of $\mathrm{PGL}(V)$ mapping such a curve to itself. (There is no copy of $\mathbb{C}^\times$ in the automorphism group of such a curve acting trivially on the Picard group of the curve.) Thus Θ is of type **Q** and that finishes the proof of the corollary. □

Proof of **Proposition 7.2.12**. Suppose first that $\dim \Theta_A \ge 2$. By **Lemma 6.1.8** we have $A \in \mathbb{X}^*_{\mathcal{W}} \cup \mathrm{PGL}(V)A_k \cup \mathrm{PGL}(V)A_h$. By **Lemma 7.2.15** we get that $[A] \in \mathbb{X}^*_{\mathcal{W}}$ and hence there exist $g \in \mathrm{PGL}(V)$ such that $gA \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ by **Remark 7.2.2**. Now suppose that $\dim \Theta_A \le 1$. By **Proposition 7.2.16** we get that there is an irreducible component $\overline{\Theta}$ of Θ_A which is projectively equivalent to $i_+(D)$ (i.e. of Type **Q**). The 1-PS $\lambda^{\mathsf{F}}_{\mathcal{F}_2}$ fixes A hence it acts on $\overline{\Theta}$: notice that the action is effective because the set of fixed points for the action of $\lambda_{\mathcal{F}_2}$ on $\mathrm{Gr}(3,V)$ is a collection of points and lines. The image

$$H := \{\rho \in \mathrm{Aut}(\overline{\Theta}) \mid \rho = \lambda_{\mathcal{F}_2}(t)|_{\overline{\Theta}} \text{ for some } t \in \mathbb{C}^\times\} \tag{7.2.42}$$

consists of the group of automorphisms fixing two points $p,q \in \Theta$. Of course $\lambda_{\mathcal{F}_2}$ acts on $R_{\overline{\Theta}}$ as well and hence also on $|\mathcal{I}_{\overline{\Theta}}(2)|$. By **Remark 7.2.4** there is a single singular quadric in $|\mathcal{I}_{\overline{\Theta}}(2)|$: it follows that there exists a smooth quadric $\overline{Q} \in |\mathcal{I}_{\overline{\Theta}}(2)|$ which is mapped to itself by $\lambda_{\mathcal{F}_2}$. On the other hand there exists $g \in \mathrm{PGL}(V)$ such that $g(\overline{\Theta}) = i_+(D) =: \Theta$ because up to projectivities there is a single curve of Type **Q**. By **Claim 7.2.5** we may choose g so that $g(p) = i_+([1,0,0,0])$, $g(q) = i_+([0,0,0,1])$ and $g(\overline{Q}) = \mathrm{Gr}(2,U)$ (recall that $\bigwedge^2 U$ is identified with V via (7.2.1) and hence $\mathrm{Gr}(2,U)$ is a smooth quadric containing R_Θ). With this choice of g the group H of (7.2.42) gets identified with the group of automorphisms of D fixing $[1,0,0,0]$ and $[0,0,0,1]$. Thus $gA \in \mathbb{W}^{\psi}_{\mathrm{fix}}$. □

7.2.6 $C_{W,A}$ for $A \in \mathbb{X}^{\psi}_{\mathcal{V}}$ and W spanned by $\alpha \in V_{01}$, $\beta \in V_{23}$ and $\gamma \in V_{45}$

Below is the main result of the present subsubsection.

Proposition 7.2.17. *Let $A \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ be a $G_{\mathcal{F}_2}$-semistable lagrangian with minimal $G_{\mathcal{F}_2}$-orbit. Suppose that there exist non-zero $\alpha \in V_{01}$, $\beta \in V_{23}$ and $\gamma \in V_{45}$ such that $\alpha \wedge \beta \wedge \gamma \in A$ and, letting $\overline{W} := \langle \alpha, \beta, \gamma \rangle$, the degeneracy locus $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map* (0.0.3). *Then $[A] \in \mathfrak{X}_{\mathcal{V}}$.*

The proof of **Proposition 7.2.17** will be given at the end of the subsubsection.

Definition 7.2.18. Let $\mathcal{E} \subset \mathrm{Gr}(3,V)$ be the subset of W such that $W = \langle \alpha,\beta,\gamma \rangle$ where $\alpha \in V_{01}$, $\beta \in V_{23}$, $\gamma \in V_{45}$. Let $\mathcal{E}_D \subset \mathcal{E}$ be the subset of W such that

$$\bigwedge^3 W \perp \langle\langle i_+(D) \rangle\rangle.$$

Remark 7.2.19. Let $A \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ and suppose that there exists $W \in \Theta_A$ which belongs to $\mathcal{E}$: then $W \in \mathcal{E}_D$.

Below we will make the identification

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \overset{\sim}{\longrightarrow} & \mathcal{E} \\ ([e_0,e_1],[e_2,e_3],[e_4,e_5]) & \mapsto & \langle e_0v_0 + e_1v_1, e_2v_2 + e_3v_3, e_4v_4 + e_5v_5 \rangle \end{array} \tag{7.2.43}$$

A straightforward computation gives the following result.

Lemma 7.2.20. *Keep notation as above. Then $([e_0,e_1],[e_2,e_3],[e_4,e_5]) \in \mathcal{E}_D$ if and only if*

$$e_0e_3e_5 - e_1e_2e_5 - e_1e_3e_4 = 0. \tag{7.2.44}$$

The group $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ - see (7.2.18) - acts on $\mathcal{E}_D$.

Proposition 7.2.21. *There are 5 orbits for the action of $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$ on $\mathcal{E}_D$ namely*

(1) An open dense orbit consisting of those $([e_0,e_1],[e_2,e_3],[e_4,e_5])$ such that $e_1e_3e_5 \neq 0$.

(2) The orbit of $([1,0],[1,0],[0,1])$.

(3) The orbit of $([1,0],[0,1],[1,0])$.

(4) The orbit of $([0,1],[1,0],[1,0])$.

(5) The orbit of $([1,0],[1,0],[1,0])$.

Proof. One checks easily that the orbit of $([0,1],[0,1],[0,1])$ is the set of $([e_0,e_1],[e_2,e_3],[e_4,e_5]) \in \mathcal{E}_D$ such that $e_1e_3e_5 \neq 0$. Now assume that $([e_0,e_1],[e_2,e_3],[e_4,e_5]) \in \mathcal{E}_D$ and that $e_1e_3e_5 = 0$. Suppose that $e_1 = 0$: then (7.2.44) gives that one among e_3, e_5 vanishes. Similarly if $e_3 = 0$ then one among e_1, e_5 vanishes, if $e_5 = 0$ then one among e_1, e_3 vanishes. The result follows from this and simple computations. □

Proposition 7.2.22. *Let $A \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ be a $G_{\mathcal{F}_2}$-semistable lagrangian with minimal $G_{\mathcal{F}_2}$-orbit. Suppose that there exists $\overline{W} \in \Theta_A$ such that*

(1) $\overline{W} \in \mathcal{E}$ and hence $\overline{W} \in \mathcal{E}_D$ by **Remark 7.2.19**.

(2) The $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$-orbit of $\overline{W}$ is not the single open orbit.

(3) $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map (0.0.3)*, i.e. $[A] \in \mathfrak{I}$.*

Then $[A] \in \mathfrak{X}_{\mathcal{W}}$.

Proof. One of Items (2) through (5) of **Proposition 7.2.21** holds. Thus we may assume that $\overline{W}$ is one of the following:

(2') $\langle v_0, v_2, v_5 \rangle$.

(3') $\langle v_0, v_3, v_4 \rangle$.

(4') $\langle v_1, v_2, v_4 \rangle$.

(5') $\langle v_0, v_2, v_4 \rangle$.

Suppose that (2') or (4') holds: we will reach a contradiction. In fact in both cases $\dim(\overline{W} \cap W_\infty) = 2$ - see (7.2.8). Thus $[A] \in \mathfrak{B}_{\mathcal{F}_1}$ and hence $[A] \notin \mathfrak{I}$ by **Proposition 6.6.1**, that is a contradiction. Suppose that (3') holds. Then Item (3) of **Proposition 7.1.1** holds for A with $\alpha = -v_0$, $\beta = v_3$ and $\gamma = v_4$ because by Table (10) we have $(v_0 \wedge v_1 \wedge v_4 - v_0 \wedge v_2 \wedge v_3) = \alpha_{(1,1,0,0)} \in A$. Now look at the proof of **Proposition 7.1.1**: since the $G_{\mathcal{F}_2}$-orbit of A is minimal we get that $\bigwedge^{10} A$ is left invariant by the 1-PS $\lambda_3 \colon \mathbb{C}^\times \to G_{\mathcal{F}_2}$ defined by (7.1.9). Let $C_{\overline{W},A} = V(P)$ where $P \in \mathrm{S}^6 \overline{W}^\vee$. Applying **Claim 3.2.4** to $C_{\overline{W},A}$ we get that P is left-invariant by the maximal torus of $\mathrm{SL}(\overline{W})$ diagonalized in the basis $\{v_0, v_3, v_4\}$ (recall that $\bigwedge^{10} A$ is left invariant by $\lambda_{\mathcal{F}_2}$): thus $P = aX_0^2 X_3^2 X_4^2$ where $\{X_0, X_3, X_4\}$ is the basis of $\overline{W}^\vee$ dual to $\{v_0, v_3, v_4\}$. By hypothesis $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map (0.0.3): it follows that $a = 0$ i.e. $C_{\overline{W},A} = \mathbb{P}(\overline{W})$. By **Proposition 6.1.9** and **Lemma 7.2.15** we get that $[A] \in \mathfrak{X}_{\mathcal{W}}$. Lastly suppose that (5') holds: we will reach a contradiction. We have $\langle v_0, v_2, v_4 \rangle = \bigwedge^2 \langle u_0, u_1, u_3 \rangle$ and hence $\dim(i_+(p) \cap \overline{W}) = 2$ for every $p \in D$. Viewing $i_+(D)$ as a subset of $\mathbb{P}(\bigwedge^3 V)$ via the Plücker embedding we get that $\langle\langle i_+(D) \rangle\rangle \subset S_{\overline{W}}$. Since $\overline{W} \in \Theta_A$ and $\dim \langle\langle i_+(D) \rangle\rangle = 5$ it follows that A is $\mathrm{PGL}(V)$-unstable (see Table (2), stratum $\mathbb{X}^{\mathsf{F}}_{\mathcal{C}_{1,+}}$), that is a contradiction. □

Let
$$W_m := \{Y_0 v_1 + Y_1 v_3 + Y_2 v_5 \mid Y_i \in \mathbb{C}\}. \tag{7.2.45}$$

Notice that $W_m \in \mathcal{E}_D$ and it belongs to the open orbit for the action of $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$. We will examine those $A \in \mathbb{W}^\psi$ such that Θ_A contains W_m and $C_{W_m,A}$ is not a sextic in the regular locus of (0.0.3). Let
$$\mathbb{M}^\psi := \{A_{\mathbf{c},\mathbf{L}} \in \mathbb{W}^\psi_{\mathrm{fix}} \mid v_1 \wedge v_3 \wedge v_5 \in A_{\mathbf{c},\mathbf{L}}\}.$$

In order to give an explicit description of $\mathbb{M}^\psi$ we introduce the following notation. Let

$$P_D^{00} := \langle \alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}, \alpha_{(0,1,1,0)} \rangle, \qquad Q_D^{00} := \langle 2\beta_{(0,2,0,0)} - \beta_{(1,0,0,1)}, \beta_{(0,1,1,0)} \rangle.$$

Thus $P_D^{00} \subset P_D^0$ and $Q_D^{00} \subset Q_D^0$. Given $\mathbf{J} \in \mathbb{LG}(P_D^{00} \oplus Q_D^{00})$ we let

$$\mathbf{L}_{\mathbf{J}} := (\langle \alpha_{(0,2,0,0)} \rangle \oplus \mathbf{J}) \in \mathbb{LG}(P_D^0 \oplus Q_D^0). \tag{7.2.46}$$

We have an isomorphism
$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) & \xrightarrow{\sim} & \mathbb{M}^\psi \\ (\mathbf{c}, \mathbf{J}) & \mapsto & A_{\mathbf{c},\mathbf{L}_{\mathbf{J}}}. \end{array} \tag{7.2.47}$$

In particular $\mathbb{M}^\psi$ is irreducible of dimension 4. Let $\mathbf{L}_M$ be as in (7.2.23): then

$$\mathbf{L}_M = \mathbf{L}_{\mathbf{J}} \text{ for some } \mathbf{J} \in \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \text{ if and only if } 0 = m_{13} = m_{23} = m_{33}. \tag{7.2.48}$$

We have $[v_1 \wedge v_3 \wedge v_5] = i_+([u_2])$; thus we have an isomorphism

$$\begin{array}{ccc} \langle u_0, u_1, u_3 \rangle & \overset{f}{\longrightarrow} & W_m \\ u & \mapsto & u \wedge u_2 \end{array} \tag{7.2.49}$$

If $p \in D \subset \mathbb{P}(\langle u_0, u_1, u_3 \rangle)$ then $[f(p)]$ belongs to the distinct planes $i_+(p)$ and to $\mathbb{P}(W_m)$. Now suppose that $A_{\mathbf{c},\mathbf{L}} \in \mathbb{M}^\psi$: then $i_+(p) \in \Theta_{A_{\mathbf{c},\mathbf{L}}}$ and hence by **Corollary 3.3.7** (and **Claim 3.2.4**) we get that

$$C_{W_m,A} = V((Y_0Y_2 + Y_1^2)^2(bY_0Y_2 + aY_1^2)) \text{ if } A \in \mathbb{M}^\psi. \tag{7.2.50}$$

(Here Y_0, Y_1, Y_2 are as in (7.2.45).)

Lemma 7.2.23. *Identify $\mathbb{M}^\psi$ with $\mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00})$ via* (7.2.47). *The set of $A \in \mathbb{M}^\psi$ such that $[v_3] \in C_{W_m,A}$ is equal to*

$$\{(\mathbf{c}, \mathbf{J}) \in \mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid c_0 = 0\} \cup \{(\mathbf{c}, \mathbf{J}) \in \mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid \mathbf{J} \cap P_D^{00} \neq \{0\}\}. \tag{7.2.51}$$

Proof. Let $\Xi \subset \mathbb{M}^\psi$ be the set of A such that $[v_3] \in C_{W_m,A}$. First we will prove that if $(\mathbf{c}, \mathbf{J})$ belongs to (7.2.51) then $A_{\mathbf{c},\mathbf{L_J}} \in \Xi$. If $c_0 = 0$ then

$$-2c_1 v_0 \wedge v_2 \wedge v_3 = (c_1\alpha_{(1,1,0,0)} + c_1\beta_{(0,0,1,1)}) \in F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}} \ni (c_1\alpha_{(0,1,0,1)} - c_1\beta_{(1,0,1,0)}) = 2c_1 v_2 \wedge v_3 \wedge v_4.$$

Since $c_1 \neq 0$ we get that $\dim(F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}) \geq 3$ and hence $[v_3] \in C_{W_m, A_{\mathbf{c},\mathbf{L_J}}}$, i.e. $A_{\mathbf{c},\mathbf{L_J}} \in \Xi$. Now suppose that $\mathbf{J} \cap P_D^{00} \neq \{0\}$ and let $0 \neq (s(\alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}) + t\alpha_{(0,1,1,0)}) \in \mathbf{J} \cap P_D^{00}$. Then

$$2sv_0 \wedge v_3 \wedge v_4 + tv_0 \wedge v_3 \wedge v_5 + tv_1 \wedge v_3 \wedge v_4 = (s(\alpha_{(0,2,0,0)} + \alpha_{(1,0,0,1)}) + s(\alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}) + t\alpha_{(0,1,1,0)}) \in F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}.$$

Thus $\dim(F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}) \geq 2$ and hence $[v_3] \in C_{W_m, A_{\mathbf{c},\mathbf{L_J}}}$, i.e. $A_{\mathbf{c},\mathbf{L_J}} \in \Xi$. We have proved that if $(\mathbf{c}, \mathbf{J})$ belongs to (7.2.51) then $A_{\mathbf{c},\mathbf{L_J}} \in \Xi$. It remains to prove that if $A_{\mathbf{c},\mathbf{L_J}} \in \Xi$ then $(\mathbf{c}, \mathbf{J})$ belongs to (7.2.51). Since v_3 generates a $\lambda_{\mathcal{F}_2}$-invariant subspace of V the intersection $F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}$ decomposes as the direct-sum of the interesections $F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}(i)$. By (7.2.9) we get that $F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}(i)$ can be non-zero only for $i = 0, \pm 1$. Looking at Tables (12) and (13) we get that $\dim(F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}(\pm 1)$ is non-zero only if $c_0 = 0$. Next we compute $\dim(F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}(0))$ for those $\mathbf{J}$ such that $\mathbf{L_J} = \mathbf{L}_M$ - see (7.2.48). Of course $v_1 \wedge v_3 \wedge v_5 \in F_{v_3} \cap A_{\mathbf{c},\mathbf{L_J}}(0)$. A straightforward computation gives that $\dim(F_{v_3} \cap A_{\mathbf{c},\mathbf{L}_M}(0)) \geq 2$ if and only if $(m_{11}m_{22} - 2m_{12}^2) = 0$ (notice: this is equivalent to requiring that $\mathbf{L}_M \cap P_D^{00} \neq \{0\}$). This shows that

$$\Xi \text{ contains } \{A_{\mathbf{c},\mathbf{L_J}} \mid [c_0, c_1] \text{ fixed}, J \in \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \text{ arbitrary}\} \text{ if and only if } c_0 = 0. \tag{7.2.52}$$

In particular Ξ is not all of $\mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00})$: it follows that it is the zero locus of a **non-zero** section of $\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{L}$ where $\mathcal{L}$ is the (ample) Plücker line-bundle on $\mathbb{LG}(P_D^{00} \oplus Q_D^{00})$ - see (3.2.23) and (3.2.26). Since Ξ contains the set of (7.2.51) we get by (7.2.52) that it is equal to that set. $\square$

By **Lemma 7.2.23** we have a rational map

$$\begin{array}{ccc} \mathbb{M}^\psi & \overset{\rho}{\dashrightarrow} & \mathbb{P}^1 \\ A & \mapsto & [a, b] \end{array} \tag{7.2.53}$$

where a, b are as in (7.2.50). Let $\widehat{\mathbb{M}}^\psi \subset \bigwedge^{10}(\bigwedge^3 V)$ be the affine cone over $\mathbb{M}^\psi$: then ρ is the projectivization of a regular map

$$\widehat{\mathbb{M}}^\psi \overset{\widehat{\rho}}{\longrightarrow} \mathbb{C}^2. \tag{7.2.54}$$

Lemma 7.2.24. *Identify $\mathbb{M}^\psi$ with $\mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00})$ via* (7.2.47). *Then the set of $A \in \mathbb{M}^\psi$ such that $[v_1 - v_5] \in C_{W_m,A}$ (i.e. $\mathbb{P}(\widehat{\rho}^{-1}\{(a, 0)\})$) is equal to*

$$\{(\mathbf{c},\mathbf{J}) \in \mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid c_0c_1 = 0\} \cup \{(\mathbf{c},\mathbf{J}) \in \mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid \mathbf{J} \cap \langle \alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}, \beta_{(0,1,1,0)} \rangle \neq \{0\}\}. \tag{7.2.55}$$

Proof. First we prove that the set of (7.2.55) is contained in $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$. Suppose that $c_0=0$. Then

$$-2(v_1-v_5)\wedge v_0\wedge v_4=\alpha_{(1,1,0,0)}-\beta_{(0,0,1,1)}+\alpha_{(0,1,0,1)}+\beta_{(1,0,1,0)}\in F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}}$$

and hence $\dim(F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}})\ge 2$: it follows that $[v_1-v_5]\in C_{W_m,A_{\mathbf{c},\mathbf{L}}}$. Now suppose that $c_1=0$. Then

$$(v_1-v_5)\wedge(v_0\wedge v_5+v_2\wedge v_3-v_4\wedge v_5)=-(\alpha_{(0,0,1,1)}+\alpha_{(1,0,1,0)})\in F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}}$$

and hence $\dim(F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}})\ge 2$: it follows that $[v_1-v_5]\in C_{W_m,A_{\mathbf{c},\mathbf{L}}}$. Lastly suppose that $\mathbf{J}\cap\langle\alpha_{(0,2,0,0)}-\alpha_{(1,0,0,1)},\beta_{(0,1,1,0)}\rangle\neq\{0\}$ and let

$$0\neq(t(\alpha_{(0,2,0,0)}-\alpha_{(1,0,0,1)})+u\beta_{(0,1,1,0)})\in\mathbf{J}\cap\langle\alpha_{(0,2,0,0)}-\alpha_{(1,0,0,1)},\beta_{(0,1,1,0)}\rangle.$$

Then

$$(v_1-v_5)\wedge((2t-u)v_2\wedge v_4+(2t+u)v_0\wedge v_2)=(-(u+2t)\alpha_{(2,0,0,0)}-t(\alpha_{(0,2,0,0)}+\alpha_{(1,0,0,1)})+$$
$$+t(\alpha_{(0,2,0,0)}-\alpha_{(1,0,0,1)})+u\beta_{(0,1,1,0)})+(u-2t)\alpha_{(0,0,0,2)})\in F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}}.$$

Thus $\dim(F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}})\ge 2$: it follows that $[v_1-v_5]\in C_{W_m,A_{\mathbf{c},\mathbf{L}}}$. It remains to prove that $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$ is contained in the set given by (7.2.51). Let $A_{\mathbf{c},\mathbf{L}}(\text{even})$ and $A_{\mathbf{c},\mathbf{L}}(\text{odd})$ be the direct sum of the $\bigwedge^3\lambda_{\mathcal{F}_2}$-isotypical summands of $A_{\mathbf{c},\mathbf{L}}$ with even and odd weights respectively. Let $\delta\in A_{\mathbf{c},\mathbf{L}}$: then $\delta\in F_{(v_1-v_5)}$ if and only if $v_1\wedge\delta=v_5\wedge\delta$. Since both v_1 and v_5 belong to $\lambda_{\mathcal{F}_2}$-isotypical summands of odd weight it follows that $F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}}$ is the direct-sum of its intersections with $A_{\mathbf{c},\mathbf{L}}(\text{even})$ and $A_{\mathbf{c},\mathbf{L}}(\text{odd})$. Going through Tables (12) and (13) we get that $F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}}(\text{odd})$ is not empty if and only if $c_0c_1=0$. Next we compute $\dim(F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}_\mathbf{J}}(\text{even}))$ for those $\mathbf{J}$ such that $\mathbf{L}_\mathbf{J}=\mathbf{L}_M$ - see (7.2.48). Of course $v_1\wedge v_3\wedge v_5\in F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}_\mathbf{J}}(\text{even})$. A straightforward computation gives that $\dim(F_{(v_1-v_5)}\cap A_{\mathbf{c},\mathbf{L}_M}(\text{even}))\ge 2$ if and only if $m_{11}=0$ (notice: this holds if and only if $(\mathbf{c},\mathbf{L}_M)$ belongs to the second set of (7.2.55)). In particular $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$ is not all of $\mathbb{P}^1\times\mathbb{LG}(P_D^{00}\oplus Q_D^{00})$. It follows that $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$ is the zero locus of a **non-zero** section of $\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{L}$ where $\mathcal{L}$ is the (ample) Plücker line-bundle on $\mathbb{LG}(P_D^{00}\oplus Q_D^{00})$ - see (3.2.23) and (3.2.26). Since $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$ contains the set of (7.2.55) we get that it is equal to that set. □

Let

$$\mathbb{N}^\psi:=\{A\in\mathbb{M}^\psi\mid a-b=0\}.\tag{7.2.56}$$

In other words $\mathbb{N}^\psi$ is the set of $A\in\mathbb{M}^\psi$ such that $C_{W_m,A}$ is not a sextic in the regular locus of the period map (0.0.3).

Proposition 7.2.25. *Identify $\mathbb{M}^\psi$ with $\mathbb{P}^1\times\mathbb{LG}(P_D^{00}\oplus Q_D^{00})$ via (7.2.47). Then*

$$\mathbb{N}^\psi=\{(\mathbf{c},\mathbf{J})\mid c_0=0\}\cup\mathbb{X}_{\mathcal{U}}^\psi\tag{7.2.57}$$

where $\mathbb{X}_{\mathcal{U}}^\psi$ is an irreducible divisor in $|\mathcal{O}_{\mathbb{P}^1}(1)\boxtimes\mathcal{L}|$ and $\mathcal{L}$ is the ample generator of the Picard group of $\mathbb{LG}(P_D^{00}\oplus Q_D^{00})$ (i.e. the Plücker line-bundle).

Proof. Let $A=A_{\mathbf{c},\mathbf{L}_\mathbf{J}}$. If $c_0=0$ then $C_{W_m,A}=\mathbb{P}(W_m)$ by **Lemma 7.2.23** and **Lemma 7.2.24**. This shows that the left-hand side of (7.2.57) contains the first set in the right-hand side of the same equation. We need to compare the two sides away from the set of $(\mathbf{c},\mathbf{J})$ such that $c_0=0$. The restriction to $\mathbb{M}^\psi$ of the Plücker (ample) line-bundle is isomorphic (via Identification (7.2.47)) to $\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{L}$. Let π and τ be the projections of $\mathbb{P}^1\times\mathbb{LG}(P_D^{00}\oplus Q_D^{00})$ to the first and second factor respectively. Both $\mathbb{P}(\widehat{\rho}^{-1}\{(0,b)\})$ and $\mathbb{P}(\widehat{\rho}^{-1}\{(a,0)\})$ are the supports of divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{L}|$: thus **Lemma 7.2.23** and **Lemma 7.2.24** give sections

$$\sigma_1,\sigma_2\in H^0(\mathbb{P}^1\times\mathbb{LG}(P_D^{00}\oplus Q_D^{00});\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{L})\tag{7.2.58}$$

such that

$$\operatorname{div}(\sigma_1)=2\pi^*(\infty)+\tau^*\Sigma_1,\qquad\operatorname{div}(\sigma_2)=\pi^*(0)+\pi^*(\infty)+\tau^*\Sigma_2\tag{7.2.59}$$

(we choose c_1/c_0 as affine coordinate on $(\mathbb{P}^1 \setminus \{[0,1]\})$) where

$$\Sigma_1 := \{\mathbf{J} \in \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid \mathbf{J} \cap P_D^{00} \neq \{0\}\} \tag{7.2.60}$$

and

$$\Sigma_2 := \{\mathbf{J} \in \mathbb{LG}(P_D^{00} \oplus Q_D^{00}) \mid \mathbf{J} \cap \langle \alpha_{(0,2,0,0)} - \alpha_{(1,0,0,1)}, \beta_{(0,1,1,0)}\rangle \neq \{0\}\}. \tag{7.2.61}$$

Now notice that away from $\pi^{-1}(\infty)$ the divisors $\mathrm{div}(\sigma_1)$ and $\mathrm{div}(\sigma_2)$ intersect properly: it follows that the rational map ρ of (7.2.53) is dominant and $\rho^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L}$. This shows that (7.2.57) holds with $\mathbb{X}^{\psi}_{\mathcal{U}}$ a divisor in $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L}|$. It remains to show that $\mathbb{X}^{\psi}_{\mathcal{U}}$ is irreducible. Now $\mathbb{X}^{\psi}_{\mathcal{U}}$ contains the base locus of the rational map ρ i.e.

$$(\pi^{-1}(\infty) \cap \tau^{-1}\Sigma_2) \cup (\pi^{-1}(0) \cap \tau^{-1}\Sigma_1) \cup (\tau^{-1}\Sigma_2 \cap \tau^{-1}\Sigma_1). \tag{7.2.62}$$

Suppose that $\mathbb{X}^{\psi}_{\mathcal{U}}$ is reducible, then it is equal to $(\pi^{-1}(s) \cup \tau^{-1}\Sigma)$ for some $s \in \mathbb{P}^1$ and $\Sigma \in |\mathcal{L}|$. Since $\mathbb{X}^{\psi}_{\mathcal{U}}$ contains the base locus i.e. (7.2.62) it follows that either $s = \infty$ and $\Sigma = \Sigma_1$ or $s = 0$ and $\Sigma = \Sigma_2$: that is absurd because for the generic $(\mathbf{c}, \mathbf{J})$ in the first set $C_{W_m, A_{\mathbf{c},\mathbf{L_J}}} = V((Y_0Y_2 + Y_1^2)^2(Y_0Y_2))$ while for the generic $(\mathbf{c}, \mathbf{J})$ in the second set $C_{W_m, A_{\mathbf{c},\mathbf{L_J}}} = V((Y_0Y_2 + Y_1^2)^2(Y_1^2))$. □

Proposition 7.2.26. $\mathbb{X}^{\psi}_{\mathcal{U}} \subset \mathbb{X}^{\psi}_{\mathcal{V}}$.

Proof. Let $\mathbb{T}^{\psi} := (\mathbb{X}^{\psi}_{\mathcal{V}} \cap \mathbb{M}^{\psi})$: thus $\mathbb{T}^{\psi}$ is a divisor in $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L}|$ by **Corollary 7.2.9** (notation as in the statement of **Proposition 7.2.25**). Since $\mathbb{X}^{\psi}_{\mathcal{U}}$ is an irreducible divisor in $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L}|$ it will suffice to prove that

$$\mathbb{T}^{\psi} \subset \mathbb{X}^{\psi}_{\mathcal{U}}. \tag{7.2.63}$$

First we notice that the restriction of the rational function ρ (see (7.2.53)) to $\mathbb{T}^{\psi}$ is constant. To see why notice that $\rho = \sigma_1/\sigma_2$ where $\sigma_i \in H^0(\mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00}); \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{L})$ are the sections appearing in the proof of **Proposition 7.2.25** - see (7.2.58). The equation of $\mathbb{T}^{\psi}$ is given by the restriction of (7.2.34) to $\mathbb{P}^1 \times \mathbb{LG}(P_D^{00} \oplus Q_D^{00})$ - see also (7.2.32): it follows that $\mathbb{T}^{\psi}$ is irreducible, smooth and

$$(\pi^*(\infty) + \tau^*\Sigma_1)|_{\mathbb{T}^{\psi}} = (\pi^*(0) + \Sigma_2)|_{\mathbb{T}^{\psi}}.$$

Looking at (7.2.59) we get that $\mathrm{div}(\sigma_1|_{\mathbb{T}^{\psi}}) = \mathrm{div}(\sigma_2|_{\mathbb{T}^{\psi}})$ and hence the restriction of ρ to $\mathbb{T}^{\psi}$ is constant. Thus it will suffice to show that

$$\text{there exists } A_0 \in \mathbb{T}^{\psi} \text{ such that } C_{W_m, A_0} = V((Y_0Y_2 + Y_1^2)^3). \tag{7.2.64}$$

Let's show that such an example is provided by the lagrangian $A_{\mathcal{R}}$ of (4.4.8). Let $Z \subset \mathbb{P}(U)$ be the smooth quadric

$$Z := \{[\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3] \mid \eta_0\eta_3 - \eta_1^2 + \eta_2^2 = 0\}.$$

Then Z contains D and is left-invariant by $\mathrm{diag}(t, 1, 1, t^{-1})$ for every $t \in \mathbb{C}^{\times}$: it follows (see the proof of **Proposition 4.4.4**) that every lagrangian $A \in \mathbb{LG}(\bigwedge^3 V)$ containing $\langle\langle i_+(Z)\rangle\rangle$ belongs to $\mathbb{W}^{\psi}$. Let $\mathcal{R}$ be the ruling of Z by lines containing the line $\langle [1,0,0,0], [0,1,-1,0]\rangle$ and let $A_{\mathcal{R}}$ be given by (4.4.8). A straightforward computation gives that

$$\overline{W} = \langle v_0 - v_1, 2v_2 - v_3, v_4 + v_5\rangle.$$

(Notation as in the definition of $A_{\mathcal{R}}$.) Thus $\overline{W} \in \mathcal{E}_D$ and it belongs to the open orbit for the action of $\mathrm{Aut}(R_{\Theta}) \cap G_{\mathcal{F}_2}$ - see **Proposition 7.2.21**. Thus there exists $g_0 \in \mathrm{Aut}(R_{\Theta}) \cap G_{\mathcal{F}_2}$ such that $A_0 := g_0 A_{\mathcal{R}} \in \mathbb{M}^{\psi}$. We have $C_{W_{\infty}, A_{\mathcal{R}}} = \mathbb{P}(W_{\infty})$ and hence $C_{W_{\infty}, A_0} = \mathbb{P}(W_{\infty})$. Thus $A_0 \in \mathbb{X}^{\psi}$. By **Corollary 7.2.9** either $A_0 \in \mathbb{X}^{\psi}_{\mathcal{V}}$ or else $A_0 = A_{[0,1],\mathbf{L_J}}$ for some $\mathbf{J}$: the latter is impossible because then A_0 would be unstable by **Proposition 7.2.11**, contradicting **Proposition 4.4.4**. Thus $A_0 \in \mathbb{X}^{\psi}_{\mathcal{V}}$ i.e. $A_0 \in \mathbb{T}^{\psi}$. On the other hand $C_{W_m, A_0} = V((Y_0Y_2 + Y_1^2)^3)$ by **Claim 4.4.6**. We have proved (7.2.64). □

The result below follows at once from **Proposition 7.2.26**.

Corollary 7.2.27. *Let $A \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ be a $G_{\mathcal{F}_2}$-semistable lagrangian with minimal $G_{\mathcal{F}_2}$-orbit. Suppose that there exists $\overline{W} \in \Theta_A$ such that*

(1) $\overline{W} \in \mathcal{E}$ and hence $\overline{W} \in \mathcal{E}_D$ by **Remark 7.2.19**.

(2) The $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2}$-orbit of $\overline{W}$ is the single open orbit.

(3) $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map (0.0.3), i.e. $[A] \in \mathfrak{I}$.

Then $[A] \in \mathfrak{X}_{\mathcal{V}}$.

Proof of **Proposition 7.2.17**. The projective plane $\mathbb{P}(\overline{W})$ belongs to one of the five $\mathrm{Aut}(R_\Theta) \cap G_{\mathcal{F}_2})$-orbits listed in **Proposition 7.2.21**. If it belongs to the open dense orbit then $[A] \in \mathfrak{X}_{\mathcal{V}}$ by **Corollary 7.2.27**, if it belongs to one of the remaining orbits then $[A] \in \mathfrak{X}_{\mathcal{W}}$ by **Proposition 7.2.22**, and hence $[A] \in \mathfrak{X}_{\mathcal{V}}$ by **Remark 7.2.2**. □

7.2.7 Proof that $\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{V}}$

By definition $\mathfrak{X}_{\mathcal{V}} \subset \mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$. It remains to prove that

$$\mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I} \subset \mathfrak{X}_{\mathcal{V}}. \tag{7.2.65}$$

Let $[A] \in \mathfrak{B}_{\mathcal{F}_2} \cap \mathfrak{I}$ and suppose that A has minimal $G_{\mathcal{F}_2}$-orbit in $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{F}_2}$. By **Proposition 7.2.12** we may assume that $A \in \mathbb{W}^{\psi}_{\mathrm{fix}}$. **Lemma 7.2.13** gives that there exists $\overline{W}$ as in (7.2.37) such that $C_{\overline{W},A}$ is not a sextic curve in the regular locus of Map (0.0.3). If $\overline{W} = W_\infty$ then $[A] \in \mathfrak{X}_{\mathcal{V}}$ by definition of $\mathfrak{X}_{\mathcal{V}}$. If $\overline{W} = \langle \alpha, \beta, \gamma \rangle$ where $\alpha \in V_{01}$, $\beta \in V_{23}$ and $\gamma \in V_{45}$ then $[A] \in \mathfrak{X}_{\mathcal{V}}$ by **Proposition 7.2.17**. Lastly suppose that $\overline{W} = W_0$. We claim that there exists $g \in \mathrm{PGL}(V)$ such that $A' := gA \in \mathbb{W}^{\psi}_{\mathrm{fix}}$ and $C_{W_\infty,A}$ is not a sextic curve in the regular locus of Map (0.0.3). In fact consider the involution

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\iota} & \mathbb{P}^1 \\ [\lambda, \mu] & \mapsto & [\mu, \lambda]. \end{array}$$

Then $g := \bigwedge^2 \iota \colon V \to V$ is an involution mapping $i_+(D)$ to itself and exchanging W_∞ and W_0. Thus $[A] = [A'] \in \mathfrak{X}_{\mathcal{V}}$. □

7.3 $\mathfrak{X}_{\mathcal{N}_3}$

We will determine the $G_{\mathcal{N}_3}$-stable points of $\mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ - notation is as in **Subsection 5.2**. We will apply the Cone Decomposition Algorithm: this makes sense because $\mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is a closed ($G_{\mathcal{N}_3}$-invariant) subset of a product of Grassmannians. Let $\{\xi_2, \xi_3\}$ be a basis of V_{23}. The isotypical summands of $\bigwedge^3 \lambda_{\mathcal{N}_3}$ with non-negative weights are the following:

$$\textstyle\bigwedge^2 V_{01} \wedge V_{23},\ \langle v_0\wedge v_1\wedge v_4,\ v_0\wedge\xi_2\wedge\xi_3\rangle,\ \langle v_0\wedge v_1\wedge v_5,\ v_0\wedge\xi_2\wedge v_4,\ v_0\wedge\xi_3\wedge v_4,\ v_1\wedge\xi_2\wedge\xi_3\rangle,\ \langle v_0\wedge\xi_2\wedge v_5,\ v_0\wedge\xi_3\wedge v_5,\ v_1\wedge\xi_2\wedge v_4,\ v_1\wedge\xi_3\wedge v_4\rangle. \tag{7.3.1}$$

The weights are (starting from the left) $3, 2, 1, 0$. Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$. Let A_i be the intersection of A and the isotypical summand of weight $(3-i)$: then $A = \sum_{i=0}^{6} A_i$. By definition

$$1 = \dim A_0 = \dim A_1 = \dim A_5 = \dim A_6, \quad 2 = \dim A_2 = \dim A_4 = \dim A_3, \quad A_i \bot A_{6-i}. \tag{7.3.2}$$

In particular

$$A_0 = [v_0 \wedge v_1 \wedge \gamma_0],\ A_6 = [\gamma_0 \wedge v_4 \wedge v_5], \quad 0 \neq \gamma_0 \in V_{23}. \tag{7.3.3}$$

We let

$$W_\infty := \langle v_0, v_1, \gamma_0 \rangle, \qquad W_0 := \langle \gamma_0, v_4, v_5 \rangle. \tag{7.3.4}$$

Thus $W_\infty, W_0 \in \Theta_A$. Let λ be a 1-PS of $G_{\mathcal{N}_3}$. There exists a basis $\{\xi_2, \xi_3\}$ of V_{23} such that

$$\lambda(t) = ((t^{m_0}, t^{m_1}, t^{m_2}), \mathrm{diag}(t^r, t^{-r})), \qquad (m_0, m_1, m_2, r) \in (\mathbb{Z}^4 \setminus \{(0,0,0,0)\}), \quad r \geq 0. \tag{7.3.5}$$

We denote such a 1-PS by (m_0, m_1, m_2, r). In the basis $\{v_0, v_1, \xi_2, \xi_3, v_4, v_5\}$ the action of $\lambda(t)$ on V is given by

$$\operatorname{diag}(t^{m_0},\ t^{2m_1},\ t^{r-m_0-m_1-m_2},\ t^{-r-m_0-m_1-m_2},\ t^{2m_2},\ t^{m_0}). \tag{7.3.6}$$

Below are the weights of the action of $\bigwedge^3 \lambda(t)$ on the isotypical summands of (7.3.1):

$$\begin{array}{cc} v_0 \wedge v_1 \wedge \xi_2 & v_0 \wedge v_1 \wedge \xi_3 \\ r + m_1 - m_2 & -r + m_1 - m_2 \end{array} \tag{7.3.7}$$

$$\begin{array}{cc} v_0 \wedge v_1 \wedge v_4 & v_0 \wedge \xi_2 \wedge \xi_3 \\ m_0 + 2m_1 + 2m_2 & -m_0 - 2m_1 - 2m_2 \end{array} \tag{7.3.8}$$

$$\begin{array}{cccc} v_0 \wedge v_1 \wedge v_5 & v_0 \wedge \xi_2 \wedge v_4 & v_0 \wedge \xi_3 \wedge v_4 & v_1 \wedge \xi_2 \wedge \xi_3 \\ 2m_0 + 2m_1 & r - m_1 + m_2 & -r - m_1 + m_2 & -2m_0 - 2m_2 \end{array} \tag{7.3.9}$$

$$\begin{array}{cccc} v_0 \wedge \xi_2 \wedge v_5 & v_0 \wedge \xi_3 \wedge v_5 & v_1 \wedge \xi_2 \wedge v_4 & v_1 \wedge \xi_3 \wedge v_4 \\ r + m_0 - m_1 - m_2 & -r + m_0 - m_1 - m_2 & r - m_0 + m_1 + m_2 & -r - m_0 + m_1 + m_2 \end{array} \tag{7.3.10}$$

In particular $I_-(\lambda) \subset \{0, 6\}$: by (5.2.5) and (2.2.9) we get that

$$\mu(A, \lambda) = 2r(2d_0^\lambda(A_0) - 1) + 2|m_0 + 2m_1 + 2m_2| \cdot (2d_0^\lambda(A_1) - 1) + 2\mu(A_2, \lambda) + \mu(A_3, \lambda).$$

Proposition 7.3.1. *$A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is not $G_{\mathcal{N}_3}$-stable if and only if one of the following holds:*

(1) $A_2 \cap \langle v_0 \wedge v_1 \wedge v_5, v_1 \wedge \xi_2 \wedge \xi_3 \rangle \neq \{0\}$.

(2) $A_2 \cap ([v_0] \wedge V_{23} \wedge [v_4]) \neq \{0\}$.

(3) $v_0 \wedge v_1 \wedge v_4 \in A_1$.

(4) $[v_1] \wedge V_{23} \wedge [v_4] = A_3$.

(5) $v_0 \wedge \xi_2 \wedge \xi_3 \in A_1$.

(6) $[v_0] \wedge V_{23} \wedge [v_5] = A_3$.

(7) $A_3 \cap \langle v_0 \wedge \gamma_0 \wedge v_5, v_1 \wedge \gamma_0 \wedge v_4 \rangle \neq \{0\}$.

(8) $A_2 \cap \langle v_0 \wedge v_1 \wedge v_5, v_0 \wedge \gamma_0 \wedge v_4 \rangle \neq \{0\}$.

(9) There exists $0 \neq \gamma \in V_{23}$ *such that* $A_2 \cap \langle v_0 \wedge v_1 \wedge v_5, v_0 \wedge \gamma \wedge v_4 \rangle \neq \{0\}$ *and* $v_0 \wedge \gamma \wedge v_5 \in A_3$.

(10) $A_2 \cap \langle v_0 \wedge \gamma_0 \wedge v_4, v_1 \wedge \xi_2 \wedge \xi_3 \rangle \neq \{0\}$.

(11) There exists $0 \neq \gamma \in V_{23}$ *such that* $A_2 \cap \langle v_0 \wedge \gamma \wedge v_4, v_1 \wedge \xi_2 \wedge \xi_3 \rangle \neq \{0\}$ *and* $v_1 \wedge \gamma \wedge v_4 \in A_3$.

Proof. We will apply the Cone Decomposition Algorithm. We choose the maximal torus $T < G_{\mathcal{N}_3}$ to be

$$T = \{(u_1, u_2, u_3), \operatorname{diag}(s, s^{-1})) \mid u_i, s \in \mathbb{C}^\times\}. \tag{7.3.11}$$

(The second entry is diagonal with respect to $\{\xi_2, \xi_3\}$.) Thus

$$\check{X}(T)_{\mathbb{R}} := \{(m_0, m_1, m_2, r) \in \mathbb{R}^5\}, \quad C := \{(m_0, m_1, m_2, r) \in \mathbb{R}^5 \mid r \geq 0\},$$

where notation is as in (7.3.5). Equations (7.3.7), (7.3.8), (7.3.9) and (7.3.10) give that $H \subset \check{X}(T)_{\mathbb{R}}$ is an ordering hyperplane if and only if is equal to the kernel of one the following linear functions:

$$r,\ m_0-m_1-m_2,\ m_0-m_1-m_2\pm r,\ m_0+2m_1+2m_2,\ 2m_0+m_1+m_2,\ 2m_0-m_1+3m_2\pm r,\ 2m_0+3m_1-m_2\pm r.$$

In particular the hypotheses of **Proposition 2.3.4** are satisfied. Notice also that if $\lambda = (m_0, m_1, m_2, r)$ is an ordering 1-PS then so are

$$\lambda' := (-m_0, -m_1, -m_2, r), \quad \lambda'' := (m_0, m_2, m_1, r). \tag{7.3.12}$$

In other words Klein's group acts on the set of ordering rays. A computation gives that the ordering rays are spanned by

$$\lambda_1 := (0,1,-1,0),\ \lambda_2 := (-1,1,1,0),\ \lambda_3 := (0,1,-1,4),\ \lambda_4 := (4,-1,-1,6), \tag{7.3.13}$$

and

$$(0,1,1,2),\ (2,1,-2,3),\ (4,5,-1,0),\ (2,1,1,6),\ (8,1,-5,0),\ (-4,1,7,12) \tag{7.3.14}$$

together with the 1-PS's obtained from them by operating with Klein's group, see (7.3.12). Table (30) lists the weights of the tensors appearing in (7.3.9) and (7.3.10) for the action of each λ_i and the 1-PS's obtained from them acting with Klein's group. (We denote $v_0 \wedge v_1 \wedge v_5$ by 015, $v_0 \wedge v_1 \wedge \xi_2$ by 012 etc.) Similarly Table (31) lists the weights of the tensors appearing in (7.3.9) and (7.3.10) for the action of the ordering 1-PS's of (7.3.14) and some of the 1-PS's obtained acting with the Klein group. Tables (30) and (31) give also the numerical function $\mu(A,\lambda)$ for λ one of the λ_i's or one of the 1-PS's obtained from them acting with Klein's group and also for ordering 1-PS's of (7.3.14) and some of their images for the Klein group. We explain our choice of ordering 1-PS's in Table (31). The sequence of weights for the action of λ' (or λ'') on the tensors appearing in (7.3.9) and (7.3.10) is obtained from that of λ by changing signs (this does not mean that the weight of a single monomial changes sign !). It follows that if the weights are symmetric about 0 then $\mu(A,\lambda) = \mu(A,\lambda') = \mu(A,\lambda'')$. This condition holds for the 1-PS's of (7.3.14) except for $\lambda \in \{(4,5,-1,0),(8,1,-5,0),\ (-4,1,7,12)\}$. That explains why we have listed the numerical function $\mu(A,\lambda')$ (which is equal to $\mu(A,\lambda'')$) for these 1-PS's. Going through Table (30) one gets the following:

- (1′) Item (1) holds if and only if $d_0^{\lambda_1}(A_2) \ge 1$, in particular if it holds then $\mu(A,\lambda_1) \ge 0$.
- (2′) Item (2) holds if and only if $d_0^{\lambda_1'}(A_2) \ge 1$, in particular if it holds then $\mu(A,\lambda_1') \ge 0$.
- (3′) Item (3) holds if and only if $d_0^{\lambda_2}(A_1) \ge 1$, in particular if it holds then $\mu(A,\lambda_2) \ge 0$.
- (4′) Item (4) holds if and only if $d_0^{\lambda_2}(A_3) \ge 2$, in particular if it holds then $\mu(A,\lambda_2) \ge 0$.
- (5′) Item (5) holds if and only if $d_0^{\lambda_2'}(A_1) \ge 1$, in particular if it holds then $\mu(A,\lambda_2') \ge 0$.
- (6′) Item (6) holds if and only if $d_0^{\lambda_2'}(A_3) \ge 2$, in particular if it holds then $\mu(A,\lambda_2') \ge 0$.
- (7′) Item (7) holds if and only if $d_0^{\lambda_3}(A_0) \ge 1$ and $d_0^{\lambda_3}(A_3) \ge 1$, in particular if it holds then $\mu(A,\lambda_3) \ge 0$ (notice that $d_0^{\lambda_3}(A_2) \ge 1$ for arbitrary A).
- (8′) Item (8) holds if and only if $d_0^{\lambda_4}(A_0) \ge 1$ and $d_0^{\lambda_4}(A_2) \ge 1$, in particular if it holds then $\mu(A,\lambda_4) \ge 0$.
- (9′) Item (9) holds if and only if $d_0^{\lambda_4}(A_2) \ge 1$ and $d_0^{\lambda_4}(A_3) \ge 1$, in particular if it holds then $\mu(A,\lambda_4) \ge 0$.
- (10′) Item (10) holds if and only if $d_0^{\lambda_4'}(A_0) \ge 1$ and $d_0^{\lambda_4'}(A_2) \ge 1$, in particular if it holds then $\mu(A,\lambda_4') \ge 0$.
- (11′) Item (11) holds if and only if $d_0^{\lambda_4'}(A_2) \ge 1$ and $d_0^{\lambda_4'}(A_3) \ge 1$, in particular if it holds then $\mu(A,\lambda_4') \ge 0$.

This proves that if one of Items(1)-(11) holds then A is not $G_{\mathcal{N}_3}$-stable. Next suppose that A is not $G_{\mathcal{N}_3}$-stable. By the Cone Decomposition Algorithm there exists an ordering 1-PS λ such that $\mu(A,\lambda) \ge 0$. Going through Tables (30) and (31) one gets that one of Items (1)-(11) holds. □

The result below follows at once from **Proposition 7.3.1**.

Corollary 7.3.2. *The generic $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is $G_{\mathcal{N}_3}$-stable.*

7.4 $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}$

7.4.1 Set-up and statement of the main results

The initial set-up is the same as in **Subsubsection 7.2.1**. Let U be a complex vector-space of dimension 4 and choose an isomorphism

$$\psi\colon \bigwedge^2 U \xrightarrow{\sim} V. \tag{7.4.1}$$

Let $\{u_0, u_1, u_2, u_3\}$ be a basis of U and F the basis of V given by

$$v_0 = u_0 \wedge u_1,\ v_1 = u_0 \wedge u_2,\ v_2 = u_0 \wedge u_3,\ v_3 = u_1 \wedge u_2,\ v_4 = u_1 \wedge u_3,\ v_5 = u_2 \wedge u_3. \tag{7.4.2}$$

Consider the action of $\mathbb{C}^\times$ on $\mathbb{P}(U)$ defined by $g(t) := \mathrm{diag}(t^3, t, t^{-1}, t^{-3})$ in the basis $\{u_0, u_1, u_2, u_3\}$: then

$$\bigwedge^2 g(t) = \lambda_{\mathcal{N}_3}(t^2). \tag{7.4.3}$$

Let $C \subset \mathbb{P}(U)$ be the rational normal cubic curve

$$C := \{[\lambda^3 u_0 + \lambda^2\mu u_1 + \lambda\mu^2 u_2 + \mu^3 u_3] \mid [\lambda, \mu] \in \mathbb{P}^1\}. \tag{7.4.4}$$

and $i_+\colon \mathbb{P}(U) \hookrightarrow \mathrm{Gr}(3, V)$ be the map of (2.4.11). Then $i_+(C)$ is an irreducible curve (of Type **R** according to the classification of [28]) parametrizing pairwise incident projective planes. Let

$$\mathbb{Y}^\psi := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \Theta_A \supset i_+(C)\}. \tag{7.4.5}$$

Let $t \in \mathbb{C}^\times$: by (7.4.3) $\lambda_{\mathcal{N}_3}(t)$ defines a projectivity of $\mathbb{P}(V)$ mapping $i_+(C)$ to itself. It follows that $\lambda_{\mathcal{N}_3}$ defines an action ρ of $\mathbb{C}^\times$ on $\mathbb{Y}^\psi$. Let $\widehat{\mathbb{Y}}^\psi \subset \bigwedge^{10}(\bigwedge^3 V)$ be the affine cone over $\mathbb{Y}^\psi$: then ρ lifts to an action $\widehat{\rho}$ on $\widehat{\mathbb{Y}}^\psi$. Let

$$\mathbb{Y}^\psi_{\mathrm{fix}} := \{A \in \mathbb{Y}^\psi \mid \bigwedge^{10} A \text{ is in the fixed locus of } \widehat{\rho}(t) \text{ for all } t \in \mathbb{C}^\times\}. \tag{7.4.6}$$

We will give an explicit description of $\mathbb{Y}^\psi_{\mathrm{fix}}$ which is analogous to the description of $\mathbb{W}^\psi_{\mathrm{fix}}$ given in **Subsubsection 7.2.3**. We start by explaining the entries in Table (15). Let $\langle\langle i_+(C)\rangle\rangle \subset A_+(U)$ be the span of the affine cone over $i_+(C)$. Going through Table (7) one gets that a basis of $\langle\langle i_+(C)\rangle\rangle$ is given by the first seven entries of Table (15). It follows by a straightforward computation that the elements of Table (15) form a basis of $i_+(C)^\perp$. Notice that each such element spans a subspace invariant under the action of $\lambda_{\mathcal{N}_3}(t)$ for $t \in \mathbb{C}^\times$: the corresponding character of $\mathbb{C}^\times$ is contained in the third column of Table (15). Let $P_C \subset A_+(U)$ be the subspace spanned by the elements of Table (15) which belong to lines 8 through 10 and $Q_C \subset A_-(U)$ be the subspace spanned by the elements belonging to lines 11 through 13. Both P_C and Q_C are isotropic for $(,)_V$ and the symplectic form identifies one with the dual of the other; thus the restriction of $(,)_V$ to $P_C \oplus Q_C$ is a symplectic form. It follows that a lagrangian $A \in \mathbb{LG}(\bigwedge^3 V)$ contains $i_+(C)$ if and only if it is equal to $\langle\langle i_+(C)\rangle\rangle \oplus R$ where $R \in \mathbb{LG}(P_C \oplus Q_C)$. Given $\mathbf{c} = [c_0, c_1] \in \mathbb{P}^1$, $\mathbf{d} = [d_0, d_1] \in \mathbb{P}^1$ we let

$$\begin{aligned} R_{\mathbf{c},\mathbf{d}} := \langle & c_0(\alpha_{(0,2,0,0)} - \alpha_{(1,0,1,0)}) + c_1(4\beta_{(0,0,2,0)} - 2\beta_{(0,1,0,1)}), \\ & d_0(\alpha_{(1,0,0,1)} - \alpha_{(0,1,1,0)}) + d_1(\beta_{(1,0,0,1)} - \beta_{(0,1,1,0)}), \\ & c_0(\alpha_{(0,0,2,0)} - \alpha_{(0,1,0,1)}) + c_1(4\beta_{(0,2,0,0)} - 2\beta_{(1,0,1,0)})\rangle \end{aligned} \tag{7.4.7}$$

and

$$A_{\mathbf{c},\mathbf{d}} := \langle\langle i_+(C)\rangle\rangle \oplus R_{\mathbf{c},\mathbf{d}}. \tag{7.4.8}$$

Looking at the action of $\lambda_{\mathcal{N}_3}(t)$ on the given bases of P_C and Q_C one gets that

$$\mathbb{Y}^\psi_{\mathrm{fix}} = \{A_{\mathbf{c},\mathbf{d}} \mid (\mathbf{c}, \mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1\}. \tag{7.4.9}$$

Table 15: Bases of $\langle\langle i_+(C)\rangle\rangle$ and of $\langle\langle i_+(C)\rangle\rangle^\perp$.

α-β notation	explicit expression	action of $\lambda_{\mathcal{N}_3}(t)$
$\alpha_{(2,0,0,0)}$	$v_0 \wedge v_1 \wedge v_2$	t^3
$\alpha_{(0,0,0,2)}$	$v_2 \wedge v_4 \wedge v_5$	t^{-3}
$\alpha_{(1,1,0,0)}$	$v_0 \wedge (v_1 \wedge v_4 - v_2 \wedge v_3)$	t^2
$\alpha_{(0,0,1,1)}$	$v_5 \wedge (v_1 \wedge v_4 + v_2 \wedge v_3)$	t^{-2}
$\alpha_{(0,2,0,0)} + \alpha_{(1,0,1,0)}$	$v_0 \wedge v_1 \wedge v_5 + v_0 \wedge v_3 \wedge v_4 - v_1 \wedge v_2 \wedge v_3$	t
$\alpha_{(1,0,0,1)} + \alpha_{0,1,1,0)}$	$v_0 \wedge v_2 \wedge v_5 + v_0 \wedge v_3 \wedge v_5 - v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_3 \wedge v_4$	1
$\alpha_{(0,0,2,0)} + \alpha_{(0,1,0,1)}$	$v_0 \wedge v_4 \wedge v_5 + v_1 \wedge v_3 \wedge v_5 + v_2 \wedge v_3 \wedge v_4$	t^{-1}
$\alpha_{(0,2,0,0)} - \alpha_{(1,0,1,0)}$	$-v_0 \wedge v_1 \wedge v_5 + v_0 \wedge v_3 \wedge v_4 + v_1 \wedge v_2 \wedge v_3$	t
$\alpha_{(1,0,0,1)} - \alpha_{(0,1,1,0)}$	$v_0 \wedge v_2 \wedge v_5 - v_0 \wedge v_3 \wedge v_5 - v_1 \wedge v_2 \wedge v_4 - v_1 \wedge v_3 \wedge v_4$	1
$\alpha_{(0,0,2,0)} - \alpha_{(0,1,0,1)}$	$-v_0 \wedge v_4 \wedge v_5 + v_1 \wedge v_3 \wedge v_5 - v_2 \wedge v_3 \wedge v_4$	t^{-1}
$4\beta_{(0,0,2,0)} - 2\beta_{(0,1,0,1)}$	$-2v_0 \wedge v_1 \wedge v_5 + 4v_0 \wedge v_2 \wedge v_4 - 2v_1 \wedge v_2 \wedge v_3$	t
$\beta_{(1,0,0,1)} - \beta_{(0,1,1,0)}$	$v_0 \wedge v_2 \wedge v_5 - v_0 \wedge v_3 \wedge v_5 + v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_3 \wedge v_4$	1
$4\beta_{(0,2,0,0)} - 2\beta_{(1,0,1,0)}$	$-2v_0 \wedge v_4 \wedge v_5 + 4v_1 \wedge v_2 \wedge v_5 + 2v_2 \wedge v_3 \wedge v_4$	t^{-1}

Notice that $A_{\mathbf{c},\mathbf{d}}$ is $\lambda_{\mathcal{N}_3}$-split of reduced type $(1,1,2)$ (look at the action of $\mathbb{C}^\times$ on the elements of the bases of $\langle\langle i_+(C)\rangle\rangle$, P_C and Q_C). Thus

$$\mathbb{Y}^\psi_{\text{fix}} \subset \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}. \tag{7.4.10}$$

We will examine $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}}$ for $(\mathbf{c},\mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1$. (See (7.3.4) for the definition of W_∞.)

Claim 7.4.1. *Let $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$. Let $\{X_0, X_1, X_2\}$ be the basis of $W_\infty^\vee$ dual to $\{v_0, v_1, \gamma_0\}$. There exist $a_i, b_i \in \mathbb{C}$ for $i = 1,2,3$ such that*

$$C_{W_\infty,A} = V((b_1X_0X_2 + a_1X_1^2)(b_2X_0X_2 + a_2X_1^2)(b_3X_0X_2 + a_3X_1^2)). \tag{7.4.11}$$

Proof. Let $t \in \mathbb{C}^\times$: then $\lambda_{\mathcal{N}_3}(t)$ fixes $\bigwedge^{10} A$, W_∞ and W_0. Applying **Claim 3.2.4** and Item (2) of **Remark 1.4.3** we get the result. □

Now let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^\psi_{\text{fix}}$: then

$$W_\infty = i_+([1,0,0,0]) = \langle v_0, v_1, v_2\rangle.$$

Let $\{X_0, X_1, X_2\}$ be as in **Claim 7.4.1**. As $[\lambda,\mu]$ varies in $\mathbb{P}^1$ the intersection $\mathbb{P}(W_\infty) \cap \mathbb{P}(i_+([\lambda,\mu])$ traces out a dense open subset of $V(X_0X_2 - X_1^2) \subset \mathbb{P}(W_\infty)$. By **Corollary 3.3.7** and **Claim 7.4.1** we get that

$$C_{W_\infty,A_{\mathbf{c},\mathbf{d}}} = V((X_0X_2 - X_1^2)^2(bX_0X_2 + aX_1^2)). \tag{7.4.12}$$

Our main object of interest is

$$\mathbb{V}^\psi := \{A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^\psi_{\text{fix}} \mid C_{W_\infty,A_{\mathbf{c},\mathbf{d}}} = V(m(X_0X_2 - X_1^2)^3),\ m \in \mathbb{C}\}.$$

In **Subsubsection 7.4.3** we will prove the following result.

Table 16: Values of $R_q = \bigwedge^3 L_q^{-1} \circ \delta_V$, I.

(012)	(013)	(014)	(015)	(023)	(024)	(025)	(034)	(035)	(045)
(012)	−(013)	−(023)	−(123)	−(014)	−(024)	−(124)	(034)	(134)	(234)

Proposition 7.4.2. *Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^{\psi}_{\text{fix}}$. Then $A_{\mathbf{c},\mathbf{d}} \in \mathbb{V}^{\psi}$ if and only if $c_1(c_0 d_1 + c_1 d_0) = 0$.*

By the above proposition $\mathbb{V}^{\psi}$ has two irreducible components. The following result gives geometric meaning to one of the components.

Claim 7.4.3. *For any $\mathbf{d} \in \mathbb{P}^1$ the lagrangian $A_{[1,0],\mathbf{d}}$ belongs to $\mathbb{X}^*_{\mathcal{W}}$. Conversely, if $A \in \mathbb{X}^*_{\mathcal{W}}$ there exist $\mathbf{d} \in \mathbb{P}^1$ and $g \in \mathrm{PGL}(V)$ such that $gA = A_{[1,0],\mathbf{d}}$.*

Proof. Let $\{\xi_0, \ldots, \xi_3\}$ be the basis of $U^{\vee}$ dual to $\{u_0, \ldots, u_3\}$ and

$$Q_0 := V(\xi_0\xi_3 - \xi_1\xi_2) \subset \mathbb{P}(U). \tag{7.4.13}$$

The span $A_0 := \langle\langle i_+(Q_0)\rangle\rangle$ in $A_+(U)$ of the affine cone over $i_+(Q_0)$ is equal to

$$\langle\langle i_+(C)\rangle\rangle \oplus \langle(\alpha_{(0,2,0,0)} - \alpha_{(1,0,1,0)}), (\alpha_{(0,0,2,0)} - \alpha_{(0,1,0,1)})\rangle.$$

(Look at (4.4.2).) Thus $A_0^{\perp} = A_0 \oplus \langle(\alpha_{(1,0,0,1)} - \alpha_{(0,1,1,0)}), (\alpha_{(1,0,0,1)} - \alpha_{(0,1,1,0)})\rangle$. It follows that $A_{[1,0],\mathbf{d}}$ belongs to $\mathbb{X}^*_{\mathcal{W}}(U)$ and that by varying $\mathbf{d}$ we get all elements of $\mathbb{X}^*_{\mathcal{W}}(U)$ up to the action of $\mathrm{PGL}(V)$. □

We will be mainly concerned with the other irreducible component of $\mathbb{V}^{\psi}$.

Definition 7.4.4. Let $\mathbb{X}^{\psi}_{\mathcal{Z}} := \{A_{\mathbf{c},\mathbf{d}} \mid c_0 d_1 + c_1 d_0 = 0\}$.

Definition 7.4.5. Let $\mathfrak{X}_{\mathcal{Z}} \subset \mathfrak{M}$ be the set of points represented by semistable lagrangians $A_{\mathbf{c},\mathbf{d}} \in \mathbb{X}^{\psi}_{\mathcal{Z}}$ (of course $\mathfrak{X}_{\mathcal{Z}}$ is independent of ψ).

By **Proposition 7.4.2** we have

$$\mathfrak{X}_{\mathcal{Z}} \subset \mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}. \tag{7.4.14}$$

Notice that $A_{[1,0],[1,0]} = A_+(U)$ and hence $\mathfrak{y} \in \mathfrak{X}_{\mathcal{Z}}$. Below is the main result of the present Subsection - it is obtained by putting together **Proposition 7.4.9** and **Subsubsection 7.4.5**.

Proposition 7.4.6. *$\mathfrak{X}_{\mathcal{Z}}$ is an irreducible curve containing $\mathfrak{y}$, $\mathfrak{x}$, $\mathfrak{x}^{\vee}$, and intersecting $\mathfrak{X}_{\mathcal{V}}$ in the single point $\mathfrak{y}$. Moreover $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{Z}} \cup \mathfrak{X}_{\mathcal{W}}$.*

7.4.2 Duality

Let $\{x_0, \ldots, x_5\}$ be the basis of $V^{\vee}$ dual to $\{v_0, \ldots, v_5\}$ and $q \in \mathrm{S}^2 V^{\vee}$ be the non-degenereate quadratic form given by $x_0x_5 - x_1x_4 + x_2x_3$: the Plücker quadric $\mathrm{Gr}(2, U) \subset \mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V)$ is the zero-set of q. Let $L_q \colon V \xrightarrow{\sim} V^{\vee}$ be the isomorphism defined by q and

$$\begin{array}{ccc} \bigwedge^3 V & \overset{R_q}{\longrightarrow} & \bigwedge^3 V \\ \alpha & \mapsto & \bigwedge^3 L_q^{-1} \circ \delta_V(\alpha) \end{array} \tag{7.4.15}$$

(See (1.3.1) for the definition of δ_V.) Tables (16) and (17) list the values of R_q on the monomials $v_i \wedge v_j \wedge v_k$ (denoted (ijk)): they give that R_q maps each of $A_{\pm}(U)$ to itself and

$$R_q|_{A_+(U)} = \mathrm{Id}_{A_+(U)}, \qquad R_q|_{A_-(U)} = -\mathrm{Id}_{A_-(U)}\,. \tag{7.4.16}$$

Proposition 7.4.7. *Let $(\mathbf{c}, \mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbf{c}' := [c_0, -c_1]$, $\mathbf{d}' := [d_0, -d_1]$. Then $A_{\mathbf{c}',\mathbf{d}'} = R_q(A_{\mathbf{c},\mathbf{d}})$.*

Table 17: Values of $R_q = \bigwedge^3 L_q^{-1} \circ \delta_V$, II.

(123)	(124)	(125)	(134)	(135)	(145)	(234)	(235)	(245)	(345)
−(015)	−(025)	−(125)	(035)	(135)	(235)	(045)	(145)	(245)	−(345)

Proof. Follows at once from (7.4.16) and the definition of $A_{\mathbf{c},\mathbf{d}}$. □

The map $\mathbb{P}(R_q)\colon \mathbb{P}(\bigwedge^3 V) \to \mathbb{P}(\bigwedge^3 V)$ given by the projectivization of R_q maps $\mathrm{Gr}(3,V)$ to itself: we will describe the image via $\mathbb{P}(R_q)$ of certain special elements of $\mathrm{Gr}(3,V)$. Let $Q \subset \mathbb{P}(U)$ be a smooth quadric. Let

$$T(Q), T'(Q) \subset \mathrm{Gr}(1,\mathbb{P}(U)) \subset \mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V) \tag{7.4.17}$$

be the two irreducible components of the family of lines on Q. Since $T(Q)$ is a smooth conic in $\mathbb{P}(V)$ the affine cone over its span in $\mathbb{P}(V)$ is a 3-dimensional vector subspace of V that we will denote $U(Q)$. Similarly we let $U'(Q)$ be the affine cone over the span of $T'(Q)$ in $\mathbb{P}(V)$.

Proposition 7.4.8. *Keeping notation as above, we have*

$$R_q(\bigwedge^3 U(Q)) = \bigwedge^3 U'(Q), \qquad R_q(\bigwedge^3 i_+(p)) = \bigwedge^3 i_+(p). \tag{7.4.18}$$

Proof. Let us prove that

$$\delta_V(\bigwedge^3 U(Q)) = \bigwedge^3 L_q(\bigwedge^3 U'(Q)). \tag{7.4.19}$$

Let $\bigwedge^3 L_q(\bigwedge^3 U'(Q)) = f_1 \wedge f_2 \wedge f_3$ where $f_i \in V^\vee$. Then (7.4.19) is equivalent to

$$U(Q) = \mathrm{Ann}\langle f_1, f_2, f_3\rangle. \tag{7.4.20}$$

Now $\mathrm{Ann}\langle f_1, f_2, f_3\rangle$ meets $\mathrm{Gr}(1,\mathbb{P}(U))$ in the set of lines meeting each line of $T'(Q)$, and that set is $T(Q)$. Since $\mathbb{P}(U(Q))$ meets $\mathrm{Gr}(1,\mathbb{P}(U))$ in the same set of lines (by definition of $U(Q)$), we get that (7.4.20) holds. This proves the first equality of (7.4.18). The proof of the second equality is similar, we leave details to the reader. □

7.4.3 Properties of $\mathfrak{X}_{\mathcal{Z}}$

In the present subsubsection we will prove **Proposition 7.4.2** and after that the result below.

Proposition 7.4.9. $\mathfrak{X}_{\mathcal{Z}}$ *is an irreducible curve containing* $\mathfrak{x}$, $\mathfrak{x}^\vee$ *and intersecting* $\mathfrak{X}_{\mathcal{V}}$ *in the single point* $\mathfrak{y}$.

In order to prove **Proposition 7.4.2** we will need to describe $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ for $(\mathbf{c},\mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1$. As a preliminary step we will show that there exist isomorphisms $\phi_i\colon \mathrm{S}^2 L \xrightarrow{\sim} V$ for $i = 1, 2$ (here L is a 3-dimensional complex vector space) such that $i_+(C)$ is contained in the image of $\Theta_{A_k(L)}$ and $\Theta_{A_h(L)}$ via the isomorphisms $\mathrm{Gr}(3, \mathrm{S}^2 L) \xrightarrow{\sim} \mathrm{Gr}(3,V)$ associated to ϕ_1 and ϕ_2 respectively. Let $\nu\colon \mathrm{Gr}(1,\mathbb{P}(U)) \hookrightarrow \mathbb{P}(\bigwedge^2 U)$ be the Plücker map. We have the embedding

$$\begin{array}{ccc} \mathbb{P}^2 \cong C^{(2)} & \xrightarrow{\kappa} & \mathbb{P}(\bigwedge^2 U) = \mathbb{P}(V) \\ z_1 + z_2 & \mapsto & \nu(\langle z_1, z_2\rangle) \end{array}$$

where $\langle z_1, z_2\rangle$ is the line spanned by z_1, z_2 (the projective tangent line to C if $z_1 = z_2$). Then $\kappa^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{O}_{\mathbb{P}^2}(2)$. It follows that there exists an isomorphism

$$\phi_2\colon \mathrm{S}^2 L \xrightarrow{\phi} V \tag{7.4.21}$$

such that

$$\mathbb{P}(\phi_2)(\mathcal{V}_1(L)) = \kappa(C^{(2)}) \tag{7.4.22}$$

where $\mathcal{V}_1(L)$ is the Veronese surface of symmetric tensors of rank 1 modulo scalars. Let $A_h(\phi_2)$ be the image of $A_h(L)$ via the isomorphism $\bigwedge^3 \phi_2\colon \bigwedge^3(\mathrm{S}^2 L) \xrightarrow{\sim} \bigwedge^3 V$; thus $A_h(\phi_2) \in \mathbb{LG}(\bigwedge^3 V)$. In order to describe the elements of $\Theta_{A_h(\phi_2)}$ we give the folowing definition.

Definition 7.4.10. Let $Q \subset \mathbb{P}(U)$ be a smooth quadric containing C. For $i = 1, 2$ we let $T_i(Q)$ be the family of lines $L \subset Q$ such that $L \cdot C = i$ (the intersection takes place in Q).

If $Q \subset \mathbb{P}(U)$ is a smooth quadric containing C then $\nu(T_2(Q))$ is a conic lying in the Veronese surface $\kappa(C^{(2)})$; it follows that

$$\Theta_{A_h(\phi_2)} = \{\langle\langle T_2(Q)\rangle\rangle \mid Q \in |\mathcal{I}_C(2)| \text{ smooth}\} \cup \{i_+(p) \mid p \in C\}. \tag{7.4.23}$$

On the other hand **Proposition 7.4.8** gives that

$$R_q(\bigwedge^3 \langle\langle T_2(Q)\rangle\rangle) = \bigwedge^3 \langle\langle T_1(Q)\rangle\rangle, \qquad R_q(\bigwedge^3 i_+(p)) = \bigwedge^3 i_+(p). \tag{7.4.24}$$

Since the right-hand side of (7.4.23) is a family of pairwise incident 3-dimensional subspaces of V it follows that also

$$\{\langle\langle T_1(Q)\rangle\rangle \mid Q \in |\mathcal{I}_C(2)| \text{ smooth}\} \cup \{i_+(p) \mid p \in C\}$$

is a family of pairwise incident 3-dimensional subspaces of V. Since $\delta_V(A_h(L)) = \delta_V(A_k(L^\vee))$ (see (2.80) of [28]), it follows that there exists an isomorphism

$$\phi_1\colon \mathrm{S}^2 L \xrightarrow{\phi} V \tag{7.4.25}$$

such that, letting $A_k(\phi_1)$ be the image of $A_k(L)$ via the isomorphism $\bigwedge^3 \phi_1$, we have

$$\Theta_{A_k(\phi_1)} = \{\langle\langle T_1(Q)\rangle\rangle \mid Q \in |\mathcal{I}_C(2)| \text{ smooth}\} \cup \{i_+(p) \mid p \in C\}. \tag{7.4.26}$$

In particular we get that

$$i_+(C) = \Theta_{A_+(U)} \cap \Theta_{A_k(\phi_1)} = \Theta_{A_+(U)} \cap \Theta_{A_h(\phi_2)} = \Theta_{A_k(\phi_1)} \cap \Theta_{A_h(\phi_2)}. \tag{7.4.27}$$

Before stating the next result we will introduce some notation. By (7.4.27) we have $i_+(C) = \bigwedge^3 \phi_1 \circ k(D_1)$ where $D_1 \subset \mathbb{P}(L)$ is a smooth conic. The 1-PS $\lambda_{\mathcal{N}_3}$ is induced by a 1-PS ρ_1 of $\mathrm{SL}(L)$ which maps the conic D_1 to itself: let $p_1, q_1, r \in \mathbb{P}(L)$ be the fixed points for the action of ρ_1 on $\mathbb{P}(L)$, with $p_1, q_1 \in D_1$. Similarly we have $i_+(C) = \bigwedge^3 \phi_2 \circ h(D_2)$ where $D_2 \subset \mathbb{P}(L)$ is a smooth conic, and $\lambda_{\mathcal{N}_3}$ is induced by a 1-PS ρ_2 of $\mathrm{SL}(L)$ mapping D_2 to itself. Let $p_2, q_2, r_2 \in \mathbb{P}(L^\vee)$ be the fixed points for the action of ρ_2 on $\mathbb{P}(L)$, with $p_2, q_2 \in D_2$. Up to reordering $\{p_1, q_1\}$ and $\{p_2, q_2\}$ we have $W_\infty = \bigwedge^3 \phi_1 \circ k(p_1) = \bigwedge^3 \phi_2 \circ h(p_2)$ and $W_0 = \bigwedge^3 \phi_1 \circ k(q_1) = \bigwedge^3 \phi_2 \circ h(q_2)$. The points r_1, r_2 are determined as follows. Let $Q_0 \subset \mathbb{P}(U)$ be the smooth quadric given by (7.4.13). Then

$$\bigwedge^3 \phi_1 \circ k(r_1) = \bigwedge^3 \langle\langle T_1(Q_0)\rangle\rangle, \qquad \bigwedge^3 \phi_2 \circ h(r_2) = \bigwedge^3 \langle\langle T_2(Q_0)\rangle\rangle.$$

Since $\mathrm{SL}(L)$ acts trivially on $\bigwedge^{10} A_k(L)$ and $\bigwedge^{10} A_h(L)$ we get that $\lambda_{\mathcal{N}_3}$ acts trivially on $A_k(\phi_1)$ and on $A_h(\phi_2)$, i.e.

$$A_k(\phi_1), A_h(\phi_2) \in \mathbb{Y}^{\psi}_{\mathrm{fix}}. \tag{7.4.28}$$

Proposition 7.4.11. *Keep notation as above. Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^{\psi}_{\mathrm{fix}}$. Then one of the following holds:*

- *(s)* $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} \geq 2$ *and*
  - *(s1)* $c_1 = 0$ - *in this case $A_{\mathbf{c},\mathbf{d}}$ belongs to $\mathbb{X}^*_{\mathcal{W}}$ by* **Claim 7.4.3**, *or*
  - *(s2)* $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$ - *in this case $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$, or*
  - *(s3)* $(\mathbf{c}, \mathbf{d}) = ([1, -1], [1, 1])$ - *in this case $A_{\mathbf{c},\mathbf{d}} = A_h(\phi_2)$.*
- *(t)* $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} = 1$ *and every irreducible component of $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ is one of the following:*

(t1) $i_+(C)$,

(t2) $\bigwedge^3 \phi_1 \circ k(\langle p_1, q_1\rangle)$, $\bigwedge^3 \phi_1 \circ k(\langle r_1, p_1\rangle)$ *or* $\bigwedge^3 \phi_1 \circ k(\langle r_1, q_1\rangle)$,

(t3) $\bigwedge^3 \phi_2 \circ h(\langle p_2, q_2\rangle)$, $\bigwedge^3 \phi_2 \circ h(\langle r_2, p_2\rangle)$ *or* $\bigwedge^3 \phi_2 \circ h(\langle r_2, q_2\rangle)$,

(t4) $i_+(\{[\xi_0 u_0 + \xi_3 u_3] \mid [\xi_0, \xi_3] \in \mathbb{P}^1\})$,

(t5) $\{\langle\langle T_1(Q_0)\rangle\rangle\}$ *where* Q_0 *is given by* (7.4.13),

(t6) $\{\langle\langle T_2(Q_0)\rangle\rangle\}$.

Moreover $\langle\langle T_1(Q_0)\rangle\rangle$ *is an element of* $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ *if and only if* $d_0 + d_1 = 0$ *and* $\langle\langle T_2(Q_0)\rangle\rangle$ *is an element of* $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ *if and only if* $d_0 - d_1 = 0$.

Proof. As is easily checked

$$\{W \in \mathrm{Gr}(3, V) \mid W \cap i_+(p) \neq \{0\} \quad \forall p \in C\} = \Theta_{A_+(U)} \cup \Theta_{A_k(\phi_1)} \cup \Theta_{A_h(\phi_2)}.$$

Since $i_+(C) \subset \Theta_{A_{\mathbf{c},\mathbf{d}}}$ it follows that

$$\Theta_{A_{\mathbf{c},\mathbf{d}}} \subset \Theta_{A_+(U)} \cup \Theta_{A_k(\phi_1)} \cup \Theta_{A_h(\phi_2)}. \tag{7.4.29}$$

Let Θ be an irreducible component of $\Theta_{A_{\mathbf{c},\mathbf{d}}}$. By (7.4.29) one of the following holds:

(A) $\Theta \subset \Theta_{A_+(U)}$,

(B) $\Theta \subset \Theta_{A_k(\phi_1)}$,

(C) $\Theta \subset \Theta_{A_h(\phi_2)}$.

Suppose that (A) holds. Then Θ is an irreducible component of $i_+^{-1}\mathbb{P}(A)$. Let $\{\xi_0, \ldots, \xi_3\}$ be the basis of $U^\vee$ dual to $\{v_0, \ldots, v_3\}$. Looking at (4.4.2) and (7.4.7) we get that $i_+^{-1}\mathbb{P}(A)$ is one of

$$C, \qquad Q_0, \qquad V(\xi_0\xi_2 - \xi_1^2, \xi_1\xi_3 - \xi_2^2).$$

It follows that Θ is one of the following:

(A1) $\Theta_{A_+(U)}$ (and hence $A_{\mathbf{c},\mathbf{d}} = A_+(U)$),

(A2) $i_+(Q_0)$,

(A3) $i_+(C)$.

(A4) $\{i_+([\xi_0 u_0 + \xi_3 u_3]) \mid [\xi_0, \xi_3] \in \mathbb{P}^1\}$,

Next suppose that (B) holds. Then $k^{-1}(\Theta_{A_{\mathbf{c},\mathbf{d}}})$ is an intersection of cubics containing the smooth conic D_1 and hence Θ is one of the following:

(B1) $\Theta_{A_k(\phi_1)}$ (and hence $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$),

(B2) $\bigwedge^3 \phi_1 \circ k(D_1) (= i_+(C))$,

(B3) $\bigwedge^3 \phi_1 \circ k(\langle p_1, q_1\rangle)$, $\bigwedge^3 \phi_1 \circ k(\langle r_1, p_1\rangle)$ or $\bigwedge^3 \phi_1 \circ k(\langle r_1, q_1\rangle)$,

(B4) $\{\bigwedge^3 \phi_1 \circ k(r_1)\} = \langle\langle T_1(Q_0)\rangle\rangle$.

Lastly suppose that (C) holds. Arguing as above we get that Θ is one of the following:

(C1) $\Theta_{A_h(\phi_2)}$ (and hence $A_{\mathbf{c},\mathbf{d}} = A_h(\phi_2)$),

(C2) $\bigwedge^3 \phi_2 \circ h(D_2) (= i_+(C))$,

(C3) $\bigwedge^3 \phi_2 \circ h(\langle p_2, q_2\rangle)$, $\bigwedge^3 \phi_2 \circ h(\langle r_2, p_2\rangle)$ or $\bigwedge^3 \phi_2 \circ h(\langle r_2, q_2\rangle)$,

(C4) $\{\bigwedge^3 \phi_2 \circ h(r_2)\} = \langle\langle T_2(Q_0)\rangle\rangle$.

A quick glance at Items (A1)-(A4), (B1)-(B4), (C1)-(C4) gives that if $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} \ge 2$ then one of (A1), (A2), (B1) or (C1) holds. A straightforward computation gives that (A1) or (A2) holds if and only if $c_1 = 0$ (see (7.4.7)). Next let's prove that (B4) or (C4) holds if and only if $\mathbf{d} = [1, -1]$ or $\mathbf{d} = [1, 1]$ respectively. Let Q_0 be as in (7.4.13): it is a smooth quadric containing C. A computation gives that

$$\langle\langle T_1(Q_0)\rangle\rangle = \langle v_1, (v_2 + v_3), v_4\rangle. \tag{7.4.30}$$

It follows that $\langle\langle T_1(Q_0)\rangle\rangle$ is an element of $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ if and only if $d_0 + d_1 = 0$. Similarly

$$\langle\langle T_2(Q_0)\rangle\rangle = \langle v_0, v_2 - v_3, v_5\rangle. \tag{7.4.31}$$

(Notice: $R_q(\bigwedge^3\langle\langle T_1(Q_0)\rangle\rangle) = \bigwedge^3\langle\langle T_2(Q_0)\rangle\rangle$.) It follows that $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ contains $\langle\langle T_2(Q_0)\rangle\rangle$ if and only if $d_0 - d_1 = 0$. Next we will prove that $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$ if and only if $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$. Suppose that $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$. Then $\langle\langle T_1(Q_0)\rangle\rangle$ is an element of $A_{\mathbf{c},\mathbf{d}}$ and hence $\mathbf{d} = [1, -1]$ by the computation above. Let

$$Q_1 = V(\xi_0\xi_2 - \xi_1^2 + \xi_1\xi_3 - \xi_2^2) \subset \mathbb{P}(U).$$

Thus Q_1 is another smooth quadric containing C. A computation shows that

$$\langle\langle T_1(Q_1)\rangle\rangle = \langle v_0 + v_2, v_1 + v_4, v_2 + v_5\rangle.$$

It follows that $\langle\langle T_1(Q_1)\rangle\rangle$ is an element of $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ if and only if

$$\begin{aligned} A_{\mathbf{c},\mathbf{d}} \ni 4(v_0+v_2)\wedge(v_1+v_4)\wedge(v_2+v_5) = \\ = 4\alpha_{(2,0,0,0)} + (\alpha_{(0,2,0,0)} + \alpha_{(1,0,1,0)}) - (\alpha_{(0,2,0,0)} - \alpha_{(1,0,1,0)}) - (4\beta_{(0,0,2,0)} - 2\beta_{(0,1,0,1)}) + \\ + (\alpha_{(0,0,2,0)} + \alpha_{(0,1,0,1)}) - (\alpha_{(0,0,2,0)} - \alpha_{(1,0,1,0)}) - (4\beta_{(0,2,0,0)} - 2\beta_{(1,0,1,0)}) + 4\alpha_{(0,0,0,2)}. \end{aligned}$$

The above holds if and only if $c_0 - c_1 = 0$. This proves that if $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$ then $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$, and by (7.4.28) there exists such a $(\mathbf{c}, \mathbf{d})$, thus $A_{\mathbf{c},\mathbf{d}} = A_k(\phi_1)$ if and only if $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$. By **Proposition 7.4.7** it follows that $A_{\mathbf{c},\mathbf{d}} = A_h(\phi_2)$ if and only if $(\mathbf{c}, \mathbf{d}) = ([1, -1], [1, 1])$. This proves that if $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} \ge 2$ then one of (s1), (s2) or (s3) holds. Now suppose that $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} = 1$. We showed above that one of (A3), (A4), (B3), (B4), (C3) or (C4) holds, thus it is clear that one of (t1) - (t6) holds. We have also shown that $\langle\langle T_1(Q_0)\rangle\rangle \in \Theta_{A_{\mathbf{c},\mathbf{d}}}$ if and only if $d_0 + d_1 = 0$ and that $\langle\langle T_2(Q_0)\rangle\rangle \in \Theta_{A_{\mathbf{c},\mathbf{d}}}$ if and only if $d_0 + d_1 = 0$. □

Corollary 7.4.12. *Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^{\psi}_{\text{fix}}$. Then $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} = \mathbb{P}(W_\infty)$ if and only if either $c_1 = 0$ or $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$.*

Proof. If $c_1 = 0$ or $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$ then $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} = \mathbb{P}(W_\infty)$ by **Proposition 7.4.11** - see **Claim 4.4.5** and (4.5.5). Thus it remains to prove the converse. Suppose that $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} = \mathbb{P}(W_\infty)$. By **Corollary 3.3.7** it follows that $\mathcal{B}(W_\infty, A_{\mathbf{c},\mathbf{d}}) = \mathbb{P}(W_\infty)$. Thus one of the following holds:

(a) Given a generic $[v] \in \mathbb{P}(W_\infty)$ there exists $W \in \Theta_{A_{\mathbf{c},\mathbf{d}}}$ containing v.

(b) For any $[v] \in \mathbb{P}(W_\infty)$ we have

$$\dim(A_{\mathbf{c},\mathbf{d}} \cap S_{W_\infty} \cap F_v) \ge 2. \tag{7.4.32}$$

If (a) holds then $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} \ge 2$. By **Proposition 7.4.11**, (4.5.5) and (4.5.6) we get that either $c_1 = 0$ or $(\mathbf{c}, \mathbf{d}) = ([1, 1], [1, -1])$. Now suppose that (a) does not hold and that (b) holds. Then

$$\dim(A_{\mathbf{c},\mathbf{d}} \cap S_{W_\infty}) \ge 4 \tag{7.4.33}$$

and of course $c_1 \ne 0$. A straightforward computation gives that (7.4.33) holds if and only if $d_1 = 0$ and in that case

$$\begin{aligned} A_{\mathbf{c},\mathbf{d}} \cap S_{W_\infty} = \langle v_0\wedge v_1\wedge v_2,\ v_0\wedge v_1\wedge v_4 - v_0\wedge v_2\wedge v_3, \\ (c_0+c_1)v_0\wedge v_1\wedge v_5 - 2c_1 v_0\wedge v_2\wedge v_4 - (c_0-c_1)v_1\wedge v_2\wedge v_3,\ v_0\wedge v_2\wedge v_5 - v_1\wedge v_2\wedge v_4\rangle. \end{aligned} \tag{7.4.34}$$

Given (7.4.34) one checks easily that the set of $[v] \in \mathbb{P}(W_\infty)$ for which (7.4.32) holds is a proper subset of $\mathbb{P}(W_\infty)$, in fact the union of a line and a singleton: that is a contradiction. □

Proof of **Proposition 7.4.2**. We start by noting that the coefficients a, b appearing on the right-hand side of (7.4.12) are bihomogeneous polynomials in $(\mathbf{c}, \mathbf{d})$ of degrees $(2,1)$ - this is a consequence of the discussion that follows (3.2.21). It follows that $\mathbb{V}^\psi$ is the zero-set of $(a+b) \in H^0(\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$. By **Proposition 7.4.11** and **Claim 4.4.5** we know that $\{(\mathbf{c}, \mathbf{d}) \mid c_1 = 0\}$ is contained in $\mathbb{V}^\psi$; it follows that there exists $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ with $m \le 1$ such that

$$\mathbb{V}^\psi = \{A_{\mathbf{c},\mathbf{d}} \mid c_1 = 0\} \cup V(\sigma).$$

Let's show that $\sigma \neq 0$. Suppose the contrary holds i.e. that $\sigma = 0$. It follows that the locus of $(\mathbf{c}, \mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} = \mathbb{P}(W)$ is the zero-set of $a \in H^0(\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$: that contradicts **Corollary 7.4.12**. This proves that $\sigma \neq 0$. By **Proposition 7.4.11** and (4.5.5), (4.5.6) we have

$$([1,1],[1,-1]),([1,-1],[1,1]) \in V(\sigma). \tag{7.4.35}$$

It follows that $m = 1$ i.e.

$$\sigma \in H^0(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)). \tag{7.4.36}$$

It remains to prove that

$$V(\sigma) = \{A_{\mathbf{c},\mathbf{d}} \mid c_0 d_1 + c_1 d_0 = 0\}. \tag{7.4.37}$$

We will show that

$$V(\sigma) \cap \{(\mathbf{c}, \mathbf{d}) \mid d_1 = 0\} = \{([1,0],[1,0])\}. \tag{7.4.38}$$

Granting the above equality we get (7.4.37) by noting that there is a single divisor in $|H^0(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))|$ whose zero-locus contains $([1,1],[1,-1])$, $([1,-1],[1,1])$ and $([1,0],[1,0])$ namely the right-hand side of (7.4.37). It remains to prove (7.4.38). By (7.4.36) the intersection number of $V(\sigma)$ and the "vertical"line $\mathbb{P}^1 \times \{[1,0]\}$ is equal to 1: thus in order to prove (7.4.38) it suffices to show that if $c_1 \neq 0$ and $d_1 = 0$ then $A_{\mathbf{c},\mathbf{d}} \notin V(\sigma)$. Let $(\mathbf{c}, \mathbf{d}) \in \mathbb{P}^1 \times \mathbb{P}^1$: as is easily checked $d_1 = 0$ if and only if

$$\Theta_{A_{\mathbf{c},\mathbf{d}}} \supset i_+(\{[\xi_0 u_0 + \xi_3 u_3]) \mid [\xi_0, \xi_3] \in \mathbb{P}^3\}). \tag{7.4.39}$$

Now suppose that $d_1 = 0$ and $c_1 \neq 0$. By **Proposition 7.4.11** we know that $\dim \Theta_{A_{\mathbf{c},\mathbf{d}}} = 1$. Thus the conic on the right-hand side of (7.4.39) is an irreducible component of $\Theta_{A_{\mathbf{c},\mathbf{d}}}$. Now let $p \in (C \setminus \{[1,0,0,0]\}$ be close to $[1,0,0,0]$ and set $W = i_+(p)$. By **Corollary 7.4.12** we know that $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} \neq \mathbb{P}(W_\infty)$. By continuity it follows that $C_{W, A_{\mathbf{c},\mathbf{d}}} \neq \mathbb{P}(W)$. On the other hand we see immediatly that $\mathcal{B}(W, A_{\mathbf{c},\mathbf{d}})$ contains a conic and a line (the "projections "from p of C and $\langle [1,0,0,0],[0,0,0,1] \rangle$ respectively). Thus $C_{W, A_{\mathbf{c},\mathbf{d}}} = 2D + 2L$ where D is a smooth conic and L is a line (intersecting D transversely). By continuity and (7.4.12) it follows that $C_{W_\infty, A_{\mathbf{c},\mathbf{d}}} = V((X_0X_2 - X_1^2)^2 X_1^2)$, in particular $(\mathbf{c}, \mathbf{d}) \notin \mathbb{V}^\psi$ and a fortiori $(\mathbf{c}, \mathbf{d}) \notin V(\sigma)$. This proves that (7.4.38) holds. □

Proposition 7.4.13. *Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^\psi_{\mathrm{fix}}$. Then $A_{\mathbf{c},\mathbf{d}}$ is not $G_{\mathcal{N}_3}$-stable if and only if*

$$c_1 d_1 (c_0^2 - c_1^2) = 0. \tag{7.4.40}$$

Proof. A straightforward application of **Proposition 7.3.1**. □

Corollary 7.4.14. *Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{X}^\psi_{\mathcal{Z}}$. Then $A_{\mathbf{c},\mathbf{d}}$ is semistable with minimal* $\mathrm{PGL}(V)$*-orbit.*

Proof. If $A_{\mathbf{c},\mathbf{d}}$ is $G_{\mathcal{N}_3}$-stable then it has minimal $\mathrm{PGL}(V)$-orbit by **Corollary 4.2.2**. By **Proposition 7.4.13** $A_{\mathbf{c},\mathbf{d}}$ is not $G_{\mathcal{N}_3}$-stable if and only if $(\mathbf{c}, \mathbf{d})$ is one of

$$([1,0],[1,0]), \qquad ([1,1],[1,-1]), \qquad ([1,-1],[1,1]).$$

Now $A_{[1,0],[1,0]}$ is equal to $A_+(U)$, $A_{[1,1],[1,-1]}$ is equal to $A_k(\phi_1)$ by **Proposition 7.4.11** and $A_{[1,-1],[1,1]}$ is equal to $A_h(\phi_2)$ by the same proposition. They all have minimal $\mathrm{PGL}(V)$-orbits by **Proposition 4.1.2**. □

Proof of **Proposition 7.4.9**. $\mathfrak{X}_{\mathcal{Z}}$ is irreducible of dimension at most 1 because $\mathbb{X}^{\psi}_{\mathcal{Z}}$ is irreducible of dimension 1. By **Proposition 7.4.11** $A_{[1,1],[1,-1]}, A_{[1,-1],[1,1]}$ are equal to $A_k(\phi_1)$ and $A_h(\phi_2)$ respectively; since $([1,1],[1,-1]),([1,-1],[1,1]) \in \mathbb{X}^{\psi}_{\mathcal{Z}}$ we get that $\mathfrak{x}, \mathfrak{x}^{\vee} \in \mathfrak{X}_{\mathcal{Z}}$. Since $\mathfrak{x} \neq \mathfrak{x}^{\vee}$ it follows that $\mathfrak{X}_{\mathcal{Z}}$ is an irreducible curve. Lastly let us prove that $\mathfrak{X}_{\mathcal{Z}} \cap \mathfrak{X}_{\mathcal{V}} = \{\mathfrak{y}\}$. Let $[A] \in \mathfrak{X}_{\mathcal{Z}} \cap \mathfrak{X}_{\mathcal{V}}$ and suppose that the $\mathrm{PGL}(V)$-orbit of A is minimal. By **Corollary 7.4.14** there exists $(\mathbf{c}, \mathbf{d}) \in \mathbb{X}^{\psi}_{\mathcal{Z}}$ such that $A_{\mathbf{c},\mathbf{d}}$ is in the $\mathrm{PGL}(V)$-orbit of A; it follows that $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ contains a rational normal curve of degree 4 (the curve $i_+(D)$ appearing in (7.2.5)). By **Proposition 7.4.11** we get that $c_1 = 0$ and hence $(\mathbf{c}, \mathbf{d}) = ([1,0],[1,0])$. Since $A_{[1,0],[1,0]} = A_+(U)$ we are done. □

7.4.4 Points of $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I}$ are represented by lagrangians in $\mathbb{Y}^{\psi}_{\mathrm{fix}}$

In the present subsubsection we will prove the result below.

Proposition 7.4.15. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is semistable with minimal orbit and $[A] \in \mathfrak{I}$. There exist $g \in \mathrm{PGL}(V)$ such that $gA \in \mathbb{Y}^{\psi}_{\mathrm{fix}}$.*

The proof of **Proposition 7.4.15** will be given at the end of the present subsubsection.

Lemma 7.4.16. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is semistable with minimal orbit and $[A] \in \mathfrak{I}$. There exists*

$$\overline{W} \in \{W_\infty, \langle v_0, \gamma, v_5\rangle, \langle v_1, \gamma, v_4\rangle, W_0\}, \qquad \gamma \in V_{23} \tag{7.4.41}$$

such that $\overline{W} \in \Theta_A$ and $C_{\overline{W},A}$ is either $\mathbb{P}(\overline{W})$ or a sextic curve in the indeterminacy locus of Map (0.0.3).

Proof. By hypothesis there exists $W_\star \in \Theta_A$ such that $C_{W_\star,A}$ is either $\mathbb{P}(W_\star)$ or a sextic curve in the indeterminacy locus of Map (0.0.3). Suppose that $C_{W_\star,A} = \mathbb{P}(W_\star)$. By **Proposition 6.1.9** we have $[A] \in \mathfrak{X}^*_{\mathcal{W}} \cup \{\mathfrak{x}\}$. By **Claim 4.4.5** and (4.5.5) we get that $C_{W,A} = \mathbb{P}(W)$ for every $W \in \Theta_A$ in particular for $W = W_\infty$ (or $W = W_0$). Thus from now on we may assume that

$$\text{for all } W \in \Theta_A \text{ we have } C_{W,A} \neq \mathbb{P}(W). \tag{7.4.42}$$

Taking $\lim_{t\to 0} \lambda_{\mathcal{N}_3}(t)W$ we get that there exists $\overline{W} \in \Theta_A$ such that $C_{\overline{W},A}$ is a sextic curve in the indeterminacy locus of Map (0.0.3) and $\overline{W}$ is fixed by $\lambda_{\mathcal{N}_3}(t)$ for all $t \in \mathbb{C}^\times$. Thus $\overline{W}$ is the direct sum of 3 irreducible summands for the representation $\lambda_{\mathcal{N}_3}\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ i.e. one of W_∞, W_0 or

$$\langle v_0,v_1,v_4\rangle, \langle v_0,v_1,v_5\rangle, \langle v_0,\gamma,v_4\rangle, \langle v_0,\gamma,v_5\rangle, \langle v_0,v_4,v_5\rangle, \langle v_1,\gamma,v_4\rangle, \langle v_1,\gamma,v_5\rangle, \langle v_1,v_4,v_5\rangle, [v_i] \oplus V_{23} \tag{7.4.43}$$

where $\gamma \in V_{23}$. Let $W_1 \neq W_2 \in \Theta_A$: by **Proposition 6.6.1** we get that $\dim(W_1 \cap W_2) = 1$. Thus we may exclude from (7.4.43) all the subspaces which intersect one of W_∞, W_0 in a 2-dimensional space. It follows that $\overline{W}$ is one of

$$W_\infty, \langle v_0,\gamma,v_4\rangle, \langle v_0,\gamma,v_5\rangle, \langle v_1,\gamma,v_4\rangle, \langle v_1,\gamma,v_5\rangle, W_0.$$

It remains to prove that we cannot have $\overline{W} = \langle v_0,\gamma,v_4\rangle$ nor $\overline{W} = \langle v_1,\gamma,v_5\rangle$. Suppose first that $\overline{W} = \langle v_0,\gamma,v_4\rangle$. Then Item (2) of **Proposition 7.3.1** holds and hence $\lim_{s\to 0} \lambda'_1(s)A$ exists and belongs to $\mathbb{LG}(\bigwedge^3 V)^{ss}$ (if ω generates $\bigwedge^{10} A$ then $\lim_{s\to 0} \lambda'_1(s)\omega$ exists and is non-zero) - see (7.3.12), (7.3.13) and Item (2′) in the proof of **Proposition 7.3.1**. By hypothesis the orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{S}^{\mathsf{F},ss}_{\mathcal{N}_3}$; thus we may replace A by $\lim_{s\to 0} \lambda'_1(s)A$ and hence we may assume that $\lambda'_1(s)$ acts trivially on $\bigwedge^{10} A$ for every $s \in \mathbb{C}^\times$. Let $C_{\overline{W},A} = V(P)$ where $0 \neq P \in \mathbb{C}[X,Y,Z]_6$ - here $\{X,Y,Z\}$ is the basis of $\overline{W}^{\vee}$ dual to $\{v_0,\gamma,v_4\}$. We know that $\lambda'_1(s)$ and $\lambda_{\mathcal{N}_3}(t)$ act trivially on $\bigwedge^{10} A$ for $(s,t) \in \mathbb{C}^\times \times \mathbb{C}^\times$. Applying **Claim 3.2.4** we get that all elements of $\mathrm{SL}(\overline{W})$ given by $\mathrm{diag}(s^{-2}t^5, s^{-2}t^{-1}, s^4t^{-4})$ act trivially on P. It follows that $P = aX^2Y^2Z^2$ and by (7.4.42) we have $a \neq 0$, that is a contradiction. Next suppose that $\overline{W} = \langle v_1,\gamma,v_5\rangle$. Then Item (1) of **Proposition 7.3.1** holds: one excludes this case arguing as above. □

Proposition 7.4.17. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is semistable with minimal orbit and $[A] \in \mathfrak{I}$. Then* $\dim \Theta_A \geq 1$.

Proof. By contradiction. Suppose that $\dim \Theta_A = 0$. In particular

$$\text{if } W_1 \neq W_2 \in \Theta_A \text{ then } \dim(W_1 \cap W_2) = 1. \tag{7.4.44}$$

Moreover $C_{W,A}$ is a sextic curve for every $W \in \Theta_A$ by **Corollary 6.1.10**. By **Lemma 7.4.16** there exists $\overline{W} \in \Theta_A$ such that (7.4.41) holds and $C_{\overline{W},A}$ is a sextic curve in the indeterminacy locus of Map (0.0.3). We claim that

$$\dim S_{\overline{W}} \leq 3. \tag{7.4.45}$$

In fact suppose that (7.4.45) does not hold. Then $A \in \mathbb{B}_{\mathcal{C}_1}$: by **Proposition 6.1.1** we get that $A \in \mathrm{PGL}(V)A_+$, that is a contradiction because $\dim \Theta_{A_+} = 3$. Let $\{w_0, w_1, w_2\}$ be the basis of $\overline{W}$ appearing in (7.3.4) or in (7.4.41): thus $w_0 = v_0$ if $\overline{W} = W_\infty$ or $\overline{W} = \langle v_0, \gamma, v_5\rangle$, $w_0 = v_1$ if $\overline{W} = \langle v_1, \gamma, v_4\rangle$, $w_0 = \gamma_0$ if $\overline{W} = W_0$ etc. Let $\{X_0, X_1, X_2\}$ be the basis of $\overline{W}^\vee$ dual to $\{w_0, w_1, w_2\}$. The 1-PS $\lambda_{\mathcal{N}_3}$ acts trivially on $\bigwedge^{10} A$; applying **Claim 3.2.4** we get that $C_{\overline{W},A} = V(P)$ where

$$P = (b_1 X_0 X_2 + a_1 X_1^2)(b_2 X_0 X_2 + a_2 X_1^2)(b_3 X_0 X_2 + a_3 X_1^2). \tag{7.4.46}$$

Since $C_{\overline{W},A}$ is a sextic curve in the indeterminacy locus of Map (0.0.3) one gets that one of the following holds:

(1) $C_{\overline{W},A} = V((bX_0X_2 + aX_1^2)^3)$.

(2) $C_{\overline{W},A} = V(X_0^2 X_2^2 (bX_0X_2 + X_1^2))$.

(3) $C_{\overline{W},A} = V(X_1^4 (bX_0X_2 + aX_1^2))$.

Let Z be the union of 1-dimensional components of $\operatorname{sing} C_{\overline{W},A}$: in all of the above cases Z is non-empty. By **Proposition 3.3.6** we have $Z \subset \mathcal{B}(\overline{W}, A)$. Arguing exactly as in the proof of **Proposition 7.2.16** one shows that

$$\dim(A \cap S_{\overline{W}}) = 3 \tag{7.4.47}$$

and that Item (1) or Item (2) leads to a contradiction. Lastly suppose that Item (3) holds. Let $V = \overline{W} \oplus U$ where U is $\lambda_{\mathcal{N}_3}$-invariant. Let $\mathcal{V} := S_{\overline{W}} \cap (\bigwedge^2 \overline{W} \wedge U)$. By (7.4.47) we have $\dim \mathcal{V} = 2$. View $\mathcal{V}$ as a subspace of $\mathrm{Hom}(\overline{W}, U)$ by choosing a volume form on $\overline{W}$: every $\phi \in \mathcal{V}$ has rank 2 and $K(\mathcal{V})$ (notation as in (A.3.6)) is the line $V(X_1)$. By **Proposition A.3.1** we get that $\mathcal{V}$ is $\mathrm{GL}(\overline{W}) \times \mathrm{GL}(U)$-equivalent to $\mathcal{V}_l$. Thus there exists a basis $\{u_0, u_1, u_2\}$ of U such that

$$\mathcal{V} = \langle w_0 \wedge w_1 \wedge u_0 + w_0 \wedge w_2 \wedge u_1,\ w_0 \wedge w_2 \wedge u_2 + w_1 \wedge w_2 \wedge u_0 \rangle. \tag{7.4.48}$$

Up to scalars there is a unique non-zero element of $\mathcal{V}$ mapping w_0 to 0 and similarly there is a unique (up to scalars) non-zero element of $\mathcal{V}$ mapping w_2 to 0: since $\mathcal{V}$, $[w_0]$ and $[w_2]$ are $\lambda_{\mathcal{N}_3}$-invariant it follows that the two elements of $\mathcal{V}$ appearing in (7.4.48) generate $\lambda_{\mathcal{N}_3}$-invariant subspaces. Since each w_i generates a $\lambda_{\mathcal{N}_3}$-invariant subspace it follows that each u_j generates a $\lambda_{\mathcal{N}_3}$-invariant subspace. Considering the possible weights of the u_j's we see that we cannot have $\overline{W} = \langle v_0, \gamma, v_5\rangle$ nor $\overline{W} = \langle v_1, \gamma, v_4\rangle$. Suppose that $\overline{W} = W_\infty$. We may (and will) choose $v_2 := w_2 = \gamma_0$ and v_3 to be a generator of the $\lambda_{\mathcal{N}_3}$-invariant subspace of U. Considering the possible weights of the u_j's we get that $u_0 \in [v_4]$, $u_1 \in [v_3]$ and $u_2 \in [v_5]$. Rescaling v_3, v_4, v_5 we get that

$$\mathcal{V} = \langle v_0 \wedge v_1 \wedge v_4 + v_0 \wedge v_2 \wedge v_3,\ v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4 \rangle.$$

Thus $(v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4) \in A \cap S_{\overline{W}}$. Now $A \cap S_{\overline{W}}$ contains a 3-dimensional subspace R dictated by the condition $A \in \mathbb{B}_{\mathcal{N}_3}$ - see Table (1) - and $(v_0 \wedge v_2 \wedge v_5 + v_1 \wedge v_2 \wedge v_4) \notin R$. Thus $\dim(A \cap S_{\overline{W}}) \geq 4$ and that contradicts (7.4.47). It remains to deal with the case $\overline{W} = W_0$: it is similar to the case $\overline{W} = W_\infty$. □

Proposition 7.4.18. *Suppose that $A \in \mathbb{S}^{\mathsf{F}}_{\mathcal{N}_3}$ is semistable with minimal orbit and that $[A] \in \mathfrak{I}$. Then Θ_A contains $i_+(C)$ for some choice of Isomorphism* (7.4.1).

Proof. By **Proposition 7.4.17** we know that $\dim\Theta_A \geq 1$. If $\dim\Theta_A \geq 2$ then by **Lemma 6.1.8** we have $[A] \in \mathfrak{X}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$, and we are done by **Claim 7.4.3** and **Proposition 7.4.9**. Thus from now on we may assume that $\dim\Theta_A = 1$. Let Θ be a 1-dimensional irreducible component of Θ_A. By Theorem 3.9 of [28] the curve Θ belongs to one of the Types

$$\mathcal{F}_1, \mathcal{D}, \mathcal{E}_2, \mathcal{E}_2^\vee, \mathbf{Q}, \mathcal{A}, \mathcal{A}^\vee, \mathcal{C}_2, \mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{T}^\vee$$

defined in [28]. Moreover if Θ if of calligraphic Type $\mathcal{X}$ then $A \in \mathbb{B}_{\mathcal{X}}$ - see Claim 3.22 of [28]. Thus if Θ has calligraphic Type then $A \in \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_2}$; by (2.6.4) we get that $[A] \in \mathfrak{B}_{\mathcal{A}} \cup \mathfrak{B}_{\mathcal{C}_1} \cup \mathfrak{B}_{\mathcal{D}} \cup \mathfrak{B}_{\mathcal{E}_1} \cup \mathfrak{B}_{\mathcal{E}_1^\vee}$ and hence $[A] \in \mathfrak{X}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$ by **Proposition 6.1.1**, **Proposition 6.2.1**, **Proposition 6.3.1**, **Proposition 6.4.2** and **Proposition 6.5.1**. As noticed above it follows that Θ_A contains $i_+(C)$ for some choice of Isomorphism (7.4.1). Thus from now on we may assume that Θ is of Type **Q**, **R**, **S**, **T** or $\mathbf{T}^\vee$. Now notice that if $t \in \mathbb{C}^\times$ then $\lambda_{\mathcal{N}_3}(t)$ acts on Θ i.e. $\lambda_{\mathcal{N}_3}(t)|_\Theta$ is an automorphism of Θ. Suppose that $\lambda_{\mathcal{N}_3}(t)|_\Theta$ is the identity for each $t \in \mathbb{C}^\times$: looking at the action of $\lambda_{\mathcal{N}_3}(t)$ on V we get that Θ is a line and hence $A \in \mathbb{B}_{\mathcal{F}_1}$. By **Proposition 6.6.1** we have $\mathfrak{B}_{\mathcal{F}_1} \cap \mathfrak{I} = \emptyset$ and hence we get a contradiction. It follows that if $t \in \mathbb{C}^\times$ is generic then $\lambda_{\mathcal{N}_3}(t)|_\Theta$ is not the identity - in particular there exist points in Θ with dense orbit and hence Θ has geometric genus 0. We claim that there does not exist a Θ of Type **Q**, **S**, **T** or $\mathbf{T}^\vee$ such that $\lambda_{\mathcal{N}_3}(t)(\Theta) = \Theta$ for $t \in \mathbb{C}^\times$. In fact suppose that Θ has type **Q**. Then we may assume that $\Theta = i_+(D)$ where $D \subset \mathbb{P}(U)$ is the conic given by (7.2.4). Arguing as in the proof of **Proposition 7.2.12** we may assume that each $\lambda_{\mathcal{N}_3}(t)$ is induced by a projectivity of $\mathbb{P}(U)$: as is easily checked that is impossible. On the other hand Θ cannot be of Type **S**, **T** or $\mathbf{T}^\vee$ because there is no 1-PS of $\mathrm{PGL}(V)$ mapping such a curve to itself. (There is no copy of $\mathbb{C}^\times$ in the automorphism group of such a curve acting trivially on the Picard group of the curve.) Thus we have proved that Θ is of Type **R**: a curve of such type is equal (up to projectivities) to $i_+(C)$ where C is given by (7.4.4) and the proposition follows. □

Proof of **Proposition 7.4.15**. Assume first that $\dim\Theta_A \geq 2$. By **Lemma 6.1.8** we have $[A] \in \mathfrak{X}_{\mathcal{W}} \cup \{\mathfrak{x}, \mathfrak{x}^\vee\}$ and the result follows from **Claim 7.4.3** and **Proposition 7.4.9**. It remains to deal with the case $\dim\Theta_A \leq 1$: by **Proposition 7.4.18** there exists an irreducible component Θ of Θ_A which is projectively equivalent to $i_+(C)$. The 1-PS $\lambda^{\mathsf{F}}_{\mathcal{N}_3}$ fixes A hence it acts on Θ: the action is effective because the set of fixed points for the action of $\lambda^{\mathsf{F}}_{\mathcal{N}_3}$ on $\mathrm{Gr}(3, V)$ is a collection of points and lines. The image H consists of the group of automorphisms fixing two (distinct) points $p, q \in \Theta$. On the other hand by Theorem 3.9 of [28] there exists $g \in \mathrm{PGL}(V)$ such that $g\Theta = i_+(C)$: we may choose g so that $g(p) = i_+([1,0,0,0])$ and $g(q) = i_+([0,0,0,1])$. With this choice of g the group H gets identified with the group of automorphisms of C fixing $[1,0,0,0]$ and $[0,0,0,1]$. Thus $gA \in \mathbb{Y}^\psi$ by definition of $\mathbb{Y}^\psi$. □

7.4.5 Proof that $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{W}} \cup \mathfrak{X}_{\mathcal{Z}}$

We will prove (at the end of the present subsubsection) the following result.

Proposition 7.4.19. *Let $A_{\mathbf{c},\mathbf{d}} \in \mathbb{Y}^\phi_{\mathrm{fix}}$. There exists $W \in \Theta_{A_{\mathbf{c},\mathbf{d}}}$ such that $C_{W,A_{\mathbf{c},\mathbf{d}}}$ is either $\mathbb{P}(W)$ or a sextic in the indeterminacy locus of the period map* (0.0.3) *if and only if $A_{\mathbf{c},\mathbf{d}} \in \mathbb{V}^\psi$.*

The equality $\mathfrak{X}_{\mathcal{N}_3} \cap \mathfrak{I} = \mathfrak{X}_{\mathcal{W}} \cup \mathfrak{X}_{\mathcal{Z}}$ follows from **Proposition 7.4.15**, **Proposition 7.4.19** and **Claim 7.4.3**. We will begin by analyzing $A_{\mathbf{c},[1,\pm 1]}$. Let

$$W_+ := \langle\langle T_2(Q_0)\rangle\rangle = \langle v_0, v_2 - v_3, v_5\rangle, \quad W_- := \langle\langle T_1(Q_0)\rangle\rangle = \langle v_1, v_2 + v_3, v_4\rangle. \tag{7.4.49}$$

By **Proposition 7.4.11** we have $W_\pm \in \Theta_{A_{\mathbf{c},[1,\pm 1]}}$.

Claim 7.4.20. *Let $\{Z_0, Z_1, Z_2\}$ be the basis of $W_\pm^\vee$ dual to the basis of $W_\pm$ appearing in* (7.4.49). *There exist homogeneous quadratic polynomials $P_\pm, Q_\pm \in \mathbb{C}[c_0, c_1]$ such that*

$$C_{W_\pm, A_{\mathbf{c},[1,\pm 1]}} = V((Z_0Z_2 - Z_1^2)^2(P_\pm(\mathbf{c})Z_0Z_2 + Q_\pm(\mathbf{c})Z_1^2)). \tag{7.4.50}$$

Proof. Applying **Claim 3.2.4** to the action of $\lambda_{\mathcal{N}_3}$ on $W_\pm$ we get that $C_{W_\pm,A_{\mathbf{c},[1,\pm 1]}}$ has equation $f_{\mathbf{c}} := \prod_{i=1}^3 (b_i(\mathbf{c})Z_0Z_2 + a_i(\mathbf{c})Z_1^2)$. Let $p \in C$; by **Corollary 3.3.7** the differential of $f_{\mathbf{c}}$ vanishes at $W_\pm \cap i_+(p)$. Since

$$\{W_\pm \cap i_+(p) \mid p \in C\} = V(Z_0Z_2 - Z_1^2) \tag{7.4.51}$$

we get that (7.4.50) holds. We may assume that $P^\pm, Q^\pm$ are homogeneous polynomials of degree 2 (beware that they are determined only up to a common scalar factor) by (3.2.22) and (3.2.23). □

Proposition 7.4.21. *Let notation be as in* **Claim 7.4.20**. *The point with* Z*-coordinates* $[0,1,0]$

(1) belongs to $C_{W_+,A_{\mathbf{c},[1,1]}}$ *if and only if* $\mathbf{c} = [3,-1]$,

(2) belongs to $C_{W_-,A_{\mathbf{c},[1,-1]}}$ *if and only if* $\mathbf{c} = [1,1]$.

Moreover

$$C_{W_+,A_{[3,-1],[1,1]}} = V((Z_0Z_2 - Z_1^2)^2 Z_0Z_2), \quad C_{W_-,A_{[1,1],[1,-1]}} = \mathbb{P}(W_-). \tag{7.4.52}$$

Proof. The point in $\mathbb{P}(W_+)$ with Z-coordinates $[0,1,0]$ is $[v_2 - v_3]$. By definition $[v_2 - v_3] \in C_{W_+,A_{\mathbf{c},[1,1]}}$ if and only if $\dim(F_{v_2-v_3} \cap A_{\mathbf{c},[1,1]}) \geq 2$. Thus the proposition is proved by a computation. A priori we need to compute the zeroes of a 9×9 determinant with entries functions of c_0, c_1. We explain why the computation breaks up into a series of trivial calculations. The intersection $F_{v_2-v_3} \cap A_{\mathbf{c},[1,1]}$ is the kernel of the multiplication map

$$\begin{array}{ccc} A_{\mathbf{c},[1,1]} & \longrightarrow & \bigwedge^4 V \\ \alpha & \mapsto & (v_2 - v_3) \wedge \alpha \end{array} \tag{7.4.53}$$

Both $A_{\mathbf{c},[1,1]}$ and $\bigwedge^4 V$ are $\mathbb{C}^\times$-modules because $\lambda_{\mathcal{N}_3}$ acts on them; let $A_{\mathbf{c},[1,1]}(t^m) \subset A_{\mathbf{c},[1,1]}$ be the weight-m susbspace. Map (7.4.53) is $\mathbb{C}^\times$-equivariant because $(v_2 - v_3)$ is $\lambda_{\mathcal{N}_3}$-invariant; hence its kernel is the direct-sum of the kernels of the multiplication maps $A_{\mathbf{c},[1,1]}(t^m) \to \bigwedge^4 V$. The kernels of these maps are readily computed. One gets that if $m \notin \{0, \pm 1\}$ the kernel is trivial for all $\mathbf{c}$,

$$F_{v_2-v_3} \cap A_{\mathbf{c},[1,1]}(t) = \begin{cases} \{0\} & \text{if } \mathbf{c} \neq [3,-1], \\ [(v_2 - v_3) \wedge (v_0 \wedge v_4 - v_1 \wedge v_3)] & \text{if } \mathbf{c} = [3,-1], \end{cases} \tag{7.4.54}$$

$$F_{v_2-v_3} \cap A_{\mathbf{c},[1,1]}(t^{-1}) = \begin{cases} \{0\} & \text{if } \mathbf{c} \neq [3,-1], \\ [(v_2 - v_3) \wedge (v_1 \wedge v_5 - v_3 \wedge v_4)] & \text{if } \mathbf{c} = [3,-1]. \end{cases} \tag{7.4.55}$$

Moreover the invariant part of $F_{v_2-v_3} \cap A_{\mathbf{c},[1,1]}$ is spanned by $(v_2 - v_3) \wedge v_0 \wedge v_5$. It follows that $[v_2 - v_3] \in C_{W_+,A_{\mathbf{c},[1,1]}}$ if and only if $\mathbf{c} = [3,-1]$. In addition we see that $[v_2 - v_3] \notin \mathcal{B}(W_+, A_{\mathbf{c},[1,1]})$: by **Proposition 3.3.6** we get that $C_{W_+,A_{[3,-1],[1,1]}}$ has an ordinary node at $[v_2 - v_3]$ and hence the first equality of (7.4.52) holds. Similar computations show that $[v_2 + v_3] \in C_{W_-,A_{\mathbf{c},[1,-1]}}$ (notice: $[v_2 + v_3]$ is the point of $\mathbb{P}(W_-)$ with Z-coordinates $[0,1,0]$) if and only if $\mathbf{c} = [1,1]$. The second equality of (7.4.52) holds because by **Proposition 7.4.11** we know that $A_{[1,1],[1,-1]} = A_k(\phi_1)$. □

Corollary 7.4.22. *Let* $\{Z_0, Z_1, Z_2\}$ *be the basis of* $W_\pm^\vee$ *dual to the basis of* $W_\pm$ *appearing in* (7.4.49). *Then*

$$C_{W_\pm,A_{[1,0],[1,\pm 1]}} = V((Z_0Z_2 - Z_1^2)^3).$$

Proof. By **Proposition 7.4.21** we know that $C_{W_\pm,A_{[1,0],[1,\pm 1]}} \neq \mathbb{P}(W_\pm)$. Thus (see **Corollary 3.2.3**) it suffices to show that

$$\dim(F_v \cap A_{[1,0],[1,\pm 1]}) \geq 4 \text{ if } [v] = W_\pm \cap i_+(p),\, p \in C. \tag{7.4.56}$$

Let $[v]$ be as above: then $v = \phi(\tau_0 \wedge \tau_1)$ where $\tau_0, \tau_1 \in U$ and $\mathbb{P}(\langle \tau_0, \tau_1 \rangle)$ is a line contained in Q_0. Given $q \in \mathbb{P}(\langle \tau_0, \tau_1 \rangle)$ we let $\alpha_q \in \bigwedge^3 V$ be a generator of $\bigwedge^3 i_+(q) = [\alpha_q]$: then $\alpha_q \in F_v \cap A_{[1,0],[1,\pm 1]}$. As q varies in $\mathbb{P}(\langle \tau_0, \tau_1 \rangle)$ the elements α_q span a 3-dimensional subsapace of $F_v \cap A_{[1,0],[1,\pm 1]}$ which does not contain a generator of $\bigwedge^3 W_\pm$; inequality (7.4.56) follows. □

Lemma 7.4.23. *If* $(\mathbf{c},\mathbf{d})\in\mathbb{P}^1\times\mathbb{P}^1$ *then* $C_{W_\infty,A_{\mathbf{c},\mathbf{d}}}$ *is projectively equivalent to* $C_{W_0,A_{\mathbf{c},\mathbf{d}}}$.

Proof. Let ι be the involution of $\mathbb{P}^1$ mapping $[\lambda,\mu]$ to $[\mu,\lambda]$. Equation (7.4.4) identifies $\mathbb{P}^1_{[\lambda,\mu]}$ with C: thus we may regard ι as an involution of C. In turn ι induces the involution on $\mathbb{P}(U)$ given by $[u_0,u_1,u_2,u_3]\mapsto[u_3,u_2,u_1,u_0]$ and also an involution $\varphi\in\mathrm{SL}(V)$ via the isomorphism $\psi\colon\bigwedge^2 U\xrightarrow{\sim}V$ of (7.4.1). A straightforward computation gives that

$$\varphi(A_{\mathbf{c},\mathbf{d}})=A_{\mathbf{c},\mathbf{d}},\qquad(\mathbf{c},\mathbf{d})\in\mathbb{P}^1\times\mathbb{P}^1.\tag{7.4.57}$$

Since $\varphi(W_\infty)=W_0$ this proves the lemma. □

Proof of **Proposition 7.4.19**. Let $A_{\mathbf{c},\mathbf{d}}\in\mathbb{V}^\psi$; then $C_{W_\infty,A_{\mathbf{c},\mathbf{d}}}$ is either $\mathbb{P}(W_\infty)$ or a sextic in the indeterminacy locus of (0.0.3) by definition of $\mathbb{V}^\psi$. Now assume that there exists $W\in\Theta_{A_{\mathbf{c},\mathbf{d}}}$ such that $C_{W,A_{\mathbf{c},\mathbf{d}}}$ is either $\mathbb{P}(W)$ or a sextic in the indeterminacy locus of (0.0.3). If $\dim\Theta_{A_{\mathbf{c},\mathbf{d}}}\ge 2$ then $A_{\mathbf{c},\mathbf{d}}\in\mathbb{V}^\psi$ by **Proposition 7.4.11** and **Claim 7.4.3**. Thus we may assume that $\dim\Theta_{A_{\mathbf{c},\mathbf{d}}}=1$. Since the 1-PS $\lambda_{\mathcal{N}_3}$ acts on $\Theta_{A_{\mathbf{c},\mathbf{d}}}$ we may assume that W is fixed by $\lambda_{\mathcal{N}_3}(t)$ for all $t\in\mathbb{C}^\times$. Going through Items (t1) - (t6) of **Proposition 7.4.11** we get that W is one of W_∞,W_0,W_+,W_-. If $W\in\{W_\infty,W_0\}$ then $A_{\mathbf{c},\mathbf{d}}\in\mathbb{V}^\psi$ by definition and by **Lemma 7.4.23**. Next let us consider W_+. By **Proposition 7.4.11** we know that $W_+\in\Theta_{W,A_{\mathbf{c},\mathbf{d}}}$ if and only if $\mathbf{d}=[1,1]$, moreover $C_{W_+,A_{\mathbf{c},[1,1]}}$ is a sextic for every $\mathbf{c}\in\mathbb{P}^1$ by **Proposition 7.4.21**. By **Claim 7.4.20** and **Corollary 7.4.12** it follows that we have a regular map

$$\begin{array}{ccc}\mathbb{P}^1 & \longrightarrow & |\mathcal{O}_{\mathbb{P}(W_+)}(6)|\\ \mathbf{c} & \mapsto & C_{W_+,A_{\mathbf{c},[1,1]}}\end{array}\tag{7.4.58}$$

with image a line and $\mathbf{c}$ has degree 2 onto its image. Let Z_0,Z_1,Z_2 be the homogeneous coordinates on $\mathbb{P}(W_+)$ introduced above. Map (7.4.58) sends $[1,0]$ to $V((Z_0Z_2-Z_1^2)^3)$ by **Corollary 7.4.22** and it sends $[1,-1]$ to the same sextic by **Proposition 7.4.11** and (4.5.6). Since Map (7.4.58) is of degree 2 onto a line it follows that no other $\mathbf{c}$ is mapped to $V((Z_0Z_2-Z_1^2)^3)$ i.e. if $\mathbf{c}\notin\{[1,0],[1,-1]\}$ then $C_{W_+,A_{\mathbf{c},[1,1]}}$ is a sextic which is not in the indeterminacy locus of the period map (0.0.3). By **Proposition 7.4.2** both $([1,0],[1,1])$ and $([1,-1],[1,1])$ belong to $\mathbb{V}^\psi$. Lastly we consider W_-. By **Proposition 7.4.11** we know that $W_-\in\Theta_{W,A_{\mathbf{c},\mathbf{d}}}$ if and only if $\mathbf{d}=[1,-1]$. By **Proposition 7.4.21** we know that $C_{W_-,A_{\mathbf{c},[1,-1]}}=\mathbb{P}(W_-)$ if and only if $\mathbf{c}=[1,1]$ moreover $C_{W_-,A_{[1,0],[1,-1]}}=V((Z_0Z_2-Z_1^2)^3)$ by **Corollary 7.4.22**. By **Claim 7.4.20** it follows that

(a) $C_{W_-,A_{\mathbf{c},[1,-1]}}=V((Z_0Z_2-Z_1^2)^3)$ for all $\mathbf{c}\neq[1,1]$ or else

(b) $C_{W_-,A_{\mathbf{c},[1,-1]}}=V((Z_0Z_2-Z_1^2)^3)$ only for $\mathbf{c}=[1,0]$.

A computation gives that the point in $\mathbb{P}(W_-)$ with Z-coordinates $[1,0,1]$ (i.e. $[v_1+v_4]$) belongs to $C_{W_-,A_{[1,-1],[1,-1]}}$: in fact

$$\begin{aligned}F_{v_1+v_4}\ni 4(v_1+v_4)\wedge(v_0\wedge v_2-v_2\wedge v_3-v_2\wedge v_5)=4\alpha_{(0,0,0,2)}-(\alpha_{(0,0,2,0)}+\alpha_{(0,1,0,1)})+((\alpha_{(0,0,2,0)}-\alpha_{(0,1,0,1)})-(4\beta_{(0,2,0,0)}-2\beta_{(1,0,1,0)}))+\\+(\alpha_{(0,2,0,0)}+\alpha_{(1,0,1,0)})-((\alpha_{(0,2,0,0)}-\alpha_{(1,0,1,0)})-(4\beta_{(0,0,2,0)}-2\beta_{(0,1,0,1)}))-4\alpha_{(2,0,0,0)}\in A_{[1,-1],[1,-1]}.\end{aligned}\tag{7.4.59}$$

Thus Item (b) holds; since $([1,0],[1,-1])\in\mathbb{V}^\psi$ this finishes the proof. □

A Elementary auxiliary results

A.1 Discriminant of quadratic forms

Let U be a complex vector-space of finite dimension d. We view $\mathrm{S}^2 U^\vee$ as the vector-space of quadratic forms on U. Given $q_* \in \mathrm{S}^2 U^\vee$ we let Φ be the polynomial on the vector-space $\mathrm{S}^2 U^\vee$ defined by $\Phi(q) := \det(q_* + q)$. Of course Φ is defined up to multiplication by a non-zero scalar, moreover it depends on q_* although that does not show up in the notation. Let

$$\Phi = \Phi_0 + \Phi_1 + \ldots + \Phi_d, \qquad \Phi_i \in \mathrm{S}^i(\mathrm{S}^2 U) \tag{A.1.1}$$

be the decomposition into homogeneous components. We will be interested in giving "intrinsic"descriptions of the loci

$$\{q \in \mathrm{S}^2 U^\vee \mid 0 = \Phi_0(q) = \ldots = \Phi_j(q)\}. \tag{A.1.2}$$

Of course all one needs to do is to expand a determinant: the point is to give a meaningful interpretation of the result. We introduce some notation. Given $q \in \mathrm{S}^2 U^\vee$ we let

$$\widetilde{q}\colon U \to U^\vee, \qquad (v,w)_q := \langle \widetilde{q}(v), w\rangle \tag{A.1.3}$$

be the associated symmetric map and symmetric bilinear form respectively (here $\langle f, v\rangle := f(v)$ for $f \in U^\vee$ and $v \in U$). Let $K := \ker q$; then $\widetilde{q}$ may be viewed as a (symmetric) map $\widetilde{q}\colon (U/K) \to \mathrm{Ann}\, K$. The **dual** quadratic form $q^\vee$ is the quadratic form associated to the symmetric map

$$\widetilde{q}^{-1}\colon\ \mathrm{Ann}\, K \to (U/K).$$

Thus $q^\vee \in \mathrm{S}^2(U/K)$. We denote by $\wedge^i q$ the quadratic form induced by q on $\bigwedge^i U$.

Remark A.1.1. If $\alpha = v_1 \wedge \ldots \wedge v_i$ is a decomposable vector of $\bigwedge^i U$ then $\wedge^i q(\alpha)$ is equal to the determinant of $q|_{\langle v_1,\ldots,v_i\rangle}$ with respect to the basis $\{v_1, \ldots, v_i\}$.

The following is well-known (it follows from a straightforward computation).

Proposition A.1.2. *Let $q_* \in \mathrm{S}^2 U^\vee$ and*

$$K := \ker(q_*), \qquad k := \dim K. \tag{A.1.4}$$

Let Φ_i be the polynomials appearing in (A.1.1)*. Then*

(1) $\Phi_i = 0$ for $i < k$, and

(2) there exists $c \neq 0$ such that $\Phi_k(q) = c \det(q|_K)$.

Keep notation and hypotheses as in **Proposition A.1.2**. Let $\mathcal{V}_K \subset \mathrm{S}^2 U^\vee$ be the subspace of quadratic forms whose restriction to K vanishes. Given $q \in \mathcal{V}_K$ we have $\widetilde{q}(K) \subset \mathrm{Ann}\, K$ and hence it makes sense to consider the restriction of $q_*^\vee$ to $\widetilde{q}(K)$.

Proposition A.1.3. *Keep notation and hypotheses as in* **Proposition A.1.2**. *There exists $c \neq 0$ such that*

$$\Phi_{2k}(q) = c \det(q_*^\vee|_{\widetilde{q}(K)}), \qquad q \in \mathcal{V}_K. \tag{A.1.5}$$

In particular by **Remark A.1.1** *we have that $\Phi_{2k}(q) = 0$ if and only if the restriction of $q_*^\vee$ to $\widetilde{q}(K)$ is degenerate.*

Proof. Choose a basis $\{u_1, \ldots, u_d\}$ of U such that $K = \langle u_1, \ldots, u_k\rangle$ and $\widetilde{q}_*(u_i) = u_i^\vee$ for $k < i \leq d$. Let $q \in \mathcal{V}_K$ and let M be the matrix of q in the chosen basis - thus the upper-left $k \times k$ subminor of M is zero. Expanding $\det(q_* + tq)$ we get that

$$\det(q_* + tq) \equiv (-1)^k t^{2k} \sum_J (\det M_{\mathbf{k},J})^2 \pmod{t^{2k+1}}$$

where $M_{\mathbf{k},J}$ is the $k \times k$ submatrix of M determined by the first k rows and the columns indicized by $J = (j_1, j_2, \ldots, j_k)$. The claim follows because

$$\sum_J (\det M_{\mathbf{k},J})^2 = \wedge^k (q_*^\vee)(\widetilde{q}(u_1) \wedge \ldots \widetilde{q}(u_k)).$$

□

Remark A.1.4. Keep notation and hypotheses as in **Proposition A.1.3**. Suppose in addition that $k = 1$ and set $K = \ker q_* = \langle e_1 \rangle$. Let $q \in \mathcal{V}_K$ i.e. $q(e_1) = 0$. Since $\ker q_* = \langle e_1 \rangle$ there exists $e_2 \in U$ (well-defined modulo $\langle e_1 \rangle$) such that $\widetilde{q}(e_1) = \widetilde{q}_*(e_2)$. An equivalent formulation of **Proposition A.1.3** (in this case) is that $\Phi_2(q) = 0$ if and only if $q_*(e_2) = 0$.

A.2 Quadratic forms of corank 2

In the present subsection $q_* \in \mathrm{S}^2 U^\vee$ will be a quadratic form such that

$$\operatorname{cork}(q_*) = 2, \qquad K := \ker(q_*). \tag{A.2.1}$$

Let $\Phi_0, \ldots, \Phi_d$ be the polynomials (well-defined up to multiplication by a non-zero scalar) associated to q_*. Let $q \in \mathrm{S}^2 U^\vee$; by **Proposition A.1.2** we know that $\Phi_i(q) = 0$ for $i \leq 1$ and moreover $\Phi_2(q) = 0$ if and only if $q|_K$ is degenerate. We will describe the loci of q (subject perhaps to some a priori condition) such that $\Phi_i(q) = 0$ for higher i.

Claim A.2.1. *Suppose that* (A.2.1) *holds. Let $q \in \mathrm{S}^2 U^\vee$ and keep notation and hypotheses as above. Suppose moreover that $\Phi_2(q) = 0$ i.e. $q|_K$ is degenerate. Then $\Phi_3(q) = 0$ if and only if there exists $0 \neq e \in K$ such that*

$$\widetilde{q}(e) \in \operatorname{Ann}(K), \qquad q_*^\vee(\widetilde{q}(e)) = 0. \tag{A.2.2}$$

(Notice that the equation makes sense because of the first condition.)

Proof. Suppose that $q|_K = 0$. Then $\Phi_3(q) = 0$ by **Proposition A.1.3**. On the other hand $\widetilde{q}(e) \in \operatorname{Ann}(K)$ for all $e \in K$ and hence we may define a quadratic form Q on K by setting $Q(v) := q_*^\vee(\widetilde{q}(v))$; since $\dim K = 2$ it follows that there exists a non-trivial zero of Q i.e. a solution of (A.2.2). Now suppose that $q|_K = 0$ has rank 1 and let $\langle e \rangle = \ker(q|_K)$. There exists a basis $\{u_1, \ldots, u_d\}$ of U such that $K = \langle u_1, u_2 \rangle$, $e = u_1$ and the matrix associated to q_* is diagonal: $\widetilde{q}_*(u_i) = u_i^\vee$ for $2 < i \leq d$. Expanding $\det(q_* + tq)$ as function of t one gets that $\Phi_3(q) = 0$ if and only if (A.2.2) holds. □

Next we assume that

$$q|_K = 0. \tag{A.2.3}$$

First we introduce some notation. Given $w \in K$ we have $\widetilde{q}(w) \in \operatorname{Ann} K$ by (A.2.3) and hence there exists $e(q; w)$ such that

$$\widetilde{q}(w) = \widetilde{q}_*(e(q; w)). \tag{A.2.4}$$

Of course $e(q; w)$ is determined modulo K.

Claim A.2.2. *Suppose that* (A.2.1) *holds. Let $q \in \mathrm{S}^2 U^\vee$ such that* (A.2.3) *holds. Let $v \in K$ and suppose that $\widetilde{q}(v) \in \ker(q_*^\vee|_{\widetilde{q}(K)})$ i.e.*

$$(e(q; v), e(q; w))_{q_*} = 0 \qquad \forall w \in K. \tag{A.2.5}$$

Then

$$(w, e(q; v))_q = 0 \qquad \forall w \in K$$

and hence $q(e(q; v))$ is well-defined although $e(q; v)$ is defined modulo K.

Proof. We have

$$(w, e(q; v))_q = \langle \widetilde{q}(w), e(q; v)\rangle = \langle \widetilde{q}_*(e(q; w)), e(q; v)\rangle = (e(q; v), e(q; w))_{q_*}.$$

The last expression vanishes by (A.2.5). $\square$

Proposition A.2.3. *Suppose that* (A.2.1) *holds. Let* $q \in \mathrm{S}^2 U^\vee$. *Assume that* $q|_K = 0$ *and hence* $\Phi_i(q) = 0$ *for* $i < 4$ *(see* **Proposition A.1.3***). Suppose moreover that* $\Phi_4(q) = 0$ *i.e.* $q_*^\vee|_{\widetilde{q}(K)}$ *is degenerate (see* **Proposition A.1.3***). Then* $\Phi_5(q) = 0$ *if and only if there exists* $0 \neq v \in K$ *such that* (A.2.5) *holds and moreover* $q(e(q; v)) = 0$.

Proof. Suppose first that $\widetilde{q}|_K$ is not injective. Then $\det(q_* + tq) = 0$ for all t, in particular $\Phi_5(q) = 0$. On the other let $v \in K$ such that $\widetilde{q}(v) = 0$. Then $e(q; v) = 0$; thus (A.2.5) holds and $q(e(q; v)) = 0$. Next suppose that $\widetilde{q}|_K$ is injective and $q_*^\vee|_{\widetilde{q}(K)}$ has rank 0. A straightforward computation gives that $\Phi_5(q) = 0$. Now (A.2.5) holds for arbitrary $v \in K$; since $\dim K = 2$ there exists $0 \neq v \in K$ such that $q(e(q; v)) = 0$. Lastly suppose that $\widetilde{q}|_K$ is injective and $q_*^\vee|_{\widetilde{q}(K)}$ has rank 1. There exists a basis $\{u_1, \ldots, u_d\}$ of U such that $K = \langle u_1, u_2\rangle$,

$$\widetilde{q}_*(u_i) = u_{7-i}^\vee \quad i = 3, 4, \qquad \widetilde{q}_*(u_i) = u_i^\vee \quad 4 < i \leq d$$

and $\widetilde{q}(u_1) = u_3^\vee$, $\widetilde{q}(u_2) = u_5^\vee$. Thus $\langle \widetilde{q}(u_1)\rangle = \ker(q_*^\vee|_{\widetilde{q}(K)})$ and $e(q; u_1) = u_4$. Let $A = (a_{ij})$ be the matrix of q with respect to the chosen basis. A straightforward computation gives that

$$\det(q_* + tq) \equiv a_{44} t^5 \pmod{t^6}$$

Since $a_{44} = q(u_4) = q(e(q; u_1))$ that finishes the proof of the proposition. $\square$

Lastly we will consider the restriction of Φ to affine planes containing q_* and subject to a certain hypothesis.

Assumption A.2.4. $r, s \in \mathrm{S}^2 U^\vee$ and the following hold:

(1) $r|_K = 0$ and $s|_K$ has rank 1 with kernel spanned by v,

(2) the subspace $\langle \widetilde{r}(v), \widetilde{s}(v)\rangle \subset \mathrm{Ann}\, K$ has dimension 2 and when we restrict $q_*^\vee$ we get a quadratic form of rank 1 with kernel spanned by $\widetilde{r}(v)$,

(3) the restriction of $q_*^\vee$ to $\widetilde{r}(K)$ is degenerate.

Suppose that r, s satisfy **Assumption A.2.4**; by **Proposition A.1.2**, **Claim A.2.1** and **Proposition A.1.3** we have

$$\det(q_* + xr + ys) \equiv c_{03} y^3 + c_{31} x^3 y + c_{22} x^2 y^2 + c_{13} x y^3 + c_{04} y^4 \pmod{(x, y)^5} \tag{A.2.6}$$

Claim A.2.5. *Suppose that* (A.2.1) *holds and moreover* r, s *satisfy* **Assumption A.2.4***, in particular* (A.2.6) *holds. Then* $c_{31} = 0$ *if and only if* $r(e(r; v)) = 0$ *where* v *is as in Item (1) of* **Assumption A.2.4** *and* $e(r; v)$ *is as in* (A.2.4) *with* q *replaced by* r.

Proof. We may choose a basis $\{u_1, \ldots, u_d\}$ of U such that the following hold

(a) $K = \langle u_1, u_2\rangle$, $\widetilde{q}_*(u_i) = u_{7-i}^\vee$ for $i = 3, 4$ and $\widetilde{q}_*(u_i) = u_i^\vee$ for $4 < i \leq d$,

(b) the matrix associated to r in the chosen basis is $A = (a_{ij})$ with $a_{1j} = \delta_{3j}$ and $a_{22} = a_{24} = 0$,

(c) the matrix associated to s in the chosen basis is $B = (b_{ij})$ with $b_{1j} = \delta_{5j}$ and $b_{22} = 1$.

Let $m_{ij} := (a_{ij}x + b_{ij}y)$; then $q_* + xr + ys$ is equal to

$$\begin{pmatrix}
0 & 0 & x & 0 & y & 0 & \cdots & 0 \\
0 & y & m_{23} & b_{24}y & m_{25} & m_{26} & \cdots & m_{2d} \\
x & m_{32} & m_{33} & 1+m_{34} & m_{35} & m_{36} & \cdots & m_{3d} \\
0 & b_{42}y & 1+m_{43} & m_{44} & m_{45} & m_{46} & \cdots & m_{4d} \\
y & m_{52} & m_{53} & m_{54} & 1+m_{55} & m_{56} & \cdots & m_{5d} \\
0 & m_{62} & m_{63} & m_{64} & m_{65} & 1+m_{66} & \cdots & m_{6d} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & m_{d2} & m_{d3} & m_{d4} & m_{d5} & m_{d6} & \cdots & 1+m_{dd}
\end{pmatrix}$$

A computation gives that

$$\det(q_* + xr + ys) = y^3 + a_{44}x^3y + \ldots$$

Now $a_{44} = r(u_4)$. On the other hand $\widetilde{q}_*(u_4) = u_3^\vee = \widetilde{r}(u_1)$ i.e. $u_4 = e(r; u_1)$; since $\langle u_1 \rangle = \ker(s|_K)$ that proves the claim. □

A.3 Pencils of degenerate linear maps

Let $\mathfrak{gl}(3)$ be the space of 3×3 complex matrices. Let $\mathfrak{gl}(3)_r \subset \mathfrak{gl}(3)$ be the closed subset of matrices of rank at most r. Let

$$P := \{\mathcal{V} \in \mathrm{Gr}(2, \mathfrak{gl}(3)) \mid \mathcal{V} \subset (\mathfrak{gl}(3)_2 \setminus \mathfrak{gl}(3)_1)\}. \tag{A.3.1}$$

In other words an element of P is a 2-dimensional space of 3×3 complex matrices whose non-zero elements have rank 2. Multiplication on the left and the right defines an action of $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$ on P; we are interested in the orbits for this action. First we give three explicit elements of P. Let

$$f := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.3.2}$$

Let

$$\mathcal{V}_l := \langle f, g \rangle, \tag{A.3.3}$$

$$\mathcal{V}_c := \langle f, h \rangle, \tag{A.3.4}$$

$$\mathcal{V}_p := \langle f^t, h^t \rangle. \tag{A.3.5}$$

Then $\mathcal{V}_l, \mathcal{V}_c, \mathcal{V}_p \in P$; we claim that the orbits of these elements are pairwise distinct. To see why we introduce a piece of notation: given $\mathcal{V} \in P$ let $K(\mathcal{V}) \subset \mathbb{P}^2$ be defined by

$$K(\mathcal{V}) := \{\ker f \mid [f] \in \mathbb{P}(\mathcal{V})\}. \tag{A.3.6}$$

(This makes sense precisely because $rk(f) = 2$ for every $[f] \in \mathbb{P}(\mathcal{V})$.) If $\mathcal{V}, \mathcal{V}' \in P$ belong to the same orbit then $K(\mathcal{V})$ and $K(\mathcal{V}')$ belong to the same $PGL_3(\mathbb{C})$-orbit. A straightforward computation shows that

$$K(\mathcal{V}_l) = V(x), \qquad K(\mathcal{V}_c) = V(x^2 - yz), \qquad K(\mathcal{V}_p) = V(x, y). \tag{A.3.7}$$

(Here $[x, y, z]$ are the standard homogeneous coordinates on $\mathbb{P}^2$.) Since the above subsets of $\mathbb{P}^2$ are pairwise not projectively equivalent we get that the orbits of $\mathcal{V}_l, \mathcal{V}_c, \mathcal{V}_p$ are pairwise distinct. One more piece of notation: if $\mathcal{V} \in P$ we let $\mathcal{V}^t := \{f^t \mid f \in \mathcal{V}\}$.

Proposition A.3.1. *Keep notation as above. Let $\mathcal{V} \in P$; then $\mathcal{V}$ is $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$-equivalent to one and only one of $\mathcal{V}_l, \mathcal{V}_c, \mathcal{V}_p$.*

Proof. It suffices to prove that if $\mathcal{V} \in P$ then $\mathcal{V}$ is equivalent to one of $\mathcal{V}_l, \mathcal{V}_c, \mathcal{V}_p$. A priori there are four possible cases:

(1) neither $K(\mathcal{V})$ nor $K(\mathcal{V}^t)$ is a singleton,

(2) $K(\mathcal{V})$ is not a singleton, $K(\mathcal{V}^t)$ is a singleton,

(3) $K(\mathcal{V})$ is a singleton, $K(\mathcal{V}^t)$ is not a singleton,

(4) both $K(\mathcal{V})$ and $K(\mathcal{V}^t)$ are singletons.

Assume that Item (1) holds. Then $\mathcal{V}$ is equivalent to $\langle\alpha, \beta\rangle$ where $Ker(\alpha) = \langle(0,0,1)\rangle$, $\mathrm{im}(\alpha) = V(z)$ and $Ker(\beta) = \langle(0,1,0)\rangle$, $\mathrm{im}(\beta) = V(y)$. Thus

$$\alpha := \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} m & 0 & n \\ 0 & 0 & 0 \\ p & 0 & q \end{pmatrix}. \tag{A.3.8}$$

Expanding $0 \equiv det(s\alpha + t\beta)$ we get that $0 = d = q$. Furthermore $bc \neq 0$ and $np \neq 0$ because $2 = rk(\alpha) = rk(\beta)$. Then it is easy to show that there exist $M, N \in GL_3(\mathbb{C})$ such that $M\alpha N = f$ and $M\beta N = g$. Thus $\mathcal{V}$ is equivalent to $\mathcal{V}_l$. Now suppose that Item (2) holds: an argument similar to that given above shows that $\mathcal{V}$ is equivalent to $\mathcal{V}_c$. On the other hand if Item (3) holds then Item (2) holds with $\mathcal{V}$ replaced by $\mathcal{V}^t$; since $\mathcal{V}_p = \mathcal{V}_c^t$ we get that $\mathcal{V}$ is equivalent to $\mathcal{V}_p$. Finally suppose that Item (4) holds. We may assume that $K(\mathcal{V}) = \langle(0,0,1)\rangle$ and $K(\mathcal{V}^t) = V(z)$. Then $\mathcal{V} \subset \mathfrak{gl}_2(\mathbb{C})$; since $\dim \mathcal{V} = 2$ there exists $0 \neq f \in \mathcal{V}$ such that $rk(f) < 2$, that is a contradiction. Thus Item (4) cannot hold. □

Remark A.3.2. Any 2-dimensional subspace of $\mathfrak{o}_3(\mathbb{C})$ is an element of P; such a subspace is equivalent to $\mathcal{V}_l$.

B Tables

Table 18: Ordering 1-PS's up to duality, I

1-PS λ	strictly positive isotypical summands of $\bigwedge^3 \lambda$					
$(1, 0_4, -1)$	$[v_0] \wedge \bigwedge^2 V_{14}$ t					
$(1_2, 0_2, -1_2)$	$[v_0] \wedge \bigwedge^2 V_{12}$ t^6	$[v_0] \wedge V_{12} \wedge V_{35}$ t^3				
$(1_2, 0_3, -2)$	$\bigwedge^2 V_{01} \wedge V_{24}$ t^2	$V_{01} \wedge \bigwedge^2 V_{24}$ t				
$(1_3, -1_3)$	$\bigwedge^3 V_{02}$ t^3	$\bigwedge^2 V_{02} \wedge V_{35}$ t				
$(1_4, -2_2)$	$\bigwedge^3 V_{03}$ t^3					
$(2, 1, 0_2, -1, -2)$	$\bigwedge^2 V_{01} \wedge V_{23}$ t^3	014, 023 t^2	015, 024, 034, 123 t			
$(2, 1_2, -1_2, -2)$	012 t^4	$[v_0] \wedge V_{12} \wedge V_{34}$ t^2	$[v_0] \wedge V_{12} \wedge [v_5] \oplus \bigwedge^2 V_{12} \wedge V_{34}$ t			
$(2, 1_2, 0, -1, -3)$	012 t^4	$[v_0] \wedge V_{12} \wedge [v_3]$ t^3	$[v_0] \wedge V_{12} \wedge [v_4] \oplus [v_1 \wedge v_2 \wedge v_3]$ t^2	034, 124 t		
$(3, 1_2, -1_2, -3)$	012 t^5	$[v_0] \wedge V_{12} \wedge V_{34}$ t^3	$[v_0] \wedge (V_{12} \wedge [v_5] \oplus \bigwedge^2 V_{34}) \oplus \bigwedge^2 V_{12} \wedge V_{34}$ t			
$(3, 1_2, 0, -2, -3)$	012 t^5	$[v_0] \wedge V_{12} \wedge [v_3]$ t^4	$[v_0] \wedge V_{12} \wedge [v_4] \oplus [v_1 \wedge v_2 \wedge v_3]$ t^2	$[v_0] \wedge V_{12} \wedge [v_5] \oplus [v_0 \wedge v_3 \wedge v_4]$ t		
$(3, 2, 1, -1, -2, -3)$	012 t^6	013 t^4	014, 023 t^3	015, 024, 123 t^2	025, 124 t	
$(3, 2, 1, 0, -2, -4)$	012 t^6	013 t^5	023 t^4	014, 123 t^3	024 t^2	015, 034, 124 t
$(3, 2, 1, 0, -1, -5)$	012 t^6	013 t^5	014, 023 t^4	024, 123 t^3	034, 124 t^2	134 t

Table 19: Ordering 1-PS's up to duality, II

1-PS λ	strictly positive isotypical summands of $\bigwedge^3 \lambda$					
$(4, 1_2, -2_3)$	$[v_0] \wedge \bigwedge^2 V_{12}$ t^6	$[v_0] \wedge V_{12} \wedge V_{35}$ t^3				
$(4, 1_3, -2, -5)$	$[v_0] \wedge \bigwedge^2 V_{13}$ t^6	$[v_0] \wedge V_{13} \wedge [v_4] \oplus [v_1 \wedge v_2 \wedge v_3]$ t^3				
$(4, 2, 1, 0, -3, -4)$	012 t^7	013 t^6	023 t^5	014, 123 t^3	015, 024 t^2	025, 034 t
$(4, 3, 1, 0, -3, -5)$	012 t^8	013 t^7	023 t^5	014, 123 t^4	015, 024 t^2	034, 124 t
$(4_2, 1, -2_2, -5)$	012 t^9	$\bigwedge^2 V_{01} \wedge V_{34}$ t^6	$[v_0 \wedge v_1 \wedge v_5] \oplus V_{01} \wedge [v_2] \wedge V_{34}$ t^3			
$(4_2, 1_2, -2, -8)$	$\bigwedge^2 V_{01} \wedge V_{23}$ t^9	$[v_0 \wedge v_1 \wedge v_4] \oplus V_{01} \wedge \bigwedge^2 V_{23}$ t^6	$V_{01} \wedge V_{23} \wedge [v_4]$ t^3			
$(5, -1_5)$	$[v_0] \wedge \bigwedge^2 V_{15}$ t^3					
$(5, 2_2, -1_2, -7)$	012 t^9	013, 014, 023, 024 t^6	034, 123, 124 t^3			
$(5, 3, 1, -1, -3, -5)$	012 t^9	013 t^7	014, 023 t^5	015, 024, 123 t^3	025, 034, 124 t	
$(5_2, -1_3, -7)$	$\bigwedge^2 V_{01} \wedge V_{24}$ t^9	$[v_0 \wedge v_1 \wedge v_5] \oplus V_{01} \wedge \bigwedge^2 V_{24}$ t^3				
$(5_2, 2, -1, -4, -7)$	012 t^{12}	013 t^9	014, 023, 123 t^6	015, 024, 124 t^3		
$(7, 4, 1, -2_2, -8)$	012 t^{12}	013, 014 t^9	023, 024 t^6	015, 034, 123, 124 t^3		
$(7, 4, 1_2, -5, -8)$	012, 013 t^{12}	023 t^9	014, 123 t^6	015, 024, 034 t^3		

Table 20: Ordering 1-PS's up to duality, III

1-PS λ	strictly positive isotypical summands of $\bigwedge^3 \lambda$					
$(7, 4, 1_2, -2, -11)$	012, 013 t^{12}	014, 023 t^{9}	024, 034, 123 t^{6}	124, 134 t^{3}		
$(7, 4_2, -2, -5, -8)$	012 t^{15}	013, 023 t^{9}	014, 024, 123 t^{6}	015, 025, 124 t^{3}		
$(7_2, 1_2, -5, -11)$	012, 013 t^{15}	014, 023, 123 t^{9}	015, 024, 034, 124, 134 t^{3}			
$(8, 5, 2, -1, -4, -10)$	012 t^{15}	013 t^{12}	014, 023 t^{9}	024, 123 t^{6}	015, 034, 124 t^{3}	
$(10, 7, 1, -2, -5, -11)$	012 t^{18}	013 t^{15}	014 t^{12}	023 t^{9}	015, 024, 123 t^{6}	034, 124 t^{3}
$(10, 7, 4, -2, -8, -11)$	012 t^{21}	013 t^{15}	023 t^{12}	014, 123 t^{9}	015, 024 t^{6}	025, 124 t^{3}
$(11, 5, 2, -1, -4, -13)$	012 t^{18}	013 t^{15}	014, 023 t^{12}	024 t^{9}	034, 123 t^{6}	015, 124 t^{3}
$(11, 5_2, -1, -7, -13)$	012 t^{21}	013, 023 t^{15}	014, 024, 123 t^{9}	015, 025, 034, 124 t^{3}		
$(11, 8, 2, -1, -7, -13)$	012 t^{21}	013 t^{18}	014, 023 t^{12}	123 t^{9}	015, 024 t^{6}	034, 124 t^{3}
$(11, 8, 5, -4, -7, -13)$	012 t^{24}	013 t^{15}	014, 023 t^{12}	024, 123 t^{9}	015, 124 t^{6}	025 t^{3}
$(13, 7, 1_2, -5, -17)$	012, 013 t^{21}	014, 023 t^{15}	024, 034, 123 t^{9}	015, 124, 134 t^{3}		
$(17, 11, 5, -1, -13, -19)$	012 t^{33}	013 t^{27}	023 t^{21}	014, 123 t^{15}	015, 024 t^{9}	025, 034, 124 t^{3}
$(19, 13, 7, -5, -11, -23)$	012 t^{39}	013 t^{27}	014, 023 t^{21}	024, 123 t^{15}	015, 124 t^{9}	025, 034 t^{3}

Table 21: Flag conditions defined by ordering 1-PS's, I

1-PS λ	$\mu(\mathbf{d},\lambda)$	subsets covering $\mathcal{P}^{\geq 0}_{\lambda}$	$\subset$
$(1, 0_4, -1)$	$2(d_0 - 3)$	$d_0 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
$(1_2, 0_2, -1_2)$	$2(2d_0 + d_1 - 4)$	$d_0 = 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 = 1$ and $d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_1 = 4$	$\mathbb{B}^*_{\mathcal{F}_1}$
$(1_2, 0_3, -2)$	$2(2d_0 + d_1 - 6)$	$d_0 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 5$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(1_3, -1_3)$	$2(3d_0 + d_1 - 6)$	$d_0 = 1$ and $d_1 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_1 \geq 6$	$\mathbb{B}^*_{\mathcal{C}_2}$
$(1_4, -2_2)$	$6(d_0 - 2)$	$d_0 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
$(2, 1, 0_2, -1, -2)$	$2(3d_0 + 2d_1 + d_2 - 7)$	$d_0 = 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 3$ or $d_0 + d_1 + d_2 \geq 5$	$\mathbb{B}^*_{\mathcal{D}}$
		$\mathbf{d} = (1, 1, 2)$	$\mathbb{X}^*_{\mathcal{N}_3}$
$(2, 1_2, -1_2, -2)$	$2(4d_0 + 2d_1 + d_2 - 8)$	$d_0 = 1$ and $d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 \geq 6$	$\mathbb{B}^*_{\mathcal{E}_2}$
$(2, 1_2, 0, -1, -3)$	$2(4d_0 + 3d_1 + 2d_2 + d_3 - 9)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 + d_2 \geq 4$ or $d_0 + d_1 + d_2 + d_3 \geq 5$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 = 1$ and $d_1 + d_2 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
$(3, 1_2, -1_2, -3)$	$2(5d_0 + 3d_1 + d_2 - 11)$	$d_0 = 1$ and $d_1 + d_2 \geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 \geq 7$	$\mathbb{B}^*_{\mathcal{A}}$
$(3, 1_2, 0, -2, -3)$	$2(5d_0 + 4d_1 + 2d_2 + d_3 - 11)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 = 1$ and $d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 + d_2 + d_3 \geq 5$	$\mathbb{B}^*_{\mathcal{D}}$
$(3, 2, 1, -1, -2, -3)$	$2(6d_0 + 4d_1 + 3d_2 + 2d_3 + d_4 - 12)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$\sum_{i=0}^{3} d_i \geq 5$ or $\sum_{i=0}^{4} d_i \geq 6$	$\mathbb{B}^*_{\mathcal{E}_2}$
$(3, 2, 1, 0, -1, -5)$	$2(6d_0 + 5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 15)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 + d_4 + d_5 \geq 5$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(4, 1_2, -2_3)$	$2(6d_0 + 3d_1 - 12)$	$d_0 = 1$ and $d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1}$
		$d_1 \geq 4$	$\mathbb{B}^*_{\mathcal{E}_2}$
$(4, 1_3, -2, -5)$	$2(6d_0 + 3d_1 - 15)$	$d_0 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 4$	$\mathbb{B}^*_{\mathcal{D}}$

Table 22: Flag conditions defined by ordering 1-PS's, II

1-PS λ	$\mu(\mathbf{d},\lambda)$	subsets covering $\mathcal{P}_\lambda^{\geq 0}$	$\subset$
$(4,2,1,0,-3,-4)$	$2(7d_0+6d_1+5d_2+3d_3+2d_4+d_5-15)$	$d_0+d_1+d_2\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1\geq 1$ and $d_2+d_3\geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0=1$ and $d_1+d_2+d_3\geq 2$	$\mathbb{B}^*_{\mathcal{E}_1}$
		$d_0=1$ and $d_1+d_2+d_3+d_4\geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0+d_1+d_2+d_3+d_4\geq 5$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
		$d_0+d_1+d_2+d_3+d_4+d_5\geq 6$	$\mathbb{B}^*_{\mathcal{A}}$
$(4,3,1,0,-3,-5)$	$2(8d_0+7d_1+5d_2+4d_3+2d_4+d_5-17)$	$d_0+d_1+d_2\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1\geq 1$ and $d_2+d_3\geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0=1$ and $d_1+d_2+d_3+d_4\geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$\sum_{i=0}^4 d_i\geq 5$ or $\sum_{i=0}^5 d_i\geq 6$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
$(4_2,1,-2_2,-5)$	$6(3d_0+2d_1+d_2-6)$	$d_0+d_1\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0=1$ and $d_1+d_2\geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0+d_1+d_2\geq 5$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
$(4_2,1_2,-2,-8)$	$6(3d_0+2d_1+d_2-8)$	$d_0\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0\geq 1$ and $d_1\geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0+d_1+d_2\geq 5$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(5,-1_5)$	$2(3d_0-15)$	$d_0\geq 5$	$\mathbb{B}^*_{\mathcal{A}}$
$(5,2_2,-1_2,-7)$	$6(3d_0+2d_1+d_2-7)$	$d_0+d_1\geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0=1$ and $d_1\geq 2$ or $d_1+d_2\geq 3$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0+d_1+d_2\geq 5$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(5,3,1,-1,-3,-5)$	$2(9d_0+7d_1+5d_2+3d_3+d_4-19)$	$d_0+d_1\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1\geq 1$ and $d_2+d_3\geq 3$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0+d_1+d_2\geq 3$ or $d_0+d_1+d_2+d_3\geq 5$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0=1$ and $d_1+d_2+d_3+d_4\geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0+d_1+d_2+d_3+d_4\geq 7$	$\mathbb{B}^*_{\mathcal{A}}$
$(5_2,-1_3,-7)$	$6(3d_0+d_1-8)$	$d_0\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1\geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(5_2,2,-1,-4,-7)$	$6(4d_0+3d_1+2d_2+d_3-8)$	$d_0+d_1\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1\geq 1$ and $d_2\geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0+d_1+d_2+d_3\geq 5$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
		$d_0=1$ and $d_1+d_2\geq 2$ or $d_1+d_2+d_3\geq 3$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
$(7,4,1,-2_2,-8)$	$6(4d_0+3d_1+2d_2+d_3-9)$	$d_0+d_1\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1+d_2\geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0+d_1+d_2+d_3\geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
		$d_0=1$ and $d_1+d_2+d_3\geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
$(7,4,1_2,-5,-8)$	$6(4d_0+3d_1+2d_2+d_3-9)$	$d_0+d_1\geq 2$ or $d_0+d_1\geq 1$ and $d_2\geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0+d_1+d_2\geq 4$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
		$d_0+d_1+d_2+d_3\geq 6$	$\mathbb{B}^*_{\mathcal{A}}$
		$d_0=1$ and $d_1+d_2\geq 2$ or $d_1+d_2+d_3\geq 4$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$

Table 23: Flag conditions defined by ordering 1-PS's, III

1-PS λ	$\mu(\mathbf{d}, \lambda)$	subsets covering $\mathcal{P}_\lambda^{\geq 0}$	$\subset$
$(7, 4, 1_2, -2, -11)$	$6(4d_0 + 3d_1 + 2d_2 + d_3 - 11)$	$d_0 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 3$ or $d_0 + d_1 + d_2 \geq 4$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 \geq 5$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(7, 4_2, -2, -5, -8)$	$6(4d_0 + 3d_1 + 2d_2 + d_3 - 11)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 \geq 1$ and $d_1 + d_2 \geq 2$ or $d_2 \geq 1$, $d_3 \geq 3$	$\mathbb{B}^*_{\mathcal{E}_1} \cup \mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0 + d_1 + d_2 \geq 4$ or $d_0 + d_1 + d_2 + d_3 \geq 6$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
$(7_2, 1_2, -5, -11)$	$6(5d_0 + 3d_1 + d_2 - 12)$	$d_0 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 \geq 1$ and $d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 + d_1 + d_2 \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(8, 5, 2, -1, -4, -10)$	$6(5d_0 + 4d_1 + 3d_2 + 2d_3 + d_4 - 11)$	$d_0 + d_1 \geq 2$ or $d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 + d_2 + d_3 \geq 4$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 + d_4 \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 \geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$\mathbf{d} = (0, 1, 1, 1, 2)$	$\mathbb{X}^*_{\mathcal{N}_3}$
$(10, 7, 1, -2, -5, -11)$	$6(6d_0 + 5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 13)$	$d_0 + d_1 + d_2 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 + d_4 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 + d_1 + d_2 + d_3 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 + d_4 + d_5 \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 + d_5 \geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$\mathbf{d} = (0, 0, 1, 1, 3, 0)$	$\mathbb{X}^*_{\mathcal{N}_3}$
$(10, 7, 4, -2, -8, -11)$	$6(7d_0 + 5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 14)$	$d_0 + d_1 + d_2 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$\sum_{i=0}^4 d_i \geq 5$ or $\sum_{i=0}^5 d_i \geq 6$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 + d_5 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
$(11, 5, 2, -1, -4, -13)$	$6(6d_0 + 5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 14)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 + d_2 + d_3 \geq 3$	$\mathbb{B}^*_{\mathcal{D}}$
		$\sum_{i=0}^4 d_i \geq 5$ or $\sum_{i=0}^5 d_i \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
		$d_0 = 1$ and $\sum_{i=1}^4 d_i \geq 3$ or $\sum_{i=1}^5 d_i \geq 4$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$\mathbf{d} = (0, 1, 1, 0, 2, 1)$	$\mathbb{X}^*_{\mathcal{N}_3}$
$(11, 5_2, -1, -7, -13)$	$6(7d_0 + 5d_1 + 3d_2 + d_3 - 15)$	$d_0 + d_1 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 + d_2 \geq 4$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
		$d_0 + d_1 + d_2 + d_3 \geq 7$	$\mathbb{B}^*_{\mathcal{A}}$
		$d_0 = 1$ and $d_1 + d_2 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
$(11, 8, 2, -1, -7, -13)$	$6(7d_0 + 6d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 15)$	$d_0 + d_1 \geq 2$ or $d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 \geq 2$ or $d_2 + d_3 + d_4 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 + d_5 \geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 + d_2 + d_3 + d_4 \geq 5$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 + d_4 + d_5 \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$

Table 24: Flag conditions defined by ordering 1-PS's, IV

1-PS λ	$\mu(\mathbf{d}, \lambda)$	subsets covering $\mathcal{P}_{\lambda}^{\geq 0}$	$\subset$
$(11, 8, 5, -4, -7, -13)$	$6(8d_0 + 5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - 16)$	$d_0 + d_1 \geq 2$ or $d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 + d_5 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$d_0 + d_1 + d_2 + d_3 \geq 4$ or $d_0 + d_1 + d_2 + d_3 + d_4 \geq 5$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
$(13, 7, 1_2, -5, -17)$	$6(7d_0 + 5d_1 + 3d_2 + d_3 - 18)$	$d_0 = 2$ or $d_0 + d_1 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 + d_2 \geq 4$	$\mathbb{B}^*_{\mathcal{D}}$
		$d_0 + d_1 + d_2 + d_3 \geq 6$	$\mathbb{B}^*_{\mathcal{A}^\vee}$
$(17, 11, 5, -1, -13, -19)$	$6(11d_0 + 9d_1 + 7d_2 + 5d_3 + 3d_4 + d_5 - 23)$	$d_0 + d_1 + d_2 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 + d_1 \geq 1$ and $d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{F}_2}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 + d_4 \geq 3$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$\sum_{i=0}^4 d_i \geq 5$ or $\sum_{i=0}^5 d_i \geq 7$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$
$(19, 13, 7, -5, -11, -23)$	$6(13d_0 + 9d_1 + 7d_2 + 5d_3 + 3d_4 + d_5 - 27)$	$d_0 + d_1 \geq 2$ or $d_0 + d_1 + d_2 \geq 3$	$\mathbb{B}^*_{\mathcal{F}_1}$
		$d_0 = 1$ and $d_1 + d_2 + d_3 \geq 2$	$\mathbb{B}^*_{\mathcal{E}_1^\vee}$
		$d_0 = 1$ and $\sum_{i=1}^4 d_i \geq 3$ or $\sum_{i=1}^5 d_i \geq 4$	$\mathbb{B}^*_{\mathcal{C}_1}$
		$\sum_{i=0}^3 d_i \geq 4$ or $\sum_{i=0}^4 d_i \geq 5$	$\mathbb{B}^*_{\mathcal{E}_2^\vee}$

Table 25: Weights of ordering 1-PS' for $G_{\mathcal{E}_1}$, I.

(m, r, s_1, s_2, s_3)												
$(1,0,0,0,0)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	0	0	0	0	0	0	3	3	3	-3	-3	-3
$(-1,0,0,0,0)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	0	0	0	0	0	0	3	3	3	-3	-3	-3
$(0,1,0,0,0)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	1	1	1	-1	-1	-1	0	0	0	0	0	0
$(0,0,6,0,-6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	6	6	0	0	-6	-6	6	6	0	0	-6	-6
$(0,3,6,0,-6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	9	3	3	-3	-3	-9	6	6	0	0	-6	-6
$(1,3,6,0,-6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	9	3	3	-3	-3	-9	9	3	3	-3	-3	-9
$(2,3,6,0,-6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	9	3	3	-3	-3	-9	12	6	0	0	-6	-12

Table 26: Weights of ordering 1-PS' for $G_{\mathcal{E}_1}$, II.

(m, r, s_1, s_2, s_3)												
$(0, 0, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	12	12	-6	-6	-6	-6	12	6	6	-6	-6	-12
$(1, 0, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	12	12	-6	-6	-6	-6	9	9	9	-9	-9	-9
$(1, 9, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	21	3	3	3	-15	-15	9	9	9	-9	-9	-9
$(4, 0, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	12	12	-6	-6	-6	-6	18	18	0	0	-18	-18
$(4, 9, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	21	3	3	3	-15	-15	18	18	0	0	-18	-18
$(-2, 0, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	12	12	-6	-6	-6	-6	18	0	0	0	0	-18
$(-2, 9, 12, -6, -6)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	21	3	3	3	-15	-15	18	0	0	0	0	-18
$(0, 0, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	6	6	6	6	-12	-12	12	6	6	-6	-6	-12
$(-1, 0, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	6	6	6	6	-12	-12	9	9	9	-9	-9	-9
$(-1, 9, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	15	15	-3	-3	-3	-21	9	9	9	-9	-9	-9
$(-4, 0, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	6	6	6	6	-12	-12	18	18	0	0	-18	-18
$(-4, 9, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$
	15	15	-3	-3	-3	-21	18	18	0	0	-18	-18
$(2, 0, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	6	6	6	6	-12	-12	18	0	0	0	0	-18
$(2, 9, 6, 6, -12)$	$v_0 \wedge \xi_1 \wedge \beta_1$	$v_0 \wedge \xi_1 \wedge \beta_2$	$v_0 \wedge \xi_1 \wedge \beta_3$	$v_0 \wedge \xi_2 \wedge \beta_1$	$v_0 \wedge \xi_2 \wedge \beta_2$	$v_0 \wedge \xi_2 \wedge \beta_3$	$v_0 \wedge \beta_1 \wedge \beta_2$	$v_0 \wedge \beta_1 \wedge \beta_3$	$v_0 \wedge \beta_2 \wedge \beta_3$	$\xi_1 \wedge \xi_2 \wedge \beta_1$	$\xi_1 \wedge \xi_2 \wedge \beta_2$	$\xi_1 \wedge \xi_2 \wedge \beta_3$
	15	15	-3	-3	-3	-21	18	0	0	0	0	-18

Table 27: Numerical functions of ordering 1-PS' for $G_{\mathcal{E}_1}$.

(m, r, s_1, s_2, s_3)	$\mu(A, \lambda)$
$(1, 0, 0, 0, 0)$	$6(d_0(A_2) - 2)$
$(-1, 0, 0, 0, 0)$	$6(d_0(A_2) - 1)$
$(0, 1, 0, 0, 0)$	$4(d_0(A_1) - 1)$
$(0, 0, 6, 0, -6)$	$12(2d_0(A_1) + d_1(A_1) + d_0(A_2) - 3)$
$(0, 3, 6, 0, -6)$	$12(3d_0(A_1) + 2d_1(A_1) + d_2(A_1) + d_0(A_2) - 4)$
$(1, 3, 6, 0, -6)$	$6(6d_0(A_1) + 4d_1(A_1) + 2d_2(A_1) + 3d_0(A_2) + d_1(A_2) - 9)$
$(2, 3, 6, 0, -6)$	$12(3d_0(A_1) + 2d_1(A_1) + d_2(A_1) + 2d_0(A_2) + d_1(A_2) - 5)$
$(0, 0, 12, -6, -6)$	$12(3d_0(A_1) + 2d_0(A_2) + d_1(A_2) - 4)$
$(1, 0, 12, -6, -6)$	$6(6d_0(A_1) + 3d_0(A_2) - 9)$
$(1, 9, 12, -6, -6)$	$6(12d_0(A_1) + 6d_1(A_1) + 3d_0(A_2) - 15)$
$(4, 0, 12, -6, -6)$	$12(3d_0(A_1) + 3d_0(A_2) - 6)$
$(4, 9, 12, -6, -6)$	$12(6d_0(A_1) + 3d_1(A_1) + 3d_0(A_2) - 9)$
$(-2, 0, 12, -6, -6)$	$12(3d_0(A_1) + 3d_0(A_2) - 3)$
$(-2, 9, 12, -6, -6)$	$36(2d_0(A_1) + d_1(A_1) + d_0(A_2) - 2)$
$(0, 0, 6, 6, -12)$	$12(3d_0(A_1) + 2d_0(A_2) + d_1(A_2) - 6)$
$(-1, 0, 6, 6, -12)$	$18(2d_0(A_1) + d_0(A_2) - 4)$
$(-1, 9, 6, 6, -12)$	$18(4d_0(A_1) + 2d_1(A_1) + d_0(A_2) - 6)$
$(-4, 0, 6, 6, -12)$	$36(d_0(A_1) + d_0(A_2) - 2)$
$(-4, 9, 6, 6, -12)$	$36(2d_0(A_1) + d_1(A_1) + d_0(A_2) - 3)$
$(2, 0, 6, 6, -12)$	$36(d_0(A_1) + d_0(A_2) - 2)$
$(2, 9, 6, 6, -12)$	$36(2d_0(A_1) + d_1(A_1) + d_0(A_2) - 3)$

Table 28: Ordering 1-PS' for $G_{\mathcal{F}_2}$, I.

(m, r_1, r_2, r_3)	weight	weight	weight	weight	weight	weight	weight	weight	$2\mu(A_0, \lambda) + \mu(A_2, \lambda)$	if neither (2) nor (4) of **Proposition 7.1.1** holds
$(m, 0, 0, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	0	
	0	0	0	0	0	0	0	0		
$(m, 0, 1, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$2(2d_0(A_0) + d_0(A_2) - 3)$	≤ 0
	1	1	1	1	-1	-1	-1	-1		
$(m, 6, 0, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(d_0(A_2) - 1)$	≤ 0
	6	6	6	6	-6	-6	-6	-6		
$(m, 0, 0, 6)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(d_0(A_2) - 1)$	≤ 0
	6	6	6	6	-6	-6	-6	-6		
$(m, 3, 0, 3)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(d_0(A_2) - 1)$	≤ 0
	6	6	0	0	0	0	-6	-6		
$(m, 6, 6, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(2d_0(A_0) + 2d_0(A_2) - 3)$	≤ -12
	12	12	0	0	0	0	-12	-12		
$(m, 0, 6, 6)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(2d_0(A_0) + 2d_0(A_2) - 3)$	≤ -12
	12	12	0	0	0	0	-12	-12		
$(m, 3, 3, 3)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$3(2d_0(A_0) + 6d_0(A_2) + 2d_1(A_2) - 7)$	≤ -3
	9	3	3	3	-3	-3	-3	-9		
$(m, 3, 6, 3)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(2d_0(A_0) + 2d_0(A_2) + d_1(A_2) - 3)$	≤ -12
	12	6	6	0	0	-6	-6	-12		
$(m, 12, 6, 6)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(2d_0(A_0) + 4d_0(A_2) + 2d_1(A_2) - 5)$	≤ -12
	24	12	12	0	0	-12	-12	-24		
$(m, 6, 6, 12)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$12(2d_0(A_0) + 4d_0(A_2) + 2d_1(A_2) - 5)$	≤ -12
	24	12	12	0	0	-12	-12	-24		
$(m, 4, 2, 2)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$4(2d_0(A_0) + 4d_0(A_2) + 2d_1(A_2) - 5)$	≤ -4
	8	4	4	0	0	-4	-4	-8		
$(m, 2, 2, 4)$	$\xi_0 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_4$	$\xi_0 \wedge \xi_3 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_5$	$\xi_1 \wedge \xi_3 \wedge \xi_5$	$4(2d_0(A_0) + 4d_0(A_2) + 2d_1(A_2) - 5)$	≤ -4
	8	4	4	0	0	-4	-4	-8		

Table 29: Ordering 1-PS' for $G_{\mathcal{F}_2}$, II.

(m, r_1, r_2, r_3)	weight	weight	weight	weight	$2\mu(A_1, \lambda)$	if (1) of **Proposition 7.1.1** does not hold
$(0, 0, r_2, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	0	
	0	0	0	0		
$(0, 3, r_2, 3)$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$12(d_0(A_1) - 1)$	≤ -12
	3	3	-3	-3		
$(1, 0, r_2, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$12(d_0(A_1) - 1)$	≤ -12
	3	3	-3	-3		
$(1, 6, r_2, 0)$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$12(2d_0(A_1) - 3)$	≤ -12
	3	3	3	-9		
$(1, 0, r_2, 6)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$12(2d_0(A_1) - 1)$	≤ -12
	9	-3	-3	-3		
$(1, 3, r_2, 3)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$12(2d_0(A_1) + d_1(A_1) - 2)$	≤ -12
	6	0	0	-6		
$(1, 12, r_2, 6)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$12(4d_0(A_1) + 2d_1(A_1) - 5)$	≤ -12
	9	9	-3	-15		
$(1, 6, r_2, 12)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$12(4d_0(A_1) + 2d_1(A_1) - 3)$	≤ -12
	15	3	-9	-9		
$(1, 4, r_2, 2)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$4(6d_0(A_1) + 4d_1(A_1) - 7)$	≤ -12
	5	1	1	-7		
$(1, 2, r_2, 4)$	$\xi_0 \wedge \xi_1 \wedge \xi_4$	$\xi_0 \wedge \xi_2 \wedge \xi_3$	$\xi_0 \wedge \xi_1 \wedge \xi_5$	$\xi_1 \wedge \xi_2 \wedge \xi_3$	$4(6d_0(A_1) + 2d_1(A_1) - 5)$	≤ -12
	7	-1	-1	-5		

Table 30: Ordering 1-PS' for $G_{\mathcal{N}_3}$, I.

(m_0, m_1, m_2, r)	weight	weight	weight	weight	weight	weight	weight	weight	$\mu(A, \lambda)$
$(0, 1, -1, 0)$	015	123	024	034	025	035	124	134	$8(d_0(A_2) - 1)$
	2	2	-2	-2	0	0	0	0	
$(0, -1, 1, 0)$	024	034	015	123	025	035	124	134	$8(d_0(A_2) - 1)$
	2	2	-2	-2	0	0	0	0	
$(-1, 1, 1, 0)$	015	024	034	123	124	134	025	035	$6(2d_0(A_1) + d_0(A_3) - 2)$
	0	0	0	0	3	3	-3	-3	
$(1, -1, -1, 0)$	015	024	034	123	025	035	124	134	$6(2d_0(A_1) + d_0(A_3) - 2)$
	0	0	0	0	3	3	-3	-3	
$(0, 1, -1, 4)$	015	024	123	034	124	025	134	035	$8(2d_0(A_0) + 2d_0(A_2) + d_0(A_3) - 5)$
	2	2	2	-6	4	4	-4	-4	
$(0, -1, 1, 4)$	024	015	123	034	124	025	134	035	$8(2d_0(A_0) + 2d_0(A_2) + d_0(A_3) - 3)$
	6	-2	-2	-2	4	4	-4	-4	
$(4, -1, -1, 6)$	015	024	123	034	025	124	035	134	$24(d_0(A_0) + d_0(A_2) + d_0(A_3) - 2)$
	6	6	-6	-6	12	0	0	-12	
$(-4, 1, 1, 6)$	024	123	015	034	124	025	134	035	$24(d_0(A_0) + d_0(A_2) + d_0(A_3) - 2)$
	6	6	-6	-6	12	0	0	-12	

Table 31: Ordering 1-PS' for $G_{\mathcal{N}_3}$, II.

(m_0, m_1, m_2, r)	weight	weight	weight	weight	weight	weight	weight	weight	$\mu(A, \lambda)$
$(0, 1, 1, 2)$	015	024	034	123	124	025	134	035	$8(d_0(A_0) + 2d_0(A_1) + d_0(A_2) + d_0(A_3) - 3)$
	2	2	-2	-2	4	0	0	-4	
$(2, 1, -2, 3)$	015	024	123	034	025	124	035	134	$12(d_0(A_0) + 2d_0(A_2) + d_1(A_2) + d_0(A_3) - 3)$
	6	0	0	-6	6	0	0	-6	
$(4, 5, -1, 0)$	015	024	123	034	124	025	134	035	$12(4d_0(A_1) + 2d_0(A_2) - 3)$
	18	-6	-6	-6	0	0	0	0	
$(-4, -5, 1, 0)$	024	123	034	015	124	025	134	035	$48(d_0(A_1) + d_0(A_2) - 2)$
	6	6	6	-18	0	0	0	0	
$(2, 1, 1, 6)$	015	024	123	034	124	025	134	035	$12(2d_0(A_0) + 2d_0(A_1) + 2d_0(A_2) + d_0(A_3) - 5)$
	6	6	-6	-6	6	6	-6	-6	
$(8, 1, -5, 0)$	015	024	123	034	025	035	134	124	$24(2d_0(A_2) + d_0(A_3) - 2)$
	18	-6	-6	-6	12	12	-12	-12	
$(-8, -1, 5, 0)$	024	123	034	015	134	124	025	035	$24(2d_0(A_2) + d_0(A_3) - 4)$
	6	6	6	-18	12	12	-12	-12	
$(-4, 1, 7, 12)$	024	015	123	034	124	025	134	035	$48(d_0(A_0) + d_0(A_1) + d_0(A_2) + d_0(A_3) - 2)$
	18	-6	-6	-6	24	0	0	-24	
$(4, -1, -7, 12)$	024	015	123	034	025	124	035	134	$48(d_0(A_0) + d_0(A_1) + d_0(A_2) + d_0(A_3) - 3)$
	6	6	6	-18	24	0	0	-24	

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