AN APPLICATION OF COLLAPSING LEVELS TO THE REPRESENTATION THEORY OF AFFINE VERTEX ALGEBRAS

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ABSTRACT. We discover a large class of simple affine vertex algebras $V_k(\mathfrak{g})$, associated to basic Lie superalgebras $\mathfrak{g}$ at non–admissible collapsing levels k, having exactly one irreducible $\mathfrak{g}$–locally finite module in the category $\mathcal{O}$. In the case when $\mathfrak{g}$ is a Lie algebra, we prove a complete reducibility result for $V_k(\mathfrak{g})$–modules at an arbitrary collapsing level. We also determine the generators of the maximal ideal in the universal affine vertex algebra $V^k(\mathfrak{g})$ at certain negative integer levels. Considering some conformal embeddings in the simple affine vertex algebras $V_{-1/2}(C_n)$ and $V_{-4}(E_7)$, we surprisingly obtain the realization of non-simple affine vertex algebras of types B and D having exactly one non-trivial ideal.

1. INTRODUCTION

Affine vertex algebras are one of the most interesting and important classes of vertex algebras. Categories of modules for simple affine vertex algebra $V_k(\mathfrak{g})$, associated to a simple Lie algebra $\mathfrak{g}$, have mostly been studied in the case of positive integer levels $k \in \mathbb{Z}_{\geq 0}$. These categories enjoy many nice properties such as: finitely many irreducibles, semisimplicity, modular invariance of characters (cf. [26], [31], [33], [41]).

In recent years, affine vertex algebras have attracted a lot of attention because of their connection with affine $\mathcal{W}$–algebras $W_k(\mathfrak{g}, f)$, obtained by quantum Hamiltonian reduction (cf. [21], [23], [34], [35]). Since the quantum Hamiltonian reduction functor $H_f(\,\cdot\,)$ maps any integrable $\widehat{\mathfrak{g}}$–module to zero (cf. [12], [34]), in order to obtain interesting $\mathcal{W}$–algebras, one has to consider affine vertex algebras $V_k(\mathfrak{g})$, for $k \notin \mathbb{Z}_{\geq 0}$.

It turns out that for certain non-admissible levels k (such as negative integer levels), the associated vertex algebras $V_k(\mathfrak{g})$ have finitely many irreducibles in category $\mathcal{O}$ (cf. [15], [17], [40]), and their characters satisfy certain modular-like properties (cf. [14]). These affine vertex algebras then give C_2–cofinite $\mathcal{W}$–algebras $W_k(\mathfrak{g}, f)$, for properly chosen nilpotent element f (cf. [36], [38]).

In this paper, we classify irreducible modules in the category KL_k (i.e. the category of $\mathfrak{g}$–locally finite $V_k(\mathfrak{g})$–modules in $\mathcal{O}^k$ (see Subsection 2.3)) for a large family of collapsing levels k. Recall from [4] that a level k is called *collapsing* if the simple $\mathcal{W}$–algebra $W_k(\mathfrak{g}, \theta)$, associated to a minimal nilpotent element $e_{-\theta}$, is isomorphic to its affine vertex subalgebra $\mathcal{V}_k(\mathfrak{g}^\natural)$ (see Definition 2.2 and (2.7)). In the present paper we keep the notation of [4]. In particular, the highest root is normalized by the condition $(\theta, \theta) = 2$. We discover a large family of vertex algebras having one irreducible module in the category KL_k, which in a way extends the results on Deligne series from [15]. Part (1) is proven there in the Lie algebra case.

Theorem 1.1. *Assume that the level k and the basic simple Lie superalgebra $\mathfrak{g}$ satisfy one of the following conditions:*

  (1) *$k = -\frac{h^\vee}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n + 8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ);*
  (2) *$k = -h^\vee/2 + 1$ and $\mathfrak{g} = osp(n + 4m + 8|n)$, $n \geq 2, m \geq 0$.*
  (3) *$k = -h^\vee/2 + 1$ and $\mathfrak{g} = D_{2m}$, $m \geq 2$.*
  (4) *$k = -10$ and $\mathfrak{g} = E_8$.*

Then $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k.

We also prove a complete reducibility result in KL_k (cf. Theorem 5.9, Theorem 5.7):

Theorem 1.2. *Assume that $\mathfrak{g}$ is a Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then KL_k is a semi-simple category in the following cases:*

- k *is a collapsing level.*
- $W_k(\mathfrak{g}, \theta)$ *is a rational vertex operator algebra.*

It is interesting that in some cases we have that KL_k is a semi-simple category, but there can exist indecomposable but not irreducible $V_k(\mathfrak{g})$–modules in the category $\mathcal{O}$. In order to prove Theorem 1.2 we modified methods from [28] and [20] in a vertex algebraic setting. In particular we prove that the contravariant functor $M \mapsto M^\sigma$ from [20] acts on the category KL_k (cf. Lemma 3.6). Then for the proof of complete reducibility in KL_k it is enough to check that every highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible (cf. Theorem 5.5).

Representation theory of a simple affine vertex algebra $V_k(\mathfrak{g})$ is naturally connected with the structure of the maximal ideal in the universal affine vertex algebra $V^k(\mathfrak{g})$. In the second part of paper we present explicit formulas for singular vectors which generate the maximal ideal in $V^{2-2\ell}(D_{2\ell})$ (which is case (3) of Theorem 1.1) and $V^{-2}(D_\ell)$. In the second case, we show that the Hamiltonian reduction functor $H_\theta(\,\cdot\,)$ gives an equivalence of the category of $\mathfrak{g}$–locally finite $V_{-2}(D_\ell)$–modules KL_{-2} and the category of modules for a rational vertex algebra $V_{\ell-4}(A_1)$. Singular vectors in $V^k(\mathfrak{g})$ for certain negative integer levels k have also been constructed in [2].

We also apply our results to study the structure of conformally embedded subalgebras of some simple affine vertex algebras.

As in [6], for a subalgebra $\mathfrak{k}$ of a simple Lie algebra $\mathfrak{g}$, we denote by $\widetilde{V}(k, \mathfrak{k})$ the vertex subalgebra of $V_k(\mathfrak{g})$ generated by $x(-1)\mathbf{1}$, $x \in \mathfrak{k}$. If $\mathfrak{k}$ is a reductive quadratic subalgebra of $\mathfrak{g}$, then we say that $\widetilde{V}(k, \mathfrak{k})$ is conformally embedded in $V_k(\mathfrak{g})$ if the Sugawara-Virasoro vectors of both algebras coincide. We also say that $\mathfrak{k}$ is conformally embedded in $\mathfrak{g}$ at level k if $\widetilde{V}(k, \mathfrak{k})$ is conformally embedded in $V_k(\mathfrak{g})$.

We are able to prove that in the cases listed in Theorem 1.3 below, $\widetilde{V}(k, \mathfrak{k})$ is not simple. On the other hand, we show that $V_{-1/2}(C_5)$ contains a *simple* subalgebra $V_{-2}(B_2) \otimes V_{-5/2}(A_1)$ (see Corollary 7.4). For the conformal embedding of $D_6 \times A_1$ into E_7 at level $k = -4$, we show that $\widetilde{V}(-4, D_6 \times A_1) = \mathcal{V}_{-4}(D_6) \otimes V_{-4}(A_1)$ where $\mathcal{V}_{-4}(D_6)$ is a quotient of the universal affine vertex algebra $V^{-4}(D_6)$ by two singular vectors of conformal weights two and three (cf. (9.6)). Moreover, $\mathcal{V}_{-4}(D_6)$ has infinitely many irreducible modules in the category of $\mathfrak{g}$–locally finite modules, which we explicitly describe. All of them appear in $V_{-4}(E_7)$ as submodules or subquotients.

Theorem 1.3. *Let $\mathcal{V}_k(D_\ell)$, $\mathcal{V}_k(B_\ell)$, be the vertex algebras defined in (6.3), (7.1), (9.6). Consider the following conformal embeddings:*

(1) $D_\ell \times A_1$ *into C_{2l} for $\ell \geq 4$ at level $k = -\frac{1}{2}$.*
(2) $B_\ell \times A_1$ *into C_{2l+1} for $\ell \geq 3$ at level $k = -\frac{1}{2}$.*
(3) $D_6 \times A_1$ *into E_7 at level $k = -4$.*

Then,

- $\widetilde{V}(-\frac{1}{2}, D_\ell \times A_1) = \mathcal{V}_{-2}(D_\ell) \otimes V_{-\ell}(A_1)$ *in case (1),*
- $\widetilde{V}(-\frac{1}{2}, B_\ell \times A_1) = \mathcal{V}_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1)$ *in case (2),*
- $\widetilde{V}(-4, D_6 \times A_1) = \mathcal{V}_{-4}(D_6) \otimes V_{-4}(A_1)$ *in case (3).*

Moreover, the algebras $\mathcal{V}_k(D_\ell)$, $\mathcal{V}_k(B_\ell)$, are non-simple, with a unique non-trivial ideal.

The decompositions of the embeddings above is still an open problem, and will be a subject of our forthcoming papers.

2. Preliminaries

We assume that the reader is familiar with the notion of vertex (super)algebra (cf. [18], [25], [32]) and of simple basic Lie superalgebras (see [30]) and their affinizations (see [31] for the Lie algebra case).

Let V be a conformal vertex algebra. Denote by $A(V)$ the associative algebra introduced in [41], called the Zhu algebra of V.

2.1. **Basic Lie superalgebras and minimal gradings.** For the reader's convenience we recall here the setting and notation of [4] regarding basic Lie superalgebras and their minimal gradings. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple finite dimensional basic Lie superalgebra. We choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_{\bar{0}}$ and let Δ be the set of roots. Assume $\mathfrak{g}$ is not $osp(3|n)$. A root $-\theta$ is called *minimal* if it is even and there exists an additive function $\varphi : \Delta \to \mathbb{R}$ such that $\varphi_{|\Delta} \neq 0$ and $\varphi(\theta) > \varphi(\eta)$, $\forall \eta \in \Delta \setminus \{\theta\}$. Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors e_θ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \qquad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad\,x$ defines a *minimal* $\frac{1}{2}\mathbb{Z}$-grading ([35, (5.1)]):

$$(2.1) \qquad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$. We thus have a bijective correspondence between minimal gradings (up to an automorphism of $\mathfrak{g}$) and minimal roots (up to the action of the Weyl group). Furthermore, one has

$$(2.2) \qquad \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{f_\theta, x, e_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$, as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition

$$(2.3) \qquad (\theta|\theta) = 2.$$

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ (equivalently, of the minimal gradation (2.1)) is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot|\cdot)$, normalized by (2.3). Since θ is the highest root, we have that $2h^\vee = (\theta|\theta + 2\rho)$ hence

$$(2.4) \qquad (\rho|\theta) = h^\vee - 1.$$

The complete list of the Lie superalgebras $\mathfrak{g}^\natural$, the $\mathfrak{g}^\natural$–modules $\mathfrak{g}_{\pm 1/2}$ (they are isomorphic and self-dual), and $h^\vee$ for all possible choices of $\mathfrak{g}$ and of θ (up to isomorphism) is given in Tables 1,2,3 of [35]. We reproduce them below. Note that in these tables $\mathfrak{g} = osp(m|n)$ (resp. $\mathfrak{g} = spo(n|m)$) means that θ is the highest root of the simple component $so(m)$ (resp. $sp(n)$) of $\mathfrak{g}_{\bar{0}}$. Also, for $\mathfrak{g} = sl(m|n)$ or $psl(m|m)$ we always take θ to be the highest root of the simple component $sl(m)$ of $\mathfrak{g}_{\bar{0}}$ (for $m = 4$ we take one of the simple roots). Note that the exceptional Lie superalgebras $\mathfrak{g} = F(4)$ and $\mathfrak{g} = G(3)$ appear in both Tables 2 and 3, which corresponds to the two inequivalent choices of θ, the first one being a root of the simple component $sl(2)$ of $\mathfrak{g}_{\bar{0}}$.

Table 1

$\mathfrak{g}$ is a simple Lie algebra.

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(n), n \geq 3$	$gl(n-2)$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	n	F_4	$sp(6)$	$\bigwedge_0^3 \mathbb{C}^6$	9
$so(n), n \geq 5$	$sl(2) \oplus so(n-4)$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	E_6	$sl(6)$	$\bigwedge^3 \mathbb{C}^6$	12
$sp(n), n \geq 2$	$sp(n-2)$	$\mathbb{C}^{n-2}$	$n/2 + 1$	E_7	$so(12)$	$spin_{12}$	18
G_2	$sl(2)$	$S^3\mathbb{C}^2$	4	E_8	E_7	$\dim = 56$	30

Table 2

$\mathfrak{g}$ is not a Lie algebra but $\mathfrak{g}^\natural$ is and $\mathfrak{g}_{\pm 1/2}$ is purely odd ($m \geq 1$).

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(2|m)$, $m \neq 2$	$gl(m)$	$\mathbb{C}^m \oplus (\mathbb{C}^m)^*$	$2-m$	$D(2,1;a)$	$sl(2) \oplus sl(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	0
$psl(2|2)$	$sl(2)$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	0	$F(4)$	$so(7)$	$spin_7$	-2
$spo(2|m)$	$so(m)$	$\mathbb{C}^m$	$2-m/2$	$G(3)$	G_2	Dim $= 0|7$	$-3/2$
$osp(4|m)$	$sl(2) \oplus sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$2-m$				

Table 3

Both $\mathfrak{g}$ and $\mathfrak{g}^\natural$ are not Lie algebras ($m, n \geq 1$).

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(m|n)$, $m \neq n, m > 2$	$gl(m-2|n)$	$\mathbb{C}^{m-2|n} \oplus (\mathbb{C}^{m-2|n})^*$	$m-n$
$psl(m|m)$, $m > 2$	$sl(m-2|m)$	$\mathbb{C}^{m-2|m} \oplus (\mathbb{C}^{m-2|m})^*$	0
$spo(n|m)$, $n \geq 4$	$spo(n-2|m)$	$\mathbb{C}^{n-2|m}$	$1/2(n-m)+1$
$osp(m|n)$, $m \geq 5$	$osp(m-4|n) \oplus sl(2)$	$\mathbb{C}^{m-4|n} \otimes \mathbb{C}^2$	$m-n-2$
$F(4)$	$D(2,1;2)$	Dim $= 6|4$	3
$G(3)$	$osp(3|2)$	Dim $= 4|4$	2

In this paper we shall exclude the case of $\mathfrak{g} = sl(n+2|n)$, $n > 0$. In all other cases the Lie superalgebra $\mathfrak{g}^\natural$ decomposes in a direct sum of all its minimal ideals, called components of $\mathfrak{g}^\natural$:

$$\mathfrak{g}^\natural = \bigoplus_{i \in I} \mathfrak{g}_i^\natural,$$

where each summand is either the (at most 1-dimensional) center of $\mathfrak{g}^\natural$ or is a basic simple Lie superalgebra different from $psl(n|n)$. Let $C_{\mathfrak{g}_i^\natural}$ be the Casimir operator of $\mathfrak{g}_i^\natural$ corresponding to $(\cdot|\cdot)_{|\mathfrak{g}_i^\natural \times \mathfrak{g}_i^\natural}$. We define the dual Coxeter number $h_{0,i}^\vee$ of $\mathfrak{g}_i^\natural$ as half of the eigenvalue of $C_{\mathfrak{g}_i^\natural}$ acting on $\mathfrak{g}_i^\natural$ (which is 0 if $\mathfrak{g}_i^\natural$ is abelian).

Denote by $V_{\mathfrak{g}}(\mu)$ (or $V(\mu)$) the irreducible finite-dimensional highest weight $\mathfrak{g}$–module with highest weight μ. Denote by P_+ the set of highest weights of irreducible finite-dimensional representations of $\mathfrak{g}$.

Since $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$, we have, in particular, that $\mu \in \mathfrak{h}^*$ can be uniquely written as

$$(2.5) \qquad \mu = \mu_{|\mathfrak{h}^\natural} + \ell\theta,$$

with $\ell \in \mathbb{C}$. If $\mu \in P_+$, then, since $\theta(\mathfrak{h}^\natural) = 0$, $\mu(\theta^\vee) = 2\ell \in \mathbb{Z}$, so $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

2.2. Affine Lie algebras, vertex algebras, $\mathcal{W}$-algebras.

Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the usual commutation relations. We let δ be the fundamental imaginary root. Let $\alpha_0 = \delta - \theta$ the affine simple root. Since θ is even, hence non-isotropic, so that $\alpha_0^\vee = K - \theta^\vee$ makes sense.

Denote by $L(\lambda)$ (or $L_{\mathfrak{g}}(\lambda)$) the irreducible highest weight $\widehat{\mathfrak{g}}$–module with highest weight λ.

Denote by $V^k(\mathfrak{g})$ the universal affine vertex algebra associated to $\widehat{\mathfrak{g}}$ of level $k \in \mathbb{C}$. We shall assume that $k \neq -h^\vee$. Then (see e.g. [32]) $V^k(\mathfrak{g})$ is a conformal vertex algebra with Segal-Sugawara conformal vector $\omega_{\mathfrak{g}}$. Let $Y(\omega_{\mathfrak{g}}, z) = \sum L_{\mathfrak{g}}(n)z^{-n-2}$ be the corresponding Virasoro field. Denote by $V_k(\mathfrak{g})$ the (unique) simple quotient of $V^k(\mathfrak{g})$. Clearly, $V_k(\mathfrak{g}) \cong L_{\mathfrak{g}}(k\Lambda_0)$ as $\widehat{\mathfrak{g}}$–modules.

Denote by $W^k(\mathfrak{g}, \theta)$ the affine $\mathcal{W}$–algebra obtained from $V^k(\mathfrak{g})$ by Hamiltonian reduction relative to a minimal nilpotent element $e_{-\theta}$. Denote by $W_k(\mathfrak{g}, \theta)$ the simple quotient of $W^k(\mathfrak{g}, \theta)$. Recall that the vertex algebra $W^k(\mathfrak{g}, \theta)$ is strongly and freely generated by elements $J^{\{a\}}$, where a runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where v runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector ω. The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and 3/2, respectively, with respect to ω.

Let $\mathcal{V}^k(\mathfrak{g}^\natural)$ be the subalgebra of the vertex algebra $W^k(\mathfrak{g}, \theta)$, generated by $\{J^{\{a\}} \mid a \in \mathfrak{g}^\natural\}$. The vertex algebra $\mathcal{V}^k(\mathfrak{g}^\natural)$ is isomorphic to a universal affine vertex algebra. More precisely, letting

$$(2.6) \qquad k_i = k + \tfrac{1}{2}(h^\vee - h_{0,i}^\vee), \ i \in I,$$

the map $a \mapsto J^{\{a\}}$ extends to an isomorphism $\mathcal{V}^k(\mathfrak{g}^\natural) \simeq \bigotimes_{i \in I} V^{k_i}(\mathfrak{g}_i^\natural)$.

We also set $\mathcal{V}_k(\mathfrak{g}^\natural)$ to be the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, \theta)$. Clearly we can write

$$(2.7) \qquad \mathcal{V}_k(\mathfrak{g}^\natural) \simeq \bigotimes_{i \in I} \mathcal{V}_{k_i}(\mathfrak{g}_i^\natural),$$

where $\mathcal{V}_{k_i}(\mathfrak{g}_i^\natural)$ is some quotient (not necessarily simple) of $V^{k_i}(\mathfrak{g}_i^\natural)$.

2.3. **Category $\mathcal{O}$ and Hamiltonian reduction functor.** Recall that $\widehat{\mathfrak{g}}$-module M is in category $\mathcal{O}^k$ if it is $\widehat{\mathfrak{h}}$-diagonalizable with finite dimensional weight spaces, K acts as $k Id_M$ and M has a finite number of maximal weights.

There is a remarkable functor H_θ from $\mathcal{O}^k$ to the category of $W^k(\mathfrak{g}, \theta)$-modules whose properties will be very important in the following. We recall them in a form suitable for our purposes (see [12] for details; there H_θ is denoted by H^0).

Theorem 2.1.

(1) H_θ is exact.
(2) If $L(\lambda)$ is a irreducible highest weight $\widehat{\mathfrak{g}}$-module, then $\lambda(\alpha_0^\vee) \in \mathbb{Z}_{\geq 0}$ implies $H_\theta(L(\lambda)) = \{0\}$. Otherwise $H_\theta(L(\lambda))$ is isomorphic to the irreducible $W^k(\mathfrak{g}, \theta)$-module with highest weight ϕ_λ defined by formula (67) in [12].

2.4. **Collapsing levels.**

Definition 2.2. *Assume $k \neq -h^\vee$. If $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^\natural)$, we say that k is a collapsing level.*

Theorem 2.3. [4, Theorem 3.3] *Let $p(k)$ be the polynomial listed in Table 4 below. Then k is a collapsing level if and only if $k \neq -h^\vee$ and $p(k) = 0$. In such cases,*

$$(2.8) \qquad W_k(\mathfrak{g}, \theta) = \bigotimes_{i \in I^*} V_{k_i}(\mathfrak{g}_i^\natural),$$

where $I^* = \{i \in I \mid k_i \neq 0\}$. If $I^* = \emptyset$, then $W_k(\mathfrak{g}, \theta) = \mathbb{C}$.

Table 4

Polynomials $p(k)$.

$\mathfrak{g}$	$p(k)$	$\mathfrak{g}$	$p(k)$
$sl(m\|n),\ n \neq m$	$(k+1)(k+(m-n)/2)$	E_6	$(k+3)(k+4)$
$psl(m\|m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$osp(m\|n)$	$(k+2)(k+(m-n-4)/2)$	E_8	$(k+6)(k+10)$
$spo(n\|m)$	$(k+1/2)(k+(n-m+4)/4)$	F_4	$(k+5/2)(k+3)$
$D(2,1;a)$	$(k-a)(k+1+a)$	G_2	$(k+4/3)(k+5/3)$
$F(4),\ \mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$	$G(3),\ \mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$F(4),\ \mathfrak{g}^\natural = D(2,1;2)$	$(k+3/2)(k+1)$	$G(3),\ \mathfrak{g}^\natural = osp(3\|2)$	$(k+2/3)(k+4/3)$

2.5. **Weyl vertex algebra.** Let M_ℓ denote the Weyl vertex algebra (also called symplectic bosons) generated by even elements $a_i^\pm$, $i = 1, \ldots, \ell$ satisfying the following λ–brackets

$$[(a_i^\pm)_\lambda (a_j^\pm)] = 0, \quad [(a_i^+)_\lambda (a_j^-)] = \delta_{i,j}.$$

Recall also that the symplectic affine vertex algebra $V_{-1/2}(C_\ell)$ is realized as a $\mathbb{Z}_2$–orbifold of M_ℓ (see [22]).

3. THE CATEGORY KL_k

Let k be a noncritical level. Note that the Casimir element of $\widehat{\mathfrak{g}}$ can be expressed as $\Omega = d + L_\mathfrak{g}(0)$; it commutes with $\widehat{\mathfrak{g}}$–action.

Consider the category $\mathcal{C}^k$ of modules for the universal affine vertex algebra $V^k(\mathfrak{g})$, i.e. the category of restricted $\widehat{\mathfrak{g}}$–modules of level k. Regard $M \in \mathcal{C}^k$ as a $\widehat{\mathfrak{g}}$–module by letting d act as $-L_\mathfrak{g}(0)$. Let KL^k be the category of modules $M \in \mathcal{C}^k$ such that, as $\widehat{\mathfrak{g}}$-modules, are in $\mathcal{O}^k$ and which admit the following weight space decomposition with respect to $L_\mathfrak{g}(0)$:

$$M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_\mathfrak{g}(0)|M(\alpha) \equiv \alpha\, Id, \ \dim M(\alpha) < \infty.$$

Our definition is related but different from the one introduced in [13]. Let KL_k be the category of all modules in KL^k which are $V_k(\mathfrak{g})$–modules.

Remark 3.1. *If $V_k(\mathfrak{g})$ has finitely many irreducible modules in the category KL^k, one can show that every $V_k(\mathfrak{g})$–module M in KL_k is of finite length. This happens when k is admissible (cf. [12]) and when $V_k(\mathfrak{g})$ is quasi-lisse (cf. [14]). But when $V_k(\mathfrak{g})$ has infinitely many irreducible modules in KL^k (as in the cases considered in [39], [11]), then one can have modules in KL_k of infinite length.*

Recall that there is a one-to-one correspondence between irreducible $\mathbb{Z}_{\geq 0}$–graded modules for a conformal vertex algebra V (with a conformal vector ω, such that $Y(\omega, z) = \sum_{i \in \mathbb{Z}} L(i) z^{-i-2}$) and irreducible modules for the corresponding Zhu algebra $A(V)$ [41]. This implies, in particular, that there is a one-to-one correspondence between irreducible finite-dimensional $A(V)$–modules and irreducible $\mathbb{Z}_{\geq 0}$–graded V–modules whose graded components, which are eigenspaces for $L(0)$, are finite-dimensional. In the case of affine vertex algebras, we have the following simple interpretation.

Proposition 3.2. *Let $\widetilde{V}_k(\mathfrak{g})$ be a quotient of $V^k(\mathfrak{g})$ (not necessary simple). Consider $\widetilde{V}_k(\mathfrak{g})$ as a conformal vertex algebra with conformal vector $\omega_{\mathfrak{g}}$. Then there is a one-to-one correspondence between irreducible $\widetilde{V}_k(\mathfrak{g})$ in the category KL^k and irreducible finite-dimensional $A(\widetilde{V}_k(\mathfrak{g}))$–modules.*

Corollary 3.3. *Assume that $\mathfrak{g}$ is a simple basic Lie superalgebra and $\widetilde{V}_k(\mathfrak{g})$ is a quotient of $V^k(\mathfrak{g})$ such that the trivial module $\mathbb{C}$ is the unique finite-dimensional irreducible $A(\widetilde{V}_k(\mathfrak{g}))$–module. Then $\widetilde{V}_k(\mathfrak{g}) = V_k(\mathfrak{g})$.*

Proof. Assume that $\widetilde{V}_k(\mathfrak{g})$ is not simple. Then it contains a non-zero graded ideal $I \neq \widetilde{V}_k(\mathfrak{g})$ with respect to $L_{\mathfrak{g}}(0)$:

$$I = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} I(n + n_0), \quad L_{\mathfrak{g}}(0)|I(r) = r\mathrm{Id}, \quad I(n_0) \neq 0.$$

Since $I \neq \widetilde{V}_k(\mathfrak{g})$, we have that $n_0 > 0$, otherwise $\mathbf{1} \in I$.

We can consider $I(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(\widetilde{V}_k(\mathfrak{g}))$.

Since the Casimir element $C_{\mathfrak{g}}$ of $\mathfrak{g}$ acts on $I(n_0)$ as the non-zero constant $2(k + h^\vee)n_0$, we conclude that $C_{\mathfrak{g}}$ acts by the same constant on any irreducible $\mathfrak{g}$–subquotient of $I(n_0)$. But any irreducible subquotient of $I(n_0)$ is an irreducible finite–dimensional $A(\widetilde{V}_k(\mathfrak{g}))$–module, and therefore it is trivial. This implies that $C_{\mathfrak{g}}$ acts non-trivially on a trivial $\mathfrak{g}$–module, a contradiction. $\square$

Take the Chevalley generators e_i, f_i, h_i, $i = 0, \ldots, \ell$, of the Kac–Moody Lie algebra $\widehat{\mathfrak{g}}$ such that e_i, f_i, h_i, $i = 1, \ldots, \ell$, are the Chevalley generators of $\mathfrak{g}$. Let σ be the Chevalley antiautomorphism of $\widehat{\mathfrak{g}}$ defined by

$$e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i, \quad d \mapsto d \quad (i = 0, \ldots, \ell).$$

Assume that M is from the category $\mathcal{O}$ of non-critical level k. Then M admits the decomposition into weight spaces $M = \bigoplus_{\mu \in \Omega(M)} M_\mu$, where $\Omega(M)$ is the set of weights of M and $\dim M_\mu < \infty$ for every $\mu \in \Omega(M)$. For a finite-dimensional vector spaces U, let U^* denote its dual space. Then we have the contravariant functor $M \mapsto M^\sigma$ [20] acting on modules from the category $\mathcal{O}$. Here $M^\sigma = \bigoplus_{\mu \in \Omega(M)} M_\mu^*$ is the $\widehat{\mathfrak{g}}$–module uniquely determined by

$$\langle yw', w \rangle = \langle w', \sigma(y)w \rangle, \quad y \in \widehat{\mathfrak{g}}, \ w' \in M^\sigma, \ w \in M.$$

It is easy to see that M admits the decomposition

(3.1) $$M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_{\mathfrak{g}}(0)|M(\alpha) \equiv \alpha \, \mathrm{Id}$$

such that :

- for any $\alpha \in \mathbb{C}$ we have $M(\alpha - n) = 0$ for $n \in \mathbb{Z}$ sufficiently large;
- for any $\mu \in \Omega(M)$ there exist $\alpha \in \mathbb{C}$ such that $M_\mu \subset M(\alpha)$.

Proposition 3.4. *Assume that a module M is in the category $\mathcal{O}^k$. Then M is in the category KL^k if and only if M is $\mathfrak{g}$-locally finite.*

Proof. If M is in KL^k then it admits a decomposition as in (3.1). Since the spaces $M(\alpha)$ are $\mathfrak{g}$–stable and finite-dimensional, M is $\mathfrak{g}$–locally finite.

Let us prove the converse. If M is a highest weight module which is $\mathfrak{g}$–locally finite, then clearly all eigenspaces for $L_{\mathfrak{g}}(0)$ are finite-dimensional. Assume now that M is an arbitrary $\mathfrak{g}$–locally finite module in the category $\mathcal{O}^k$. Take $\alpha \in \mathbb{C}$ such that $M(\alpha) \neq \{0\}$. Then from [20, Proposition 3.1] we see that M has an increasing filtration (possibly infinite)

$$(3.2) \qquad \{0\} = M_0 \subset M_1 \subset \cdots \subset M$$

such that for every $j \in \mathbb{Z}_{>0}$, $M_j/M_{j-1} \cong \widetilde{L}(\lambda_j)$ is a highest weight $V^k(\mathfrak{g})$–module with highest weight λ_j, which is $\mathfrak{g}$–locally finite. Let h_{λ_j} denotes the lowest conformal weight of $\widetilde{L}(\lambda_j)$. Since the factors M_i/M_{i-1} $(i \leq j)$ of M_j are highest weight modules, their $L_{\mathfrak{g}}(0)$–eigenspaces are finite-dimensional. This implies that the $L_{\mathfrak{g}}(0)$–eigenspaces of M_j is finite-dimensional. By using the properties of the category $\mathcal{O}$ one sees the following:

- There exists a finite subset $\{d_1, \cdots, d_s\} \subset \mathbb{C}$ such that $\alpha \in \bigcup_{i=1}^s (d_i + \mathbb{Z}_{\geq 0})$.
- For $d \in \mathbb{C}$ there exist only finitely many subquotients $\widetilde{L}(\lambda_j)$ in (3.2) such that $h_{\lambda_j} = d$.

This implies that there is $j_0 \in \mathbb{Z}_{>0}$ such that $\alpha < h_{\lambda_j}$ for $j \geq j_0$. Therefore $M(\alpha) \subset M_{j_0}$. This proves that $M(\alpha)$ is finite-dimensional. $\qquad\square$

Remark 3.5. *We will use several times the following fact, which is a consequence of the previous proposition: for any $k \notin \mathbb{Z}_{\geq 0}$ and any irreducible highest weight module $L(\lambda)$ in the category KL^k, one has $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$.*

Since $\sigma(L_{\mathfrak{g}}(0)) = L_{\mathfrak{g}}(0)$, if M is in the category KL^k, then M^σ is also in the category KL^k. The next result shows that this functor acts on the category KL_k. In the proof we find an explicit relation of M^σ with the contragradient modules, defined for ordinary modules for vertex operator algebras [24].

Lemma 3.6.
(1) *Assume that M is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Then M^σ is also a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$.*
(2) *Assume that M is a $V_k(\mathfrak{g})$–module in the category KL_k. Then M^σ is also in KL_k.*

Proof. Assume that M is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Take the weight decomposition $M = \bigoplus_{\mu \in \Omega(M)} M_\mu$, and set $M^c = \bigoplus_{\mu \in \Omega(M)} M_\mu^*$. By applying the same approach as in the construction of the contragredient module from [24, Section 5], we get a $V_k(\mathfrak{g})$–module $(M^c, Y_{M^c}(\cdot, z))$, with vertex operator map

$$(3.3) \qquad \langle Y_{M^c}(v, z)w', w\rangle = \langle w', Y_M(e^{zL_{\mathfrak{g}}(1)}(-z^{-2})^{L_{\mathfrak{g}}(0)}v, z)w\rangle,$$

where $w' \in M^c$, $w \in M$. The $\widehat{\mathfrak{g}}$–action on M^c is uniquely determined by

$$\langle x(n)w', w\rangle = -\langle w', x(-n)w\rangle \quad (x \in \mathfrak{g}).$$

As a vector space $M^c = M^\sigma$, but we have different actions of $\widehat{\mathfrak{g}}$. (Note that, in general, M^c can be outside of the category $\mathcal{O}$.)

Take the Lie algebra automorphism $h \in Aut(\mathfrak{g})$ such that

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \quad h_i \mapsto -h_i \quad (i = 1, \ldots, \ell).$$

Then h can be lifted to an automorphism of $V^k(\mathfrak{g})$. Since the maximal ideal of $V^k(\mathfrak{g})$ is unique, then it is h–invariant, thus h is also an automorphism of $V_k(\mathfrak{g})$. Then we can define a $V_k(\mathfrak{g})$–module $(M_h^c, Y_{M_h^c}(\cdot, z))$ where

$$M_h^c := M^c, \quad Y_{M_h^c}(v, z) = Y_{M^c}(hv, z).$$

On M_h^c we have

$$\langle e_i(n)w', w\rangle = \langle w', f_i(-n)w\rangle$$

$$\langle f_i(n)w', w\rangle = \langle w', e_i(-n)w\rangle$$

$$\langle h_i(n)w', w\rangle = \langle w', h_i(-n)w\rangle$$

where $i = 1, \ldots, \ell$. This implies that $M_h^c = M^\sigma$. This proves the assertion (1).

Assume now that M is in the category KL_k. Then all $L_{\mathfrak{g}}(0)$–eigenspaces are finite-dimensional, thus

$$M^c = \bigoplus_{\mu \in \Omega(M)} M_\mu^* = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha)^*.$$

This implies the $V_k(\mathfrak{g})$–module $(M^c, Y_{M^c}(\cdot, z))$ coincides with the contragredient module [24], realized on the restricted dual space $\bigoplus_{\alpha \in \mathbb{C}} M(\alpha)^*$, with the vertex operator map (3.3). Since the $L_{\mathfrak{g}}(0)$–eigenspaces of M^c are finite-dimensional, we conclude that M^c and $M^\sigma = M_h^c$ are $V_k(\mathfrak{g})$–modules in KL_k. Claim (2) follows. □

4. Constructions of vertex algebras with one irreducible module in KL_k via collapsing levels

By [4], if k is a collapsing level, then either $W_k(\mathfrak{g}, \theta) = \mathbb{C}$, $W_k(\mathfrak{g}, \theta) = M(1)$, or $W_k(\mathfrak{g}, \theta) = V_{k'}(\mathfrak{a})$ for a unique simple component $\mathfrak{a}$ of $\mathfrak{g}^\natural$. Here the level k' is computed with respect to the invariant bilinear form of $\mathfrak{a}$ normalized so that the minimal root has squared length 2. For $\mathfrak{a} = sl(m|n)$, $m \geq 2$, the minimal root is always chosen to be the lowest root of $sl(m)$. For $\mathfrak{a} = osp(m|n)$ we write $spo(n|m)$ vs. $osp(m|n)$ to specify the choice of the minimal root. In all other cases the minimal root of $\mathfrak{a}$ is unique.

To simplify notation define $V_{k'}(\mathfrak{g}^\natural)$ to be as follows:

$$V_{k'}(\mathfrak{g}^\natural) = \begin{cases} \mathbb{C} & \text{if } W_k(\mathfrak{g}, \theta) = \mathbb{C}; \text{ in this case we set } k' = 0; \\ M(1) & \text{if } W_k(\mathfrak{g}, \theta) = M(1); \text{ in this case we set } k' = 1; \\ V_{k'}(\mathfrak{a}) & \text{otherwise.} \end{cases}$$

In Table 5 we summarize all the relevant data.

Assume that $k \notin \mathbb{Z}_{\geq 0}$ and that:

(1) k is a collapsing level for $\mathfrak{g}$;
(2) $V_{k'}(\mathfrak{g}^\natural)$ is the unique irreducible $V_{k'}(\mathfrak{g}^\natural)$–module in the category $KL_{k'}$.

Assume that $L(\widehat{\Lambda})$ is an irreducible $V_k(\mathfrak{g})$-module in the category KL_k. Set $\mu = \widehat{\Lambda}_{|\mathfrak{h}}$. By Proposition 3.4 we have $\mu \in P_+$, hence, by (2.5), the weight μ has the form $\mu = \mu^\natural + \ell\theta$ with $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, where $\mu^\natural = \mu_{|\mathfrak{h}^\natural}$.

Since $k \notin \mathbb{Z}_{\geq 0}$, by Theorem 2.1, $H_\theta(L(\widehat{\Lambda}))$ is a non-trivial irreducible module for $W_k(\mathfrak{g}, \theta)$. Since $L(\widehat{\Lambda})$ is a quotient of the Verma module $M(\widehat{\Lambda})$, then, by exactness of H_θ, $H_\theta(L(\widehat{\Lambda}))$ is the quotient of a Verma module for $W_k(\mathfrak{g}, \theta) = V_{k'}(\mathfrak{g}^\natural)$ hence it is an irreducible highest weight module. By [35, (6.14)] its highest weight as $\mathcal{V}_k(\mathfrak{g}^\natural)$-module is $\widehat{\Lambda}^\natural$ with $\widehat{\Lambda}^\natural(K) = k'$ and $\widehat{\Lambda}^\natural_{|\mathfrak{h}^\natural} = \mu^\natural$. Therefore

$$H_\theta(L(\widehat{\Lambda})) = L_{\mathfrak{g}^\natural}(\widehat{\Lambda}^\natural).$$

In particular $H_\theta(L(\widehat{\Lambda}))$ is in the category $KL_{k'}$.

Moreover, under the identification of the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ with $\mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}$ via $ad(f)$ (see Example 6.2 of [35]), we get that x acts on $H_\theta(L(\widehat{\Lambda}))$ via $J_0^{\{f\}}$, and $J^{\{f\}}$ is the conformal vector of $W(k, \theta)$ (see the proof of Theorem 5.1 of [35]). Since the level is collapsing we know, by Proposition 4.1 of [4], that the conformal vector of $W_k(\mathfrak{g}, \theta)$ coincides with the Segal-Sugawara vector conformal

Table 5

Values of k and k'.

$\mathfrak{g}$	$V_{k'}(\mathfrak{g}^\natural)$	k	k'
$sl(m\|n),\, m \neq n, m > 3, m - 2 \neq n$	$V_{k'}(sl(m-2\|n))$	$\frac{n-m}{2}$	$\frac{n-m+2}{2}$
$sl(3\|n),\, n \neq 3, n \neq 1, n \neq 0$	$V_{k'}(sl(1\|n))$	$\frac{n-3}{2}$	$\frac{1-n}{2}$
$sl(3)$	$\mathbb{C}$	$-\frac{3}{2}$	0
$sl(2\|n),\, n \neq 2, n \neq 1, n \neq 0$	$V_{k'}(sl(n))$	$\frac{n-2}{2}$	$-\frac{n}{2}$
$sl(2\|1) = spo(2\|2)$	$\mathbb{C}$	$-\frac{1}{2}$	0
$sl(m\|n),\, m \neq n, n+1, n+2, m \geq 2$	$M(1)$	-1	1
$psl(m\|m),\, m \geq 2$	$\mathbb{C}$	-1	0
$spo(n\|m),\, m \neq n, n+2, n \geq 4$	$V_{k'}(spo(n-2\|m))$	$\frac{m-n-4}{4}$	$\frac{m-n-2}{4}$
$spo(2\|m),\, m \geq 5$	$V_{k'}(so(m))$	$\frac{m-6}{4}$	$\frac{4-m}{2}$
$spo(2\|3)$	$V_{k'}(sl(2))$	$-\frac{3}{4}$	1
$spo(2\|1)$	$\mathbb{C}$	$-\frac{5}{4}$	0
$spo(n\|m),\, m \neq n+1, n \geq 2$	$\mathbb{C}$	$-1/2$	0
$osp(m\|n),\, m \neq n, m \neq n+8, m \geq 7$	$V_{k'}(osp(m-4\|n))$	$\frac{n-m+4}{2}$	$\frac{8-m+n}{2}$
$osp(m\|n),\, n \neq m, 0; 4 \leq m \leq 6$	$V_{k'}(osp(m-4\|n))$	$\frac{n-m+4}{2}$	$\frac{m-n-8}{4}$
$osp(m\|n),\, m \neq n+4, n+8; m \geq 4$	$V_{k'}(sl(2))$	-2	$\frac{m-n-8}{2}$
$osp(n+8\|n),\, n \geq 0$	$\mathbb{C}$	-2	0
$D(2,1;a)$	$V_{k'}(sl(2))$	a	$-\frac{1+2a}{1+a}$
$D(2,1;a)$	$V_{k'}(sl(2))$	$-a-1$	$-\frac{1+2a}{a}$
$F(4)$	$V_{k'}(D(2,1;2))$	-1	$\frac{1}{2}$
$F(4)$	$\mathbb{C}$	$-3/2$	0
$F(4)$	$V_{k'}(so(7))$	$\frac{2}{3}$	-2
$F(4)$	$\mathbb{C}$	$-\frac{2}{3}$	0
E_6	$V_{k'}(sl(6))$	-4	-1
E_6	$\mathbb{C}$	-3	0
E_7	$V_{k'}(so(12))$	-6	-2
E_7	$\mathbb{C}$	-4	0
E_8	$V_{k'}(E_7)$	-10	-4
E_8	$\mathbb{C}$	-6	0
F_4	$V_{k'}(sp(6))$	-3	$-\frac{1}{2}$
F_4	$\mathbb{C}$	$-5/2$	0
G_2	$V_{k'}(sl(2))$	$-\frac{4}{3}$	1
G_2	$\mathbb{C}$	$-\frac{5}{3}$	0
$G(3)$	$V_{k'}(G_2)$	$\frac{1}{2}$	$-\frac{5}{3}$
$G(3)$	$\mathbb{C}$	$-\frac{3}{4}$	0
$G(3)$	$V_{k'}(osp(3\|2))$	$-\frac{2}{3}$	1
$G(3)$	$\mathbb{C}$	$-\frac{4}{3}$	0

$\omega_{\mathfrak{g}^{\natural}}$ of $V_{k'}(\mathfrak{g}^{\natural})$ hence, by (6.14) of [35] again, we obtain that the $(\omega_{\mathfrak{g}^{\natural}})_0$ acts on the lowest component of $H_{\theta}(L(\widehat{\Lambda}))$ by cI with

$$(4.1) \qquad c = \frac{(\mu + 2\rho, \mu)}{2(k + h^{\vee})} - \mu(x).$$

Now condition (2) implies that $\mu^{\natural} = 0$, so $\mu = \ell\theta$ and

$$\frac{(\mu + 2\rho, \mu)}{2(k + h^{\vee})} - \mu(x) = \frac{(\ell\theta + 2\rho, \ell\theta)}{2(k + h^{\vee})} - \ell = 0.$$

By using formula (2.4), we get

$$(4.2) \qquad \frac{2\ell^2 + (2h^{\vee} - 2)\ell}{2(k + h^{\vee})} - \ell = \frac{\ell^2 - (k+1)\ell}{k + h^{\vee}} = 0.$$

- Consider first the case $k = -h^{\vee}/2 + 1$ (this holds for $\mathfrak{g} = D_{2n}$, $n \geq 2$ and $\mathfrak{g} = osp(n+4m+8|n)$, $n \geq 0$). Then (4.2) gives that

$$(4.3) \qquad \frac{2\ell^2 + (h^{\vee} - 4)\ell}{h^{\vee} + 2} = 0.$$

  We get $\ell = 0$ or $2\ell + h^{\vee} - 4 = 0$.
- Next we consider the case $k = -h^{\vee}/6 - 1$. We get

$$(4.4) \qquad \frac{6\ell^2 + h^{\vee}\ell}{5h^{\vee} - 6} = 0.$$

  We conclude that $\ell = 0$ or $\ell = -\frac{h^{\vee}}{6}$.

By using the above analysis and properties of Hamiltonian reduction, we get the following lemma, which extends a result of [15] for Lie algebras to the super case.

Lemma 4.1. *Assume that $k = -\frac{h^{\vee}}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$), $osp(n + 8|n)$ ($n \geq 2$), $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ).*
Assume that $L(\lambda)$ is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Then one of the following condition holds:

- (1) $\lambda(\alpha_0^{\vee}) \in \mathbb{Z}_{\geq 0}$;
- (2) $\bar{\lambda}$ is either 0 or $\frac{-h^{\vee}}{6}\theta$, where $\bar{\lambda}$ is the restriction of λ to $\mathfrak{h}$.

Proof. By Theorem 2.1, if $L(\lambda)$ is a $V_k(\mathfrak{g})$–module for which $\lambda(\alpha_0^{\vee}) \notin \mathbb{Z}_{\geq 0}$, then $H_{\theta}(L(\lambda))$ is an irreducible $W_k(\mathfrak{g}, \theta) = H_{\theta}(V_k(\mathfrak{g}))$–module. The conditions on $\mathfrak{g}$ exactly correspond to the cases when $W_k(\mathfrak{g}, \theta)$ is one-dimensional (cf. [4], [15]), so the discussion that precedes the Lemma and relation (4.4) imply that $\bar{\lambda}$ is as in (2). □

Lemma 4.1 implies:

Theorem 4.2. *Assume that the level k and the Lie superalgebra $\mathfrak{g}$ satisfy one of the following conditions:*

- (1) $k = -\frac{h^{\vee}}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$), $osp(n + 8|n)$ ($n \geq 2$), $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ);
- (2) $k = -h^{\vee}/2 + 1$ and $\mathfrak{g} = osp(n + 4m + 8|n)$, $n \geq 2, m \geq 0$.
- (3) $k = -h^{\vee}/2 + 1$ and $\mathfrak{g} = D_{2m}$, $m \geq 2$.
- (4) $k = -10$ and $\mathfrak{g} = E_8$.

Then $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k.

Proof. If the Lie superalgebra $\mathfrak{g}$ is as in (1), then Lemma 4.1 and Remark 3.5 imply that $\bar{\lambda}$ is either 0 or $\frac{-h^{\vee}}{6}\theta$. Since in all cases in (1) we have that $h^{\vee} \in \mathbb{Z}_{\geq 0}$, one obtains that the irreducible highest weight $\mathfrak{g}$–module with highest weight $\bar{\lambda} = \frac{-h^{\vee}}{6}\theta$ cannot be finite-dimensional. Therefore $L(\lambda)$ can not be a

module in KL_k. This proves that $\bar{\lambda} = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k.

Let us consider the case $\mathfrak{g} = osp(n + 4m + 8|n)$. Then for every $m \in \mathbb{Z}_{\geq 0}$ we have:

$$(4.5) \qquad\qquad\qquad\qquad h^\vee = 4m + 6,$$

$$(4.6) \qquad\qquad\qquad\qquad k = -h^\vee/2 + 1 = -2(m+1),$$

$$(4.7) \qquad\qquad\qquad\qquad 2\ell + h^\vee - 4 \neq 0 \quad \forall \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

We prove the claim by induction. In the case $m = 0$, the claim was proved in (1). Assume now that the claim holds for $\mathfrak{g}' = osp(n + 4(m-1) + 8, n)$, and $k' = -2m$.

By Theorem 2.3, $k = -2(m+1)$ is a collapsing level and $W_k(\mathfrak{g}, \theta) = V_{k'}(\mathfrak{g}')$.

By inductive assumption $V_{k'}(\mathfrak{g}')$ is the unique irreducible $V_{k'}(\mathfrak{g}')$ in the category $KL_{k'}$. By applying (4.3) and (4.7) we get that $\ell = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k. The assertion now follows by induction on m.

(3) is a special case of (2), by taking $n = 0$.

(4) follows from the fact that $H_\theta(V_{-10}(E_8)) = V_{-4}(E_7)$ and case (1) by applying formula (4.2). $\qquad\square$

Remark 4.3. *Theorem 4.2 can be also proved by non-cohomological methods, using explicit formulas for singular vectors and Zhu algebra theory. As an illustration, we shall present in Theorem 8.6 a direct proof in the case of D_{2n} at level $k = -h^\vee/2 + 1$.*

In the following sections we shall study some other applications of collapsing levels. We shall restrict our analysis to the case of Lie algebras. In what follows we let $\omega_1, \ldots, \omega_n$ be the fundamental weights for $\mathfrak{g}$ and $\Lambda_0, \ldots, \Lambda_n$ the fundamental weights for $\widehat{\mathfrak{g}}$.

5. On complete reducibility in the category KL_k

In this Section we prove complete reducibility results in the category KL_k when $\mathfrak{g}$ is a Lie algebra. We start with a preliminary result, which also holds in the super setting.

Lemma 5.1. *Assume that the Lie superalgebra $\mathfrak{g}$ and level k satisfy the conditions of Theorem 4.2. Assume that M is a highest weight $V_k(\mathfrak{g})$–module from the category KL_k. Then M is irreducible.*

Proof. By using the classification of irreducible modules from Theorem 4.2 we know that the highest weight of M is necessary $k\Lambda_0$, and therefore M is a $\mathbb{Z}_{\geq 0}$–graded with respect to $L_{\mathfrak{g}}(0)$. Denote a highest weight vector by $w_{k\Lambda_0}$. We have that

$$L_{\mathfrak{g}}(0)v = 0 \quad \Longleftrightarrow \quad v = \nu w_{k\Lambda_0} \quad (\nu \in \mathbb{C}).$$

Assume that M is not irreducible. Then it contains a non-zero graded submodule $N \neq M$ with respect to $L_{\mathfrak{g}}(0)$:

$$N = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} N(n + n_0), \quad L_{\mathfrak{g}}(0)_{|N(r)} = r\mathrm{Id}, \quad N(n_0) \neq 0.$$

Since $N \neq M$, we have that $n_0 > 0$, otherwise $w_{k\Lambda_0} \in M$.

We can consider $N(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(V_k(\mathfrak{g}))$. Note that Theorem 4.2 and Proposition 3.2 imply that any irreducible finite-dimensional $A(V_k(\mathfrak{g}))$–module is trivial. Since the Casimir element $C_{\mathfrak{g}}$ of $\mathfrak{g}$ acts on $N(n_0)$ as the non-zero constant $2(k + h^\vee)n_0$, we conclude that $C_{\mathfrak{g}}$ acts by the same constant on any irreducible $\mathfrak{g}$–subquotient of $N(n_0)$. But any irreducible subquotient of $N(n_0)$ is an irreducible finite–dimensional $A(V_k(\mathfrak{g}))$–module, and therefore it is trivial. This implies that $C_{\mathfrak{g}}$ acts non-trivially on a trivial $\mathfrak{g}$–module, a contradiction. $\qquad\square$

The following Lemma is a consequence of [28, Theorem 0.1].

Lemma 5.2. [28] *Assume that $\mathfrak{g}$ is a simple Lie algebra and k is a rational number, $k > -h^\vee$. Then, in the category of $V_k(\mathfrak{g})$–modules, we have: $\mathrm{Ext}^1(V_k(\mathfrak{g}), V_k(\mathfrak{g})) = (0)$.*

Theorem 5.3. *Assume that $\mathfrak{g}$ is a simple Lie algebra and that the level k satisfies the conditions of Theorem 4.2. Then any $V_k(\mathfrak{g})$–module M from the category KL_k is completely reducible.*

Proof. Since M is in KL_k we have that any irreducible subquotient of M is isomorphic to $V_k(\mathfrak{g})$. M has finite length. This implies that M is $\mathbb{Z}_{\geq 0}$–graded:

$$M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n), \quad L_{\mathfrak{g}}(0)_{|M(r)} = r\mathrm{Id}.$$

Assume that $M(0) = \mathrm{span}_{\mathbb{C}}\{w_1, \ldots, w_s\}$. Then by Lemma 5.1 we have that $V_k(\mathfrak{g})w_i \cong V_k(\mathfrak{g})$ for every $i = 1, \ldots, s$. Now using Lemma 5.2 we get $M \cong \oplus V_k(\mathfrak{g})w_i$ and therefore M is completely reducible. $\qquad\square$

Remark 5.4. *We expect that the previous theorem holds in the case when $\mathfrak{g}$ is the Lie superalgebra from Theorem 4.2. We shall study this case in* [7].

We shall now prove much more general result on complete reducibility in KL_k.

Theorem 5.5. *Assume that level $k \in \mathbb{Q}$, $k > -h^{\vee}$, and the simple Lie algebra $\mathfrak{g}$ satisfy the following property:*

(5.1)   *Every highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible.*

Then the category KL_k is semi-simple.

Proof. We shall present a sketch of the proof and omit some standard representation theoretic arguments which can be found in [20] and [28].

- Since every irreducible $V_k(\mathfrak{g})$-module in KL_k is isomorphic to $L(\lambda)$ for certain rational, non-critical weight λ, then [28, Theorem 0.1] implies that $\mathrm{Ext}^1(L(\lambda), L(\lambda)) = (0)$ in the category KL_k.
- We prove that in the category KL_k we have

(5.2) $$Ext^1(L_1, L_2) = (0)$$

  for any two irreducible modules L_1 and L_2 from KL_k.

  It remains to consider the case $L_1 \neq L_2$. Take an exact sequence in KL_k:

  $$0 \to L(\lambda_1) \to M \to L(\lambda_2) \to 0,$$

  where $\lambda_1 \neq \lambda_2$. Then M contains a singular vector w_{λ_1} of highest weight λ_1 and a subsingular vector w_{λ_2} of weight λ_2 and w_{λ_1} generates a submodule isomorphic to $L(\lambda_1)$. Consider the case $\lambda_1 - \lambda_2 \notin Q_+$. Then λ_2 is a maximal element of the set $\Omega(M)$ of weights of M, and therefore the subsingular vector w_{λ_2} in M of weight λ_2 is a singular vector. By (5.1), it generates an irreducible module isomorphic to $L(\lambda_2)$ and we conclude that $M \cong L(\lambda_1) \oplus L(\lambda_2)$.

  If $\lambda_1 - \lambda_2 \in Q_+$ we can use the contravariant functor $M \mapsto M^{\sigma}$ and get an exact sequence

  $$0 \to L(\lambda_2) \to M^{\sigma} \to L(\lambda_1) \to 0.$$

  Since M^{σ} is again a $V_k(\mathfrak{g})$–module in KL_k (cf. Lemma 3.6) by the first case we have that $M^{\sigma} = L(\lambda_1) \oplus L(\lambda_2)$. This implies that

  $$M = L(\lambda_1)^{\sigma} \oplus L(\lambda_2)^{\sigma} = L(\lambda_1) \oplus L(\lambda_2).$$

- Assume now that M is a finitely generated module from KL_k. Then from [20, Proposition 3.1] we see that M has an increasing filtration

(5.3) $$(0) = M_0 \subseteq M_1 \subseteq \cdots$$

  such that
  (1) for every $j \in \mathbb{Z}_{>0}$, M_j/M_{j-1} is an highest weight module in category $\mathcal{O}$;
  (2) for any weight λ of M, there exists r such that $(M/M_r)_{\lambda} = 0$.

Since M is finitely generated as $\widehat{\mathfrak{g}}$–module, we can assume that its generators are weight vectors of weights say $\mu_1, ... \mu_p$. Since they are a finite number there certainly exists t such that $(M/M_t)_{\mu_i} = 0$ for all $i = 1, .., p$. Hence the filtration (5.3) is finite and stops at $M = M_t$. Since M is in category KL_k, we have that the factors of (5.3) are in category KL_k. Hence, by our assumption, they are irreducible. Therefore (5.3) is a composition series of finite length. Using assumption (5.1), relation (5.2) and induction on t we get that

$$M \cong \bigoplus_{j=1}^{t} L(\lambda_j).$$

- Finally, we shall consider the case when M is not finitely generated. Since M is in KL_k, it is countably generated. So $M = \cup_{n=1}^{\infty} M^{(n)}$ such that each $M^{(n)}$ is finitely generated $V_k(\mathfrak{g})$–module. By previous case $M^{(n)}$ is completely reducible, so:

(5.4)
$$M^{(n)} = \bigoplus_{i=1}^{n_i} L(\lambda_{i,n}).$$

Therefore M is a sum of irreducible modules from KL_k and by using classical algebraic arguments one can see that M is a direct sum of countably many irreducible modules from KL_k appearing in decompositions (5.4).

The claim follows. $\qquad\square$

In order to apply Theorem 5.5, the basic step is to check relation (5.1). We have the following method.

Lemma 5.6. *Let $k \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. Assume that $H_\theta(U)$ is an irreducible, non-zero $W_k(\mathfrak{g}, \theta) = H_\theta(V_k(\mathfrak{g}))$–module for every non-zero highest weight $V_k(\mathfrak{g})$–module U from the category KL_k. Then every highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible.*

Proof. Assume that M is a highest weight $V_k(\mathfrak{g})$–module in KL_k. Then $H_\theta(M)$ is an irreducible $H_\theta(V_k(\mathfrak{g}))$–module. If M is not irreducible, then it contains a highest weight submodule U such that $\{0\} \subsetneqq U \subsetneqq M$. Modules U and M/U are again highest weight modules in KL_k. By the assumption of the Lemma we have that $H_\theta(U)$ is a non-trivial submodule of $H_\theta(M)$. Irreducibility of $H_\theta(M)$ implies that $H_\theta(U) = H_\theta(M)$, and therefore $H_\theta(M/U) = \{0\}$, a contradiction. $\qquad\square$

Theorem 5.7. *Assume that $\mathfrak{g}$ is a simple Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ such that $W_k(\mathfrak{g}, \theta)$ is rational. Then KL_k is a semi-simple category.*

Proof. Assume that $\widetilde{L}(\lambda)$ is a highest weight $V_k(\mathfrak{g})$–module in KL_k. Clearly $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$ and by Theorem 2.1 $H_\theta(\widetilde{L}(\lambda)) \neq (0)$. Since $H_\theta(\widetilde{L}(\lambda))$ is non-zero highest weight module for the rational vertex algebra $W_k(\mathfrak{g}, \theta)$, we conclude that $H_\theta(\widetilde{L}(\lambda))$ is irreducible. Now assertion follows from Theorem 5.5 and Lemma 5.6. $\qquad\square$

Remark 5.8. *The previous theorem proves that the category KL_k is semisimple in the following (non-admissible) cases:*

- $\mathfrak{g} = D_4, E_6, E_7, E_8$ *and* $k = -\frac{h^\vee}{6}$ *using results from* [38].

Moreover, using Theorem 5.5 and Lemma 5.6 we can prove the semi-simplicity of KL_k for all collapsing levels not accounted by Theorem 1.1. We list here only non-admissible levels, since in admissible case KL_k is semi-simple by [12].

Theorem 5.9. *The category KL_k is semisimple in the following cases:*
1. $\mathfrak{g} = D_\ell$, $\ell \geq 3$ *and* $k = -2$;
2. $\mathfrak{g} = B_\ell$, $\ell \geq 2$ *and* $k = -2$;
3. $\mathfrak{g} = A_\ell$, $\ell \geq 2$ *and* $k = -1$;
4. $\mathfrak{g} = A_{2\ell-1}$, $\ell \geq 2$, $k = -\ell$;

(5) $\mathfrak{g} = D_{2\ell-1}$, $\ell \geq 3$ *and* $k = -2\ell + 3$;
(6) $\mathfrak{g} = C_\ell$, $k = -1 - \ell/2$;
(7) $\mathfrak{g} = E_6$, $k = -4$;
(8) $\mathfrak{g} = E_7$, $k = -6$;
(9) $\mathfrak{g} = F_4$, $k = -3$.

Proof. We will give a proof of relations (1) and (2) in Corollaries 6.8 and 7.7, respectively. Case (1) for $\ell \neq 3$ will follow from Theorem 5.7. Note also that case (1) for $\ell = 3$ is a special case of case (4), and that case (2) for $\ell = 2$ is a special case of (6). The proof in cases (3) – (6) is similar, and it uses the classification of irreducible modules from [10], [11], [16] and the results on collapsing levels [4]. Cases (7) – (9) are reduced to cases we have already treated. Here are some details.
   Case (3):

- [16], [4] $H_\theta(V_{-1}(A_\ell))$ is isomorphic to the Heisenberg vertex algebra $M(1)$ of central charge $c = 1$
- By using the fact that every highest weight $M(1)$–module is irreducible, we see that if U is a highest weight $V_{-1}(A_\ell)$–module in KL_{-1}, then $H_\theta(U)$ is a non-trivial irreducible $M(1)$–module.

Case (4):

- [16], [4] $H_\theta(V_{-\ell}(A_{2\ell-1})) = V_{-\ell+1}(A_{2\ell-3})$.
- For $\ell = 2$, we have that every highest weight $V_{-\ell+1}(A_{2\ell-3}) = V_{-1}(sl(2))$–module $\widetilde{L}(\lambda)$ in KL_{-1} with highest weight $\lambda = -(1+j)\Lambda_0 + j\Lambda_1$, $j \in \mathbb{Z}_{\geq 0}$, is irreducible.
- By induction, we see that for every highest weight $V_{-\ell}(A_{2\ell-1})$–module U in $KL_{-\ell}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-\ell+1}(A_{2\ell-3})$–module.

Case (5)

- $H_\theta(V_{-2\ell+3}(D_{2\ell-1})) \cong V_{-2\ell+5}(D_{2\ell-3})$.
- By induction we see that for or every highest weight $V_{-2\ell+3}(D_{2\ell-1})$–module U in $KL_{-2\ell+3}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-2\ell+5}(D_{2\ell-3})$–module.

Case (6)

- $H_\theta(V_{-1-\ell/2}(C_\ell)) \cong V_{-1/2-\ell/2}(C_{\ell-1})$.
- For $\ell = 2$, we have that every highest weight $V_{-1/2-\ell/2}(C_{\ell-1}) = V_{-3/2}(sl(2))$–module in $KL_{-3/2}$ is irreducible.
- By induction, we see that for every highest weight $V_{-1-\ell/2}(C_\ell)$–module U in $KL_{-1-\ell/2}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-1/2-\ell/2}(C_{\ell-1})$–module.

The proof follows by applying Theorem 5.5 and Lemma 5.6.

   Cases (7) – (8)

   We have

$$H_\theta(V_{-4}(E_6)) = V_{-1}(A_3), \quad H_\theta(V_{-6}(E_7)) = V_{-2}(D_6),$$

and these cases are settled in (3) and Theorem 1.1 (3) respectively. Case (9) follows from the fact that $H_\theta(V_{-3}(F_4))$ is isomorphic to the admissible affine vertex algebra $V_{-\frac{1}{2}}(C_3)$ which is semisimple in $KL_{-1/2}$ (cf. [1]). □

Remark 5.10. *The problem of complete-reducibility of modules in KL_k when $\mathfrak{g}$ is a Lie superalgebra will be also studied in* [7]. *An important tool in the description of the category KL_k will be the conformal embedding of $\widetilde{V}_k(\mathfrak{g}_0)$ to $V_k(\mathfrak{g})$ where $\mathfrak{g}_0$ is the even part of $\mathfrak{g}$.*

   Note that in the category $\mathcal{O}$ we can have indecomposable $V_k(\mathfrak{g})$–modules in some cases listed in Theorem 5.9. See [10, Remark 5.8] for one example.

6. The vertex algebra $V^{-2}(D_\ell)$ and its quotients

In this section we exploit Hamiltonian reduction and the results on conformal embeddings from [4] to investigate the quotients of the vertex algebra $V^{-2}(D_\ell)$. In particular we are interested in a non-simple quotient $\mathcal{V}_{-2}(D_\ell)$ which appears in the analysis of certain dual pairs (see [6]) as well as in the simple quotient $V_{-2}(D_\ell)$. We will show that the vertex algebra $\mathcal{V}_{-2}(D_\ell)$ has infinitely many irreducible modules in the category KL^{-2}, while by [15], $V_{-2}(D_\ell)$ has finitely many irreducible modules in KL_{-2}. Recall that -2 is a collapsing level for D_ℓ [4].

Consider the vector

$$(6.1) \qquad w_1 := (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1) + e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}.$$

It is a singular vector in $V^{-2}(D_\ell)$ (cf. [15]). Note that this vector is contained in the subalgebra $V^{-2}(D_4)$ of $V^{-2}(D_\ell)$.

By using the explicit expression for singular vectors v_n in $V^{n-\ell+1}(D_\ell)$ (see (8.1)), we have that

$$(6.2) \qquad w_2 := v_{\ell-3} = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1) \Big)^{\ell-3} \mathbf{1}$$

is a singular vector in $V^{-2}(D_\ell)$.

For $\ell = 4$ we also have a third singular vector (cf. [40])

$$w_3 := (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3-\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2-\epsilon_4}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}.$$

6.1. The vertex algebra $\mathcal{V}_{-2}(D_\ell)$ for $\ell \geq 4$. Define the vertex algebra

$$(6.3) \qquad \mathcal{V}_{-2}(D_\ell) = V^{-2}(D_\ell)/J_\ell,$$

where

$$J_\ell = \langle w_1, w_3 \rangle \quad (\ell = 4), \qquad J_\ell = \langle w_1 \rangle \quad (\ell \geq 5).$$

The following proposition is essentially proven in [6].

Proposition 6.1.
(1) *There is a non-trivial vertex algebra homomorphism $\overline{\Phi} : \mathcal{V}_{-2}(D_\ell) \to M_{2\ell}$ where $M_{2\ell}$ the Weyl vertex algebra of rank ℓ.*
(2) *$\mathcal{V}_{-2}(D_\ell)$ is not simple, and $L((-2 - t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $\mathcal{V}_{-2}(D_\ell)$–modules.*

Proof. The homomorphism $\Phi : V^{-2}(D_\ell) \to M_{2\ell}$ was constructed in [6, Section 7]. By direct calculation one proves that $\Phi(w_1) = 0$ for $\ell \geq 4$ and $\Phi(w_3) = 0$ for $\ell = 4$. Finally [6, Lemma 7.1] implies that $L((-2-t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $\mathcal{V}_{-2}(D_\ell)$–modules. Since the simple vertex algebra $V_{-2}(D_\ell)$ has only finitely many irreducible modules in the category $\mathcal{O}$ [15], we have that $\mathcal{V}_{-2}(D_\ell)$ is not simple. $\qquad \square$

Next, we exploit the fact that in the case $\mathfrak{g} = D_\ell$, $k = -2$ is a collapsing level, i.e., in the affine W-algebra $W^k(\mathfrak{g}, \theta)$, all generators $G^{\{u\}}$ at conformal weight $3/2$, $u \in \mathfrak{g}_{-1/2}$, belong to the maximal ideal (see [4] for details). This implies that there exists a non-trivial ideal I in $V^{-2}(\mathfrak{g})$ such that $G^{\{u\}} \in H_\theta(I)$ for all $u \in \mathfrak{g}_{-1/2}$.

Note also that $\mathfrak{g}^\natural = A_1 \oplus D_{\ell-2}$, so we have that $V^{\ell-4}(A_1) \otimes V^0(D_{\ell-2})$ is a subalgebra of $W^{-2}(D_\ell, \theta)$. In the case $\ell = 4$ we identify D_2 with $A_1 \oplus A_1$.

Lemma 6.2. *We have*
- *$x_{(-1)}\mathbf{1} \in H_\theta(J_\ell)$ for all $x \in D_{\ell-2} \subset \mathfrak{g}^\natural$,*
- *$G^{\{u\}} \in H_\theta(J_\ell)$ for all $u \in \mathfrak{g}_{-1/2}$.*

Proof. Assume that $\ell \geq 5$. Since w_1 is a singular vector in $V^{-2}(D_\ell)$, the ideal J_ℓ is a highest weight module of highest weight $\lambda = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$. Now, the Main Theorem from [12] implies that $H_\theta(J_\ell)$ is a non-trivial highest weight module. By formula [35, (6.14)] the highest weight is $(0, \omega_2)$ and, by (4.1), the conformal weight of its highest weight vector is 1. Up to a non-zero constant,

there is only one vector in $W^{-2}(D_\ell, \theta) = V^{\ell-4}(A_1) \otimes V^0(D_{\ell-2})$ that has these properties, namely $J_{(-1)}^{\{e_{\epsilon_3+\epsilon_4}\}}\mathbf{1}$, and therefore $H_\theta(J_\ell)$ contains all generators of $V^0(D_{\ell-2})$.

In the case $\ell = 4$, w_1 and w_3 generate submodules N_1 and N_3 of highest weights $\lambda_1 = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$, $\lambda_3 = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4$, respectively. Applying the same arguments as above we get that $J_{(-1)}^{\{e_{\epsilon_3 \pm \epsilon_4}\}}\mathbf{1} \in H_\theta(I)$, which implies that $H_\theta(J_\ell)$ contains all generators of $V^0(D_2) = V^0(A_1) \otimes V^0(A_1)$.

Now, claim follows by applying the action of generators of $V^0(D_{\ell-2})$ to $G^{\{u\}}$ (see [4]). $\qquad\square$

Proposition 6.3. *We have*
(1) $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$.
(2) $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell - 4 - t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$.
(3) *The set* $\{L((-2-t)\Lambda_0 + t\Lambda_1) \mid t \in \mathbb{Z}_{\geq 0}\}$ *provides a complete list of irreducible* $\mathcal{V}_{-2}(D_\ell)$*–modules from the category* KL^{-2}.

Proof. By Lemma 6.2 we see that the vertex algebra $H_\theta(\mathcal{V}_{-2}(D_\ell))$ is generated only by $x_{(-1)}\mathbf{1}$, $x \in A_1 \subset D_\ell^\natural$. So there are only two possibilities: either $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$ or $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V_{\ell-4}(A_1)$. Moreover, for every $t \in \mathbb{Z}_{\geq 0}$, $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1))$ must be the irreducible $H_\theta(\mathcal{V}_{-2}(D_\ell))$–module with highest weight $t\omega_1$ with respect to A_1. So $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell - 4 - t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$. Therefore, $H_\theta(\mathcal{V}_{-2}(D_\ell))$ contains infinitely many irreducible modules, which gives that $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$. In this way we have proved claims (1) and (2).

Let us now prove claim (3).

Assume that $L(k\Lambda_0 + \mu)$ ($\mu \in P_+$, $k = -2$) is an irreducible $\mathcal{V}_k(D_\ell)$–module in the category KL^k. Then $H_\theta(L(k\Lambda_0 + \mu))$ is a non-trivial irreducible $V^{\ell-4}(A_1)$–module. The representation theory of $V^{\ell-4}(A_1)$ implies that:
$$H_\theta(L(k\Lambda_0 + \mu)) = L_{A_1}((\ell - 4 - j)\Lambda_0 + j\Lambda_1) \qquad \text{for } j \in \mathbb{Z}_{\geq 0}.$$
Since $D_\ell^\natural = A_1 \times D_{\ell-2}$, we conclude that $\mu^\natural = j\omega_1$ and therefore, by (2.5),
$$\mu = j\omega_1 + s\omega_2 = (s+j)\epsilon_1 + s\epsilon_2 \qquad (s \in \mathbb{Z}_{\geq 0}).$$
By using the action of $L(0) = \omega_0$ on the lowest component of $H_\theta(L(k\Lambda_0 + \mu))$ we get
$$\frac{(\mu + 2\rho, \mu)}{2(k + h^\vee)} - \mu(x) = \frac{j(j+2)}{4(\ell-2)} \qquad (x = \theta^\vee/2).$$
Since $2(k + h^\vee) = 2(-2 + 2\ell - 2) = 4(\ell - 2)$ and $\mu(x) = (2s + j)/2$ we get
$$(\mu + 2\rho, \mu) - (h^\vee - 2)(2s + j) = j(j+2).$$
By direct calculation we get
$$(\mu + 2\rho, \mu) = (s+j)^2 + s^2 + h^\vee(s+j) + (h^\vee - 2)s,$$
which gives an equation:
$$\begin{aligned} & (s+j)^2 + s^2 + h^\vee(s+j) + (h^\vee - 2)s - (h^\vee - 2)(2s+j) = j(j+2). \\ \iff \quad & (s+j)^2 + s^2 + h^\vee(s+j) - (h^\vee - 2)(s+j) = j(j+2). \\ \iff \quad & (s+j)(s+j+2) = j(j+2) \\ \iff \quad & s = 0 \quad \text{or} \quad s = -2j - 2. \end{aligned}$$
Since $\mu \in P_+$ we conclude that $s = 0$. Therefore $\mu = j\omega_1$ for certain $j \in \mathbb{Z}_{\geq 0}$. The proof of claim (3) is now complete. $\qquad\square$

6.2. **The simple vertex algebra** $V_{-2}(D_\ell)$**.** Next we use the fact that the simple affine W-algebra $W_{-2}(D_\ell, \theta)$ is isomorphic to the simple affine vertex algebra $V_{\ell-4}(A_1)$, for $\ell \geq 4$.

Proposition 6.4. *The set* $\{L((-2-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq \ell - 4\}$ *provides a complete list of irreducible* $V_{-2}(D_\ell)$*–modules from the category* KL_{-2}.

Proof. Assume that N is an irreducible $V_{-2}(D_\ell)$–module from the category KL_{-2}. Then N is also irreducible as $\mathcal{V}_{-2}(D_\ell)$–module, and therefore $N \cong L((-2-j)\Lambda_0 + j\Lambda_1)$ for certain $j \in \mathbb{Z}_{\geq 0}$. Since $H_\theta(N)$ must be an irreducible $H_\theta(V_{-2}(D_\ell)) = W_{-2}(D_\ell, \theta) = V_{\ell-4}(A_1)$–module, we get $j \leq \ell - 4$, as desired. □

Now we want to describe the maximal ideal in $V^{-2}(D_\ell)$. The next lemma states that any non-trivial ideal in $\mathcal{V}_{-2}(D_\ell)$ is automatically maximal.

Lemma 6.5. *Let* $\{0\} \neq I \subsetneq \mathcal{V}_{-2}(D_\ell)$ *be any non-trivial ideal in* $\mathcal{V}_{-2}(D_\ell)$*. Then we have*
(1) $H_\theta(I)$ *is the maximal ideal in* $V^{\ell-4}(A_1)$*.*
(2) I *is a maximal ideal in* $\mathcal{V}_{-2}(D_\ell)$ *and* $I = L(-2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1)$*.*

Proof. Assume that I is a non-trivial ideal in $\mathcal{V}_{-2}(D_\ell)$. Then I can be regarded as a $\mathcal{V}_{-2}(D_\ell)$–module in the category KL^{-2} and therefore, by Proposition 6.3, (3), it contains a non-trivial subquotient isomorphic to $L((-2-j)\Lambda_0 + j\Lambda_1)$ for some $j \in \mathbb{Z}_{\geq 0}$. Since, by part (2) of the aforementioned Proposition, $H_\theta(L((-2-j)\Lambda_0 + j\Lambda_1)) \neq 0$ for every $j \in \mathbb{Z}_{\geq 0}$, we conclude that $H_\theta(I)$ is a non-trivial ideal in $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$. But since $V^{\ell-4}(A_1)$, $\ell \geq 4$, contains a unique non-trivial ideal, which is automatically maximal, we have that $H_\theta(I)$ is a maximal ideal in $V^{\ell-4}(A_1)$. So

$$H_\theta(\mathcal{V}_{-2}(D_\ell)/I) \cong V_{\ell-4}(A_1).$$

Assume now that $\mathcal{V}_{-2}(D_\ell)/I$ is not simple. Then it contains a non-trivial singular vector v' of weight $-(2+j)\Lambda_0 + j\Lambda_1$ for $j \in \mathbb{Z}_{>0}$. By [12], we have that $H_\theta(V^{-2}(D_\ell).v')$ is a non-trivial ideal in $V_{\ell-4}(A_1)$ generated by a singular vector of A_1–weight $j\omega_1$. This is a contradiction. So I is the maximal ideal.

Since the maximal ideal in $V^{\ell-4}(A_1)$ is generated by a singular vector of A_1–weight $2(\ell-3)\omega_1$ and since the maximal ideal is simple, we conclude that $I = \mathcal{V}_{-2}(D_\ell).v_{sing}$ for a certain singular vector v_{sing} of weight $\lambda = -2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1$. It is also clear that this singular vector is unique, up to scalar factor. Therefore, $I = L(-2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1)$. □

Note that in the previous lemma we proved the existence of a singular vector which generates the maximal ideal without presenting a formula for such a singular vector. Since the vector in (6.2) has the correct weight, we also have an explicit expression for this singular vector:

$$\left(\sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1) e_{\epsilon_1 + \epsilon_i}(-1) \right)^{\ell-3} \mathbf{1}$$

Corollary 6.6.
(1) *The maximal ideal in* $V^{-2}(D_\ell)$ *is generated by the vectors* w_1 *and* w_2 *for* $\ell \geq 5$ *and by the vectors* w_1, w_2, w_3 *for* $\ell = 4$*.*
(2) *The homomorphism* $\overline{\Phi} : \mathcal{V}_{-2}(D_\ell) \to M_{2\ell}$ *is injective. In particular, the vertex algebra* $\mathcal{V}_{-2}(D_\ell) \otimes V_{-\ell}(A_1)$ *is conformally embedded into* $V_{-1/2}(C_{2\ell})$*.*
(3) $ch(\mathcal{V}_{-2}(D_\ell)) = ch(V_{-2}(D_\ell)) + chL(-2(\ell-2))\Lambda_0 + 2(\ell-3)\Lambda_1)$*.*

Remark 6.7. *D. Gaiotto in [27] has started a study of the decomposition of* $M_{2\ell}$ *as a* $V^{-2}(D_\ell) \otimes V_{-\ell}(A_1)$*–module in the case* $\ell = 4$*. By combining results from [6, Section 8] and results from this Section we get that*

$$Com(V_{-\ell}(A_1), M_{2\ell}) \cong \mathcal{V}_{-2}(D_\ell).$$

So the vertex algebra responsible for the decomposition of $M_{2\ell}$ *is exactly* $\mathcal{V}_{-2}(D_\ell)$*. Therefore in the decomposition of* $M_{2\ell}$ *only modules for* $\mathcal{V}_{-2}(D_\ell)$ *can appear. In our forthcoming papers we plan to apply the representation theory of* $\mathcal{V}_{-2}(D_\ell)$ *to the problem of finding branching rules.*

Corollary 6.8. *For* $\ell \geq 3$ *the category* KL_{-2} *is semi-simple.*

Proof. The assertion in the case $\ell \geq 4$ follows from Theorem 5.7 since then $W_{-2}(D_\ell, \theta) = V_{\ell-4}(sl(2))$ is a rational vertex algebra.

In the case $\ell = 3$, we have that a highest weight $V_{-2}(D_3)$–module M is isomorphic to $\widetilde{L}((-2-j)\Lambda_0 + j\Lambda_1)$ where $j \in \mathbb{Z}_{\geq 0}$. The irreducibility of M follows easily from the fact that $H_\theta(M)$ is isomorphic

to an irreducible $V_{-1}(sl(2))$–module $L_{A_1}(-1-j)\Lambda_0 + j\Lambda_1)$. Now claim follows from Theorem 5.5 and Lemma 5.6. $\qquad\square$

7. THE VERTEX ALGEBRA $V^{-2}(B_\ell)$ AND ITS QUOTIENTS

In this section let $\ell \geq 2$. Note that $k = -2$ is a collapsing level for B_ℓ [4], and that the simple affine W-algebra $W_{-2}(B_\ell, \theta)$ is isomorphic to $V_{\ell-\frac{7}{2}}(A_1)$. This implies that $H_\theta(V_{-2}(B_\ell)) = V_{\ell-\frac{7}{2}}(A_1)$. But as in the case of the affine Lie algebra of type D, we can construct an intermediate vertex algebra $\mathcal{V}$ so that $H_\theta(\mathcal{V}) = V^{\ell-7/2}(A_1)$.

Remark 7.1. *The formula for a singular vector of conformal weight two in $V^{-2}(B_\ell)$ was given in* [15, Theorem 4.2] *for $\ell \geq 3$, and in* [15, Remark 4.3] *for $\ell = 2$. Note that, for $\ell \geq 4$, the vector $\sigma(w_2)$ from* [15] *is equal to the vector w_1 from relation (6.1), i.e. it is contained in the subalgebra $V^{-2}(D_4)$. For $\ell = 3$, we have*

$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2}(-1) + e_{\epsilon_1}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}.$$

For $\ell = 2$, the singular vector of conformal weight two in $V^{-2}(B_2)$ is equal to

$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{-\epsilon_2}(-1) + \frac{1}{2}h_{\epsilon_2}(-1)e_{\epsilon_1}(-1) - e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_2}(-1))\mathbf{1}.$$

Consider the singular vector in $V^{-2}(B_\ell)$ denoted by $\sigma(w_2)$ in [15, Theorem 4.2] and [17, Section 7]. Let us denote that singular vector by w_1 in this paper (see Remark 7.1 for explanation).

Then we have the quotient vertex algebra

$$(7.1) \qquad\qquad \mathcal{V}_{-2}(B_\ell) = V^{-2}(B_\ell)/\langle w_1\rangle.$$

As in the case of the vertex algebra $\mathcal{V}_{-2}(D_\ell)$, we have the non-trivial homomorphism $\mathcal{V}_{-2}(B_\ell) \to M_{2\ell+1}$.

The proof of the following result is completely analogous to the proof of Proposition 6.3 and it is therefore omitted.

Proposition 7.2. *We have*
(1) *There is a non-trivial homomorphism $\overline{\Phi} : \mathcal{V}_{-2}(B_\ell) \to M_{2\ell+1}$.*
(2) *$H_\theta(\mathcal{V}_{-2}(B_\ell)) = V^{\ell-7/2}(A_1)$.*
(3) *$H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell - 7/2 - t)\Lambda_0 + t\Lambda_1), \ t \in \mathbb{Z}_{\geq 0}$.*
(4) *The set*

$$(7.2) \qquad\qquad \{L((-2-t)\Lambda_0 + t\Lambda_1) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of irreducible $\mathcal{V}_{-2}(B_\ell)$–modules from the category KL^{-2}.

We have the following result on classification of irreducible modules.

Proposition 7.3. *Assume that $\ell \geq 3$. Then the set $\{L((-2-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell-3)+1\}$ provides a complete list of irreducible $\mathcal{V}_{-2}(B_\ell)$–modules from the category KL_{-2}.*

Proof. The proof is analogous to the proof of Proposition 6.4: it uses the exactness of the functor H_θ and the representation theory of affine vertex algebras. In particular, we use the result from [8] which gives that the set

$$\{L(-(\ell - 7/2) - j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell - 3) + 1\}$$

provides a complete list of irreducible $V_{\ell-7/2}(A_1)$–modules from the category $KL_{\ell-7/2}$. $\qquad\square$

An important consequence is the simplicity of the vertex algebra $\mathcal{V}_{-2}(B_2)$.

Corollary 7.4. *The vertex algebra $\mathcal{V}_{-2}(B_\ell)$ is simple if and only if $\ell = 2$. In particular, the set (7.2) provides a complete list of irreducible modules for $V_{-2}(B_2)$ in KL_{-2}.*

Proof. Since by Proposition 7.2, $\mathcal{V}_{-2}(B_\ell)$ has infinitely many irreducible modules in the category KL^{-2}, and, by Proposition 7.3, $V^{-2}(B_\ell)$ has finitely many irreducible modules in the category KL^{-2} (if $\ell \geq 3$), we conclude that $\mathcal{V}_{-2}(B_\ell)$ cannot be simple for $\ell \geq 3$.

Let us consider the case $\ell = 2$. Assume that $\mathcal{V}_{-2}(B_2)$ is not simple. Then it must contain an ideal I generated by a singular vector of weight $\lambda = (-2 - j)\Lambda_0 + j\Lambda_1$ for certain $j > 0$. By applying the functor H_θ, we get a non-trivial ideal in $V^{-3/2}(A_1)$, against the simplicity of $V^{-3/2}(A_1)$. $\square$

Next we notice that $V^{\ell-7/2}(A_1)$ has a unique non-trivial ideal J which is generated by a singular vector of A_1–weight $2(\ell-2)\omega_1$. The ideal J is maximal and simple (cf. [5]). By combining this with properties of the functor H_θ from [12], one proves the existence of a unique maximal ideal I (which is also simple) in $\mathcal{V}_{-2}(B_\ell)$ such that $I \cong L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1)$.

Remark 7.5. *The explicit expression for a singular vector which generates I is more complicated that in the case D, and it won't be presented here.*

In [6] we constructed a homomorphism $\mathcal{V}_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1) \to M_{2\ell+1}$. The results of this section enable us to find the image of this homomorphism.

Corollary 7.6. *We have:*
(1) *The vertex algebra $\mathcal{V}_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1)$ is conformally embedded into $V_{-1/2}(C_{2\ell+1})$.*
(2) *The vertex algebra $\mathcal{V}_{-2}(B_\ell)$ for $\ell \geq 3$ contains a unique ideal $I \cong L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1))$ and*

$$ch(\mathcal{V}_{-2}(B_\ell)) = ch(V_{-2}(B_\ell)) + ch(L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1)).$$

Finally, we apply Theorem 5.5 and prove that KL_{-2} is a semi-simple category.

Corollary 7.7. *If $\ell \geq 2$, then every $V_{-2}(B_\ell)$–module in KL_{-2} is completely reducible.*

Proof. It suffices to prove that every highest weight $V_{-2}(B_\ell)$–module in KL_{-2} is irreducible. Assume that $\ell \geq 3$. If $M \cong \widetilde{L}(\lambda)$ is a highest weight module in KL_{-2} then the highest weight is $\lambda = -(2 + j)\Lambda_0 + j\Lambda_1$ where $0 \leq j \leq 2(\ell-3)j + 1$. Since $H_\theta(L(\lambda))$ is a non-zero highest weight $V_{-\ell+7/2}(sl(2))$–module, then the complete reducibility result from [8] implies that $H_\theta(L(\lambda))$ is irreducible. The assertion now follows from Lemma 5.6. The proof in the case $\ell = 2$ is similar, and it uses the classification of irreducible $V_{-2}(B_2)$–modules from Corollary 7.4 and the fact that every highest weight $V_{-3/2}(sl(2)) = H_\theta(V_{-2}(B_2))$–module in $KL_{-3/2}$ is irreducible. $\square$

8. On the representation theory of $V_{2-\ell}(D_\ell)$

8.1. **The vertex algebra $\overline{V}_{2-\ell}(D_\ell)$.** Let $\mathfrak{g}$ be a simple Lie algebra of type D_ℓ. Recall that $2 - \ell = -h^\vee/2 + 1$ is a collapsing level [4]. We have the singular vector

$$(8.1) \qquad v_n = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1) \Big)^n \mathbf{1}$$

in $V^{n-\ell+1}(D_\ell)$, for any $n \in \mathbb{Z}_{>0}$. As in [40], we consider the vertex algebra

$$(8.2) \qquad \overline{V}_{2-\ell}(D_\ell) = V^{2-\ell}(D_\ell)/\langle v_1 \rangle,$$

where $\langle v_1 \rangle$ denotes the ideal in $V^{2-\ell}(D_\ell)$ generated by the singular vector v_1. We recall the following result on the classification of irreducible $\overline{V}_{2-\ell}(D_\ell)$–modules in the category $KL^{2-\ell}$.

Proposition 8.1. [40]
(1) *The set*

$$\{V(t\omega_\ell), V(t\omega_{\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of irreducible finite-dimensional modules for the Zhu algebra $A(\overline{V}_{2-\ell}(D_\ell))$.
(2) *The set*

$$\{L((2-t-\ell)\Lambda_0 + t\Lambda_\ell), L((2-t-\ell)\Lambda_0 + t\Lambda_{\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of irreducible $\overline{V}_{2-\ell}(D_\ell)$–modules from the category $KL^{2-\ell}$.

In the odd rank case $D_{2\ell-1}$, the modules from Proposition 8.1 (2) provide a complete list of irreducible $V_{3-2\ell}(D_{2\ell-1})$–modules from the category $KL_{3-2\ell}$ (cf. [11]). The paper [11] also contains a fusion rules result in the category $KL_{3-2\ell}$. Detailed fusion rules analysis will be presented elsewhere.

On the other hand, Theorem 4.2 implies that in the even rank case $D_{2\ell}$, $V_{2-2\ell}(D_{2\ell})$ is the unique irreducible $V_{2-2\ell}(D_{2\ell})$–module from the category $KL_{2-2\ell}$. In the next section we will give an explanation of this difference using singular vectors existing in the even rank case $D_{2\ell}$.

8.2. **Singular vectors in $V^{n-2\ell+1}(D_{2\ell})$.** In this section, we construct more singular vectors in $V^{n-2\ell+1}(D_{2\ell})$. In the case $n = 1$, we show that the maximal submodule of $V^{2-2\ell}(D_{2\ell})$ is generated by three singular vectors. We present explicit formulas for these singular vectors.

Let $\mathfrak{g}$ be a simple Lie algebra of type $D_{2\ell}$. Denote by $S_{2\ell}$ the group of permutations of 2ℓ elements. Let
$$\Pi_\ell = \left\{ p \in S_{2\ell} \mid p^2 = 1, \ p(i) \neq i, \forall i \in \{1, \ldots, 2\ell\} \right\}$$
be the set of fixed-points free involutions, which is well known to have $(2\ell-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2\ell-1)$ elements. For $i \neq j$, denote by $(i\,j) \in S_{2\ell}$ the transposition of i and j. Then, any $p \in \Pi_\ell$ admits a unique decomposition of the form:
$$p = (i_1\,j_1) \cdots (i_\ell\,j_\ell),$$
such that $i_h < j_h$ for $1 \leq h \leq \ell$, and $i_1 < \ldots < i_\ell$. Define a permutation $\bar{p} \in S_{2\ell}$ by:
$$\bar{p}(2h-1) = i_h, \ \bar{p}(2h) = j_h, \ 1 \leq h \leq \ell.$$
Thus, we have a well defined map $p \mapsto \bar{p}$ from Π_ℓ to $S_{2\ell}$. Define the function $s : \Pi_\ell \to \{\pm 1\}$ as follows:
$$s(p) = \mathrm{sign}(\bar{p}),$$
where $\mathrm{sign}(q)$ denotes the sign of the permutation $q \in S_{2\ell}$.

We have:

Theorem 8.2. *The vector*
$$(8.3) \qquad w_n = \left(\sum_{p \in \Pi_\ell} s(p) \prod_{\substack{i \in \{1, \ldots, 2\ell\} \\ i < p(i)}} e_{\epsilon_i + \epsilon_{p(i)}}(-1) \right)^n \mathbf{1}$$

is a singular vector in $V^{n-2\ell+1}(D_{2\ell})$, for any $n \in \mathbb{Z}_{>0}$.

Proof. Direct verification of relations $e_{\epsilon_k - \epsilon_{k+1}}(0)w_n = 0$, for $k = 1, \ldots, 2\ell-1$, $e_{\epsilon_{2\ell-1}+\epsilon_{2\ell}}(0)w_n = 0$ and $e_{-(\epsilon_1+\epsilon_2)}(1)w_n = 0$. $\qquad\square$

Remark 8.3. *The vector w_n has conformal weight $n\ell$ and its $\mathfrak{g}$–highest weight equals $2n\omega_{2\ell} = n(\epsilon_1 + \ldots + \epsilon_{2\ell})$. In particular, for $n = 1$, the vector w_1 has conformal weight ℓ and highest weight $2\omega_{2\ell} = \epsilon_1 + \ldots + \epsilon_{2\ell}$.*

Example 8.4. *Set $n = 1$ for simplicity. For $\ell = 2$ we recover the singular vector*
$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1) + e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}$$
in $V^{-2}(D_4)$ of conformal weight 2 from [40]. For $\ell = 3$, the formula for the singular vector in $V^{-4}(D_6)$ of conformal weight 3 is more complicated. It is a sum of $5!! = 15$ monomials:

$$\begin{aligned}
w_1 = (&e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_5+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_5}(-1)e_{\epsilon_4+\epsilon_6}(-1) \\
+&e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_6}(-1)e_{\epsilon_4+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_5+\epsilon_6}(-1) \\
+&e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_4+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_4+\epsilon_5}(-1) \\
+&e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_5+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_3+\epsilon_6}(-1) \\
+&e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_3+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_4+\epsilon_6}(-1) \\
+&e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_3+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_3+\epsilon_4}(-1) \\
+&e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_4+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_3+\epsilon_5}(-1) \\
+&e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_3+\epsilon_4}(-1))\mathbf{1}.
\end{aligned}$$

Denote by ϑ the automorphism of $V^{n-2\ell+1}(D_{2\ell})$ induced by the automorphism of the Dynkin diagram of $D_{2\ell}$ of order two such that

(8.4)
$$\vartheta(\epsilon_k - \epsilon_{k+1}) = \epsilon_k - \epsilon_{k+1}, \ k = 1, \ldots, 2\ell - 2,$$

(8.5)
$$\vartheta(\epsilon_{2\ell-1} - \epsilon_{2\ell}) = \epsilon_{2\ell-1} + \epsilon_{2\ell}, \ \vartheta(\epsilon_{2\ell-1} + \epsilon_{2\ell}) = \epsilon_{2\ell-1} - \epsilon_{2\ell}.$$

Theorem 8.2 now implies that $\vartheta(w_n)$ is a singular vector in $V^{n-2\ell+1}(D_{2\ell})$, for any $n \in \mathbb{Z}_{>0}$, also. The vector $\vartheta(w_n)$ has conformal weight $n\ell$ and its highest weight for $\mathfrak{g}$ is $2n\omega_{2\ell-1} = n(\epsilon_1 + \ldots + \epsilon_{2\ell-1} - \epsilon_{2\ell})$.

We consider the associated quotient vertex algebra

(8.6)
$$\widetilde{V}_{n-2\ell+1}(D_{2\ell}) := V^{n-2\ell+1}(D_{2\ell})/\langle v_n, w_n, \vartheta(w_n)\rangle,$$

where v_n is given by relation (8.1) (for $D_{2\ell}$):

$$v_n = \Big(\sum_{i=2}^{2\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^n \mathbf{1}.$$

In particular, for $n = 1$ we have the vertex algebra

$$\widetilde{V}_{2-2\ell}(D_{2\ell}) = V^{2-2\ell}(D_{2\ell})/\langle v_1, w_1, \vartheta(w_1)\rangle.$$

Clearly, $\widetilde{V}_{2-2\ell}(D_{2\ell})$ is a quotient of vertex algebra $\overline{V}_{2-2\ell}(D_{2\ell})$ from Subsection 8.1. The associated Zhu algebra is

$$A(\widetilde{V}_{2-2\ell}(D_{2\ell})) = U(\mathfrak{g})/\langle \bar{v}, \bar{w}, \vartheta(\bar{w})\rangle,$$

where

$$\bar{v} = \sum_{i=2}^{2\ell} e_{\epsilon_1-\epsilon_i} e_{\epsilon_1+\epsilon_i}, \qquad \bar{w} = \sum_{p \in \Pi_\ell} s(p) \prod_{\substack{i \in \{1,\ldots,2\ell\} \\ i < p(i)}} e_{\epsilon_i+\epsilon_{p(i)}}.$$

Lemma 8.5. *We have:*

    (1) $\bar{w}V(t\omega_{2\ell}) \neq 0$, *for* $t \in \mathbb{Z}_{>0}$.
    (2) $\vartheta(\bar{w})V(t\omega_{2\ell-1}) \neq 0$, *for* $t \in \mathbb{Z}_{>0}$.

Proof. (1) Let $t = 1$. Denote by $v_{\omega_{2\ell}}$ the highest weight vector of $V(\omega_{2\ell})$, and by $v_{-\omega_{2\ell}}$ the lowest weight vector of $V(\omega_{2\ell})$. One can easily check, using the spinor realization of $V(\omega_{2\ell})$, that there exists a constant $C \neq 0$ such that

$$\bar{w}(v_{-\omega_{2\ell}}) = Cv_{\omega_{2\ell}}.$$

For general $t \in \mathbb{Z}_{>0}$, the claim follows using the embedding of $V(t\omega_{2\ell})$ into $V(\omega_{2\ell})^{\otimes t}$. Claim (2) follows similarly. $\square$

Theorem 8.6. *We have:*
(i) *The trivial module* $\mathbb{C}$ *is the unique finite-dimensional irreducible module for* $A(\widetilde{V}_{2-2\ell}(D_{2\ell}))$.
(ii) $V_{2-2\ell}(D_{2\ell})$ *is the unique irreducible* $\mathfrak{g}$*-locally finite module for* $\widetilde{V}_{2-2\ell}(D_{2\ell})$.
(iii) *The vertex operator algebra* $\widetilde{V}_{2-2\ell}(D_{2\ell})$ *is simple, i.e.*

$$V_{2-2\ell}(D_{2\ell}) = V^{2-2\ell}(D_{2\ell})/\langle v_1, w_1, \vartheta(w_1)\rangle.$$

Proof. (i) Proposition 8.1 implies that the set

$$\{V(t\omega_{2\ell}), V(t\omega_{2\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of finite-dimensional irreducible modules for the algebra $U(\mathfrak{g})/\langle \bar{v}\rangle = A(\overline{V}_{2-2\ell}(D_{2\ell}))$. Lemma 8.5 shows that $V(t\omega_{2\ell})$ and $V(t\omega_{2\ell-1})$ are not modules for $A(\widetilde{V}_{2-2\ell}(D_{2\ell}))$, for $t \in \mathbb{Z}_{>0}$. Claim (i) follows. Claims (ii) and (iii) follow from (i) by applying Proposition 3.2 and Corollary 3.3. $\square$

Remark 8.7. *A general character formula for certain simple affine vertex algebras at negative integer levels has been recently presented by V. G. Kac and M. Wakimoto in* [37], *(more precisely,* $\mathfrak{g} = A_n, C_n$ *for* $k = -1$ *and* $\mathfrak{g} = D_4, E_6, E_7, E_8$ *for* $k = -2, -3, -4, 6$*). Note that conditions (i)-(iii) of* [37, *Theorem 3.1] hold for vertex algebras* $V_{-b}(D_n)$, $n > 4$, $b = 1, \ldots, n - 2$, *too. We conjecture that condition (iv) of this theorem holds as well; therefore formula (3.1) in* [37] *gives the character formula.*

9. Conformal embedding of $\widetilde{V}(-4, D_6 \times A_1)$ into $V_{-4}(E_7)$

In this section, we apply the results on representation theory of $V_{-4}(D_6)$ from previous sections to the conformal embedding of $\widetilde{V}(-4, D_6 \times A_1)$ into $V_{-4}(E_7)$. This gives us an interesting example of a maximal semisimple equal rank subalgebra such that the associated conformally embedded subalgebra is not simple.

We use the construction of the root system of type E_7 from [19], [29], and the notation for root vectors similar to the notation for root vectors for E_6 from [9].

For a subset $S = \{i_1, \ldots, i_k\} \subseteq \{1, 2, 3, 4, 5, 6\}, i_1 < \ldots < i_k$, with odd number of elements (so that $k = 1, 3$ or 5), denote by $e_{(i_1 \ldots i_k)}$ a suitably chosen root vector associated to the positive root

$$\frac{1}{2}\left(\epsilon_8 - \epsilon_7 + \sum_{i=1}^{6}(-1)^{p(i)}\epsilon_i\right),$$

such that $p(i) = 0$ for $i \in S$ and $p(i) = 1$ for $i \notin S$. We will use the symbol $f_{(i_1 \ldots i_k)}$ for the root vector associated to corresponding negative root.

Note now that the subalgebra of E_7 generated by positive root vectors

(9.1) $e_{\epsilon_6+\epsilon_5}, e_{\alpha_1} = e_{(1)}, e_{\alpha_3} = e_{\epsilon_2-\epsilon_1}, e_{\alpha_4} = e_{\epsilon_3-\epsilon_2}, e_{\alpha_2} = e_{\epsilon_1+\epsilon_2}, e_{\alpha_5} = e_{\epsilon_4-\epsilon_3}$

and the associated negative root vectors is a simple Lie algebra of type D_6. There are 30 root vectors associated to positive roots for D_6:

$$e_{\epsilon_6+\epsilon_5}, \ e_{\epsilon_8-\epsilon_7},$$
$$e_{(i)}, \ i \in \{1, 2, 3, 4\},$$
$$e_{(ijk)}, \ i, j, k \in \{1, 2, 3, 4\}, \ i < j < k,$$
$$e_{(i56)}, \ i \in \{1, 2, 3, 4\},$$
$$e_{(ijk56)}, \ i, j, k \in \{1, 2, 3, 4\}, \ i < j < k,$$
(9.2) $$e_{\pm\epsilon_i+\epsilon_j}, \ i, j \in \{1, 2, 3, 4\}, i < j.$$

Furthermore, the subalgebra of E_7 generated by $e_{\epsilon_6-\epsilon_5}$ and the associated negative root vector is a simple Lie algebra of type A_1. Thus, $D_6 \oplus A_1$ is a semisimple subalgebra of E_7.

It follows from [3], [9] that the affine vertex algebra $\widetilde{V}(-4, D_6 \times A_1)$ is conformally embedded in $V_{-4}(E_7)$. Remark that $\widetilde{V}(-4, A_1) = V_{-4}(A_1)$ (since $V^{-4}(A_1) = V_{-4}(A_1)$). This implies that $\widetilde{V}(-4, D_6 \times A_1) \cong \widetilde{V}(-4, D_6) \otimes V_{-4}(A_1)$.

It was shown in [15] that

$$v_{E_7} = (e_{\epsilon_8-\epsilon_7}(-1)e_{\epsilon_6+\epsilon_5}(-1) + e_{(156)}(-1)e_{(23456)}(-1) +$$
$$+e_{(256)}(-1)e_{(13456)}(-1) + e_{(356)}(-1)e_{(12456)}(-1) +$$
(9.3) $$+e_{(456)}(-1)e_{(12356)}(-1))\mathbf{1}$$

is a singular vector in $V^{-4}(E_7)$. Moreover,

$$V_{-4}(E_7) \cong V^{-4}(E_7)/\langle v_{E_7}\rangle.$$

Vectors $(e_{(12346)}(-1))^s\mathbf{1}$, for $s \in \mathbb{Z}_{>0}$ are (non-trivial) singular vectors for the affinization of $D_6 \oplus A_1$ in $V_{-4}(E_7)$ of highest weights $-(s+4)\Lambda_0 + s\Lambda_6$ for $D_6^{(1)}$ and $-(s+4)\Lambda_0 + s\Lambda_1$ for $A_1^{(1)}$. Thus there exist highest weight modules $\widetilde{L}_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6)$ and $\widetilde{L}_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$, for $D_6^{(1)}$ and $A_1^{(1)}$,

respectively such that $(\widetilde{V}(-4, D_6) \otimes V_{-4}(A_1)).(e_{(12346)}(-1))^s\mathbf{1}$ is isomorphic to $\widetilde{L}_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \otimes \widetilde{L}_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$. This implies that

$$L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \otimes L_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$$

are irreducible $\widetilde{V}(-4, D_6 \times A_1)$–modules, for $s \in \mathbb{Z}_{>0}$.

In particular, $L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6)$ are irreducible (D_6–locally finite) $\widetilde{V}(-4, D_6)$–modules, for $s \in \mathbb{Z}_{>0}$. In the next proposition, we use the notation from (8.2), (8.3), (8.4), (8.5).

Proposition 9.1. *We have:*
(1) *Assume that* $\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6)$ *and* $\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5)$ *are highest weight* $\overline{V}_{-4}(D_6)$*–modules from the category* KL^{-4}*, not necessarily irreducible. Then*

$$\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6) \boxtimes \widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5) = 0,$$

where $\boxtimes$ *is the tensor functor for* KL^{-4}*–modules. In other words, we cannot have a non-zero* $\overline{V}_{-4}(D_6)$*–module* M *from* KL^{-4} *and a non-zero intertwining operator of type*

(9.4) $$\begin{pmatrix} M \\ \widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6) \quad \widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5) \end{pmatrix}.$$

(2) *Relations* $w_1 \neq 0$ *and* $\vartheta(w_1) = 0$ *hold in* $V_{-4}(E_7)$*. In particular,* $\widetilde{V}(-4, D_6)$ *is not simple.*

Proof. For the proof of assertion (1) we first notice that the following decomposition of D_6–modules holds:

$$V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5) \quad = \quad V_{D_6}(2\omega_5 + 2\omega_6) \oplus V_{D_6}(\omega_3 + \omega_5 + \omega_6) \oplus V_{D_6}(2\omega_3)$$
(9.5) $$\oplus V_{D_6}(\omega_1 + \omega_5 + \omega_6) \oplus V_{D_6}(\omega_1 + \omega_3) \oplus V_{D_6}(2\omega_1).$$

Assume that M is a non-zero $\overline{V}_{-4}(D_6)$–module in the category KL^{-4} such that there is a non-trivial intertwining operator of type (9.4). Then the Frenkel-Zhu formula for fusion rules implies that M must contain a non-trivial subquotient whose lowest graded component appears in the decomposition of $V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5)$. But by Proposition 8.1, the D_6–modules appearing in (9.5) cannot be lowest components of any $\overline{V}_{-4}(D_6)$–module. This proves assertion (1).

Assertion (1) implies that if $w_1 \neq 0$ and $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then

$$Y(w_1, z)\vartheta(w_1) = 0,$$

a contradiction since $V_{-4}(E_7)$ is a simple vertex algebra. The same fusion rules argument shows that if $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then

$$Y(\vartheta(w_1), z)e_{(12346)}(-1)^2\mathbf{1} = 0,$$

which again contradicts the simplicity of $V_{-4}(E_7)$. So, $\vartheta(w_1) = 0$.

But if $w_1 = 0$, then, by Theorem 8.6 (iii), we have that $\widetilde{V}(-4, D_6) = V_{-4}(D_6)$. Theorem 4.2 implies that $\widetilde{V}(-4, D_6)$ is not simple, since the simple vertex operator algebra $V_{-4}(D_6)$ has only one irreducible D_6–locally finite module, a contradiction. So $w_1 \neq 0$ and claim (2) follows. $\qquad\square$

Set

(9.6) $$\mathcal{V}_{-4}(D_6) = \frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}.$$

Theorem 9.2. *We have:*
(1) $\widetilde{V}(-4, D_6) \cong \mathcal{V}_{-4}(D_6)$.
(2) *The set* $\{L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \mid s \in \mathbb{Z}_{\geq 0}\}$ *provides a complete list of irreducible* $\mathcal{V}_{-4}(D_6)$*–modules.*

Proof. We first notice that $\widetilde{V}(-4, D_6)$ is a certain quotient of $\frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}$, and that

$$H_\theta\left(\frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}\right) = \mathcal{V}_{-2}(D_4).$$

Since $\mathcal{V}_{-2}(D_4)$ contains a unique non-trivial ideal which is maximal and simple, we conclude that $\frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}$ also contains a unique ideal, and it must be the ideal generated by w_1. Since in $\widetilde{V}(-4, D_6)$ we have that $w_1 \neq 0$, we conclude that

$$\widetilde{V}(-4, D_6) \cong \frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}.$$

The proof of assertion (2) follows from (1), the classification result of $\overline{V}_{-4}(D_6)$–modules from Proposition 8.1 and Lemma 8.5. $\qquad\square$

10. Funding

References

[1] Adamović, D. "Some rational vertex algebras." *Glas. Mat. Ser. III* 29 (1994): 25–40.

[2] Adamović, D. "A construction of some ideals in affine vertex algebras." *Int. J. Math. Math. Sci.* (2003): 971–980.

[3] Adamović D., Kac V. G., Möseneder Frajria P., Papi P., and Perše O. "Finite vs infinite decompositions in conformal embeddings." *Comm. Math. Phys.* 348 (2016): 445–473.

[4] Adamović D., Kac V. G., Möseneder Frajria P., Papi P., and Perše O. "Conformal embeddings of affine vertex algebras in minimal *W*-algebras I: Structural results." *J. Algebra* 500 (2018): 117–152.

[5] Adamović D., Kac V. G., Möseneder Frajria P., Papi P., and Perše O. "Conformal embeddings of affine vertex algebras in minimal *W*-algebras II: decompositions." *Jpn. J. Math.* 12 (2017): 261–315.

[6] Adamović D., Kac V. G., Möseneder Frajria P., Papi P., and Perše O. "On classification of non-equal rank affine conformal embeddings and applications." *Selecta Math. (N.S.)* 24 (2018): 2455–2498.

[7] Adamović D., Kac V. G., Möseneder Frajria P., Papi P., and Perše O., Conformal embeddings in affine vertex superalgebras, to appear.

[8] Adamović D., and Milas A. "Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$." *Math. Res. Lett.* 2 (1995): 563–575.

[9] Adamović D., and Perše O. "Some general results on conformal embeddings of affine vertex operator algebras." *Algebr. Represent. Theory* 16 (2013): 51–64.

[10] Adamović D., and Perše O. "Representations of certain non-rational vertex operator algebras of affine type." *J. Algebra* 319 (2008): 2434–2450.

[11] Adamović D., and Perše O. "Fusion rules and complete reducibility of certain modules for affine Lie algebras." *J. Algebra Appl.* 13 (2014): 1350062, 18 pp.

[12] Arakawa T. "Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture." *Duke Math. J.* 130 (2005): 435–478.

[13] Arakawa T. "Associated varieties of modules over Kac–Moody algebras and C_2-cofiniteness of W-algebras." *Int. Math. Res. Not. IMRN* (2015): 11605–11666.

[14] Arakawa T., and Kawasetsu K. "Quasi-lisse vertex algebras and modular linear differential equations." to appear in *Kostant Memorial Volume, Progress in Math.* 326, Birkhauser 2018., arXiv:1610.05865.

[15] Arakawa T., and Moreau A. "Joseph ideals and lisse minimal *W*-algebras." *J. Inst. Math. Jussieu* 17 (2018): 397–417.

[16] Arakawa T., and Moreau A. "Sheets and associated varieties of affine vertex algebras." *Adv. Math.* 320 (2017): 157–209.

[17] Arakawa T., and Moreau A. "On the irreducibility of associated varieties of W-algebras." *J. Algebra* 500 (2018): 542–568.

[18] Borcherds R. E. "Vertex algebras, Kac-Moody algebras, and the Monster." *Proc. Nat. Acad. Sci. U.S.A.* 83 (1986): 3068–3071.

[19] Bourbaki. *Groupes et algèbras de Lie*, Paris: Hermann, 1975.

[20] Deodhar V. V., Gabber O., and Kac V. G. "Structures of some categories of representations of infinite-dimensional Lie algebras." *Adv. Math.* 45 (1982): 92–116.

[21] Feigin B., and Frenkel E. "Quantization of the Drinfel'd-Sokolov reduction." *Phys. Lett. B* 246 (1990): 75–81.

[22] Feingold A. J., and Frenkel I. B. "Classical affine algebras." *Adv. Math.* 56 (1985): 117–172.

[23] Frenkel E., Kac V. G., and Wakimoto M. "Characters and fusion rules for W-algebras via quantized Drinfel'd-Sokolov reduction." *Comm. Math. Phys.* 147 (1992): 295–328.

[24] Frenkel I., Huang Y. Z., and Lepowsky J. "On axiomatic approaches to vertex operator algebras and modules." *Mem. Amer. Math. Soc.* 104 (1993): viii+64 pp.

[25] Frenkel I., Lepowsky J., and Meurman A. *Vertex Operator Algebras and the Monster*, Pure and Appl. Math. 134, Boston: Academic Press, 1988.

[26] Frenkel I., and Zhu Y.-C. "Vertex operator algebras associated to representations of affine and Virasoro algebras" *Duke Math. J.* 66 (1992): 123–168.

[27] Gaiotto D. "Twisted compactifications of 3d N = 4 theories and conformal blocks." arXiv:1611.01528.

[28] Gorelik M., and Kac V. G. "On complete reducibility for infinite-dimensional Lie algebras." *Adv. Math.* 226 (2011): 1911–1972.

[29] Humphreys J. E. *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics 9, New York-Berlin: Springer-Verlag, 1972.

[30] Kac V. G. "Representations of classical Lie superalgebras." in *Differential geometrical methods in Mathematical Physics, II (Proc. Conf. Univ. Bonn, Bonn 1977)*, Springer, Berlin, 1978, 597–626.

[31] Kac V. G. *Infinite dimensional Lie algebras*, 3rd ed. Cambridge: Cambridge Univ. Press, 1990.

[32] Kac V. G. *Vertex Algebras for Beginners*, University Lecture Series, 2nd ed. AMS, Providence, RI, 1998.

[33] Kac V. G., and Peterson D. H. "Infinite-dimensional Lie algebras, theta functions and modular forms." *Adv. in Math.* 53 (1984): 125–264.

[34] Kac V. G., Roan S.-S., and Wakimoto M. "Quantum reduction for affine superalgebras." *Comm. Math. Phys.* 241 (2003): 307–342.

[35] Kac V. G., and Wakimoto M. "Quantum reduction and representation theory of superconformal algebras." *Adv. Math.* 185 (2004): 400–458.

[36] Kac V. G., and Wakimoto M. "On Rationality of W–algebras." *Transform. Groups* 13 (2008): 671–713.

[37] Kac V. G., and Wakimoto M. "On characters of irreducible highest weight modules of negative integer level over affine Lie algebras." to appear in *Kostant Memorial Volume, Progress in Math.* 326, Birkhauser 2018., arXiv:1706.08387.

[38] Kawasetsu K. "W-algebras with non-admissible levels and the Deligne exceptional series." *Int. Math. Res. Not. IMRN* (2018): 641–676.

[39] Kazhdan D., and Lusztig G. "Tensor structures arising from affine Lie algebras, I, II." *J. Amer. Math. Soc.* 6 (1993): 905–947, 949–1011.

[40] Perše O. "A note on representations of some affine vertex algebras of type D." *Glas. Mat. Ser. III* 48 (68) (2013): 81–90.

[41] Zhu Y.-C. "Modular invariance of characters of vertex operator algebras." *J. Amer. Math. Soc.* 9 (1996): 237–302.