Neighboring Optimal Guidance and Attitude Control of Low-Thrust Earth Orbit Transfers

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ABSTRACT

Recently, low-thrust propulsion is gaining a strong interest by the research community, and already found application in some mission scenarios. This paper proposes an integrated guidance and control methodology, termed VTD-NOG & PD-RM, and applies it to orbit transfers from a low Earth orbit (LEO) to a geostationary orbit (GEO), using low-thrust. The variable time-domain neighboring optimal guidance (VTD-NOG) is a closed-loop guidance approach based on minimization of the second differential of the objective functional along the perturbed path, and avoids the singularities that occur using alternate neighboring optimal guidance algorithms. VTD-NOG finds the trajectory corrections considering the thrust direction as the control input. A proportional-derivative scheme based on rotation matrices (PD-RM) is used in order to drive the actual thrust direction toward the desired one determined by VTD-NOG. Reaction wheels are tailored to actuate attitude control. In the numerical simulations, thrust magnitude oscillations, displaced initial conditions, and gravitational perturbations are modeled. Extensive Monte Carlo campaigns show that orbit insertion at GEO occurs with excellent accuracy, thus proving that VTD-NOG & PD-RM represents an effective architecture for guidance and control of low-thrust Earth orbit transfers.

INTRODUCTION

Low-thrust propulsion has recently been established as a valuable option for a variety of mission scenarios, spanning from interplanetary missions to Earth orbit transfers. Low-thrust systems have very high specific impulses, much larger than those available using chemical propulsion. This circumstance implies that low-thrust propulsion can usually outperform high-thrust engines with regard to propellant mass requirements. Optimization of low-thrust orbit transfers is aimed at minimizing the propellant mass, and leads to identifying the nominal trajectory associated with the mission specifications. However, in practical scenarios, the space vehicle is subject to perturbations, related either to unpredictable (environmental) phenomena or to imperfect modeling of the space vehicle. As a result, driving a spacecraft toward the desired final conditions requires the identification of the corrective maneuvers aimed at compensating the displacements due to perturbations, while minimizing the propellant needed to perform these corrective actions.

The present study aims at describing and applying a guidance and control methodology, capable of driving the spacecraft along a perturbed trajectory sufficiently close to the nominal path, which is assumed to be optimal. Specifically, the minimum-time transfer from a low-altitude Earth orbit (LEO) to a geostationary orbit (GEO) found in Pontani 2018 is selected as the nominal path.

Driving the space vehicle in the proximity of the optimal trajectory in nonnominal flight conditions requires defining the feedback corrective actions aimed at compensating the perturbations, on the basis of the displaced state, evaluated at specified sampling times. Two major classes of guidance schemes exist. Explicit algorithms redefine the transfer path (leading to the desired final conditions) at each guidance interval (cf. Teofilatto and De Pasquale 1999, Calise et al. 1998, and Lu et al. 2003). Implicit schemes evaluate the deviations from a specified nominal path, and identify the feedback control actions aimed at maintaining the vehicle in the neighborhood of the nominal path (cf. Hull 2003, Lu 1991, and Townsend et al. 1968). Neighboring Optimal Guidance (NOG) represents an implicit guidance algorithm based on the second-order optimality conditions. A few researches have been devoted to investigating neighboring optimal (cf. Kugelmann and Pesch 1990, Afshari et al. 2009, Seywald and Cliff 1994, Yan et al. 2002, Naidu et al. 1993, Hull and Helfrich 1991), and a usual difficulty consisted in the fact that the gain matrices, which play a crucial role in the guidance scheme, become singular while approaching the final time.

This research is focused on a unified guidance and control architecture for low-thrust Earth orbit transfers, based on the iterated, joint use of two techniques: (i) the variable-time-domain neighboring optimal guidance (VTD-NOG), and (ii) a proportional-derivative algorithm that uses rotation matrices (PD-RM) for attitude control. The adoption of a normalized time scale represents a major feature of VTD-NOG (cf. Pontani et al. 2015a and Pontani et al. 2015b), and leads to avoiding the singularities that affect the numerical performance of alternative NOG schemes. Moreover, the updating formula for the flight time and the guidance ending criterion are derived in a way that is consistent with the optimality conditions. VTD-NOG determines the corrective control actions by considering the thrust direction as the control input. Because the thrust has fixed direction in the spacecraft body axes, the actual spacecraft orientation must be modified so that the actual thrust direction is driven toward the desired one determined by VTD-NOG, and this is the specific objective of the attitude control system. In this research, reaction wheels are assumed as the attitude actuators. This technological solution is often employed onboard spacecraft equipped with low-thrust propulsion systems (Berge et al. 2009, Garulli et al. 2011). The attitude control law proposed in this study is proportional-derivative-like and uses the rotation matrices (PD-RM), for the purpose of avoiding singularities and sign ambiguities inherent to other representations. Alternative combinations of VTD-NOG and different types of attitude control were implemented in Pontani and Celani 2018a and Pontani and Celani 2019.

This work employs VTD-NOG & PD-RM for guidance and control of the low-thrust orbit transfer starting from a low Earth orbit (LEO) and ending at insertion into a coplanar geostationary orbit (GEO). Nonnominal flight conditions are considered, related to (i) gravitational perturbations, (ii) unpredictable oscillations of the propulsive thrust magnitude, and (iii) errors on the initial conditions. Extensive Monte Carlo campaigns are performed, with the objective of proving effectiveness and efficiency (in terms of propellant budget) of VTD-NOG & PD-RM for low-thrust Earth orbit transfers, in the presence of perturbed flight conditions. A preliminary version of the present work can be found in Pontani and Celani 2018b. Several remarkable novelties are introduced in this research with respect to former publications on a similar subject (Pontani and Celani 2018a and Pontani and Celani 2019). First, gravitational perturbations are modeled, i.e. those due to a relevant number of harmonics of the geopotential, as well as the attraction of Sun and Moon as third bodies. In fact, while in previous studies (Pontani and Celani 2018a and Pontani and Celani 2019) the time of flight was relatively short, low-thrust orbit transfers have considerable durations, therefore the previously mentioned gravitational perturbations yield nonnegligible effects. Second, the control algorithm considers different actuation modality and devices (i.e. reaction wheels instead of thrust vectoring), as well as a different representation for orientation, i.e. rotation matrices (instead of quaternions). The latter choice is related to a non-ambiguous representation of the commanded spacecraft orientation, and is accompanied by an effective attitude control law that employs directly the rotation matrices. Lastly, because the flight time is long for the orbit transfer studied in this work, a non-uniform sampling time for feedback guidance and control is adopted. This is proposed as an effective approach with the potential of joining computational efficiency and accuracy at orbit injection.

NOMINAL LEO-GEO ORBIT TRANSFER

This research is focused on the problem of transferring a space vehicle from an equatorial low Earth circular orbit (LEO) to a final, coplanar geostationary orbit (GEO), in the presence of perturbed flight conditions. The initial altitude equals 400 km. Both trajectory and attitude dynamics are considered. This section addresses the definition of the nominal transfer path, and the space vehicle is modeled as a point mass. In the succeeding sections, attitude dynamics is introduced.

Continuous low-thrust propulsion is used to complete the orbit transfer of interest. Under the assumption of constant, continuous thrust, if c and n_0 represent the (constant) effective exhaust velocity of the propulsion system and the thrust acceleration at the initial time, the instantaneous thrust acceleration $\left(T/\tilde{m}\right)$ is given by

$$\frac{T}{\tilde{m}} = \frac{n_0 c}{c - n_0 t} \tag{1}$$

where t denotes time. The following (nominal) values are assumed: $n_0 = 0.0001 g_0$ and $c = 30\ \text{km/sec}$ $\left(g_0 = 9.8\ \text{m/sec}^2\right)$.

Formulation of the problem

The spacecraft trajectory is described in the Earth-centered inertial frame (ECI), defined through the right-hand triad of unit vectors $\left(\hat{c}_1,\hat{c}_2,\hat{c}_3\right)$, where $\left(\hat{c}_1,\hat{c}_2\right)$ corresponds to the equatorial plane, $\hat{c}_1$ is the vernal axis, and $\hat{c}_3$ is directed toward the Earth rotation axis (cf. Figure 1(a)). The time-dependent position is associated with the radius *r*, the latitude ϕ, and the absolute longitude ξ, depicted in Figure 1(a). The velocity is described in terms of components in the rotating frame $\left(\hat{r},\hat{t},\hat{n}\right)$, where $\hat{r}$ points toward the position vector $\boldsymbol{r}$ and $\hat{t}$ is parallel to the $\left(\hat{c}_1,\hat{c}_2\right)$-plane (cf. Figure 1(a)). Inspection of Figure 1(a) leads to

$$\begin{bmatrix}\hat{r} & \hat{t} & \hat{n}\end{bmatrix}^T = \mathbf{R}_2\left(-\phi\right)\mathbf{R}_3\left(\xi\right)\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}^T \tag{2}$$

where $\mathbf{R}_j\left(\eta\right)$ is a counterclockwise elementary rotation about axis *j* by (the generic) angle η. The symbols $\left(v_r,v_t,v_n\right)$ denote the components of the velocity and are referred to as radial, transverse, and normal component. The state vector $\boldsymbol{x}$ (with components x_k $\left(k=1,\ldots,6\right)$) of the space vehicle is given by $\boldsymbol{x}:=\begin{bmatrix}r & \xi & \phi & v_r & v_t & v_n\end{bmatrix}^T$. The thrust direction represents the control, and is defined by the out-of-plane angle β and the in-plane angle α, both portrayed in Figure 1(b) (in which $\hat{T}$ points toward the thrust direction). Hence, the control vector $\boldsymbol{u}$ is $\boldsymbol{u}:=\begin{bmatrix}u_1 & u_2\end{bmatrix}^T=\begin{bmatrix}\alpha & \beta\end{bmatrix}^T$. The motion equations, also termed state equations henceforth, are

$$\frac{dr}{dt}=v_r \qquad \frac{d\xi}{dt}=\frac{v_t}{r\cos\phi} \qquad \frac{d\phi}{dt}=\frac{v_n}{r} \tag{3}$$

$$\frac{dv_r}{dt}=-\frac{\mu}{r^2}+\frac{v_t^2+v_n^2}{r}+\frac{T}{\tilde{m}}\sin\alpha\cos\beta+a_r \tag{4}$$

$$\frac{dv_t}{dt}=\frac{v_t}{r}\left(v_n\tan\phi-v_r\right)+\frac{T}{\tilde{m}}\cos\alpha\cos\beta+a_t \tag{5}$$

$$\frac{dv_n}{dt}=-\frac{v_t^2}{r}\tan\phi-\frac{v_r v_n}{r}+\frac{T}{\tilde{m}}\sin\beta+a_n \tag{6}$$

where $\left(T/\tilde{m}\right)$ is given by Equation (1) and $\mu\ \left(=398600.4\ \text{km}^3/\text{sec}^2\right)$ is the terrestrial gravitational parameter. The symbols a_r, a_t, and a_n represent the acceleration components related to the presence of perturbations. In general, these terms have very limited magnitude and are functions of the state of the space vehicle in a complicated fashion. For this reason, perturbations are neglected while finding the optimal trajectory, whereas they are being considered while applying VTD-NOG & PD-RM. Equations (3)-(6) (with $a_r = a_t = a_n = 0$) can be written as

$$\frac{d\boldsymbol{x}}{dt} = \tilde{\boldsymbol{f}}\left(\boldsymbol{x},\boldsymbol{u},t\right) \tag{7}$$

The terminal conditions (at the initial and final time, denoted respectively with subscripts "0" and "f") are

$$r_0 = R_{LEO} \quad \xi_0 = \xi_i \quad \phi_0 = 0 \quad v_{r0} = 0 \quad v_{t0} = \sqrt{\frac{\mu}{R_{LEO}}} \quad v_{n0} = 0 \tag{8}$$

$$r_f = R_{GEO} \quad \phi_f = 0 \quad v_{rf} = 0 \quad v_{tf} = \sqrt{\frac{\mu}{R_{GEO}}} \quad v_{nf} = 0 \tag{9}$$

where R_{LEO} and R_{GEO} are respectively the radii of the initial LEO and the final GEO, whereas ξ_i denotes the (prescribed) initial absolute longitude. The previous conditions (8)-(9) can be written in compact form as

$$\boldsymbol{\psi}\left(\boldsymbol{x}_0,\boldsymbol{x}_f,t_f\right) = \boldsymbol{0} \tag{10}$$

The problem under consideration can be formulated also by using the normalized time τ,

$$\tau := t/t_f \qquad \Rightarrow \qquad \tau_0 \equiv 0 \le \tau \le 1 \equiv \tau_f \tag{11}$$

If the dot denotes the derivative with respect to τ, Equation (7) is rewritten as

$$\dot{\boldsymbol{x}} = t_f\,\tilde{\boldsymbol{f}}\left(\boldsymbol{x},\boldsymbol{u},t_f\tau\right) =: \boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{a},\tau\right) \tag{12}$$

where $\boldsymbol{a}$ contains all the time-independent unknown quantities ($\boldsymbol{a} = t_f$ for the present problem).

Because continuous propulsion is employed, minimization of the propellant expenditure is achieved if the flight time $\left(t_f - t_0\right)$ is minimized. Therefore, letting $t_0 = 0$, the objective function J is

$$J = t_f \tag{13}$$

Optimal LEO-GEO orbit transfer

The analytical necessary conditions (cf. Hull 2003) for an optimal solution can be written after introducing a Hamiltonian H and a boundary condition function Φ,

$$H\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{a}\right):=\boldsymbol{\lambda}^T\boldsymbol{f} \quad \text{and} \quad \Phi\left(\boldsymbol{x}_0,\boldsymbol{x}_f,\boldsymbol{a}\right):=J+\boldsymbol{\upsilon}^T\boldsymbol{\psi} \tag{14}$$

where $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\upsilon}$ denote the adjoint vectors associated to the state equations (12) and to the conditions (10), respectively; their dimension is appropriate to the context and the respective components are $\lambda_k\left(t\right)$ υ_k . Specifically, the first-order (local) optimality conditions (cf. Hull 2003) include the costate (or adjoint) equations, together with the respective boundary conditions, as well as the Pontryagin minimum principle and the parameter condition (cf. Hull 2003). Their explicit expressions are omitted for the sake of brevity. It is worth mentioning that the Pontryagin minimum principle leads to writing the optimal control $\boldsymbol{u}^*$ as a function of the adjoint variables, whereas the parameter condition is proven to be equivalent to

$$\dot{\boldsymbol{\mu}}=-\left[\frac{\partial H}{\partial \boldsymbol{a}}\right]^T \quad \text{with} \quad \boldsymbol{\mu}_0=\boldsymbol{0} \quad \text{and} \quad \boldsymbol{\mu}_f-\left[\frac{\partial \Phi}{\partial \boldsymbol{a}}\right]^T=\boldsymbol{0} \tag{15}$$

where $\boldsymbol{\mu}$ is an auxiliary time-varying vector.

In the context of orbit transfer optimization, the gravitational geopotential is modeled as spherical. The spacecraft is not subject to any other external force, and the optimal transfer can be assumed to belong to the $\left(\hat{c}_1,\hat{c}_2\right)$-plane. In fact, any alternate three-dimensional path would imply an out-of-plane thrust component and a waste of propellant as a result. This means that the out-of-plane variables can be set to zero, i.e.

$$\phi=0 \qquad v_n=0 \qquad \lambda_3=0 \qquad \lambda_6=0 \qquad \beta=0 \tag{16}$$

Only the state equations for r, ξ, v_r, and v_t, in conjunction with the respective adjoint equations and the Pontryagin minimum principle for α, are needed in order to determine the minimum-time transfer. The remaining adjoint equations, accompanied by the respective boundary conditions, are identically satisfied. In addition, the state equation for x_2 $\left(\equiv\xi\right)$ is ignorable, as the absolute longitude x_2 does not appear in any final condition and is not contained in any right-hand-side the equations of motion.

The optimal transfer path is obtained in Pontani 2018 using the indirect heuristic method (cf. Pontani and Conway 2014 and Pontani and Conway 2015). The optimal time histories of the state variables r, v_r, and v_t are portrayed in Figures 2 through 4, whereas Figure 5 illustrates the optimal thrust direction; the total time of flight equals 50.33 days. The indirect heuristic method employs the first-order necessary conditions to identify the optimal solution. Nevertheless, the second-order sufficient conditions are also to be met in order to apply VTD-NOG using the optimal path as the reference, nominal solution. Evaluation of matrix $\hat{\mathbf{S}}$ (cf. Hull 2003) and the Hessian matrix $H_{\boldsymbol{uu}}$ along the optimal trajectory proves that the second-order sufficient conditions are fulfilled. This is the fundamental prerequisite for applying VTD-NOG.

ORBIT PERTURBATIONS

The spacecraft orbital motion is primarily subject to the gravitational attraction of Earth, therefore the perturbed two-body-problem model represents the appropriate dynamical framework for the study of the orbit transfer in nonnominal flight conditions. First, the real geopotential differs from that yielded by a spherical mass distribution. As a result, some meaningful harmonic terms of the Earth gravitational field must be considered in dynamical modeling. Second, the gravitational pull of Moon and Sun is a further contribution. This section is focused on describing and modeling these perturbations of a gravitational nature.

Earth gravitational harmonics

This study utilizes the EGM2008 model (cf. Pavlis et al. 2008), which provides the coefficients of zonal, sectorial, and tesseral harmonics of the geopotential up to order 2160. These coefficients ($J_{l,m}$ and λ_{lm}) appear in the expression of gravitational potentials of celestial bodies,

$$U=\frac{\mu}{r}-\frac{\mu}{r}\sum_{l=2}^{\infty}\left(\frac{R_E}{r}\right)^l J_l P_{l0}\left(\sin\phi\right)+\sum_{l=2}^{\infty}\sum_{m=1}^{l}\left(\frac{R_E}{r}\right)^l J_{l,m}P_{lm}\left(\sin\phi\right)\cos\left[m\left(\lambda_g-\lambda_{lm}\right)\right] \quad (17)$$

where the terms P_{lm} are Legendre polynomials, R_E is the Earth equatorial radius, whereas λ_g denotes the spacecraft geographical longitude. If θ_G represents the Greenwich sidereal time (taken counterclockwise from $\hat{c}_1$), then $\lambda_g = \xi - \theta_G$. The gravitational acceleration in the $(\hat{r},\hat{t},\hat{n})$-frame is

$$\boldsymbol{G} = \nabla U \quad \text{where} \quad \nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{t}}{r\cos\phi}\frac{\partial}{\partial \lambda_g} + \frac{\hat{n}}{r}\frac{\partial}{\partial \phi} \tag{18}$$

Equations (17) and (18) allow attaining the three components (G_r, G_t, G_n) in the rotating frame $(\hat{r},\hat{t},\hat{n})$. As G_r includes the main term of the gravitational acceleration, the disturbing contributions are $a_r^{(H)} = G_r + \mu/r^2$, $a_t^{(H)} = G_t$, and $a_n^{(H)} = G_n$. These three components contribute to the terms (a_r, a_t, a_n) in Equations (3)-(6).

Third body perturbation

Third body gravitational perturbations are related to the Moon and Sun gravitational pull. A third body yields an acceleration that can be conveniently written as

$$\boldsymbol{a}_{3B} = -\frac{\mu_3}{s_3^3\left(1+q_3\right)^{3/2}}\left[\boldsymbol{r} + \boldsymbol{s}_3 q_3 \frac{3+3q_3+q_3^2}{1+\left(1+q_3\right)^{3/2}}\right] \qquad \text{where} \qquad q_3 := \frac{r^2 - 2\boldsymbol{r}^T\boldsymbol{s}_3}{s_3^2} \tag{19}$$

The symbol μ_3 represents the gravitational parameter of the third body, $\boldsymbol{s}_3$ denotes its position vector with respect to the Earth, and $s_3 = |\boldsymbol{s}_3|$. Equation (19) employs the Battin-Giorgi approach (cf. Battin 1987 and Giorgi 1964) to the Encke's approach. Then, the components of $\boldsymbol{a}_{3B}$ along the $(\hat{r},\hat{t},\hat{n})$-frame must be obtained for their use in the equations of motion. The term $\boldsymbol{s}_3$ is written in the $(\hat{r},\hat{t},\hat{n})$-frame with this intent.

The Moon orbit about Earth is approximated as circular, therefore its position vector $\boldsymbol{r}_M$ can be written in the ECI-frame as a function of of Ω_M, i_M, and θ_M, i.e. the right ascension of the ascending node (RAAN), inclination, and (instantaneous) argument of latitude θ_M of the lunar orbit (Prussing and Conway 1993),

$$\boldsymbol{r}_M = r_M \begin{bmatrix} \cos\Omega_M \cos\theta_M - \sin\Omega_M \sin\theta_M \cos i_M \\ \sin\Omega_M \cos\theta_M + \cos\Omega_M \sin\theta_M \cos i_M \\ \sin\theta_M \sin i_M \end{bmatrix}^T \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} \tag{20}$$

where the (constant) Moon orbit radius r_M is approximated to 384400 km. The position vector of the Moon is $\boldsymbol{s}_3^{(M)} = \boldsymbol{r}_M$. The two angles Ω_M and i_M are time-varying, with a period of 18.6 years due to precession of $\boldsymbol{h}_M$. Combination of Equations (2) and (20) allows attaining the projections of $\boldsymbol{s}_3^{(M)}$ along $(\hat{r}, \hat{t}, \hat{n})$, and finally the components $\left(a_r^{(M)}, a_t^{(M)}, a_n^{(M)}\right)$.

As a further step, also the Earth motion about the Sun is described by using the two-body-problem model. The heliocentric inertial system (HCI) is aligned with the unit vectors $\left(\hat{c}_1^{(S)}, \hat{c}_2^{(S)}, \hat{c}_3^{(S)}\right)$, where $\hat{c}_1^{(S)}$ is the vernal axis (associated with the line where the Earth equatorial plane and the plane of ecliptic intersect) and $\hat{c}_3^{(S)}$ is directed toward the orbital angular momentum of Earth (cf. Prussing and Conway 1993). The ECI-frame is obtained from the HCI-frame through a single clockwise rotation about axis 1 by the ecliptic obliquity angle δ_E $(= 23.45 \text{ deg})$,

$$\begin{bmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_2 \end{bmatrix}^T = \mathbf{R}_1(-\delta_E) \begin{bmatrix} \hat{c}_1^{(S)} & \hat{c}_2^{(S)} & \hat{c}_3^{(S)} \end{bmatrix}^T \tag{21}$$

Under the assumption of approximating the Earth orbit as circular, its position vector $\boldsymbol{r}_E$ can be expressed in terms of Earth ecliptic longitude θ_E in the HCI-frame,

$$\boldsymbol{r}_E = r_E \begin{bmatrix} \cos\theta_E & \sin\theta_E & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_1^{(S)} & \hat{c}_2^{(S)} & \hat{c}_3^{(S)} \end{bmatrix}^T \tag{22}$$

where the (constant) radius of the Earth orbit, r_E, is set to 1 AU. The Sun position with respect to the Earth is $\boldsymbol{s}_3^{(S)} = -\boldsymbol{r}_E$. Combination of Equations (2), (21), and (22) allows attaining the projections of $\boldsymbol{s}_3^{(S)}$ along $(\hat{r}, \hat{t}, \hat{n})$, and, as a final step, the components $\left(a_r^{(S)}, a_t^{(S)}, a_n^{(S)}\right)$.

VARIABLE-TIME-DOMAIN NEIGHBORING OPTIMAL GUIDANCE

The Variable-Time-Domain Neighboring optimal guidance (VTD-NOG) is an implicit algorithm that employs the minimum-time path as the reference solution, for the purpose of attaining the control correction at each sampling time $\{t_k\}_{k=0,\dots,n_S}$ $(t_0 = 0)$. The state displacement along the actual trajectory (corresponding to $\boldsymbol{x}$) with respect to the nominal path (associated with $\boldsymbol{x}^*$) is evaluated at these sampling times,

$$d\boldsymbol{x}_k \equiv \delta\boldsymbol{x}_k = \boldsymbol{x}(t_k) - \boldsymbol{x}^*(t_k) \tag{23}$$

The overall number of sampling times, n_S, is unspecified, while $\Delta t_S^{(k)} = t_{k+1} - t_k$ $(k = 0,\dots,n_S - 1)$ is the interval between two subsequent sampling times. In general, the actual sampling interval $\Delta t_S^{(k)}$ can be programmed offline, in relation to the nominal trajectory, and can vary adaptively, in order to ameliorate the performance of the guidance and control algorithm. In this study, two prescribed values for $\Delta t_S^{(k)}$ are employed: a large value for the great majority of the transfer path and a reduced value for the terminal arc that ends at final orbit injection. Details on the specific values adopted in this research are reported in the succeeding section. A key ingredient of VTD-NOG is represented by the formula for updating $t_f^{(k)}$, i.e. the (corrected) time of flight, which is computed at each sampling time t_k.

Time-to-go and termination criterion

At each sampling time, VTD-NOG is intended to define the updated time of flight $t_f^{(k)}$ and the correction $\delta\boldsymbol{u}(\tau)$ to the control in the normalized interval $[\tau_k, \tau_{k+1}]$, corresponding to the actual interval $[t_k, t_{k+1}]$. The fundamental relations of VTD-NOG derive from minimizing the second differential of the objective functional J (cf. Pontani et al. 2015a), while enforcing the first-order expansions of the state and costate equations, the respective final conditions, and the parameter condition. Minimization of the second differential of J is equivalent to the solution of the accessory optimization problem in the interval $[\tau_k, 1]$. The classical relations that hold in optimal control theory refer to the overall interval $[0,1]$ and are reported in Hull 2003. They are generalized to the interval $\left[\tau_k,1\right]$ in Pontani et al. 2015a, focused on the analytical foundations of VTD-NOG. Among all the relations that form the core of VTD-NOG (omitted in this work for the sake of brevity), it is worth reporting the feedback law that yields the control correction as a function of the parameter vector correction $d\boldsymbol{a}$ and the state and costate displacements $\delta\boldsymbol{x}(\tau)$ and $\delta\boldsymbol{\lambda}(\tau)$,

$$\delta\boldsymbol{u} = -H_{\boldsymbol{uu}}^{-1}\left(H_{\boldsymbol{ux}}\delta\boldsymbol{x} + H_{\boldsymbol{ua}}d\boldsymbol{a} + H_{\boldsymbol{u\lambda}}\delta\boldsymbol{\lambda}\right) \qquad \tau_k \le \tau \le \tau_{k+1} \tag{24}$$

Specifically, $d\boldsymbol{a}$ is given by (cf. Pontani et al. 2015a)

$$\begin{bmatrix} d\boldsymbol{\upsilon} \\ d\boldsymbol{a} \end{bmatrix} = -\mathbf{V}_k^{-1}\mathbf{U}_k^T\delta\boldsymbol{x}_k - \mathbf{V}_k^{-1}\boldsymbol{\Theta}\delta\boldsymbol{\mu}_k \quad \text{with} \quad \boldsymbol{\Theta} := \begin{bmatrix} \mathbf{0}_{qxp} \\ \mathbf{I}_{pxp} \end{bmatrix} \tag{25}$$

where $\delta\boldsymbol{\mu}_k$ is the final value of $\delta\boldsymbol{\mu}$ in the preceding interval $\left[\tau_{k-1},\tau_k\right]$ (with $\delta\boldsymbol{\mu}_0 = \mathbf{0}$), whereas $\delta\boldsymbol{x}(\tau)$ and $\delta\boldsymbol{\lambda}(\tau)$ are obtained by integrating the following linear differential system:

$$\delta\dot{\boldsymbol{x}} = \mathbf{A}\delta\boldsymbol{x} - \mathbf{B}\delta\boldsymbol{\lambda} + \mathbf{D}d\boldsymbol{a} \tag{26}$$

$$\delta\dot{\boldsymbol{\lambda}} = -\mathbf{C}\delta\boldsymbol{x} - \mathbf{A}^T\delta\boldsymbol{\lambda} - \mathbf{E}d\boldsymbol{a} \tag{27}$$

$$\delta\dot{\boldsymbol{\mu}} = -\mathbf{E}^T\delta\boldsymbol{x} - \mathbf{D}^T\delta\boldsymbol{\lambda} - \mathbf{F}d\boldsymbol{a} \tag{28}$$

In each interval $\left[\tau_k,\tau_{k+1}\right]$, the initial condition for $\delta\boldsymbol{x}$ is given by Equation (23), while for $\delta\boldsymbol{\lambda}$ the following relation (to evaluate at τ_k) is obtained (cf. Pontani et al. 2015a):

$$\delta\boldsymbol{\lambda}_k = \left(\hat{\mathbf{S}} - \mathbf{W}\mathbf{m}^T\right)\delta\boldsymbol{x}_k - \mathbf{W}\mathbf{n}^T d\boldsymbol{\upsilon} - \mathbf{W}\boldsymbol{\alpha}d\boldsymbol{a} \tag{29}$$

In Equations (26)-(29) several matrices appear, i.e. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\hat{\mathbf{S}}$, $\mathbf{W}$, $\mathbf{m}$, $\mathbf{n}$, and $\boldsymbol{\alpha}$. All of them are evaluated along the nominal path.

The updated time of flight is $t_f^{(k)} = t_f^* + dt_f^{(k)}$, where t_f^* is the nominal time of flight and $dt_f^{(k)}$ derives directly from the analytical conditions for optimality, because it is included as a component of $d\boldsymbol{a}$ (cf. Pontani et al. 2015a). Because the actual sampling interval $\Delta t_S^{(k)}$ is specified (and depends, in general, only on the nominal trajectory), while $t_f^{(k)}$ is updated at each iteration, the general formula for τ_{k+1} is

$$\tau_{k+1} = \sum_{j=0}^{k} \frac{\Delta t_S^{(j)}}{t_f^{(j)}} \qquad \left(k = 0,\ldots,n_S - 1;\ t_f^{(0)} = t_f^*\right) \tag{30}$$

Finally, the total number of intervals n_S corresponds to occurrence of the condition

$$\sum_{j=0}^{n_S-1} \frac{\Delta t_S^{(j)}}{t_f^{(j)}} \geq 1 \quad \Rightarrow \quad \tau_{n_S} = 1 \tag{31}$$

In the end, the adoption of the normalized time τ has remarkable consequences. First, all the gain matrices do not become singular, because they are defined in the interval [0,1]. Second, the values $\{\tau_k\}$ are calculated at each sampling time through Equation (30). The guidance and control algorithm terminates when τ reaches the upper bound of the interval where τ is defined (i.e., when $\tau = 1$).

Modified sweep method

The backward numerical integration of the sweep equations (cf. Hull 2003) represents a necessary step, in order to obtain the gain matrices associated with neighboring optimal paths. However, unlike the accessory minimization problem, VTD-NOG refers to the interval $\left[\tau_k, 1\right]$. This circumstance implies the need of deriving modified sweep equations.

Lengthy analytical developments (presented in Pontani et al. 2015a, and not reported in this work for the sake of brevity) lead to the equations that follow,

$$\dot{\hat{\mathbf{S}}} = -\hat{\mathbf{S}}\mathbf{A} + \hat{\mathbf{S}}\mathbf{B}\hat{\mathbf{S}} + \left[\hat{\mathbf{S}}\mathbf{D}\boldsymbol{\alpha}^{-1} + \mathbf{W}\mathbf{F}\boldsymbol{\alpha}^{-1} + \mathbf{E}\boldsymbol{\alpha}^{-1}\right]\mathbf{m}^T - \mathbf{W}\mathbf{E}^T - \mathbf{W}\mathbf{D}^T\hat{\mathbf{S}} - \mathbf{C} - \mathbf{A}^T\hat{\mathbf{S}} \qquad \dot{\mathbf{Q}} = -\mathbf{R}^T\mathbf{B}\mathbf{W}\mathbf{n}^T \tag{32}$$

$$\dot{\mathbf{R}}^T = \mathbf{R}^T\mathbf{B}\hat{\mathbf{S}} - \mathbf{R}^T\mathbf{A} - \mathbf{R}^T\mathbf{B}\mathbf{W}\mathbf{m}^T \qquad \dot{\mathbf{n}} = -\mathbf{R}^T\left(\mathbf{D} + \mathbf{B}\mathbf{W}\boldsymbol{\alpha}\right) \qquad \dot{\boldsymbol{\alpha}} = \mathbf{D}^T\mathbf{W}\boldsymbol{\alpha} - \mathbf{F} - \mathbf{m}^T\mathbf{B}\mathbf{W}\boldsymbol{\alpha} - \mathbf{m}^T\mathbf{D} \tag{33}$$

$$\dot{\mathbf{m}}^T = -\mathbf{m}^T\mathbf{A} + \mathbf{m}^T\mathbf{B}\hat{\mathbf{S}} - \mathbf{m}^T\mathbf{B}\mathbf{W}\mathbf{m}^T - \mathbf{E}^T - \mathbf{D}^T\hat{\mathbf{S}} + \mathbf{D}^T\mathbf{W}\mathbf{m}^T \tag{34}$$

Hence, the gain matrices **S**, $\hat{\mathbf{S}}$, **R**, **Q**, **n**, **m**, and $\boldsymbol{\alpha}$, must be integrated backward, from $\tau = 1$ to $\tau = 0$, in two steps:

(a) the equations of the classical sweep method (cf. Hull 2003), with the associated boundary conditions are employed in the interval $\left[\tau_{sw}, 1\right]$

(b) Equations (32)-(34) are used in the interval $\left[0,\tau_{sw}\right]$. Matrices **R**, **Q**, **n**, **m**, and $\boldsymbol{\alpha}$ are continuous across the switching time τ_{sw}, whereas $\hat{\mathbf{S}}$ is given by $\hat{\mathbf{S}} := \mathbf{S} - \mathbf{U}\mathbf{V}^{-1}\mathbf{U}^T$ (cf. Pontani et al. 2015a); in this work τ_{sw} is set to 0.99.

Offline computations and algorithm structure

This subsection first summarizes the preliminary steps to complete offline before running VTD-NOG. Then, the overall architecture of VTD-NOG & PD-RM is illustrated in a block diagram.

In order to implement VTD-NOG & PD-RM, the optimal solution must be identified, together with the related state, control, and adjoint variables. These are available as equally-spaced sets of discrete values, e.g. $\boldsymbol{u}_i^* = \boldsymbol{u}^*\left(\tau_i\right)$ $\left(i = 0,\ldots,n_D;\ \tau_0 = 0 \text{ and } \tau_{n_D} = 1\right)$. However, because VTD-NOG evaluates the control corrections $\delta\boldsymbol{u}\left(\tau\right)$ at times τ not coincident with $\left\{\tau_i\right\}$, interpolation is mandatory, for the control $\boldsymbol{u}^*$, as well as all the remaining nominal quantities, $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, **A**, **B**, **C**, **D**, **E**, **F**, $\boldsymbol{f_x}$, $\boldsymbol{f_u}$, $\boldsymbol{f_a}$, $H_{\boldsymbol{xx}}$, $H_{\boldsymbol{xu}}$, $H_{\boldsymbol{x\lambda}}$, $H_{\boldsymbol{xa}}$, $H_{\boldsymbol{ux}}$, $H_{\boldsymbol{uu}}$, $H_{\boldsymbol{ua}}$, $H_{\boldsymbol{u\lambda}}, H_{\boldsymbol{ax}}, H_{\boldsymbol{au}}, H_{\boldsymbol{aa}}, H_{\boldsymbol{a\lambda}}, \boldsymbol{\psi}_{x_f}, \boldsymbol{\psi}_{x_0}, \boldsymbol{\psi}_{\boldsymbol{a}}, \Phi_{x_0 x_0}, \Phi_{x_0 \boldsymbol{a}}, \Phi_{x_f x_f}, \Phi_{x_f \boldsymbol{a}}, \Phi_{\boldsymbol{a} x_f}, \Phi_{\boldsymbol{aa}}$. Then, the backward integration of the sweep equations yields the matrices $\hat{\mathbf{S}}$, **R**, **m**, **Q**, **n**, and $\boldsymbol{\alpha}$. The preliminary computations end with the interpolation of all the gain matrices. If a suitable number of times $\left\{\tau_i\right\}$ is adopted (e.g., 10000), linear interpolation is a simple and effective option and is adopted in this study.

At time τ_k, using the nominal variables and gain matrices (evaluated offline), VTD-NOG computes the flight time $t_f^{(k)}$, the value τ_{k+1}, and the correction $\delta\boldsymbol{u}\left(\tau\right)$. In particular, the guidance methodology at hand is interpolated through the following steps:

1. Specifiy the sampling interval Δt_S
2. At each time τ_k $\left(k = 0,\ldots,n_S - 1;\ \tau_0 = 0\right)$
    a. Evaluate $\delta\boldsymbol{x}_k$ thorugh Equation (23)
    b. Assume the value of $\delta\boldsymbol{\mu}$ obtained at the end of the previous interval $\left[\tau_{k-1},\tau_k\right]$ as $\delta\boldsymbol{\mu}_k$

c. Calculate $dt_f^{(k)}$ and update the flight time $t_f^{(k)}$

d. Obtain the upper value τ_{k+1} for the current interval

e. Evaluate $\delta\boldsymbol{\lambda}_k$ and integrate the linear differential system composed of Equations (26)-(28)

f. Obtain the correction $\delta\boldsymbol{u}(\tau)$ in $[\tau_k, \tau_{k+1}]$ by means of Equation (24)

3. If Equation (31) is met, then VTD-NOG terminates, otherwise point 2 is repeated (after increasing *k* by 1).

Figure 6 depicts a block diagram that shows the sampled-data feedback structure of VTD-NOG. The corrections on control and flight time are obtained after evaluating the state deviation $\delta\boldsymbol{x}$, using the gain matrices. The attitude control loop is encircled by the dotted line, and is being described in the following section.

PD-LIKE ATTITUDE CONTROL BASED ON ROTATION MATRICES

The objective of the attitude control system is ensuring that the actual orientation of the spacecraft is sufficiently close to the commanded orientation obtained from VTD-NOG. The actual spacecraft attitude is associated with the actual control $\boldsymbol{u}_a$ (cf. Figure 6). The control torque is generated by reaction wheels

Commanded attitude

VTD-NOG determines $\boldsymbol{u}$ i.e. the thrust angles α and β that yield the thrust direction. Because the thrust direction does not vary with respect to the spacecraft body axes, the attitude control system must modify the spacecraft orientation so that the desired thrust direction is achieved. Thus, the two thrust angles yielded by VTD-NOG actually represent the commanded values for the desired thrust angles denoted by α_c and β_c.

Consider the body frame $(\hat{x}_b, \hat{y}_b, \hat{z}_b)$ whose origin coincides with the current mass center of the spacecraft, its axes are principal inertia axes, and $\hat{x}_b$ is directed along the longitudinal axis. The commanded angles α_c and β_c define the commanded direction for $\hat{x}_b$, denoted with $\hat{x}_b^{(c)}$ and expressed by

$$\hat{x}_b^{(c)} = \begin{bmatrix} \cos\beta_c \sin\alpha_c \\ \cos\beta_c \cos\alpha_c \\ \sin\beta_c \end{bmatrix}^T \begin{bmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{bmatrix} = \begin{bmatrix} \cos\beta_c \sin\alpha_c \\ \cos\beta_c \cos\alpha_c \\ \sin\beta_c \end{bmatrix}^T \mathbf{R}_2(-\phi)\mathbf{R}_3(\xi) \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} \tag{35}$$

The commanded direction for $\hat{z}_b^{(c)}$ is defined as

$$\hat{z}_b^{(c)} = \frac{\hat{c}_3 \times \hat{x}_b^{(c)}}{\left|\hat{c}_3 \times \hat{x}_b^{(c)}\right|} \tag{36}$$

and $\hat{y}_b^{(c)}$ completes the right-handed coordinate system $\left(\hat{x}_b^{(c)}, \hat{y}_b^{(c)}, \hat{z}_b^{(c)}\right)$. In nominal flight conditions $\hat{z}_b^{(c)}$ is in the equatorial plane and coincides with the nadir direction if a circular orbit is traveled. Using Equations (35) and (36), the commanded rotation matrix $\mathbf{R}_c$ can be determined,

$$\left[\hat{x}_b^{(c)} \quad \hat{y}_b^{(c)} \quad \hat{z}_b^{(c)}\right]^T = \mathbf{R}_c \left[\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3\right]^T \tag{37}$$

Attitude dynamics

The spacecraft attitude is controlled through a reaction wheel assembly. The current attitude is determined by the rotation matrix **R** defined as

$$\left[\hat{x}_b \quad \hat{y}_b \quad \hat{z}_b\right]^T = \mathbf{R}\left[\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3\right]^T \tag{38}$$

Let $\boldsymbol{\omega} = \left[\omega_x \quad \omega_y \quad \omega_z\right]^T$ denote the spacecraft angular velocity, with components written along the body axes. Thus, the attitude kinematics are described by

$$\dot{\mathbf{R}} = -\boldsymbol{\omega}^{\times}\mathbf{R} \qquad \text{where} \qquad \boldsymbol{\omega}^{\times} := \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{39}$$

Let $\mathbf{I} = \text{diag}\left\{I_x, I_y, I_z\right\}$ be the inertia matrix, whereas $\boldsymbol{M}_c = \left[M_{cx} \quad M_{cy} \quad M_{cz}\right]^T$ and $\boldsymbol{M}_e = \left[M_{ex} \quad M_{ey} \quad M_{ez}\right]^T$ represent respectively the control torque and the environmental torque both resolved along the body axes. Thus, the attitude dynamics are given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \boldsymbol{M}_c + \boldsymbol{M}_e \tag{40}$$

The control torques are generated by the reaction wheel assembly, and the amplitude of each component is bounded by the maximum values $\overline{M_{cx}}$, $\overline{M_{cy}}$, $\overline{M_{cz}}$. These limits are taken into account by introducing the variable $\tilde{\boldsymbol{M}}_c = \begin{bmatrix} \tilde{M}_{cx} & \tilde{M}_{cy} & \tilde{M}_{cz} \end{bmatrix}^T$, whose relation with $\boldsymbol{M}_c$ is given by

$$M_{cx} = \mathrm{sat}_{\overline{M_{cx}}}\left(\tilde{M}_{cx}\right) = \begin{cases} -\overline{M_{cx}} & \text{if} \quad \tilde{M}_{cx} < -\overline{M_{cx}} \\ \tilde{M}_{cx} & \text{if} \quad -\overline{M_{cx}} \le \tilde{M}_{cx} \le \overline{M_{cx}} \\ \overline{M_{cx}} & \text{if} \quad \tilde{M}_{cx} > \overline{M_{cx}} \end{cases} \tag{41}$$

Similar relations hold for M_{cy} and M_{cz}. Thus, the spacecraft attitude control input is given by $\boldsymbol{M}_c$. The effect of the reaction wheel assembly can be neglected for practical purposes (cf. Sidi 1997). As a result, no model for the assembly is considered here. Environmental torques are typically due to residual magnetization, gravity-gradient, aerodynamics, and solar radiation.

Attitude control

The torque that the reaction wheel assembly must provide is determined by a control law that uses the rotation matrix $\mathbf{R}_c$, which specifies the commanded attitude, addressed in a preceding subsection. Using rotation matrices implies the advantage of avoiding singularities and ambiguities that would be otherwise introduced by other attitude parameterizations, such as sequences of angles and Euler parameters.

The following PD-like control law is employed (Chaturvedi et al. 2011):

$$\tilde{\boldsymbol{M}}_c = -\mathbf{K}_p \sum_{i=1}^{3} \left(\boldsymbol{e}_i \times \mathbf{R}_c \mathbf{R}^T \boldsymbol{e}_i \right) - \mathbf{K}_d \boldsymbol{\omega} \tag{42}$$

In the preceding equations $\mathbf{K}_p = \mathrm{diag}\left\{k_{px}, k_{py}, k_{pz}\right\}$ and $\mathbf{K}_d = \mathrm{diag}\left\{k_{dx}, k_{dy}, k_{dz}\right\}$ are diagonal matrices containing positive control gains, and $\left\{\boldsymbol{e}_i\right\}_{i=1,2,3}$ are the columns of the 3 by 3 identity matrix. For convergence analysis of the proposed control law, $\mathbf{R}_c$ can be approximated as constant since its rate of variation is small compared to that of $\mathbf{R}$. Thus, neglecting $\boldsymbol{M}_e$ with respect to $\boldsymbol{M}_c$, one obtains that $\mathbf{R}$ converges to $\mathbf{R}_c$ locally and exponentially, by means of Proposition 2 in Chaturvedi et al. 2011. In fact, in Chaturvedi et al. 2011 it is shown that the linearization of the closed-loop system given by Equations (39) through (42) about $\mathbf{R} = \mathbf{R}_c$ and $\boldsymbol{\omega} = \mathbf{0}$ leads to the following equation:

$$\ddot{\boldsymbol{\zeta}} + \mathbf{I}^{-1}\mathbf{K}_d\dot{\boldsymbol{\zeta}} + 2\mathbf{I}^{-1}\mathbf{K}_p\boldsymbol{\zeta} = \mathbf{0} \tag{43}$$

where $\boldsymbol{\zeta} := \begin{bmatrix}\Phi & \Theta & \Psi\end{bmatrix}^T$ and Ψ, Θ, and Φ is the current 3-2-1 Euler sequence of the body reference with respect to the commanded attitude. Thus, it is immediate to obtain that $\boldsymbol{\zeta} \to \mathbf{0}$.

Selection of gains

The gains k_{px}, k_{py}, k_{pz}, k_{dx}, k_{dy}, and k_{dz} are determined by adopting the following approach which is presented only for gains k_{px} and k_{dx} since it is straightforward to adapt it to the remaining gains. The first equation of the linearized closed-loop system in Equation (43) is given by

$$\ddot{\Phi} + \frac{k_{dx}}{I_x}\dot{\Phi} + 2\frac{k_{px}}{I_x}\Phi = 0 \tag{44}$$

The corresponding characteristic equation is

$$s^2 + \frac{k_{dx}}{I_x}s + 2\frac{k_{px}}{I_x} = 0 \tag{45}$$

The principal moment of inertia I_x changes during flight. Let $\underline{I_x}$ and $\overline{I_x}$ denote the minimum and maximum of I_x. Thus, gains k_{px} and k_{dx} are determined so that the solutions of Equation (44) have damping ratio $\zeta_x \geq \underline{\zeta_x}$ and natural angular frequency $\omega_{nx} \geq \underline{\omega_{nx}}$ for $\underline{I_x} \leq I_x \leq \overline{I_x}$. The lower bounds $\underline{\zeta_x}$ and $\underline{\omega_{nx}}$ are selected through experience and trial-and-error. Since $2k_{px}/I_x = \omega_{nx}^2$ and $k_{dx}/I_x = 2\zeta_x\omega_{nx}$, it is immediate to verify that inequalities $\zeta_x \geq \underline{\zeta_x}$ and $\omega_{nx} \geq \underline{\omega_{nx}}$ hold for all $\underline{I_x} \leq I_x \leq \overline{I_x}$ by setting

$$k_{px} = \frac{\underline{\omega_{nx}}^2\,\overline{I_x}}{2} \qquad \text{and} \qquad k_{dx} = 2\underline{\zeta_x}\,\underline{\omega_{nx}}\,\overline{I_x} \tag{46}$$

Actual attitude

The current attitude of the spacecraft is determined by matrix **R**. Thus, the actual orientation of axis $\hat{x}_b$ can be obtained by using Equation (38). Combining the latter equation with Equation (2) leads to

$$\begin{bmatrix}\hat{x}_b & \hat{y}_b & \hat{z}_b\end{bmatrix}^T = \mathbf{R}\begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \hat{c}_3\end{bmatrix}^T = \mathbf{R}\mathbf{R}_3^T(\xi)\mathbf{R}_2(\phi)\begin{bmatrix}\hat{r} & \hat{t} & \hat{n}\end{bmatrix}^T \tag{47}$$

The actual thrust direction coincides with $\hat{x}_b$, and can be resolved in the $(\hat{r},\hat{t},\hat{n})$-frame as follows,

$$\hat{T}_a \equiv \hat{x}_b = \begin{bmatrix}\cos\beta_a \sin\alpha_a & \cos\beta_a \cos\alpha_a & \sin\beta_a\end{bmatrix}\begin{bmatrix}\hat{r} & \hat{t} & \hat{n}\end{bmatrix}^T \tag{48}$$

The two angles α_a and β_a can be expressed as functions of **R**, ϕ, and ξ, by comparing Equations (47) and (48).

VTD-NOG & PD-RM APPLIED TO LOW-THRUST LEO-GEO TRANSFER

The guidance and control architecture termed VTD-NOG & PD-RM is tested on the low-thrust orbit transfer from LEO to GEO. The minimum-time path is obtained in a preceding section and requires more than 50 days.

The spacecraft has initial mass $\tilde{m}_0 = 2400$ kg and maximal torque yielded by the reaction wheels $\overline{M_{cx}} = \overline{M_{cy}} = \overline{M_{cz}} = 0.5$ N m (about each body axis). The time-dependent inertia moments I_x, I_y, and I_z are given by

$$I_x = I_{x0} + \dot{I}_x t \qquad I_y = I_{y0} + \dot{I}_y t \qquad I_z = I_{z0} + \dot{I}_z t \tag{49}$$

where

$$\begin{aligned} I_{x0} &= 1200 \text{ kg m}^2 & \dot{I}_x &= -3.92\cdot 10^{-4} \text{ kg m}^2/\text{sec} \\ I_{y0} = I_{y0} &= 800 \text{ kg m}^2 & \dot{I}_y = \dot{I}_z &= -2.61\cdot 10^{-4} \text{ kg m}^2/\text{sec} \end{aligned} \tag{50}$$

In addition, the following values are chosen for VTD-NOG & PD-RM. A non-uniform sampling interval is employed: $\Delta t_S^{(k)}$ = 2 hours if $0 \le \tau_k \le 0.995$ and $\Delta t_S^{(k)}$ = 3 min if $0.995 \le \tau_k \le 1$. Therefore, the state deviations are evaluated more frequently while reaching the final time, with the intent of improving accuracy at final orbit injection. The control gains are identified on the basis of the following considerations. First, it is worth noting that the nominal thrust is $T = n_0 \tilde{m}_0$, whereas the maximum values for I_x, I_y, and I_z are $\overline{I_x} = I_{x0}$, $\overline{I_y} = I_{y0}$, and $\overline{I_z} = I_{z0}$. The lower bounds for the natural frequencies are set to $\underline{\omega_{nx}} = \underline{\omega_{ny}} = \underline{\omega_{nz}} = 0.03 \text{ rad sec}^{-1}$. In fact, a spectral analysis of the commanded attitude in the nominal case shows that using those values the attitude control loop should be fast enough to track the commanded attitude even in perturbed conditions. The lower bounds of the damping coefficients have been selected as $\underline{\zeta_x} = \underline{\zeta_y} = \underline{\zeta_z} = 0.7$. In fact, the latter value represents a good compromise between fast response and low overshoot. Thus, by Equation (46) and similar equations for the remaining gains, one obtains $k_{px} = 11.76$, $k_{dx} = 151.2$, $k_{py} = k_{pz} = 7.84$, $k_{dy} = k_{dz} = 100.8$.

The first reason for the existence of nonnominal flight conditions is due to the fact that the actual attitude differs from the commanded attitude. In fact, in general the commanded angles, determined by VTD-NOG, are discontinuous across successive guidance intervals, unlike the real steering direction. The latter is driven by the attitude control system, and converges toward the desired one with some delay. This fact is apparent also in Figure 6, which points out that the adjusted commanded control $\boldsymbol{u}$ differs from the actual control $\boldsymbol{u}_a$.

Moreover, (modest) displacements from the nominal trajectory arise also as an effect of the gravitational perturbations. These are related to the harmonics of the geopotential, as well as to the Moon and Sun pull. Albeit these deviations were neglected while determining the optimal trajectory, these terms are retained while testing VTD-NOG & PD-RM. As a result, (a_r, a_t, a_n) contain all the mentioned perturbations of a gravitational nature,

$$a_r = a_r^{(H)} + a_r^{(M)} + a_r^{(S)} \qquad a_t = a_t^{(H)} + a_t^{(M)} + a_t^{(S)} \qquad a_n = a_n^{(H)} + a_n^{(M)} + a_n^{(S)} \tag{51}$$

where superscripts *H*, *M*, and *S* denote respectively the contributions of the geopotential harmonics, the Moon, and the Sun. The numerical simulations consider all the harmonics with magnitude $|J_{lm}| > 10^{-6}$, i.e. J_2 J_3, J_4, J_{22}, and J_{31}.

In the following, an estimate of the maximum amplitude of the enviromantal torque $\boldsymbol{M}_e$ is determined. The magnitude of the residual magnetization torque is bounded by $B_{max}\, m_0$ where B_{max} is the maximum amplitude of the geomagnetic field during spaceflight, and m_0 is the magnitude of the magnetic dipole moment due to residual magnetization. The maximum amplitude of the geomagnetic field is achieved when the spacecraft is closest to the Earth and is of the order of 10^{-7} T. A residual magnetization on the spacecraft may have an order of 1 A m^2. Thus, the maximum amplitude of the torque due to residual magnetization moment has an order of magnitude of 10^{-7} N m. Using the well-known expression for the gravity-gradient torque reported in Sidi 1997, it is easy to obtain that each component of that torque is bounded by

$$\frac{3\mu}{R_{LEO}^{3}}\left(I_{x0}-I_{y0}\right)=1.54\cdot10^{-3}\ \mathrm{N\,m} \tag{52}$$

An estimate of the amplitude of the aerodynamic torque is given by (cf. Pisacane 2005)

$$\frac{1}{2}c_D A\rho v_R^2 r_{p,m} \tag{53}$$

where c_D is the drag coefficient, A is the aerodynamics reference surface of the spacecraft, ρ is the atmospheric density, v_R is the spacecraft velocity relative to the atmosphere, and $r_{p,m}$ is the distance that separates the center of pressure from the mass center. Magnitudes ρ and v_R take their largest values at the initial time (at LEO), which corresponds to ρ of the order of $10^{-12}\ \mathrm{kg\ m^{-3}}$ and $v_R = 7.174\ \mathrm{km/sec}$. Thus, setting $c_D = 2$, $A = 1\ \mathrm{m}^2$, $r_{p,m} = 1\ \mathrm{m}$ one obtains the maximum amplitude of the aerodynamic torque, which has order of 10^{-4} N m. The magnitude of the solar radiation torque is at most of the order of 10^{-5} N m (cf. Pisacane 2005). Thus, after summing all these disturbing torques, each component of the total environmental torque $\boldsymbol{M}_e$ has at most an order of 10^{-3} N m. Since the magnitude of each component of the control torque $\boldsymbol{M}_c$ can reach 0.5 N m, the effects of $\boldsymbol{M}_e$ are negligible. As a result, $\boldsymbol{M}_e$ is not included in the simulations being presented.

As a first step, the guidance and control architecture of interest has been tested with the inclusion of the nonnominal conditions exclusively related to gravitational perturbations and to attitude motion. The reference epoch, corresponding to the initial time, is set to 1 January 2020 at 12 UTC, whereas the initial absolute ξ_i is set to 0. The acronym GP ("gravitational perturbation") labels the first column of Table 1, which collects the related results attained in a single simulation, i.e. the final displacements regarding the state components of interest. The numerical results demonstrate the excellent accuracy of the guidance and control methodology in this context.

However, further perturbations are responsible of nonnominal flight conditions. Monte Carlo (MC) campaigns are run for the purpose of getting several statistical information on accuracy of VTD-NOG & PD-RM, with stochastic inclusion of the most relevant perturbations. In particular, the initial conditions are perturbed by introducing an error on the initial latitude and radius, with Gaussian distribution, zero average value and standard deviations $\phi_0^{(\sigma)}$ (for ϕ_0) and $r_0^{(\sigma)}$ (for r_0) respectively equal to 0.085 deg and 10 km. The former value corresponds to an out-of-plane distance of 10 km. The velocity deviation has direction distributed uniformly over a unit sphere, whereas the velocity magnitude has displacement with zero mean value and standard deviation $v_0^{(\sigma)} = 30 \text{ m/sec}$. Moreover, errors on the initial attitude angles and rates are introduced. All these displacements have Gaussian distribution and zero mean. Their standard deviation equals 10 deg for the initial Euler angles and 10 deg/sec for the initial attitude rates. Finally, because the thrust magnitude can exhibit reduced (although nonnegligible) fluctuations, the perturbation of the thrust acceleration is modeled using a trigonometric function with stochastic coefficients,

$$n_0^p = n_0 \left[1 + \sum_{k=1}^{5} \tilde{a}_k \sin\left(\frac{2k\pi t}{t_f^*} \right) + \sum_{k=1}^{5} \tilde{a}_{k+5} \cos\left(\frac{2k\pi t}{t_f^*} \right) \right] \tag{54}$$

where n_0^p represents the perturbed value of n_0, while the coefficients $\{\tilde{a}_k\}_{k=1,\ldots,10}$ have a Gaussian distribution with zero mean and 0.01 as the standard deviation. Let $\tilde{m}_0$ denote the initial spacecraft mass. Because the thrust magnitude is no longer constant, Equation (1) is replaced by

$$\frac{T}{\tilde{m}} = \frac{n_0^p \tilde{m}_0}{\tilde{m}} \quad \text{where} \quad \frac{\dot{\tilde{m}}}{\tilde{m}_0} = -\frac{n_0^p}{c} \tag{55}$$

At the end of VTD-NOG & PD-RM, two parameters are calculated, i.e. the average value and the standard deviation for all of the quantities of interest. $\overline{\Delta\chi}$ and $\chi^{(\sigma)}$ represent the average error and standard deviation of χ hence forward. A MC campaign is performed, including 100 numerical simulations: and assuming all the previously perturbations. Figures 7 through 16 illustrate the time histories of n_0^p, the state variables of interest, the torque components, and the principal angle (cf. Schaub and Junkins 2003) that relates the actual and the commanded rotation matrices, $\mathbf{R}$ and $\mathbf{R}_c$ (for a single case). Both the altitude and the velocity components exhibit nonnegligible deviations from the respective nominal values. Specifically, the time histories of altitude and transverse velocity (cf. Figs. 8 and 11) resemble the respective optimal time histories, although the flight times of perturbed paths vary. Moreover, from inspection of Figs. 9, 10, and 12, it is apparent that the perturbed time histories of latitude, radial velocity, and normal velocity exhibit nonnegligible deviations from the respective nominal values; these displacements decrease in time, up to reaching modest values at orbit injection. The torque component M_y (corresponding to pitch motion) has an oscillating time behavior, with average amplitude that decreases in time, as shown in Fig. 14. Instead, the remaining two torque components reach modest values (never exceeding 0.03 Nm). Component M_y in Fig. 14 shows nonnegligible oscillations that are closely related to the oscillations presented by the commanded attitude. The oscillations of M_y have a decreasing amplitude, with the exception of the time interval that precedes orbit injection. This is due to the fact that while approaching the final time the commanded control, yielded by VTD-NOG and directly related to the commanded attitude (cf. Fig. 6), exhibits fast time variations, aimed at reducing the injection errors. This is consistent with the time history of the displacement angle (cf. Fig. 16), which also shows an increase in amplitude while approaching the final time. Overall, the final errors at injection are quite satisfactory, as shown in Table 1, which reports also the statistics on the time of flight. From inspection of this table it is apparent that VTD-NOG & PD-RM ensures orbit insertion with great accuracy, in spite of the relatively large sampling time. In addition, the mean time of flight approaches the nominal value, while the related standard deviation is modest.

On an Intel i7-4700MQ @ 2.40 GHz, the runtime of the guidance and control algorithm at hand takes 1.91 hours. Because the nominal time of flight exceeds 50 days, this relatively short runtime ensures that VTD-NOG & PD-RM can be run in real time.

CONCLUSION

This research addresses a new, real-time, feedback guidance and control architecture tailored to space vehicles and applied to a low-thrust orbit transfer, from a low Earth orbit to the geostationary orbit, in the presence of nonnominal flight conditions. Two main parts, which interact iteratively, compose the architecture at hand, i.e. (a) the variable-time-domain neighboring optimal guidance (VTD-NOG) and (b) a proportional-derivative-like attitude control scheme that uses rotation matrices (PD-RM). VTD-NOG is a guidance algorithm aimed at identifying the corrective maneuvers, based on the second-order optimality conditions. The introduction of a normalized time domain for the nominal path avoids the numerical difficulties related to the gain matrices, which do not diverge for the entire flight. Both the updating relation for the flight time and the guidance ending condition are consistent with the optimality conditions. VTD-NOG identifies the desired path corrections, which are yielded by a commanded thrust direction with discontinuous time history. Because the steering direction coincides with the spacecraft longitudinal axis, the actual attitude (and thrust direction) must converge to the desired (commanded) attitude as quickly as possible. PD-RM, a proportional-derivative algorithm that employs rotation matrices, is intended to perform this task. The attitude control action is actuated by means of reaction wheels. The guidance and control architecture at hand is tested on a LEO-to-GEO orbit transfer. Gravitational perturbations, fluctuations of the propulsive thrust, and initial condition errors are introduced in the numerical simulations, and yield three-dimensional perturbed transfers. Extensive Monte Carlo campaigns show that orbit insertion at GEO occurs with excellent accuracy, thus proving that VTD-NOG & PD-RM is an effective guidance and control technique for the low-thrust transfer of interest. VTD-NOG & PD-RM is formulated with reference to a quite general dynamical system, thus it may be regarded as an effective architecture for spacecraft guidance and control, even in different mission scenarios.

DATA AVAILABILITY STATEMENT

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

Afshari, H. H., Novinzadeh, A. B., and Roshanian, J. (2009). "Determination of Nonlinear Optimal Feedback Law for Satellite Injection Problem Using Neighboring Optimal Control." *Amer. J. Appl. Sciences*, 6(3), 430-438

Battin, R. H. (1987). *An Introduction to the Mathematics and Methods of Astrodynamics*. AIAA Educ. Series, New York, NY, 448-450

Berge, S., A., E., Olsson, T., Pionnier, G., Björk, M., Cbasset, C., Nordebäck, T., Rieschel, M., Lübke-Ossenbeck, B., and Zentgraf, P. (2009). "Advanced AOCS design on the first small GEO telecom satellite." Proc. 60th Intern. Astron. Congr., Daejon, South Korea; paper IAC-09.C1.6.10

Calise, A. J., Melamed N., and Lee, S. (1998). "Design and Evaluation of a Three-Dimensional optimal Ascent Guidance Algorithm." *J. Guid., Contr., and Dyn.*, 21(6), 867-875

Chaturvedi, N. A., Sanyal, A. K., and McClamroch, N. H. (2011). "Rigid-Body Attitude Control." *IEEE Contr. Syst.*, 31(3), 30-51

Garulli, A., Giannitrapani, A., Leomanni, M., and Scortecci, F. (2011). "Autonomous low-earth-orbit station-keeping with electric propulsion." *J. Guid., Contr., and Dyn.*, 34(6), 1683-1693

Giorgi, S. (1964). "Una Formulazione Caratteristica del Metodo di Encke in Vista dell'Applicazione Numerica." Scuola di Ingegneria Aerospaziale, Rome, Italy

Hull, D. G. and Helfrich, C. E. (1991). "Neighboring Extremal Guidance for Systems with a Piecewise Linear Control." AIAA Third Aerosp. Pl. Conf., Orlando, FL

Hull, D. G. (2003). *Optimal Control Theory for Applications*. Springer International Edition, New York, NY, 199-254

Kugelmann, B. and Pesch, H. J. (1990). "New General Guidance Method in Constrained Optimal Control, Part 1: Numerical Method." *J. Opt. Theory Appl.*, 67(3), 421-435

Lu, P. (1991). "Optimal Feedback Control Laws Using Nonlinear Programming." *J. Opt. Theory Appl.*, 71(3), 599-611

Lu, P., Sun, H., and Tsai, B. (2003). "Closed-Loop Endoatmospheric Ascent Guidance." *J. Guid., Contr., and Dyn.*, 26(2), 283-294

Naidu, S. N., Hibey, J. L., and Charalambous, C. B. (1993). "Neighboring Optimal Guidance for Aeroassisted Orbital Transfer." *IEEE Trans. Aerosp. Electr. Syst.*, 29(3), 656-663.

Pavlis, N. K., Holmes, S. A., Kenyon, S. C., and Factor, J. K. (2008). "An Earth Gravitational Model to Degree 2160: EGM2008." General Assembly of the European Geosciences Union, Vienna, Austria

Pisacane, V. L. (ed.) (2005). *Fundamentals of Space Systems*. Oxford Univ. Press, New York, 281-287

Pontani, M. and Celani, F. (2018a). "Lunar Ascent and orbit Injection via Neighboring Optimal Guidance and Constrained Attitude Control." *J. Aerosp. Eng.*, 31(5)

Pontani, M. and Celani, F. (2018b). "Variable-Time-Domain Neighboring Optimal Guidance and Attitude Control for Low-Thrust Orbit Transfers." Space Fl. Mech. Meeting, Kissimmee, FL; paper AIAA 2018-1456

Pontani, M. and Celani, F. (2019). "Neighboring optimal guidance and constrained attitude control applied to three-dimensional lunar ascent and orbit injection." *A. Astron.*, 156, 78-91

Pontani, M. and Conway, B. A. (2014). "Optimal Low-Thrust Orbital Maneuvers via Indirect Swarming Method." *J. Opt. Theory Appl.*, 162(1), 272-292

Pontani, M. and Conway, B.A. (2015). "Minimum-Fuel Finite-Thrust Relative Orbit Maneuvers via Indirect Heuristic Method." *J. Guid., Contr., and Dyn.*, 38(5), 913-924

Pontani, M., Cecchetti, G., and Teofilatto, P. (2015a). "Variable-Time-Domain Neighboring Optimal Guidance, Part 1: Algorithm Structure." *J. Opt. Theory Appl.*, 166(1), 76-92

Pontani, M., Cecchetti, G., and Teofilatto, P. (2015b). "Variable-time-domain neighboring optimal guidance applied to space trajectories." *A. Astron.*, 115, 102-120

Pontani, M. (2018). "Optimal Low-Thrust Trajectories Using Nonsingular Equinoctial Orbit Elements." 1st IAA SciTech, Moscow, Russian Federation

Prussing, J. E. and Conway, B. A. (1993). *Orbital Mechanics*. Oxford Univ. Press, New York, NY, 20, 139-154

Schaub, H. and Junkins, J. L. (2003). *Analytical Mechanics of Space Systems*. AIAA Educ. Series, Reston, VA, 72-78

Seywald, H. and Cliff, E. M. (1994). "Neighboring Optimal Control Based Feedback Law for the Advanced Launch System." *J. Guid., Contr., and Dyn.*, 17(3), 1154-1162

Sidi, M. J. (1997). *Spacecraft Dynamics and Control.* Cambridge Univ. Press, New York, 164

Teofilatto, P. and De Pasquale, E. (1999). "A non-linear adaptive guidance algorithm for last-stage launcher control." Proceed. Instit. of Mech. Eng., Part G, *J. Aerosp. Eng.*, 213, 45-55

Townsend, G. E., Abbott, A. S., and Palmer, R. R. (1968). "Guidance, Flight Mechanics and Trajectory Optimization, Volume VIII, Boost Guidance Equations." NASA Contr. Rep., Washington, D.C.

Yan, H., Fahroo, F., and Ross, I. M. (2002). "Real-Time Computation of Neighboring Optimal Control Laws." *Proc. AIAA Guid., Nav. and Contr. Conf. and Exhibit*, Monterey, CA. AIAA Paper 2002-4657

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TABLE 1. Outputs using VTD-NOG & PD-RM

	$\overline{\Delta r_f}$ (km)	$\overline{\Delta \phi_f}$ (deg)	$\overline{\Delta v_{rf}}$ (m/sec)	$\overline{\Delta v_{tf}}$ (m/sec)	$\overline{\Delta v_{nf}}$ (m/sec)	$\overline{t}_f$ (days)
GP	-0.010	-5.9 e -9	-0.050	-0.375	3.3 e -5	50.36
MC	-1.687	-4.2 e -8	-0.120	-0.856	1.9 e -3	50.40
	$r_f^{(\sigma)}$ (km)	$\phi_f^{(\sigma)}$	$v_{rf}^{(\sigma)}$ (m/sec)	$v_{tf}^{(\sigma)}$ (m/sec)	$v_{nf}^{(\sigma)}$ (m/sec)	$t_f^{(\sigma)}$ (days)
MC	11.945	3.1 e -7	1.102	1.774	1.6 e -3	0.21

Legend. GP = nominal conditions, MC = Monte Carlo campaign with all nonnominal flight conditions; $\overline{t}_f$ = average time of flight

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