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MAGNETO-ELASTICITY ON THE DISK

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ABSTRACT. A model problem of magneto-elastic body is considered. Specifically, the case of a two dimensional circular disk is studied. The functional which represents the magneto-elastic energy is introduced. Then, the minimisation problem, referring to the simplified two-dimensional model under investigation, is analysed. The existence of a minimiser is proved and its dependence on the eigenvalues of the problem is investigated. A bifurcation result is obtained corresponding to special values of the parameters.

1. Introduction

The interest in magneto-elastic materials finds its motivation in the growing variety of new materials among which magneto-rheological elastomers or magneto-sensitive polymeric composites (Hossain *et al.* 2015a,b) may be mentioned. A special issue of the journal *Materials* devoted to *Magnetoelastic Materials* is currently being published (Szewczyk 2020). Many applications of magneto-elastic materials, covering a wide area of interest from technological to biomedical devices (see, *e.g.*, Ren *et al.* 2019), can be listed. In particular, also two dimensional problems are subject of applicative investigations (Hadda and Tilioua 2012). The model we consider is a two dimensional simplified one, however, we believe that, it might open the way to further applications, possibly, via perturbative methods (Bernard *et al.* 2019).

We study the functional energy of a magneto-elastic material, that is a material which is capable of deformation and magnetisation. The magnetisation is a phenomenon that does not appear at a macroscopic level, it is characterised by the *magnetisation vector* whose magnitude is independent of the position while its direction which can vary from one point to another. In this context, the magnetisation vector $\mathbf{m}$ is a map from Ω (a bounded open set of $\mathbb{R}^2$) to S^2 (the unit sphere of $\mathbb{R}^3$). In particular, here we assume Ω is the unit disk of $\mathbb{R}^2$. The magnetisation distribution is well described by a free energy functional which we assume composed of three terms, namely the *exchange* energy E_{ex}, the *elastic* energy E_{el} and the *elastic-magnetic* energy E_{em}. In Section **2** we detail the three energetic terms and, after some simplifications, derive the proposed functional for describing some phenomena.

Assuming the hypothesis of radially symmetric maps, *i.e.*,

$$\mathbf{m} = (\cos\theta \sin h(r), \sin\theta \sin h(r), \cos h(r)),$$

we get to the analysis of a one-dimensional energy functional that can be expressed in terms of the only scalar function h. The effect of the elastic deformation reveals through a positive parameter μ which characterizes the connection between the magnetic and elastic processes. In Section **3** the minimisation of the energy functional, namely

$$E(h) = \pi \int_0^1 \left[h_r^2 + \left(\frac{\sin h}{r}\right)^2 - \frac{\mu}{2}(\sin 2h)^2 \right] r dr,$$

is the aim of our paper. In particular, we prove that there exists a critical value μ^0 such that for $\mu \le \mu^0$ the functional energy is not negative and there is only a global minimiser that is the trivial solution $h \equiv 0$; for $\mu > \mu^0$ other nontrivial minimisers appear, moreover the energy takes negative values. The local bifurcation analysis is carried out. More precisely we prove that at the point μ^0, two branches of minimisers, with small norm, bifurcate from the trivial stable solution. This local analysis does not exclude the existence of other solutions of the minimisation problem even for $\mu = 0$ (see also the results by Brezis and Coron 1983, concerning the solutions of harmonic maps from the unit disk in $\mathbb{R}^2$ to the sphere S^2).

For the modelling of magneto-elastic interactions see also Brown (1966), He (1999), Bertsch *et al.* (2001), Valente and Vergara Caffarelli (2007), Cerimele *et al.* (2008), and Chipot *et al.* (2008, 2009). Magneto-viscoelastic problems were studied by Carillo *et al.* (2011, 2012, 2017). Moreover we recall that the phenomenon of bifurcation of minimising harmonic maps has been studied by Bethuel *et al.* (1992) in a different physical context.

2. The model

We start with the general three-dimensional theory. We assume $\Omega \subset \mathbb{R}^3$ is the volume of the magneto-elastic material and $\partial\Omega$ its boundary. Let x_i, $i = 1,2,3$, be the position of a point $\mathbf{x}$ of Ω and denote by

$$u_i = u_i(\mathbf{x}), \qquad i = 1,2,3$$

the components of the displacement vector $\mathbf{u}$ and by

$$\varepsilon_{kl}(\mathbf{u}) = \frac{1}{2}(u_{k,l} + u_{l,k}), \qquad k,l = 1,2,3$$

the deformation tensor where, as a common praxis, $u_{k,l}$ stands for $\frac{\partial u_k}{\partial x_l}$. Moreover we denote by

$$m_j = m_j(\mathbf{x}), \qquad j = 1,2,3$$

the components of the magnetisation vector $\mathbf{m}$ that we assume of unit modulus, *i.e.*, $|\mathbf{m}| = 1$. In the sequel, where not specified, the Latin indices vary in the set $\{1,2,3\}$ and the summation over repeated indices is assumed. We first define the exchange energy which arises from exchange neighbourhood interactions as

$$E_{\text{ex}}(\mathbf{m}) = \frac{1}{2}\int_\Omega a_{ij} m_{k,i} m_{k,j} d\Omega \tag{1}$$

where $a_{ijkl} = a_1\delta_{ijkl} + a_2\delta_{ij}\delta_{kl}$ with $a_1, a_2 \geq 0$ and $\delta_{ijkl} = \delta_{ik}\delta_{jl}$ is the fourth-order identity tensor. This integral represents the interface energy between magnetised domains with different orientations. For most magnetic materials $\operatorname{div}\mathbf{m} = \delta_{ij}m_{i,j} = 0$, so hereafter we assume $a_1 = a > 0$ and $a_2 = 0$ (see Landau and Lifshitz 1935). The magneto-elastic energy is due to the coupling between the magnetic moments and the elastic lattice. For cubic crystals it is assumed to be

$$E_{\mathrm{em}}(\mathbf{m},\mathbf{u}) = \frac{1}{2}\int_\Omega \lambda_{ijkl}m_i m_j \varepsilon_{kl}(\mathbf{u})d\Omega \tag{2}$$

where $\mathbb{L} = \{\lambda_{klmn}\}$ denotes the *magneto-elasticity tensor* whose entries $\lambda_1, \lambda_2, \lambda_3 \geq 0$, and $\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. Moreover we introduce the elastic energy

$$E_{\mathrm{el}}(\mathbf{u}) = \frac{1}{2}\int_\Omega \sigma_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u})\,d\Omega \tag{3}$$

where $\mathbb{E} = \{\varepsilon_{lm}\}$ indicates the *strain tensor* σ_{ijkl} satisfying the following symmetry property

$$\sigma_{ijkl} = \sigma_{klij} = \sigma_{jilk}$$

and moreover the inequality

$$\sigma_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \beta\varepsilon_{ij}\varepsilon_{ij}$$

holds for some $\beta > 0$. In the isotropic case

$$\sigma_{ijkl} = \tau_1\delta_{ijkl} + \tau_2\delta_{ij}\delta_{kl}, \qquad \tau_1, \tau_2 \geq 0.$$

The resulting energy functional E is given by

$$E(\mathbf{m},\mathbf{u}) = E(m,u) = E_{ex}(m) + E_{em}(m,u) + E_{ve}(u), \tag{4}$$

which, after some manipulations (Bertsch *et al.* 2001; Carillo *et al.* 2012), under the assumption the material is isotropic, reads

$$\begin{aligned} E(\mathbf{m},\mathbf{u}) = \frac{1}{2}\int_\Omega a|\nabla\mathbf{m}|^2 d\Omega + \frac{1}{2}\int_\Omega \left[\tau_1|\nabla\mathbf{u}|^2 + \tau_2(\operatorname{div}\mathbf{u})^2\right] d\Omega + \\ + \frac{1}{2}\int_\Omega \left[\lambda_1\delta_{klij}u_{j,i}m_k m_l + \lambda_2|\mathbf{m}|^2\operatorname{div}\mathbf{u} + 2\lambda_3(\nabla u_i\cdot\mathbf{m})m_i\right] d\Omega\,. \end{aligned} \tag{5}$$

2.1. A simplified 2D model. To get the proposed model we make some approximations. First of all we assume $\Omega \subset \mathbb{R}^2$ and neglect the components in plane of the displacement vector $\mathbf{u}$, *i.e.*, we assume $\mathbf{u} = (0,0,w)$, which implies $\operatorname{div} u = 0$ since w depends only on the plane coordinates. Let $\lambda_3 = \lambda$ be a positive constant, setting[1] $\tau_1 = 1$ and $a = 1$, the functional E reduces to

$$E(\mathbf{m},w) = \frac{1}{2}\int_\Omega \left(|\nabla\mathbf{m}|^2 + 2\lambda m_3(m_\alpha w_{,\alpha}) + |\nabla w|^2\right) d\Omega \tag{6}$$

where the Greek indices vary in the set $\{1,2\}$.

[1]No need to prescribe nor λ_2 nor $\tau_2 \geq 0$ since they both appear only as factors of $\operatorname{div} u$; also λ_1 can be left arbitrary; indeed, $\delta_{klij}u_{j,i} = 0$ since $u_{j,i} \neq 0$ only if $j = 3$ and $i = 1,2$ but $\delta_{klij} = 0$ when $j \neq i$.

Setting $\Omega \equiv D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and assuming radial symmetry, further to $w = w(r)$, we can express the components of the vector $\mathbf{m}$ in terms of r, that is

$$\mathbf{m} = \left(\frac{x}{r}\sin h(r), \frac{y}{r}\sin h(r), \cos h(r)\right), \qquad r = \sqrt{x^2+y^2},$$

where $h : (0,1) \subset \mathbb{R} \to \mathbb{R}$ is an unknown regular function. Using the fact that $\partial_x r = \frac{x}{r}$ and $\partial_y r = \frac{y}{r}$ we deduce by the chain rule, where $h_r := \dfrac{dh}{dr}$ denotes the derivatives of the h with respect to the variable r:

$$\partial_x \mathbf{m} = \left(\frac{\sin h}{r} + \frac{x^2}{r}\left(\frac{\sin h}{r}\right)_r, \ \frac{xy}{r}\left(\frac{\sin h}{r}\right)_r, \ \frac{x}{r}(\cos h)_r\right)$$

$$\partial_y \mathbf{m} = \left(\frac{xy}{r}\left(\frac{\sin h}{r}\right)_r, \ \frac{\sin h}{r} + \frac{y^2}{r}\left(\frac{\sin h}{r}\right)_r, \ \frac{y}{r}(\cos h)_r\right).$$

Thus we get

$$\begin{aligned}
|\nabla \mathbf{m}|^2 &= \left[\frac{\sin h}{r} + \frac{x^2}{r}\left(\frac{\sin h}{r}\right)_r\right]^2 + \left[\frac{\sin h}{r} + \frac{y^2}{r}\left(\frac{\sin h}{r}\right)_r\right]^2 + 2\left[\frac{xy}{r}\left(\frac{\sin h}{r}\right)_r\right]^2 + [(\cos h)_r]^2 \\
&= 2\left(\frac{\sin h}{r}\right)^2 + \frac{x^4+2x^2y^2+y^4}{r^2}\left[\left(\frac{\sin h}{r}\right)_r\right]^2 + 2\frac{x^2+y^2}{r^2}\sin h\left(\frac{\sin h}{r}\right)_r + h_r^2(\sin h)^2 \\
&= 2\left(\frac{\sin h}{r}\right)^2 + r^2\left[\left(\frac{\sin h}{r}\right)_r\right]^2 + 2\sin h\left(\frac{\sin h}{r}\right)_r + h_r^2(\sin h)^2 \\
&= \left(\frac{\sin h}{r}\right)^2 + \left[\frac{\sin h}{r} + r\left(\frac{\sin h}{r}\right)_r\right]^2 + h_r^2(\sin h)^2 \\
&= \left(\frac{\sin h}{r}\right)^2 + \left[\frac{\sin h}{r} + r\left(\frac{rh_r\cos h - \sin h}{r^2}\right)\right]^2 + h_r^2(\sin h)^2 \\
&= \left(\frac{\sin h}{r}\right)^2 + h_r^2.
\end{aligned}$$

So the energy (6), when we recall the assumed radial symmetry implies also $w = w(r)$, adopting the notation $w_r := \dfrac{dw}{dr}$, becomes

$$E(h,w) = \pi\int_0^1 \left[h_r^2 + \left(\frac{\sin h}{r}\right)^2 + \lambda \sin 2h\, w_r + w_r^2\right] r dr$$

and from that we deduce the governing equations

$$\begin{cases} h_{rr} + \dfrac{h_r}{r} - \dfrac{\sin 2h}{2r^2} - \lambda \cos 2h\, w_r = 0 \\[2ex] w_{rr} + \dfrac{w_r}{r} + \dfrac{\lambda}{2}\left[(\sin 2h)_r + \dfrac{\sin 2h}{r}\right] = 0. \end{cases} \tag{7}$$

We prescribe the following boundary conditions

$$w_r(0) = 0,\ w(1) = 0, \tag{8}$$

where the first condition is motivated by the symmetry assumptions, while the second one corresponds to prescribe the boundary of Ω is fixed, and

$$h_r(1) = 0. \tag{9}$$

Solving the second equation of (7) which can be written

$$(rw_r)_r + \frac{\lambda}{2}(r\sin 2h)_r = 0 \Leftrightarrow w_r = -\frac{\lambda}{2}\sin 2h,$$

where the double implication is guaranteed when we set $h(0) = 0$, then letting $\mu = \lambda^2/2$ we get the equation

$$h_{rr} + \frac{h_r}{r} - \frac{\sin 2h}{2r^2} + \mu \sin 2h \cos 2h = 0 \tag{10}$$

and the energy E becomes

$$E(h) = \pi \int_0^1 \left[h_r^2 + \left(\frac{\sin h}{r}\right)^2 - \frac{\mu}{2}(\sin 2h)^2 \right] rdr. \tag{11}$$

The variational analysis of the functional $E(h)$ is the objective of the following section.

3. The minimisation problem

Lemma 3.1. *Let us define*

$$V = \{v \mid v_r, \frac{v}{r} \in L^2(0,1;rdr)\}. \tag{12}$$

V is a Hilbert space equipped with the norm

$$\|v\|^2 = \int_0^1 ({v_r}^2 + \frac{v^2}{r^2})rdr\ . \tag{13}$$

Proof. Let v_n be a Cauchy sequence in V, $\{(v_n)_r\}, \left\{\frac{v_n}{r}\right\}$ are Cauchy sequences in

$$L^2(rdr) = L^2(0,1;rdr)$$

and there exist h, g such that

$$\{(v_n)_r\}, \left\{\frac{v_n}{r}\right\} \to g, h \ \text{ in } \ L^2(rdr) \tag{14}$$

Set $\tilde{h} = hr$. Since

$$\int_0^1 \left(\frac{v_n}{r} - h\right)^2 rdr \ \to 0 \tag{15}$$

one has

$$v_n \to rh \ \text{ in } \ L^2\big(0,1;\frac{dr}{r}\big)\ . \tag{16}$$

but also in $\mathscr{D}'(0,1)$ so that

$$(v_n)_r \to \tilde{h}_r \ \text{ in } \ \mathscr{D}'\ . \tag{17}$$

We deduce from (15) that $\tilde{h}_r = g$ and thus $\tilde{h} \in V$ and since

$$(v_n)_r, \frac{v_n}{r} \to \tilde{h}_r, \frac{\widetilde{h}}{r} \text{ in } L^2(rdr) \tag{18}$$

one has $v_n \to \tilde{h} \in V$. This completes the proof of Lemma **3.1**. □

Lemma 3.2.

$$V \subset \{v \in C([0,1]) \mid v(0) = 0\}$$

Proof. For $x, y \in (0,1]$ one has

$$|xv(x) - yv(y)| = \left| \int_x^y (rv)_r dr \right| = \left| \int_x^y r(v_r + \frac{v}{r}) dr \right| \leq \int_x^y r(|v_r| + \frac{|v|}{r}) dr \, . \tag{19}$$

Using the Cauchy-Young inequality $a \leq \frac{1}{2}a^2 + \frac{1}{2}$ one gets

$$|xv(x) - yv(y)| \leq \int_x^y \left\{ \frac{r}{2} \left({v_r}^2 + \frac{v^2}{r^2} \right) + r \right\} dr \to 0 \text{ when } y \to x. \tag{20}$$

It follows that v is continuous at any point where $r \neq 0$ on $[0,1]$. Now, one has also

$$\begin{aligned} v(x)^2 - v(y)^2 &= \int_x^y \frac{d}{dr} v(r)^2 dr = \int_x^y 2 v_r \, v \, dr = \int_x^y 2\sqrt{r} \, v_r \frac{v}{\sqrt{r}} dr \\ &\leq \int_x^y \left\{ r({v_r}^2 + \frac{v^2}{r^2}) \right\} \, dr \leq \varepsilon \end{aligned} \tag{21}$$

for x, y small enough (we used again the Cauchy-Young inequality). Thus, when $x \to 0$, $v(x)^2$ is a Cauchy sequence and there exist $l \geq 0$ such that

$$\lim_{x \to 0} v(x)^2 = l. \tag{22}$$

If $l > 0$ one has for ε small enough

$$\|v\|^2 \geq \int_0^1 \frac{v^2}{r} dr \geq \int_{\varepsilon^2}^{\varepsilon} \left(\frac{l}{2} \right)^2 \frac{dr}{r} = \frac{l^2}{4} (\ln \varepsilon - 2 \ln \varepsilon) = -\frac{l^2}{4} \ln \varepsilon \tag{23}$$

and a contradiction when $\varepsilon \to 0$. Thus, $l = 0$ and this completes the proof of the Lemma **3.2**. □

Remark Since $V \subset H^1(\varepsilon, 1)$, it follows that $V \subset C^{1/2}(\varepsilon, 1)$ for every ε.

One sets

$$E(h) = \pi \int_0^1 \left\{ h_r^2 + (\frac{\sin h}{r})^2 - \frac{\mu}{2} (\sin 2h)^2 \right\} rdr. \tag{24}$$

One would like to show that $E(h)$ possesses a minimiser on V for any μ.

Lemma 3.3. *The energy $E(h)$ is bounded from below on V and one can find a minimising sequence v_n such that*

$$0 \leq v_n \leq \frac{\pi}{2} \ . \tag{25}$$

Proof. One has clearly for every $h \in V$

$$E(h) \geq -\pi \frac{|\mu|}{2} \int_0^1 r dr = -\pi \frac{|\mu|}{4} \ . \tag{26}$$

Thus

$$I = \inf_{h \in V} E(h)$$

exists. Let us denote by v_n a sequence such that

$$E(v_n) \to I \ .$$

If $v_n \in V$, then also $|v_n| \in V$ and one has

$$E(v_n) = E(|v_n|)$$

so, without loss of generality, we assume $v_n \geq 0$.

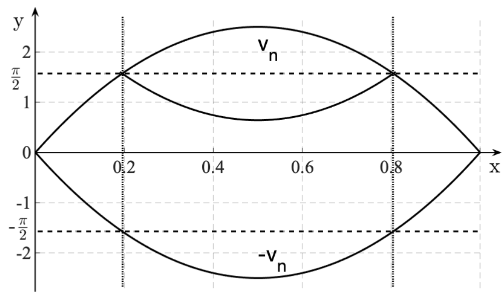

FIGURE 1. Graphical representation.

Then on $v_n > \frac{\pi}{2}$, we replace v_n by $-v_n + \pi$ (cfr. Fig. 1). It is clear that

$$\tilde{v}_n = v_n X_{\{v_n \leq \frac{\pi}{2}\}} + (-v_n + \pi) X_{\{v_n > \frac{\pi}{2}\}} \tag{27}$$

satisfies $\tilde{v}_n \in V$ and

$$E(\tilde{v}_n) = E(v_n).$$

This completes the proof of the Lemma **3.3**. □

Remark 3.4. *It could be that $-v_n + \pi$ achieves negative values, but clearly, after a finite number of operations like the one we just did we get a v_n satisfying* (25).

Lemma 3.5. *There exists a minimiser $\tilde{h}$ of E in V satisfying*

$$0 \le \tilde{h} \le \frac{\pi}{2} \, . \tag{28}$$

Proof. We consider the sequence $\{v_n\}$ constructed in Lemma **3.3**. We claim that $\{v_n\}$ is bounded in V independently of n. Indeed, one has, since for some constant $\lambda > 0$ one has $\left(\dfrac{\sin x}{x}\right)^2 \ge \lambda$, $\forall x \in [0, \frac{\pi}{2}]$,

$$\begin{aligned}\int_0^1 r\left\{(v_n)_r{}^2 + \frac{v_n{}^2}{r^2}\right\} dr &\le \int_0^1 r\left\{(v_n)_r{}^2 + \frac{1}{\lambda}\left(\frac{\sin v_n}{r}\right)^2\right\} dr \\ &\le \left(1 \vee \frac{1}{\lambda}\right)\int_0^1 r\left\{(v_n)_r{}^2 + \left(\frac{\sin v_n}{r}\right)^2\right\} dr \le C\end{aligned} \tag{29}$$

where C is a constant independent of n and $\vee$ denotes the maximum of two numbers. Recall that since v_n is a minimising sequence one has, for n large enough,

$$E(v_n) \le E(0) = 0$$

i.e., see the definition of E

$$\pi \int_0^1 \left\{(v_n)_r{}^2 + \left(\frac{\sin v_n}{r}\right)^2\right\} rdr \le \pi\frac{|\mu|}{2}\int_0^1 \sin^2(2v_n) rdr \le \pi\frac{|\mu|}{4} \, . \tag{30}$$

Since $\{(v_n)_r\}, \{\dfrac{v_n}{r}\}$ are bounded in $L^2(rdr)$ one finds a subsequence, still labelled by n, such that

$$\frac{v_n}{r} \rightharpoonup h \;\; , \;\; (v_n)_r \rightharpoonup g \text{ in } L^2(rdr) \, .$$

Set $\tilde{h} = hr$. The first weak convergence above reads

$$\int_0^1 \frac{v_n}{r}\Psi rdr \to \int_0^1 h\Psi rdr \quad , \quad \forall \Psi \in L^2(rdr).$$

In particular, taking $\Psi \in \mathscr{D}(0,1)$ one see that

$$v_n \to \tilde{h} = hr \text{ in } \mathscr{D}'(0,1)$$

and thus, by the continuity of the derivative in $\mathscr{D}'$

$$(v_n)_r \to \tilde{h}_r = g \;\text{ in }\; \mathscr{D}'(0,1) \, .$$

Thus, we have $\tilde{h} \in V$. For any $k \ge 2$ one has also, thank to (29), that v_n is bounded in $H^1\left(\dfrac{1}{k}, 1\right)$. Thus, by induction, one can find a subsequence $\{n_k\}$ extracted from $\{n_{k-1}\}$ such that

$$v_{n_k} \to \tilde{h} \text{ in } L^2\left(\frac{1}{k}, 1\right) \text{ and a. e..}$$

Then clearly

$$v_{n_k} \to \tilde{h} \;\; \text{a.e. on } (0,1).$$

By the dominated Lebesgue theorem one has then that

$$r\sin 2v_{n_k} \to r\sin 2\tilde{h} \quad \text{in } L^2(0,1)$$
$$\sin v_{n_k} \to \sin\tilde{h} \quad \text{a.e. on } (0,1)\ .$$

Then, since $x \mapsto x^2$ is convex by the Fatou lemma one has

$$\begin{aligned}
I = \underline{\lim}\, E(v_{n_k}) = \pi\, \underline{\lim} \int_0^1 \left\{ (v_{n_k})_r{}^2 + \left(\frac{\sin v_{n_k}}{r}\right)^2 \right\} rdr - \pi \int_0^1 \frac{\mu}{2} (\sin(2v_{n_k}))^2 rdr \geq \\
\geq \pi\ \underline{\lim} \int_0^1 (v_{n_n})_r{}^2 rdr + \pi\ \underline{\lim} \int_0^1 \left(\frac{\sin v_{n_k}}{r}\right)^2 rdr - \frac{\pi\mu}{2} \int_0^1 (\sin(2\tilde{h}))^2 rdr \geq \\
\geq \int_0^1 (\tilde{h}_r)^2 rdr + \pi \int_0^1 \underline{\lim} \left(\frac{\sin v_n}{r}\right)^2 rdr - \frac{\pi\mu}{2} \int_0^1 (\sin(2\tilde{h}))^2 rdr = \\
= E(\tilde{h}) = I\ .
\end{aligned} \tag{31}$$

This shows that $\tilde{h}$ is the minimiser that we are looking for. □

Lemma 3.6. *The Euler equation of the minimising problem is given by*

$$\begin{cases} -h_{rr} - \dfrac{h_r}{r} + \dfrac{\sin 2h}{r^2} = \mu \sin 2h \cos 2h \quad \textit{in } (0,1) \\ h(0) = h_r(1) = 0 \end{cases} \tag{32}$$

Proof. If h is a minimiser of E on V one has

$$\frac{d}{d\lambda} E(h+\lambda v)|_0 = 0 \quad , \quad \forall v \in V$$

Since

$$E(h+\lambda v) = \pi \int_0^1 \left\{ (h+\lambda v)_r^2 + \frac{\sin(h+\lambda v)^2}{r^2} - \frac{\mu}{2} \sin(2(h+\lambda v))^2 \right\} rdr\ . \tag{33}$$

One gets $\forall v$

$$\begin{aligned}
\int_0^1 \left\{ 2h_r v_r + 2\frac{\sin h \cos h}{r^2} v - 2\mu \sin(2h)\cos(2h)\, v \right\} rdr = 0 \\
\Longleftrightarrow \int_0^1 \left\{ h_r v_r + \frac{\sin(2h)}{2r^2} v - \mu \sin(2h)\cos(2h)\, v \right\} rdr = 0 \quad , \quad \forall v \in V\ .
\end{aligned} \tag{34}$$

Thus, in the distributional sense

$$
\begin{aligned}
&-(rh_r)_r+\frac{\sin(2h)}{2r}-\mu\, r\sin(2h)\cos(2h)=0\\
&\Longrightarrow -rh_{rr}-h_r+\frac{\sin(2h)}{2r}-\mu\, r\sin(2h)\cos(2h)=0\ .
\end{aligned}
\tag{35}
$$

Dividing by r we get the first equation of (32). Integrating by parts in (34) and using (35) we get

$$
\int_0^1 (rh_rv)_r-(rh_r)_rv+\frac{\sin 2hv}{2r}-\mu\sin(2h)\cos 2hrv=0 \quad , \quad \forall v\in V
$$

i.e.,

$$
\int_0^1 (rh_rv)_r=0 \quad , \quad \forall v\in V
$$

which gives

$$
h_r(1)=0\ .
$$

(in a weak sense) $h(0)=0$ follows from $h\in V$. This completes the proof of the Lemma. □

Lemma 3.7. *If $h\neq 0$ is a nonnegative minimiser of E on V then $h>0$ on $(0,1)$.*

Proof. Indeed, if h vanishes at $r_0\in(0,1)$ then, since h is smooth and r_0 is a minimum for h, one would have

$$
h(r_0)=h_r(r_0)=0
$$

then from the theory of o.d.e's (see [1]), $h\equiv 0$. □

Lemma 3.8. *If h is a positive minimiser of E then $0<h\leq\frac{\pi}{2}$.*

Proof. If not then h constructed as in the figure before (Fig.1) is a minimiser but it has a jump in the derivative unless this one is 0. But then $h=\frac{\pi}{2}$ is solution of the o.d.e. on $h>\frac{\pi}{2}$ and a contradiction follows. Note that the solution of the elliptic equation (14) is smooth on $(0,1)$. □

Lemma 3.9. *A minimiser cannot vanish on $(0,1)$ unless it vanishes identically.*

Proof. If h is a minimiser, $|h|$ is also a minimiser. But, then, $|h|$ would have a jump discontinuity in its derivative unless when it vanishes so does h_r. This implies (theory of o.d.e's), $h=0$. □

Lemma 3.10. *If $h\in V$ then $\sin(kh)\in V$, $\forall k\in\mathbb{R}$.*

Proof. One has

$$
\sin(kh)_r=kh_r\cos(kh) \quad , \quad |\sin(kh)|\leq|kh|. \tag{36}
$$

Therefore one has

$$
\begin{aligned}
\|\sin(kh)\|^2 &= \int_0^1 \left\{ \sin(kh)_r^2 + \left(\frac{\sin(kh)}{r}\right)^2 \right\} r dr \\
&\le \int_0^1 \left\{ k^2 \cos(kh)^2 h_r^2 + k^2 \frac{h^2}{r^2} \right\} r dr \le k^2 \|h\|^2
\end{aligned}
\tag{37}
$$

□

It easy to check that $h \equiv 0$ solves (9), (10) and hence it is a stationary point of the functional (11).

Let γ_0 be the first eigenvalue of the problem

$$
\begin{cases}
-\phi_{rr} - \dfrac{\phi_r}{r} + \dfrac{\phi}{r^2} = \gamma\phi \\
\phi(0) = 0, \quad \phi_r(1) = 0 \; .
\end{cases}
\tag{38}
$$

Lemma 3.11.

$$\gamma_0 > 1 \; .$$

Proof. Suppose not, *i.e.*, $\gamma_0 \le 1$. Let ϕ be the corresponding positive (or nonnegative) eigenfunction. One has

$$
\begin{aligned}
&-\phi_{rr} - \frac{\phi_r}{r} = \phi(\gamma_0 - \frac{1}{r^2}) \le 0 \quad \text{since } r \in (0,1) \\
&(r\phi_r)_r \ge 0 \Longrightarrow r\phi_r \nearrow \implies r\phi_r \le 0 \;\text{ since }\; \phi_r(1) = 0 \; .
\end{aligned}
\tag{39}
$$

Thus, the maximum of ϕ is achieved at 0 but, since $\phi(0) = 0$, we get a contradiction *i.e.*, $\phi \equiv 0$. □

We have the following bifurcation lemma.

Lemma 3.12. *If $\mu \le \gamma_0/2$ we have $E(h) \ge 0$ and the global minimum is attained only for $h \equiv 0$. For $\mu > \gamma_0/2$ the global minimum is negative.*

Proof. The first equation of (38) can also be written after a multiplication by r as

$$-(r\phi_r)_r + \frac{\phi}{r} = \gamma\phi r.$$

Multiplying by ϕ and integrating over $(0,1)$ we derive by definition of γ_0 that

$$\int_0^1 \left(\phi_r^2 + \frac{\phi^2}{r^2}\right) r dr \ge \gamma_0 \int_0^1 \phi^2 r dr \;\; \forall\phi \;\text{ with }\; \phi(0) = 0, \quad \phi_r(1) = 0. \tag{40}$$

We divide the proof in two parts:

(i) $\quad \mu \le \gamma_0/2$

In this case we have (using (40) with $\phi = \sin h$)

$$
\begin{aligned}
E(h) &= \pi \int_0^1 \left[(\cos h)^2 h_r^2 + \left(\frac{\sin h}{r} \right)^2 - 2\mu (\sin h)^2 (\cos h)^2 + (1 - (\cos h)^2) h_r^2 \right] r dr \\
&\geq \int_0^1 \left[\gamma_0 (\sin h)^2 - 2\mu (\sin h)^2 (\cos h)^2 + (1 - (\cos h)^2) h_r^2 \right] r dr \\
&= \int_0^1 \left[(\gamma_0 - 2\mu)) (\sin h)^2 + (1 - (\cos h)^2)(2\mu (\sin h)^2 + h_r^2) \right] r dr \\
&\geq 0 = E(0)
\end{aligned}
$$

the equality taking place only for $h = 0$.

$(ii\,)$ $\quad \mu > \gamma_0/2$

Let us denote by ϕ_0 the first positive normalised eigenfuntion to (38). One has, for $\varepsilon > 0$:

$$
\begin{aligned}
E(\varepsilon\phi_0) &\leq \pi \int_0^1 \left[(\varepsilon\phi_0)_r^2 + \frac{(\varepsilon\phi_0)^2}{r^2} - \frac{\mu}{2} (\sin(2\varepsilon\phi_0))^2 \right] r dr \\
&= \pi \int_0^1 \left[\gamma_0 (\varepsilon\phi_0)^2 - \frac{\mu}{2} (\sin(2\varepsilon\phi_0))^2 \right] r dr.
\end{aligned}
$$

Using with $x = 2\varepsilon\phi_0$ the formula

$$
\sin x = x - \int_0^1 (1 - \cos(tx)) x dt
$$

$E(\varepsilon\phi_0)$ can be written as

$$
\begin{aligned}
E(\varepsilon\phi_0) &= \pi \int_0^1 \left[(\varepsilon\phi_0)^2 \left\{ \gamma_0 - 2\mu (1 - \int_0^1 (1 - \cos(2t\varepsilon\phi_0)) dt)^2 \right\} \right] r dr \\
&< 0 = E(0)
\end{aligned}
$$

for ε small, since

$$
\int_0^1 (1 - \cos(2t\varepsilon\phi_0)) dt \to 0
$$

when $\varepsilon \to 0$. $\square$

Alternative proof of (ii)

Suppose $h \neq 0$ is a minimiser of E one has

$$
E(h) < E(0) \tag{41}
$$

i.e.,

$$\int_0^1 \left\{ h_r^2 + \left(\frac{\sin h}{r}\right)^2 \right\} rdr < \frac{\mu}{2} \int_0^1 (\sin 2h)^2\, rdr = 2\mu \int_0^1 \sin h^2 \cos h^2 rdr$$

$$\Longrightarrow \int_0^1 \left\{ \cos h^2 h_r^2 + \left(\lambda \frac{\sin h}{r}\right)^2 \right\} rdr < \gamma_0 \int_0^1 \sin h^2 rdr \tag{42}$$

$$\Longrightarrow \gamma_0 > \frac{\int_0^1 \left\{ \sin h_r^2 + \left(\frac{\sin h}{r}\right)^2 \right\} rdr}{\int_0^1 \sin h^2 rdr}$$

and a contradiction since $\sin h \in V$ with the definition of γ_0.

Consider the problem

$$\begin{cases} -h_{rr} - \dfrac{h_r}{r} + \dfrac{\sin 2h}{r^2} = \mu \sin 2h \cos 2h \quad \text{on } (0,1) \\ h(0) = h_r(1) = 0 \end{cases} \tag{43}$$

Lemma 3.13. *If $\mu \leq \gamma_0/2$ the only solution of (43) such that $h \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $h \equiv 0$.*

Proof. Recall that for any $\phi \in V$ one has by definition of γ_0

$$\gamma_0 \int_0^1 \phi^2 rdr \leq \int_0^1 \left(\phi_r^2 + \frac{\phi^2}{r^2}\right) rdr \ . \tag{44}$$

Let us write the equation (43) as

$$(rh_r)_r + \frac{\sin 2h}{2r} = \mu \sin 2h \cos 2h\, r\ . \tag{45}$$

Multiply both sides by $\sin 2h$ and integrate on $(0,1)$. It comes

$$\int_0^1 r \left\{ h_r (\sin 2h)_r + \frac{(\sin 2h)^2}{2r^2} \right\} dr = \mu \int_0^1 (\sin 2h)^2 \cos 2h\ rdr\ . \tag{46}$$

One has

$$(\sin 2h)_r = 2\cos 2h\ h_r \Longleftrightarrow h_r = \frac{(\sin 2h)_r}{2\cos 2h}\ . \tag{47}$$

Thus, the equation above becomes

$$\int_0^1 r \left\{ (\sin 2h)_r^2 \frac{1}{\cos 2h} + \frac{(\sin 2h)^2}{r^2} \right\} dr = 2\mu \int_0^1 (\sin 2h)^2 \cos 2h rdr\ . \tag{48}$$

Suppose that h is such that $h \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then since $-1 \leq \cos 2h \leq 1$ one gets

$$\int_0^1 r\left\{(\sin 2h)_r^2 + \frac{(\sin 2h)^2}{r^2}\right\} dr < \gamma_0 \int_0^1 (\sin 2h)^2 r dr \tag{49}$$

i.e, $\sin 2h \in V$ and satisfies an inequality contradicting (44), except if $h \equiv 0$. □

Each minimiser of $E(h)$ solves the problem (9), (10). For the solutions of this problem we can give the following existence result around the bifurcation point.

Lemma 3.14. *There exist two positive numbers ρ_0 and δ_0 such that, the problem (9), (10) does not have non-zero solutions for $\mu \in (\gamma_0/2 - \delta_0, \gamma_0/2]$ and $\|h\|_0 \leq \rho_0$. The problem has exactly two solutions h_1 and $h_2 = -h_1$ in the sphere $\|h\|_0 \leq \rho_0$ for $\mu \in (\gamma_0/2, \gamma_0/2 + \delta_0)$.*

Proof. The proof follows from Krasnosel'skii (1964, Theorem 6.12). Indeed the equation (10) can be written in the form

$$2\mu h = L(h,r) + C(h,r,\mu) + D(h,r,\mu) \tag{50}$$

where L is the linear operator

$$L(h,r) = -h_{rr} - \frac{h_r}{r} + \frac{h}{r^2}$$

and C, D are given by

$$C(h,r,\mu) = -\frac{2}{3}\frac{h^3}{r^2} + \frac{16}{3}\mu h^3,$$

$$D(h,r,\mu) = -\left(\frac{2h}{2r^2} - \frac{\sin 2h}{2r^2}\right) + \frac{2}{3}\frac{h^3}{r^2} + \frac{\mu}{2}(4h - \sin 4h) - \frac{16}{3}\mu h^3.$$

It is easy to check that

$$C(th,r,\mu) = t^3 C(h,r,\mu), \quad (-\infty < t < \infty) \tag{51}$$

and

$$\|D(h,r,\mu)\|_0 = o(\|h\|^3). \tag{52}$$

Moreover we have

$$((C(\phi^0,r,\mu),\phi^0))_0 = \int_0^1 \left[-\frac{2}{3}\frac{(\phi^0)^4}{r} + \frac{16}{3}\mu(\phi^0)^4 r\right] dr > 0, \quad \text{for } \mu \geq \frac{\gamma_0}{8}. \tag{53}$$

Indeed from $((L(\phi^0,r) - \gamma_0\phi^0, (\phi^0)^3))_0 = 0$ it follows that

$$\int_0^1 \left[-\frac{d}{dr}(\phi_r^0 r)(\phi^0)^3 + \frac{(\phi^0)^4}{r} - \gamma(\phi^0)^4 r\right] dr = 0 \tag{54}$$

The latter, on by parts integration

$$-(\phi_r^0 r)(\phi^0)^3\Big|_0^1 + \int_0^1 3(\phi_r^0)^2(\phi^0)^2 r dr + \int_0^1 \left[\frac{(\phi^0)^4}{r} - \gamma_0(\phi^0)^4 r\right] dr = 0$$

that is
$$\int_0^1 \frac{(\phi^0)^4}{r} - \gamma_0(\phi^0)^4 r\,dr \leq 0,$$
and the inequality (53) can be easily derived. The statements (50)- (53), together to the local Lipschitz condition on the operators C and D, assure (see Krasnosel'skii 1964) the existence of exactly two branch of non-zero solutions bifurcating from the point $\gamma_0/2$. Finally, we remark that the existence of two opposite branch follows from the odd functions in (50). □

Remark 3.15. *In order to establish the stability of the solutions to (9), (10) around the point $\mu_0 = \gamma_0/2$, we perform a qualitative analysis of the bifurcation equation to the lowest order (see equation (55) below). From (50) setting*
$$G(h,r,\mu) = -2\mu h + L(h,r) + C(h,r,\mu) + D(h,r,\mu) = 0$$
and
$$2\mu = \gamma_0 + \delta, \qquad |\delta| << 1,$$
assuming that each element $h \in \mathscr{H}^1(0,1)$ has the unique representation
$$h = \beta\phi^0 + Ph, \qquad (Ph,\phi^0)_0 = 0, \quad \beta \in \mathbb{R},$$
we have
$$(G(h,r,\mu),\phi^0)_0 = -\delta\beta + (C(\beta\phi^0,r,\mu),\phi^0)_0 + \dots$$
Moreover from (51), (53) we can get to the simple l.o. bifurcation equation, namely
$$-\delta\beta + \beta^3\bar{C} = 0, \qquad \bar{C} = (C(\phi^0,r,\mu),\phi^0)_0 \geq 0. \tag{55}$$

It is easy to check that:

for $\delta \leq 0$ there is the only solution $\beta = 0$ and this solution is stable (indeed in this case: $-\delta + 3\beta^2\bar{C} \geq 0$);

for $\delta > 0$ the trivial solution is no more stable but other two stable solutions appear, i.e., $\beta = \pm\sqrt{\delta/\bar{C}}$.

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