Journal of Mathematical Inequalities
Volume 12, Number 2 (2018), 517–530 doi:10.7153/jmi-2018-12-38

INEQUALITIES FOR SHEPARD–TYPE OPERATORS

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(*Communicated by I. Raşa*)

Abstract. Direct and converse approximation error estimates for generalized Shepard operators are given, improving analogous inequalities for well-known Shepard operators. As application in CAGD, generalized degree elevation algorithms for modeling the shape of Shepard-type curves are presented, improving previous techniques.

1. Introduction

For $n \in N$ and $f \in C([0,1])$ denote by $\mathscr{S}_n$ the Shepard operator defined by

$$\mathscr{S}_n(f;x) = \frac{\sum_{k=0}^{n} \dfrac{f(x_k)}{(x-x_k)^2}}{\sum_{k=0}^{n} \dfrac{1}{(x-x_k)^2}}, \quad 0 \leqslant x \leqslant 1, \tag{1}$$

with $x_k = \frac{k}{n}$, $k = 0,\dots,n$. From (1) it follows that $\mathscr{S}_n$ is a linear, positive operator, preserving constants, interpolating f at the equispaced knots x_k, $k = 0,\dots,n$ and $\mathscr{S}_n$ is a rational function of degree $(2n,2n)$. Shepard-type operators are widely used in classical approximation theory and in scattered data interpolation problems (see, e.g., [1, 6, 22]). In particular if $\|g\|$ denotes the usual supremum norm of $g \in C([0,1])$ and $\omega(g)$ the usual modulus of continuity of g, then the following approximation error estimate is well-known for $\mathscr{S}_n$ (see, e.g., [7, 17])

$$\|f - \mathscr{S}_n(f)\| \leqslant C\omega\left(f; \frac{\log n}{n}\right). \tag{2}$$

Here and in the following C denotes a positive constant which may assume different values even in the same formula. Converse results and saturation statements present some complications (see, e.g., [9, 18]).

The aim of the present paper is to modify slightly Shepard operators to improve (2), by dropping the factor $\log n$ in the relative error estimate, and to overcome above

Mathematics subject classification (2010): 41A36, 41A25, 65D17.
Keywords and phrases: Shepard-type operators, direct and converse results, saturation, Shepard-type curves, degree-elevation formula, progressive iterative approximation technique.

Paper JMI-12-38

complications. More generally in Section 2 Theorem 1 establishes uniform and pointwise approximation error estimates for generalized Shepard-type operators. Then Theorem 2 gives corresponding Markov-Bernstein inequalities, Steckin inequalities and direct and converse results. Furthermore Theorem 3 solves the saturation problem.

Then an application of Shepard-type operators in CAGD is discussed in Section 3. In [4] Shepard-type curves were introduced and studied, overcoming some of the original Shepard operator' s drawbacks and having some advantages with respect to the Bézier case. A progressive iterative approximation (PIA in short) technique for Shepard-type curves was also developed in [4], giving an intuitive and straightforward way to generate a sequence of curves with finer and finer precision for data fitting, by adjusting the control points of a blending curve iteratively. The limit of the curves sequence interpolates the initial control points. In [4] a degree-elevation-type formula was also showed for Shepard-type curves.

Here we present a modelling technique for Shepard-type curves, combining above degree-elevation formula and PIA format. Indeed by a shape vector we generate sequences of Shepard-type curves based on two independent modelling parameters, giving as extreme cases the original Shepard-type curve, global interpolating Shepard-type curve and degree-elevation Shepard-type curve (see Theorem 5). Therefore by our algorithm we can model the basic shape to satisfy the designer' s requirements, with more flexibility than the above techniques. A fairing algorithm for Shepard-type curves combining a degree-reduction formula and PIA process is also deduced. Numerical experiments are presented in Section 4, confirming the outperformance of our methods. Finally Section 5 contains proofs of main results, based on direct estimates and interesting inequalities for Shepard-type operators and their eigenstructure.

2. Generalized Shepard operators

Let $\phi \in C([0,1])$, with $\phi \geqslant 0$ and $\phi(0)=0$. The following cases

a) $\phi(x)=|x|^{\alpha}$, $1<\alpha$;

b) $\phi(x)=\exp(-|x|^{\alpha})$, $0<\alpha$;

c) $\phi(x)=x^2\log^{\alpha}\left(x^2+1\right)$, $0<\alpha$;

d) $\phi(x)=x^2\left(\exp\left(x^2\right)-1\right)^{\alpha}$, $0<\alpha$;

e) $\phi(x)=\log^{\alpha}\left(x^2+1\right)$, $1<\alpha$;

f) $\phi(x)=(\exp(x^2)-1)^{\alpha}$, $1<\alpha$,

are interesting for our study. Then consider the Shepard-type operator $\overline{\mathscr{S}}_n$ defined by

$$\overline{\mathscr{S}}_n(f;x)=\frac{\sum_{k=0}^{n}\dfrac{f(x_k)}{\phi(x-x_k)}}{\sum_{k=0}^{n}\dfrac{1}{\phi(x-x_k)}},\quad 0\leqslant x\leqslant 1. \tag{3}$$

From (3) $\overline{\mathscr{S}}_n$ is a linear, positive operator, preserving constants, interpolating f at $x_k, k=0,\ldots,n$. If ϕ is given as in Case a), with $\alpha=2$, then (3) gives back original S_n operator. The general case $\alpha>1$ was deeply studied (see, e.g., [1] and references therein). In Case b) we find back an exponential Shepard-type operator studied in [11, 12]. Cases of type c), d), e) and f) were mentioned in [2, 10]. Compare case e) with radial basis functions of Duchon's Thin Plate Splines, used for example in visualising 2D survey data (see, e.g., [22]).

Now we prove that if ϕ is given as in Case c) or d) or e) or f), then we get approximation results, improving (2). Indeed we have

THEOREM 1. *Let $\overline{\mathscr{S}}_n$ be the operator given by (3), with ϕ as in Cases c) or d) or e) or f). Then for every $f \in C([0,1])$*

$$\left\| f-\overline{\mathscr{S}}_n(f)\right\| \leqslant C\omega\left(f;\frac{1}{n}\right). \tag{4}$$

Furthermore,

THEOREM 2. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned} \left\|\overline{\mathscr{S}'}_n(f)\right\| &\leqslant Cn\left\|f\right\|, \\ \left\|\overline{\mathscr{S}'}_n(f)\right\| &\leqslant C\left\|f'\right\|, \text{ if } f'\in C([0,1]). \end{aligned} \tag{5}$$

Hence if $K(f;t)$ denotes the usual $K-$functional of f, then

$$K\left(f;\frac{1}{n}\right) \leqslant \left\|\overline{\mathscr{S}}_m(f)-f\right\| + C\frac{m}{n}K\left(f;\frac{1}{m}\right), \tag{6}$$

$$\left\|\overline{\mathscr{S}}_n(f)-f\right\| = O\left(n^{-\alpha}\right) \iff \omega(f;h)=O\left(h^{\alpha}\right), \quad 0<\alpha<1, \tag{7}$$

and

$$\omega\left(f;\frac{1}{n}\right) \leqslant C_\gamma n^{\gamma-1}\sum_{k=1}^{n} k^{-\gamma}\left\|f-\overline{\mathscr{S}}_k(f)\right\| + C_\gamma n^{\gamma-1}\left\|f\right\|, \quad \gamma>0. \tag{8}$$

Letting $\nu \sim \mu$, for ν and μ two quantities depending on some parameters, if $|\nu/\mu|^{\pm 1} \leqslant C$, with C independent of the parameters, we state

THEOREM 3. *If $f \neq$ constant, then*

$$\limsup_{n\to\infty} \frac{\left\|\overline{\mathscr{S}}_n(f)-f\right\|}{\omega(f;1/n)} \sim 1, \tag{9}$$

where the sign $\sim$ does not depend on f. Also

$$\left\|\overline{\mathscr{S}}_n(f)-f\right\| = o\left(\frac{1}{n}\right) \iff f=\text{constant}, \tag{10}$$

$$\left\|\overline{\mathscr{S}}_n(f)-f\right\| = O\left(\frac{1}{n}\right) \iff \omega(f;t)\leqslant Ct. \tag{11}$$

REMARK 1. From Theorem 1 we deduce the uniform convergence of $\overline{\mathscr{S}}_n(f)$ to f, $\forall f \in C([0,1])$. A pointwise approximation error estimate is also derived (see (23)), showing both the interpolatory character of $\overline{\mathscr{S}}_n$ at x_k, $k = 0,\ldots,n$, both the constants preservation property. Hence from (4) we deduce that a slight modification of the original Shepard operator (namely the operator $\overline{\mathscr{S}}_n$), does not change its main properties and allows to drop the $\log n$ factor in the corresponding error estimate. Direct estimate (4) cannot be improved because of (9). Equivalence (7) gives a converse result for $\overline{\mathscr{S}}_n$ operator.

Estimate (8) gives the Steckin-Marchaud type inequality for $\overline{\mathscr{S}}_n$ operator.

Estimate (9) is a counterpart of (4) and has a character similar to the result (see [19])

$$\|B_n(f) - f\| \sim \omega_\psi^2\left(f; \frac{1}{\sqrt{n}}\right),$$

with B_n the $n-$th Bernstein polynomial and ω_ψ^2 the second order modulus of smoothness of Ditzian and Totik, where $\psi(x) = \sqrt{x(1-x)}$. However, due to the interpolatory character of $\overline{\mathscr{S}}_n$, we cannot get the estimation (9) with "lim" (instead of "lim sup") as a consequence of a result stated in [9], p. 77 (cf. also [21], Theorem 2.1, p. 320).

Estimation (9) combined with the equivalence relation $\omega(f;t) \sim K(f;t)$, with $K(f)$ the $K-$ functional, can serve as a characterization of such $K-$functionals. Relations (10)–(11) handle the saturation problem for $\overline{\mathscr{S}}_n$.

3. A generalized degree elevation technique

In this section we discuss an application of Shepard-type operators in CAGD. Let $A_n(t) = [A_{n,0}(t), A_{n,1}(t), \ldots, A_{n,n}(t)]^T$, where

$$A_{n,i}(t) = \frac{1/\left((t-t_i)^s + \lambda\right)}{\sum_{i=0}^n 1/\left((t-t_i)^s + \lambda\right)}, \tag{12}$$

for $0 \leqslant i \leqslant n$, $i,n \in N$, $t \in [0,1]$, $t_i = i/n$, $i = 0,\ldots,n$, s even > 2 and $0 < n^s\lambda \leqslant C$. In Lemma 5 in [4] we showed that $A_{n,i}(t)$, $0 \leqslant i \leqslant n$, form a basis generating a subspace of rational functions of degree (sn, sn), with

$$0 \leqslant A_{n,i}(t) \leqslant 1, \;\; i = 0,\ldots,n, \;\; \sum_{i=0}^n A_{n,i}(t) = 1.$$

Hence in the following the functions $A_{n,i}(t)$, $i = 0,\ldots,n$, are called blending functions. Given the blending functions $A_{n,i}(t)$ defined by (12) and a control vector $P = [P_0, P_1, \ldots, P_n]^T$, $P_i \in R^d$, $i = 0,\ldots,n$, $d \geqslant 2$, in [4] we introduced the Shepard-type curve $S_n[P,t]$ defined by

$$S_n[P,t] = \sum_{i=0}^n A_{n,i}(t) P_i = A_n(t) P. \tag{13}$$

It is easy to check that $S_n[P,t]$ is a rational curve of degree (sn, sn), it reproduces points, it is smooth, it is non degenerate, it lies in the convex-hull of the control points, it satisfies the pseudo-local control property (indeed each function $A_{n,j}(t)$, $0 \leqslant j \leqslant n$, attains its maximum value close to 1 at $t = t_j$ and is very small for $|t - t_j| > 1/n$, in other words the point P_j influences strongly the shape of the curve in a neighborhood of $t = t_j$) and it interpolates at the control points, as λ tends to 0.

In [4] curves $S_n[P]$ were studied, overcoming some of the original Shepard operator's drawbacks and having some advantages with respect to Bézier case. Further generalizations and improvements for such curves were discussed in [5].

In [4] the following raising degree formula was also derived for S_n curves

$$S_{n+1}[Q,t] = \frac{1}{\overline{D}_{n+1}(t)} \left[S_n[P,t]D_n(t) + \frac{Z}{(t-\bar{t})^s + \lambda} \right], \tag{14}$$

with $t_0 < t_1 < \ldots < t_j < \bar{t} < t_{j+1} < \ldots < t_n$, $Q = \left[P_0, P_1, \ldots, P_j, Z, P_{j+1}, \ldots, P_n\right]$, $Z \in R^d$, $d \geqslant 2$,

$$D_n(t) = \sum_{k=0}^{n} \frac{1}{(t-t_k)^s + \lambda}, \quad \text{and} \quad \overline{D}_{n+1}(t) = \sum_{k=0}^{n} \frac{1}{(t-t_k)^s + \lambda} + \frac{1}{(t-\bar{t})^s + \lambda}.$$

In [4] the PIA process for S_n curves was also introduced. Indeed given the control polygon P and the basis functions $A_{n,i}(x)$, $i = 0, \ldots, n$, defined by (12), we generate the initial Shepard-type curve

$$\gamma^1[P,t] = \sum_{i=0}^{n} A_{n,i}(t) P_i^1 := S_n[P,t],\ t \in [0,1],$$

with $P_i^1 = P_i$, $i = 0, \ldots, n$. Then we calculate the successive Shepard-type curves of the sequence $\gamma^{m+1}[P,t]$, for $m \geqslant 1$, as follows

$$\gamma^{m+1}[P,t] = \sum_{i=0}^{n} A_{n,i}(t) P_i^{m+1}, \qquad P_i^{m+1} = P_i^m + \Delta_i^m, \tag{15}$$

with

$$\Delta_i^m = P_i - \gamma^m[P,t_i], \ \ i = 0, \ldots, n,$$

the corresponding error vectors.

We say that γ^1 curve satisfies the PIA property $\iff$ $\lim_m \gamma^m[P,t_i] = P_i$, $i = 0, \ldots, n$. In [4] by some results on the iterates of Shepard-type operators (cfr., e.g., [14, 16]) and on their eigenstructure, the following statement was proved

THEOREM 4. *Curve γ^1 satisfies the PIA property.*

PIA property makes possible to construct a sequence of control polygons converging to the control polygon of an interpolating curve of Shepard type. Moreover the parameter k can be used as a shape parameter in order to model different shapes, obtaining as an extreme case the S_n curve and the global Shepard-type interpolating curve (see [4]).

Now we present a generalized degree-elevation technique for Shepard-type curves.

Assume we want to model the shape of an object in 2D (analogously if we are in 3D). First we can consider $S_n[P,t]$ curve. Then assume we want to modify slightly that shape by a modeling vector. By the following algorithm we get pencils of curves acting as a sculptor to model that shape and we can choose among several curves approaching the desired shape. Indeed given a control vector P, draw $S_n[P]$ curves for various λ and choose a proper λ giving a satisfactory shape. Then assume we want to modify $S_n[P]$ curve by a modeling vector

$$M = \left(m_0, m_1, \ldots, m_{j+1}, \ldots, m_{n+1}\right),\ m_i \in R^2,\ i = 0, \ldots, n+2,$$

with $m_{j+1} \neq 0$ and $m_i = 0$, $i \neq j$ and associated knots vector $T = [t_0, t_1, \ldots, t_j,\ \bar{t},\ t_{j+1}, \ldots, t_n]$, with $t_j < \bar{t} < t_{j+1}$. The condition $m_{j+1} \neq 0$ guarantees that we are going to model our object next to a new knot $\bar{t}$ between two consecutive knots t_j and t_{j+1}. The case of a modeling vector modifying our object next to more new knots can be treated similarly. In other words by the degree-elevation formula (14) for S_n curves we construct the new curve

$$T_{n+1}[t] := S_n[P,t] + S_n\left[-P + \overline{M}, t\right] S_{n+1}[F,t], \tag{16}$$

with $\overline{M} = (m_{j+1}, m_{j+1}, \ldots, m_{j+1})$ a vector of $n+1$ components all equal to m_{j+1} and $F = [0, \ldots, 1, \ldots, 0]^T$, the vector of $n+2$ components, with the $j+2$-nd one equal to 1 and the remaining ones equal to 0.

It is interesting for the designer to have a pencil of intermediate curves to go from S_n to T_{n+1}. To this aim for $\ell \geqslant 1$ introduce the pencil of curves

$$U_{n+1,\ell}(t) := S_n[P,t] + \frac{\ell - 1}{\ell + 1} S_n\left[-P + \overline{M}, t\right] S_{n+1}[F,t]. \tag{17}$$

Obviously from (17) $U_{n+1,1}(t) = S_n[P,t]$ and $U_{n+1,\infty}(t) = T_{n+1}[t]$. Hence acting on the parameter ℓ we can go from a Shepard-type curve to the degree-elevated corresponding one.

Furthermore the designer could be interested in modeling the shape near the original control points and eventually the new one. Therefore we introduce the curves

$$V_{n+1,k,\ell}(t) := \gamma^k[P,t] + \frac{\ell - 1}{\ell + 1} \gamma^k\left[-P + \overline{M}, t\right] \gamma^1[F,t], \tag{18}$$

with $k, \ell \geqslant 1$ and γ^k given by (15).

We have

THEOREM 5. *Let $V_{n+1,k,\ell}(t)$ be as in (18). Then*

$$V_{n+1,k,1}(t) = \gamma^k[P,t], \quad V_{n+1,1,\infty}(t) = T_{n+1}[t]. \tag{19}$$

Moreover

$$V_{n+1,\infty,1}(t_i) = P_i,\ i = 0, \ldots, n. \tag{20}$$

REMARK 2. From (18) and Theorem 5 acting independently on two shape parameters we can model the shape of the curve reaching closer and closer the original control points and approximating the new point. Therefore such technique allows a wider choice in modifying the form of our object than the previous methods (cfr. [4, 5]). In other words such algorithm is a sort of generalized degree elevation technique, because it increases the degree of the curve and at the same time models its shape. The extension of such algorithm to the tensor product surfaces case is immediate (cfr. [4]) and we omit it.

From (18) we also deduce a fairing algorithm for Shepard-type curves. Indeed denote by $T_{n+1}[t]$ the Shepard-type curve based on $n+2$ control points. Assume that the shape of T_{n+1} is not enough satisfactory for the designer near a certain point, say the $j+2-$nd one. Then from (16) we can immediately construct the reduction-degree formula

$$S_n[P,t] = T_{n+1}[t] - S_n\left[-P+\overline{M},t\right] S_{n+1}[F,t],$$

in other words we can delete the bad point from the original control polygon and consider S_n instead of T_{n+1}. On the other hand for the designer it could be useful to have a pencil of curves to go from T_{n+1} to S_n. i.e. we consider for $\ell \geqslant 1$

$$\overline{U}_{n,\ell}[t] = T_{n+1}[t] - \frac{\ell-1}{\ell+1} S_n\left[-P+\overline{M},t\right] S_{n+1}[F,t]. \tag{21}$$

Obviously $\overline{U}_{n,1}[t] = T_{n+1}(t)$ and $\overline{U}_{n,\infty}(t) = S_n[P,t]$.

In addition the designer could be interested in fairing the shape near the original control points or the good ones only. Hence working as for the generalized degree elevation method in (18) we can use the PIA process at the r.h.s. in (21) and get intermediate curves by acting on two independent parameters.

4. Examples

Consider a helix of radius 5 given by (cfr. [4, 5, 16])

$$(x(t),y(t),z(t)) = (5\cos t, 5\sin t, t),\ t \in [0,6\pi].$$

A sequence of 19 control points P_i, $i=0,\ldots,18$, is sampled from the helix as

$$(x(s_i),y(s_i),z(s_i)),\ s_i = \frac{\pi}{3}i,\ i=0,1,\ldots,18.$$

Starting with these control points we fit the helix by the corresponding Shepard-type curve defined in (13) with $s=4$ and $\lambda = 2\cdot 10^{-5}$ (see [4, 5]). Now assume we want to change slightly the shape of the helix between P_8 and P_9 by the modeling vector $M = [M_0, M_1, \ldots, M_{19}]^T$, with $M_9 = [5\cos(6\pi\bar{t}) - 15, 5\sin(6\pi\bar{t}), 6\pi\bar{t}]$, $\bar{t} = 0.475$ and $M_i = [0,0,0]$, $i \neq 9$. Then we fit the helix by a sequence of curves generated by the process defined by (18). Figures 1–5 show curves corresponding to the first, second, fourth and tenth iteration level for $\ell = 1,2,3,5,10$, respectively.

So we modeled the helix by new intermediate curves.

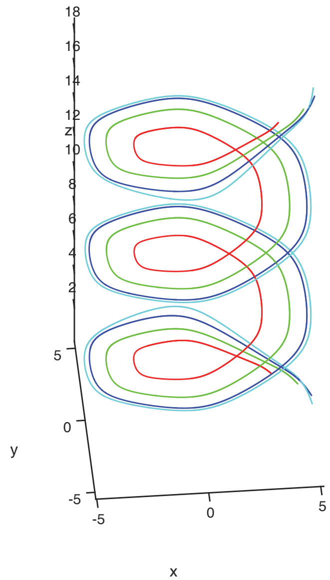

Figure 1. *Sequence of curves generated by algorithm* (18) *for* $\ell = 1$ *at the first (red), second (green), fourth (blue) and tenth (cyan) iteration.*

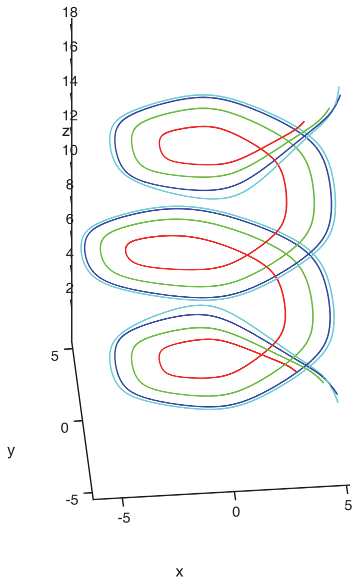

Figure 2. *Sequence of curves generated by algorithm* (18) *for* $\ell = 2$ *at the first (red), second (green), fourth (blue) and tenth (cyan) iteration.*

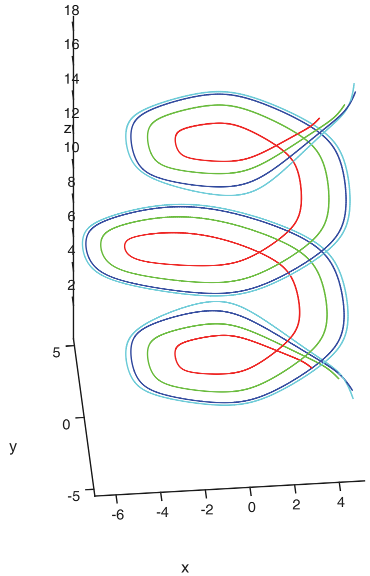

Figure 3. *Sequence of curves generated by algorithm* (18) *for* $\ell = 3$ *at the first (red), second (green), fourth (blue) and tenth (cyan) iteration.*

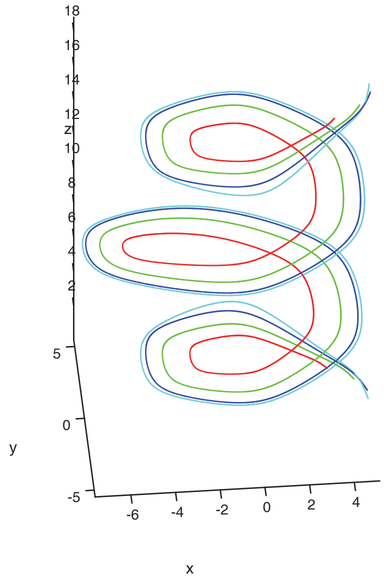

Figure 4. *Sequence of curves generated by algorithm* (18) *for* $\ell = 5$ *at the first (red), second (green), fourth (blue) and tenth (cyan) iteration*

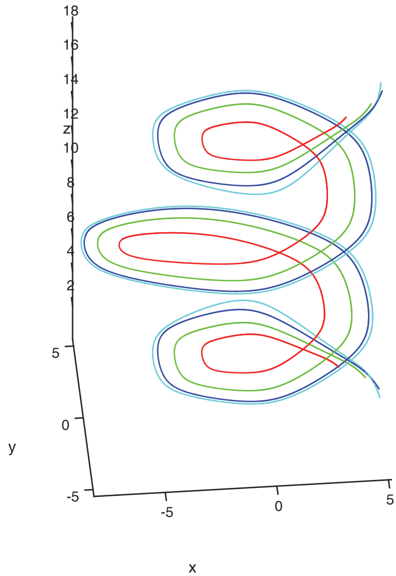

Figure 5. *Sequence of curves generated by algorithm* (18) *for* $\ell = 10$ *at the first (red), second (green), fourth (blue) and tenth (cyan) iteration*

5. Proofs of main results

Proof of Theorem 1. Let $\phi(x) = x^2 \log^{\alpha}\left(x^2+1\right)$, $\alpha > 0$. (Analogously we can work in other cases.) Because of the interpolatory behavior of $\overline{\mathscr{S}}_n$ operator, we may assume $x \neq x_k$, $k = 0,\dots,n$. Then denote by x_j the closest knot to x, with $x_j < x < x_{j+1}$, i.e. $|x - x_j| \leqslant 1/(2n)$. (Analogously if x_{j+1} is the closest knot to x.) We have

$$\left|f(x) - \overline{\mathscr{S}}_n(f;x)\right| \leqslant \frac{\dfrac{\left|f(x)-f(x_j)\right|}{\phi(x-x_j)}}{\sum_{k=0}^{n} \dfrac{1}{\phi(x-x_k)}} + \frac{\dfrac{\left|f(x)-f(x_{j+1})\right|}{\phi(x-x_{j+1})}}{\sum_{k=0}^{n} \dfrac{1}{\phi(x-x_k)}} + \frac{\sum_{k\neq j,j+1} \dfrac{\left|f(x)-f(x_k)\right|}{\phi(x-x_k)}}{\sum_{k=0}^{n} \dfrac{1}{\phi(x-x_k)}}$$
$$:= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Obviously

$$\Sigma_1 \leqslant \phi(x-x_j)\frac{\left|f(x)-f(x_j)\right|}{\phi(x-x_j)} \leqslant \omega\left(f;\frac{1}{n}\right)$$

and

$$\Sigma_2 \leqslant \phi(x-x_{j+1})\frac{|f(x)-f(x_{j+1})|}{\phi(x-x_{j+1})} \leqslant \omega\left(f;\frac{1}{n}\right).$$

Since

$$\frac{1}{\sum_{k=0}^{n}\dfrac{1}{\phi(x-x_k)}} \leqslant (x-x_j)^2\log^{\alpha}\left((x-x_j)^2+1\right) \leqslant \frac{C}{n^{2+2\alpha}}, \tag{22}$$

it follows that

$$\begin{aligned}\Sigma_3 \leqslant \frac{C}{n^{2+2\alpha}}&\left[\sum_{k=0}^{j-1}\frac{\omega(f;x-x_k)}{(x-x_k)^2\log^{\alpha}((x-x_k)^2+1)}\right.\\ &\left.+\sum_{k=j+2}^{n}\frac{\omega(f;x_k-x)}{(x_k-x)^2\log^{\alpha}((x-x_k)^2+1)}\right]\\ &:=\Sigma_4+\Sigma_5.\end{aligned}$$

Now

$$\Sigma_4 \leqslant C\frac{\omega(f;1/n)}{n^{2+2\alpha}}\sum_{k=0}^{j-1}\frac{(j-k)n^{2+2\alpha}}{(j-k)^{2+2\alpha}} \leqslant C\omega\left(f;\frac{1}{n}\right).$$

Analogously

$$\Sigma_5 \leqslant C\frac{\omega(f;1/n)}{n^{2+2\alpha}}\sum_{k=j+2}^{n}\frac{(k-j-1)n^{2+2\alpha}}{(k-j-1)^{2+2\alpha}} \leqslant C\omega\left(f;\frac{1}{n}\right).$$

Collecting all the above estimates, (4) follows. □

REMARK 3. From the proof of Theorem 1, we deduce the following pointwise approximation error estimate

$$\begin{aligned}\left|\overline{\mathscr{S}}_n(f;x)-f(x)\right| \leqslant \omega(f;|x-x_j|)+\phi(x-x_j)\frac{\omega(f;|x_{j+1}-x|)}{\phi(x_{j+1}-x)}\\ +C\phi(x-x_j)n^{2+2\alpha}\omega\left(f;\frac{1}{n}\right).\end{aligned} \tag{23}$$

Proof of Theorem 2. Working as in [8] we get (5). Then following [13], Theorem 9.3.2, p. 117, we obtain (6). From [13], Lemma 9.3.4, p. 122, (7) follows. Finally from (5) working as in [20] we deduce (8). □

Proof of Theorem 3. Let $\phi(x) = x^2 \log\left(x^2+1\right)$. Other cases can be treated similarly. First we prove (9). From (12) the operator $\overline{\mathscr{S}}_n$ can be written as

$$\overline{\mathscr{S}}_n(f;x) = \sum_{k=0}^{n} s_k(x) f(x_k),$$

with

$$s_k(x) = \frac{\dfrac{1}{(x-x_k)^2 \log\left((x-x_k)^2+1\right)}}{\sum_{k=0}^{n} \dfrac{1}{(x-x_k)^2 \log\left((x-x_k)^2+1\right)}}.$$

If we prove that

$$\overline{\mathscr{S}}_n(f;x) = f, \quad \text{if } f = \text{constant}, \tag{24}$$

$$\sum_{|x-x_k|\geqslant d_0} s_k(x) = o\left(\frac{1}{n}\right), \; d_0 > 0, \text{ arbitrarily fixed}, \tag{25}$$

$$s_j(x) \geqslant \frac{1}{2}, \quad \text{if } |x-x_j| \leqslant \frac{\delta}{n}, \quad 0<\delta<d_1<1, \tag{26}$$

$$\sum_{|x-x_k|} s_k(x) \leqslant d_2 \frac{\delta^{1+\varepsilon}}{n}, \quad \delta \quad \text{as} \quad \text{above}, \tag{27}$$

with x_j again the closest knot to x and with certain positive fixed reals d_1, d_2, ε, then by [15], Theorem 2.1 (see also [3])

$$\limsup_n \left\|f - \overline{\mathscr{S}}_n(f)\right\| > M(f),$$

with

$$M(f) = \sup_x \left(M(f;x),\; M(f;x) := \limsup_{\tau \to x} \frac{|f(\tau)-f(t)|}{|\tau - x|} \right).$$

Now we prove (24)–(27). Relation (24) holds true by definition. From (22) it follows that

$$\sum_{|x-x_k|>d_0} s_k(x) \leqslant \frac{C}{n^4} \sum_{|x-x_k|>d_0} \frac{1}{(x-x_k)^2 \log\left((x-x_k)^2+1\right)} \leqslant \frac{Cn}{n^4 d_0^4} = o\left(\frac{1}{n}\right),$$

that is (25). Now we prove (26). Again by (22)

$$\begin{aligned}
\sum_{k\neq j} s_k(x) &\leqslant \sum_{|x_k-x|\leqslant 1/2} s_k(x) + \sum_{|x-x_k|>1/2} s_k(x) \\
&\leqslant \frac{4n^4}{2^4 n^4} \sum_{k=1}^{n} \frac{1}{k^4} + \frac{2^5 n}{2^4 n^4} \\
&\leqslant \frac{1}{2}, \; n \geqslant 3,
\end{aligned}$$

and by $s_k(x) \geqslant 0$ and $\sum s_k(x) = 1$ we get (26). Now we prove (27). Indeed

$$\sum_{k \neq j} |x-x_k| s_k(x) \leqslant (x-x_j)^2 \log\left((x-x_j)^2+1\right) \times \left[\sum_{|x-x_k|\leqslant 1/2} + \sum_{|x-x_k|>1/2} \right] \frac{|x-x_k|}{(x-x_k)^2 \log\left((x-x_k)^2+1\right)} \leqslant C \frac{\delta^4}{n^4}\left[n^3+n\right] \leqslant C \frac{\delta^{1+\varepsilon}}{n}, \tag{28}$$

i.e. (27) holds true. Then working as in [21], we get (9). The proof of (10) and (11) is similar to the proof of Theorem 2.2 p. 316 in [21] and we omit it. □

Proof of Theorem 5. From (18) (19) immediately follows. Moreover from (18), by Theorem 4, we get (20). □

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(Received August 15, 2016)

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