

# Stability of the stationary solutions of the Allen–Cahn equation with non–constant stiffness

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## Abstract

We study the solutions of a generalized Allen–Cahn equation deduced from a Landau energy functional, endowed with a non–constant higher order stiffness. We assume the stiffness to be a positive function of the field and we discuss the stability of the stationary solutions proving both linear and local non–linear stability.

*Keywords:* Allen–Cahn equation, gradient equation, phase coexistence, interface, stability

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## 1. Introduction

The behavior of homogeneous phases simultaneously present in a physical system can be described via a *phase–field*  $\phi(x, t)$  depending on the space variable  $x \in \Omega \subset \mathbb{R}^3$  and on the time variable  $t \in [0, \infty)$ . Two particular values of the field represent the two homogeneous phases. These models have been widely used in the study of the *spinodal decomposition* phenomenon [1, 2, 3], namely, the process in which a system undergoing a second order phase transition is suddenly quenched from the disordered high temperature phase into a

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broken-symmetry low temperature state and the evolving field  $\phi(x, t)$  encodes  
 10 the process of separation of the low temperature phases [4].

The evolution equation can be obtained as a gradient equation [5, 6] of a  
 suitable Landau energy functional, namely,

$$\frac{\partial \phi}{\partial t} = -\text{grad } H(\phi) \quad \text{with} \quad H(\phi) := \int_{\Omega} \left[ \frac{1}{2} \varepsilon |\nabla \phi|^2 + W(\phi) \right] dx, \quad (1)$$

where the symbol  $\text{grad}$  means the variational derivative of the functional  $H$ .  
 The function  $W \in C^2(\mathbb{R})$  is the bulk energy of the field  $\phi$  and the *higher-order  
 stiffness* (or, simply, stiffness)  $\varepsilon$  weights the interface contribution represented  
 by the squared-gradient term. The bulk energy  $W$  is usually chosen as a  
 double well with the two minima corresponding to the two phases 0 and 1. If  
 the stiffness is constant and no constraint to the total value of the field  $\phi$  is  
 imposed, the *standard* Allen-Cahn or Ginzburg-Landau equation

$$\frac{\partial \phi}{\partial t} = \varepsilon \Delta \phi - W'(\phi) \quad (2)$$

is found, see, e.g., [7], which was firstly introduced to describe the motion  
 of anti-phase boundaries in crystalline solids. Recently a similar model has  
 been introduced also in the framework of the mechanics of (partially) saturated  
 porous media, see, e.g., [8, 9, 10].

In this paper we consider the case in which the higher-order stiffness is not  
 constant [11, 12, 13], but it is a sufficiently regular positive function of the field,  
 namely,  $\varepsilon \in C^\infty(\mathbb{R})$  such that there is  $\bar{C} > 1$  for which

$$\frac{1}{\bar{C}} \leq \varepsilon(\phi) \leq \bar{C}. \quad (3)$$

15 The fact that  $\varepsilon$  is strictly positive will be crucial in this paper, indeed, it has  
 been proven in different contexts, see, e.g., [13, 14, 15], that if the stiffness  
 is allowed to vanish in correspondence of the homogeneous states, not regular  
 stationary solutions, often called *compactons* [16], can be found.

We also assume that the double well function  $W \in C^\infty(\mathbb{R})$  and satisfies, for

some  $\bar{r} \in (0, 1)$ ,

$$\begin{aligned}
W(0) &= W(1) = 0, \\
W'(r) &> 0 \text{ for } r \in (0, \bar{r}), \\
W'(r) &< 0 \text{ for } r \in (\bar{r}, 1), \\
W''(0) &> 0, \quad W''(1) > 0.
\end{aligned} \tag{4}$$

In the not constant stiffness case, the gradient equation (1) provides the generalized Allen–Cahn equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \varepsilon'(\phi) |\nabla \phi|^2 + \varepsilon(\phi) \Delta \phi - W'(\phi) \tag{5}$$

see, for instance, [13, Appendix A].

It is useful to reformulate the problem in terms of the field

$$u(x, t) := g(\phi(x, t)), \quad g(\phi) := \int_{1/2}^{\phi} \sqrt{\varepsilon(s)} \, ds.$$

Indeed, a straightforward computation shows that the Landau energy functional (1) and the evolution equation (5) expressed in terms of the new field  $u$  read,

$$F(u) = \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + \tilde{W}(u) \right] dx$$

and

$$\frac{\partial u}{\partial t} = -\sigma(u) \operatorname{grad} F(u) = \sigma(u) [\Delta u - \tilde{W}'(u)], \tag{6}$$

20 where  $\tilde{W} := W \circ g^{-1}$  is still double well function and  $\sigma := \varepsilon \circ g^{-1}$  plays the role of a *mobility* for the dynamics.

In the sequel, we shall consider the problem formulated in terms of the field  $u$ . Moreover, via an affine transformation of  $u$ , we can assume that the double well function, again denoted by  $W$ , still assumes its minima in  $u = 0, 1$  and  
25 equations (4) hold (with a different  $\bar{r}$ ).

We focus on the one-dimensional case  $\Omega = \mathbb{R}$  and study the evolution problem

$$\begin{cases} \frac{\partial u}{\partial t} = \sigma(u) \left[ \frac{\partial^2 u}{\partial x^2} - W'(u) \right] \\ u(x, t) \xrightarrow{x \rightarrow -\infty} 0 \quad \text{and} \quad u(x, t) \xrightarrow{x \rightarrow +\infty} 1 \end{cases} \tag{7}$$

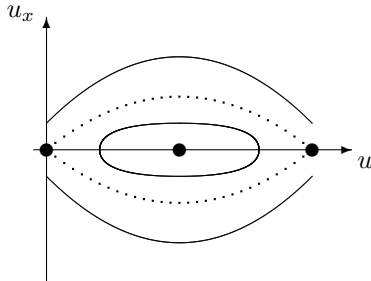


Figure 1: Phase portrait of the stationary equation (8). The dotted lines represent the separatrix.

namely, we solve the equation with Dirichlet boundary conditions 0 and 1 respectively at minus and plus infinity. As  $\sigma > 0$ , the associated stationary problem reads,

$$\begin{cases} u''(x) - W'(u(x)) = 0 \\ u(x) \xrightarrow{x \rightarrow -\infty} 0 \quad \text{and} \quad u(x) \xrightarrow{x \rightarrow +\infty} 1 \end{cases} \quad (8)$$

and is often addressed to in the literature as the problem of finding a *connection* between the two phases on the line [17, 18].

The existence of the solution to the connection problem is an immediate consequence of a simple qualitative analysis of the equation (8) based on the conservation law,

$$\frac{d}{dx} \left[ \frac{1}{2} (u')^2 - W(u) \right] = 0. \quad (9)$$

It is easily seen that there exists a unique solution up to space translations, precisely the function  $\bar{u}(x)$  defined by

$$x = \int_{1/2}^{\bar{u}(x)} \sqrt{\frac{1}{2W(s)}} ds, \quad x \in \mathbb{R}, \quad (10)$$

together with its translates  $\bar{u}_z(x) = \bar{u}(x-z)$ . The phase portrait of the solutions to (9) in the interval  $u \in [0, 1]$  is plotted in figure 1. The separatrix is represented  
30 by a dotted line and its portion in the half-plane  $u' > 0$  corresponds to the solution of the connection problem.

In this paper we discuss the stability of such stationary solutions, hereafter named standing waves or *fronts*, following ideas developed in [19] in the case of the standard Allen–Cahan evolution, see, also, [20]. We prove both linear stability in Section 2 and local non-linear stability in Section 3. Some numerical simulations will be finally discussed in the closing Section 4.

## 2. Linear stability

We first derive the linearized problem associated to (8). We consider the case of the reference solution  $\bar{u}$ , the adaptation to the case of its translates  $\bar{u}^z$  is immediate.

We first emphasize some relevant properties of the connection  $\bar{u}$ . In view of the assumptions on  $W$  and  $\sigma$ , by (10) it follows that  $\bar{u} \in C^\infty(\mathbb{R})$  with  $\bar{u}'(x) > 0$  for any  $x \in \mathbb{R}$ . Moreover,  $\bar{u}$  approaches its limit exponentially fast as  $x \rightarrow \pm\infty$ ; more precisely, there exist  $A_0, A_1 > 0$  such that

$$\lim_{x \rightarrow -\infty} e^{-\alpha_0 x} \bar{u}(x) = A_0, \quad \lim_{x \rightarrow +\infty} e^{\alpha_1 x} (1 - \bar{u}(x)) = A_1, \quad (11)$$

where  $\alpha_0 = \sqrt{W''(0)}$  and  $\alpha_1 = \sqrt{W''(1)}$ .

Finally, as  $\bar{u}' = \sqrt{2W(\bar{u})}$ , by (11) a similar estimate holds for the derivative:

$$\lim_{x \rightarrow -\infty} e^{-\alpha_0 x} \bar{u}'(x) = A'_0, \quad \lim_{x \rightarrow +\infty} e^{\alpha_1 x} \bar{u}'(x) = A'_1, \quad (12)$$

with  $A'_0 = A_0 \sqrt{W''(0)}$  and  $A'_1 = A_1 \sqrt{W''(1)}$ .

Letting  $u = \bar{u} + v$  and using that  $\bar{u}$  is a stationary solution, to the first order in  $v$  the problem (7) yields,

$$\begin{cases} \frac{\partial v}{\partial t} = \sigma(\bar{u}) \left[ \frac{\partial^2 v}{\partial x^2} - W''(\bar{u})v \right] \\ v(x, t) \xrightarrow{x \rightarrow -\infty} 0 \quad \text{and} \quad v(x, t) \xrightarrow{x \rightarrow +\infty} 0 \end{cases} \quad (13)$$

By using again that  $\bar{u}$  is a solution of the stationary problem, it is possible to cancel the term depending on the potential energy  $W$  in the linearized problem. Indeed, by differentiating the stationary equation (8) we have

$$\bar{u}'''(x) = W''(\bar{u}(x))\bar{u}'(x). \quad (14)$$

By substituting this expression in the equation (13) we have that the linearized equation can be written as

$$\frac{\partial v}{\partial t} = Lv \quad (15)$$

with  $L$  the linear operator

$$L = a(x) \left[ \frac{d^2}{dx^2} - \frac{\bar{u}'''(x)}{\bar{u}'(x)} \right], \quad a(x) := \sigma(\bar{u}(x)), \quad (16)$$

on the space  $L^2(\mathbb{R}; dx) \cap C^2(\mathbb{R})$  or  $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ , where  $C_b(\mathbb{R})$  is the space of bounded continuous functions. We remark that, by the assumptions on  $\sigma$ , the  
 45 function  $a(x)$  is smooth, bounded, and such that  $\inf_x a(x) > 0$ .

By (14) it follows that  $\bar{u}'$  is an eigenfunction of  $L$  with eigenvalue zero. Our task is to show that the rest of the spectrum is strictly negative. To this end, we take advantage of the fact that  $\bar{u}' > 0$  to reduce the problem to estimating the rate of convergence to equilibrium of a suitable diffusion process.

We notice that  $L\bar{u}' = \bar{u}'\mathcal{L}$ , where

$$\mathcal{L} = a(x) \left[ \frac{d^2}{dx^2} - U'(x) \frac{d}{dx} \right], \quad (17)$$

with

$$U(x) = -2 \log \bar{u}'(x). \quad (18)$$

The operator  $\mathcal{L}$  is the generator of a diffusion. More precisely, it is the generator of the Markov semigroup associated to the dynamics on  $\mathbb{R}$  given by the solution to the stochastic differential equation,

$$dx = a(x)U'(x) dt + \sqrt{2a(x)} dB,$$

where  $B = B(t)$  is a 1-dimensional Wiener process. Since  $a > 0$ , by elliptic regularity the transition probability  $P_t(x, dy)$  admits a smooth density  $p_t(x, y)$  which is the fundamental solution associated to  $\mathcal{L}$ , i.e.,

$$\begin{cases} \frac{\partial p_t}{\partial t}(x, \cdot) = \mathcal{L}p_t(x, \cdot) \\ p_0(x, y) = \delta_y(x) \end{cases} \quad (19)$$

50 Moreover,  $p_t(x, y) > 0$  for any  $x, y \in \mathbb{R}$  and  $t > 0$ , hence the associated Markov process is irreducible aperiodic.

Under these conditions, the process has a unique (up to multiplication with a constant) invariant measure with a smooth positive density [21]. This density is solution to the stationary Fokker-Plank equation  $\mathcal{L}^* \varrho = 0$ , where  $\mathcal{L}^*$  is the  $L^2$ -adjoint of  $\mathcal{L}$ . An explicit computation shows that  $\varrho(x) = \bar{u}'(x)^2 a(x)^{-1} = e^{-U(x)} a(x)^{-1}$ . By (12),  $\varrho$  is integrable so that we obtain an invariant probability measure,

$$\mu(dx) = \frac{\bar{u}'(x)^2 a(x)^{-1} dx}{\int_{\mathbb{R}} \bar{u}'(x)^2 a(x)^{-1} dx'}.$$

It is worthwhile to notice that the process is in fact reversible with respect to  $\mu$ , since by integration by parts,

$$\int \psi \mathcal{L} \varphi d\mu = \int (\mathcal{L} \psi) \varphi d\mu = - \int \sigma(\bar{u}) \psi' \varphi' d\mu.$$

To study the convergence  $P_t \rightarrow \mu$  as  $t \rightarrow \infty$  we apply a classical result based on the existence of a Liapunov function. By definition, a Liapunov function is a function  $V \geq 1$  with compact level sets (i.e.,  $V(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ ). We claim there exists a Liapunov function  $V$  for which there are a compact set  $K \subset \mathbb{R}$  and constants  $r > 0$  and  $b < \infty$  such that

$$\mathcal{L}V(x) \leq -rV(x) + b\mathbb{1}_K(x). \quad (20)$$

Then, see, e.g., [21, Thm. 8.7], introducing the weighted norms,

$$\|\varphi\|_{V,\infty} = \sup_{x \in \mathbb{R}} \frac{|\varphi(x)|}{V(x)}, \quad \|\varphi\|_{V,p} = \left\| \frac{\varphi}{V} \right\|_{L^p(\mathbb{R}; dx)}$$

and letting

$$P_t \varphi(x) = \int p_t(x, y) \varphi(y) dy, \quad \mu(\varphi) = \int \varphi d\mu,$$

there are constants  $C > 0$  and  $\alpha > 0$  such that the bounds

$$\begin{aligned} \|P_t \varphi - \mu(\varphi)\|_V &\leq C e^{-\alpha t} \|\varphi - \mu(\varphi)\|_V, \\ \|P_t \varphi - \mu(\varphi)\|_{V,p} &\leq C e^{-\alpha t} \|\varphi - \mu(\varphi)\|_{V,p} \end{aligned} \quad (21)$$

hold for every measurable function  $\varphi$  with finite weighted norm.

We now prove the claim on the existence of  $V$  for which (20) holds. By (12), the function  $V = \gamma e^{U/2} = \gamma(\bar{u}')^{-1}$ , with  $\gamma = \max \bar{u}'$ , is Liapunov and, by an

explicit computation,

$$\mathcal{L}V = \frac{a}{2} \left( U'' - \frac{1}{2}(U')^2 \right) V. \quad (22)$$

Now, by (8), (14), in view of (11), and (12),

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\bar{u}''(x)}{\bar{u}'(x)} &= \sqrt{W''(0)}, & \lim_{x \rightarrow +\infty} \frac{\bar{u}''(x)}{\bar{u}'(x)} &= \sqrt{W''(1)}, \\ \lim_{x \rightarrow -\infty} \frac{\bar{u}'''(x)}{\bar{u}'(x)} &= W''(0), & \lim_{x \rightarrow +\infty} \frac{\bar{u}'''(x)}{\bar{u}'(x)} &= W''(1). \end{aligned}$$

By computing  $U'$  and  $U''$  and using the above limits, a straightforward computation shows,

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{a(x)}{2} \left( U''(x) - \frac{1}{2}U'(x)^2 \right) &= -\sigma(0)W''(0), \\ \lim_{x \rightarrow +\infty} \frac{a(x)}{2} \left( U''(x) - \frac{1}{2}U'(x)^2 \right) &= -\sigma(1)W''(1). \end{aligned}$$

Therefore, in view of (22), the function  $V$  clearly satisfies (20), e.g., with  $r = \frac{1}{2}W''(0) \wedge W''(1)$  and  $K = [-R, R]$  so large that  $V > 4$  and  $\frac{1}{2}a \left( U'' - \frac{1}{2}(U')^2 \right) > r$  outside  $K$ .

The desired spectral properties of the operator  $L$  can be now easily deduced from (21). To this end, it is useful to work in the Hilbert space  $L^2(\mathbb{R}; a(x)^{-1}dx)$ , where  $L$  is symmetric. We denote the inner product by  $\langle \cdot, \cdot \rangle_a$  and the corresponding norm by  $\| \cdot \|_{a, L^2}$ . We observe that since  $V = \gamma(\bar{u}')^{-1}$ , for every  $v \in C_b(\mathbb{R})$  or  $v \in L^2(\mathbb{R}; dx)$  the function  $\varphi_v := v/\bar{u}'$  has the corresponding weighted norms finite. Moreover, as  $e^{Lt} = \bar{u}' P_t(\bar{u}')^{-1}$ ,

$$e^{Lt}v - \langle v, q \rangle_a q = [P_t \varphi_v - \mu(\varphi_v)] \bar{u}',$$

where  $q(x) = \|\bar{u}'\|_{a, L^2}^{-1} \bar{u}'(x)$  is the normalized eigenfunction of  $L$  with eigenvalue zero. Therefore, by (21), it is easily deduced that there is a constant  $C_0 > 0$  such that

$$\begin{aligned} \|e^{Lt}v - \langle v, q \rangle_a q\|_{L^2} &\leq C_0 e^{-\alpha t} \|v - \langle v, q \rangle_a q\|_{L^2}, \\ \|e^{Lt}v - \langle v, q \rangle_a q\|_{\infty} &\leq C_0 e^{-\alpha t} \|v - \langle v, q \rangle_a q\|_{\infty}, \end{aligned} \quad (23)$$

where, in the first estimate, we used that, in view of the assumptions on  $\sigma$ , the norms  $\| \cdot \|_{a, L^2}$  and  $\| \cdot \|_{L^2}$  are equivalent.



An estimate like (23) holds also with respect to the  $H^1$ -norm. This is quite standard but we sketch the proof for the sake of completeness. To this purpose, we shall use the following estimates on the fundamental solution  $e^{Lt}(x, y)$ , which can be deduced by exploiting the parametric method to construct the fundamental solution, see [22]. Letting

$$e^{Lt}(x, y) = \sqrt{\frac{a(y)}{4\pi t}} \exp\left(-\frac{a(y)(x-y)^2}{4t}\right) + Q_t(x, y), \quad (24)$$

for any  $T > 0$  and  $\kappa > \sup_x a(x)$  there exists a constant  $C_T > 0$  such that, for any  $t \in (0, T]$  and  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} Q_t(x, y) &\leq C_T \exp\left(-\kappa \frac{(x-y)^2}{4t}\right), \\ \left|\frac{\partial Q_t(x, y)}{\partial x}\right| &\leq \frac{C_T}{\sqrt{t}} \exp\left(-\kappa \frac{(x-y)^2}{4t}\right). \end{aligned} \quad (25)$$

Let now  $v \in H^1$  that, without loss of generality, we identify with its continuous representative. Clearly  $e^{Lv} - \langle v, q \rangle_a q = e^{Lv}_\perp$ , where  $v_\perp = v - \langle v, q \rangle_a q$  is the component of  $v$  orthogonal to  $q$  in  $L^2(\mathbb{R}; a(x)^{-1} dx)$ . We estimate  $\|e^{Lv}_\perp\|_{H^1}$  by analyzing separately the case  $t \in (0, 1]$  and  $t > 1$ . In view of (23), we only need to study the  $L^2$ -norm of  $\frac{\partial(e^{Lv}_\perp)}{\partial x}$ .

If  $t \in (0, 1]$ , in view of the identity

$$\frac{\partial}{\partial x} \frac{e^{-\frac{a(y)(x-y)^2}{4t}}}{\sqrt{4\pi a(y)t}} + \frac{\partial}{\partial y} \frac{e^{-\frac{a(y)(x-y)^2}{4t}}}{\sqrt{4\pi a(y)t}} = \frac{a'(y)e^{-\frac{a(y)(x-y)^2}{4t}}}{a(y)\sqrt{4\pi a(y)t}} \left(\frac{1}{2} - \frac{a(y)^2(x-y)^2}{4t}\right),$$

by (24) and an integration by parts we get,

$$\begin{aligned} \frac{\partial(e^{Lv}_\perp)(x)}{\partial x} &= \int \frac{e^{-\frac{a(y)(x-y)^2}{4t}}}{\sqrt{4\pi a(y)t}} v'_\perp(y) dy \\ &+ \int \frac{a'(y)e^{-\frac{a(y)(x-y)^2}{4t}}}{a(y)\sqrt{4\pi a(y)t}} \left(\frac{a(y)^2(x-y)^2}{4t} - \frac{1}{2}\right) v_\perp(y) dy \\ &+ \int \frac{\partial Q_t(x, y)}{\partial x} v_\perp(y) dy. \end{aligned}$$

By (25)<sub>1</sub>, for some positive constants  $C_1$  and  $\lambda$ ,

$$\left|\frac{\partial(e^{Lv}_\perp)(x)}{\partial x}\right| \leq \int \frac{C_1 e^{-\frac{(x-y)^2}{4\lambda t}}}{\sqrt{4\pi\lambda t}} (|v_\perp(y)| + |v'_\perp(y)|) dy,$$

which implies, recalling that the heat kernel is a contraction in  $L^2$ ,

$$\left\| \frac{\partial(e^{Lt}v_\perp)}{\partial x} \right\|_{L^2} \leq C_1 \|( |v_\perp| + |v'_\perp| )\|_{L^2} \leq 2C_1 \|v\|_{H^1}.$$

If  $t > 1$  we write,

$$\frac{\partial(e^{Lt}v_\perp)(x)}{\partial x} = \int \frac{\partial e^L(x,y)}{\partial x} (e^{L(t-1)}v_\perp)(y) dy$$

and observe that, for any  $f \in L^2(\mathbb{R}; dx)$ , by (24) and (25)<sub>1</sub> for  $t = 1$ , for some  $C_2 > 0$ ,

$$\left\| \frac{\partial(e^L f)}{\partial x} \right\|_{L^2} \leq C_2 \|f\|_{L^2}.$$

Therefore, in view of (23)<sub>1</sub>, setting  $C_3 = C_0 C_2$ ,

$$\left\| \frac{\partial(e^{Lt}v_\perp)}{\partial x} \right\|_{L^2} \leq C_2 \|e^{L(t-1)}v_\perp\|_{L^2} \leq C_3 e^{-\alpha(t-1)} \|v_\perp\|_{L^2} \leq C_3 e^{-\alpha t} \|v_\perp\|_{H^1}.$$

In conclusion, there is a constant  $C_4 > 0$  such that

$$\|e^{Lt}v - \langle v, q \rangle_a q\|_{H^1} \leq C_4 e^{-\alpha t} \|v - \langle v, q \rangle_a q\|_{H^1}. \quad (26)$$

### 3. Non-linear stability

In this section we prove the one-dimensional manifold  $\mathcal{M} := \{\bar{u}_z : z \in \mathbb{R}\}$  of  
65 the translates of  $\bar{u}$  is asymptotically stable. This is the content of two theorems.

The first result follows from the linear analysis done in the previous section. Indeed,  $\lambda = 0$  is a semisimple eigenvalue of the (unbounded) operator  $L$  on  $H^1$ , and the estimate (26) shows further that the remaining part of the spectrum is strictly contained in  $\{\lambda : \Re \lambda < 0\}$ . This implies, see, e.g., [23, Proposition 4.1],  
70 the following stability theorem.

**Theorem 1.** *Consider the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \sigma(u) \left[ \frac{\partial^2 u}{\partial x^2} - W'(u) \right], \\ u(x, 0) = u_0(x). \end{cases} \quad (27)$$

For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if the initial datum  $u_0$  satisfies  $\|u_0 - \bar{u}_{\zeta_0}\|_{H^1} \leq \delta$  for some  $\zeta_0 \in \mathbb{R}$  then the solution  $u(t) = u(\cdot, t)$  exists for all

$t \geq 0$  and  $\text{dist}_{H^1}(u(t), \mathcal{M}) \leq \varepsilon$  for any  $t \geq 0$ . Moreover, if  $\delta$  is small enough, there is  $z_0 \in \mathbb{R}$  such that, for some  $\beta > 0$ ,

$$\lim_{t \rightarrow +\infty} e^{\beta t} \|u(t) - \bar{u}_{z_0}\|_{H^1} = 0$$

and  $|z_0 - \zeta_0| \rightarrow 0$  if  $\delta \rightarrow 0$ .

In Theorem 2 below, we instead prove convergence in sup-norm to a suitable front of those solutions whose initial datum approximately resembles a front. The proof is an adaptation of the argument in the seminal paper [19], where  
75 the case of constant mobility is concerned. In [19] it is also shown that the limit is reached exponentially fast in time. Due to the non-constant mobility, we are able to prove this rate of convergence only under the more restrictive assumption on the initial datum of Theorem 1.

**Theorem 2.** *Consider the Cauchy problem Eq. (27) and assume that  $u_0$  is a piecewise continuous function such that  $0 \leq u_0 \leq 1$ . Then, there exists a unique bounded classical solution  $u(x, t)$  and  $0 \leq u(x, t) \leq 1$ . Furthermore, if*

$$\limsup_{x \rightarrow -\infty} u_0(x) < \bar{r}, \quad \liminf_{x \rightarrow +\infty} u_0(x) > \bar{r}, \quad (28)$$

with  $\bar{r}$  as in (4), there is  $z_0 \in \mathbb{R}$  such that

$$\lim_{t \rightarrow +\infty} \|u(\cdot, t) - \bar{u}_{z_0}\|_{\infty} = 0.$$

*Proof.* We split the proof into three steps.

80 *Step 1: Existence and uniqueness.* The existence and uniqueness in large of bounded classical solution  $u(x, t)$  and that  $0 \leq u(x, t) \leq 1$  follow from a priori estimates and comparison theorems for quasilinear parabolic equations, see, e.g., [24].

*Step 2: A priori estimates.* There are  $z_1, z_2 \in \mathbb{R}$  and  $q_0, \mu > 0$  such that

$$\bar{u}_{z_1}(x) - q_0 e^{-\mu t} \leq u(x, t) \leq \bar{u}_{z_2}(x) + q_0 e^{-\mu t}. \quad (29)$$

We prove the lower bound, the other can be obtained similarly. We fix  $q_0 > 0$  such that

$$q_0 < \bar{r} < 1 - q_0 < \liminf_{x \rightarrow +\infty} u_0(x),$$

so that there exists  $z_*$  large enough for which  $\bar{u}_{z_*}(x) - q_0 \leq u_0(x)$  for any  $x \in \mathbb{R}$ .

We then look for  $z_1 > z_*$  and  $\mu > 0$  such that, setting

$$q(t) = q_0 e^{-\mu t}, \quad z(t) = z_1 + (z_* - z_1) e^{-\mu t},$$

the function

$$u_-(x, t) := \max\{0; \bar{u}_{z(t)}(x) - q(t)\}$$

is a subsolution. This clearly proves the estimate since  $u_-(x, t) \leq u(x, t)$  by the

85 Comparison Theorem and  $\bar{u}_{z_1}(x) - q_0 e^{-\mu t} \leq u_-(x, t)$  by construction.

Since  $u = 0$  is a solution,  $u_-$  is a subsolution if  $u_- > 0$  implies

$$N[u_-] := \frac{\partial u_-}{\partial t} - \sigma(u_-) \left[ \frac{\partial^2 u_-}{\partial x^2} - W'(u_-) \right] \leq 0.$$

Now, if  $u_- > 0$ , we have  $\bar{u}_z > q$  and

$$N[u_-] = -\dot{z}\bar{u}'_z - \dot{q} + \sigma(\bar{u}_z - q)[W'(\bar{u}_z - q) - W'(\bar{u}_z)],$$

where we used that  $\bar{u}''_z = W'(\bar{u}_z)$  and omitted the explicit dependence on  $x$  and  $t$ . Since  $q_0 < \bar{r} < 1 - q_0$ ,  $W'(0) = W'(1) = 0$ , and  $W''(0)$  and  $W''(1)$  are positive, there is  $C > 0$  such that  $W'(1 - q) - W'(1) < -2Cq$  and  $W'(0) - W'(q) \leq -2Cq$  for any  $q \in [0, q_0]$ . Therefore, by continuity, there exists  $\delta < q_0$  such that  $W'(u - q) - W'(u) \leq -Cq$  for any  $(u, q) \in [1 - \delta, 1] \times [0, q_0]$  and for any  $0 \leq q \leq u \leq \delta$ . We conclude that whenever  $u_z \in [1 - \delta, 1]$  or  $q \leq \bar{u}_z \leq \delta$  we have,

$$N[u_-] \leq -\dot{z}\bar{u}'_z - \dot{q} - (\bar{C}/C)q \leq (\mu - \bar{C}/C)q,$$

with  $\bar{C}$  as in Eq. (3) and where we used that  $\dot{q} = -\mu q$ ,  $\dot{z} > 0$ , and  $\bar{u}'_z > 0$ . On the other hand, from the properties of  $\bar{u}$  and the smoothness of  $W$ , there is  $C_5 > 0$  such that  $\bar{u}' \geq C_5$  whenever  $\delta \leq \bar{u} \leq 1 - \delta$  and  $W'(\bar{u}_z - q) - W'(\bar{u}_z) \leq C_5 q$  for any  $q \in [0, q_0]$ . Therefore, if  $\delta \leq \bar{u}_z \leq 1 - \delta$ ,

$$N[u_-] \leq -C_5 \dot{z} - \dot{q} + \bar{C}C_5 q \leq [C_5 \mu (z_* - z_1) + (\mu + \bar{C}C_5)q_0] e^{-\mu t},$$

where we used the upper bound in Eq. (3) and that  $\dot{z} = -\mu(z_* - z_1)e^{-\mu t}$ . Therefore, choosing  $\mu = \bar{C}/C$  and  $z_1 = z_* + q_0(\mu + C_5)/(\mu C_5)$  we obtain  $N[u_-] \leq 0$  whenever  $u_- > 0$ .

An immediate corollary of the proof of Eq. (29) is the stability of fronts.

90 Indeed, suppose that the initial datum satisfies also  $|u_0(x) - \bar{u}_{z_0}(x)| \leq \varepsilon$  for some  $z_0 \in \mathbb{R}$  and small  $\varepsilon > 0$ . Then, it is possible to choose  $q_0 = O(\varepsilon)$  and  $|z_* - z_0| = O(\varepsilon)$ , so that also  $|z_1 - z_0| + |z_2 - z_0| = O(\varepsilon)$ . In conclusion,  $|u(x, t) - \bar{u}_{z_0}(x)| = O(\varepsilon)$  independently of  $x \in \mathbb{R}$  and  $t \geq 0$ .

*Step 3: Convergence to fronts.* We now prove that there exists  $z_0 \in \mathbb{R}$  such that, uniformly in  $x \in \mathbb{R}$ ,

$$\lim_{t \rightarrow +\infty} |u(x, t) - \bar{u}_{z_0}(x)| = 0. \quad (30)$$

We first observe that, for a suitable constant  $C_6 > 0$ ,

$$|1 - u(x, t)| + \left| \frac{\partial u}{\partial t}(x, t) \right| + \left| \frac{\partial u}{\partial x}(x, t) \right| + \left| \frac{\partial^2 u}{\partial x^2}(x, t) \right| \leq C_6 (e^{-\alpha_1 x} + e^{-\mu t}), \quad (31)$$

for  $x > 0$ ,  $t \geq 0$ , and

$$|u(x, t)| + \left| \frac{\partial u}{\partial t}(x, t) \right| + \left| \frac{\partial u}{\partial x}(x, t) \right| + \left| \frac{\partial^2 u}{\partial x^2}(x, t) \right| \leq C_6 (e^{\alpha_0 x} + e^{-\mu t}), \quad (32)$$

for  $x < 0$  and  $t \geq 0$ . The estimates in Eqs. (31) and (32) concerning the undif-  
95 ferentiated function  $u$  come from Eqs. (11) and (29). Since  $W'(0) = W'(1) = 0$ , these estimates also imply that  $|W'(u(x, t))| \leq C(e^{\alpha_0 x} + e^{-\mu t})$  for some  $C > 0$ . The bounds on the differentiated function then follow from standard Schauder estimates for quasilinear parabolic equations, see, e.g., [24].

Then, by arguing exactly as in [19, Lemma 4.4], we conclude that for each  
100  $\delta > 0$  the orbit  $\{u(\cdot, t) : t \geq \delta\}$  is relatively compact in  $C^2(\mathbb{R})$ .

To prove Eq. (30), we let  $v$  be the following truncation of  $u$ ,

$$v(x, t) = \begin{cases} u(x, t) & \text{for } |x| \leq t, \\ 0 & \text{for } x \leq -t - 1, \\ 1 & \text{for } x \geq t + 1, \end{cases}$$

where the interpolation is done smoothly in such a way that the estimates Eqs. (31) and (32) hold also for  $v$  (possibly, with a larger constant  $C_6$ ).

We next introduce the Lyapunov function

$$F(v) = \int_{\mathbb{R}} \left[ \frac{1}{2} \left| \frac{\partial v}{\partial x} \right|^2 + W(v) \right] dx.$$

It is bounded uniformly in  $t$  in view of Eqs. (31), (32), and since  $W(0) = W(1) = 0$ . Moreover, letting  $f(t) := F(v(\cdot, t))$ , after integration by parts it is easily shown that  $f$  is differentiable and

$$\dot{f}(t) = - \int_{\mathbb{R}} \left( \frac{\partial^2 v}{\partial x^2} - W'(v) \right) \frac{\partial v}{\partial t} dx.$$

From the definition of  $v$  it follows that  $\dot{f}(t) = Q(t) + R(t)$  with

$$Q(t) = - \int_{\mathbb{R}} \sigma(v) \left( \frac{\partial^2 v}{\partial x^2} - W'(v) \right)^2 dx \leq 0$$

and, for a some  $C_7 > 0$ ,

$$\begin{aligned} |R(t)| &= \left| \int_{t \leq |x| \leq t+1} \left( \frac{\partial^2 v}{\partial x^2} - W'(v) \right) \frac{\partial v}{\partial t} dx \right| \\ &\leq C_7 \int_{-t-1}^{-t} (e^{\alpha_0 x} + e^{-\mu t})^2 dx + C_7 \int_t^{t+1} (e^{-\alpha_1 x} + e^{-\mu t})^2 dx. \end{aligned}$$

Therefore  $\lim_{t \rightarrow +\infty} R(t) = 0$  so that  $\limsup_{t \rightarrow +\infty} \dot{f}(t) \leq 0$ . Hence, since  $f(t)$  is bounded uniformly in time, there must exist a diverging sequence  $\{t_n\}$  such that  $\lim_{n \rightarrow \infty} \dot{f}(t_n) =$

105 0. Since  $\lim_{t \rightarrow +\infty} R(t) = 0$ , we finally obtain  $\lim_{n \rightarrow \infty} Q(t_n) = 0$ .

Now, by compactness, we can extract a subsequence  $\{t_{n_k}\}$  such that  $u(\cdot, t_{n_k})$ , and hence also  $v(\cdot, t_{n_k})$ , converges in  $C^2(\mathbb{R})$  to some function  $\bar{v}$ . Since  $\lim_{k \rightarrow \infty} Q(t_n) =$   
0 we deduce that

$$\begin{aligned} &\int_{|x| \leq K} \sigma(v) (\bar{v}'' - W'(\bar{v}))^2 dx \\ &= \lim_{k \rightarrow \infty} \int_{|x| \leq K} \sigma(v) \left( \frac{\partial^2 v(\cdot, t_{n_k})}{\partial x^2} - W'(v(\cdot, t_{n_k})) \right)^2 dx \leq \lim_{k \rightarrow \infty} Q(t_n) = 0 \end{aligned}$$

for all  $K > 0$ , so that  $\bar{v}'' - W'(\bar{v}) = 0$ . Since we also have  $\bar{v}(-\infty) = 0$  and  $\bar{v}(+\infty) = 1$ , by the uniqueness of fronts we deduce  $\bar{v} = \bar{u}_{z_0}$  for some  $z_0 \in \mathbb{R}$ .

To conclude the proof of Eq. (30), we simply recall that we have already proved the stability of fronts, so that the convergence of  $u(\cdot, t_{n_k})$  to  $\bar{u}_{z_0}$  implies that of

110 the whole trajectory  $u(\cdot, t)$ .  $\square$

#### 4. Numerical check of stability results

In this section we discuss the result of some numerical simulations aiming to check the stability properties stated in Theorems 1 and 2. The strategy

will be the following: i) we consider the particular solution  $\bar{u} \in \mathcal{M}$  defined in  
 115 Eq. (10) of the stationary equation Eq. (8) associated with the problem Eq. (7);  
 ii) we consider a profile obtained by modifying locally  $\bar{u}$  (type I) and a profile  
 obtained by adding an oscillatory term to  $\bar{u}$  in such a way that the assumptions  
 of Theorem 2 are satisfied (type II); iii) the problem Eq. (7) is solved numerically  
 via the finite elements method and the  $H^1$ ,  $L^2$ , and  $L^\infty$  norms of the difference  
 120 between the solution at time  $t$  and  $t - \Delta t$  are computed as a function of time.

Our results will be plotted in figures made of two panels: in the left one we  
 will show the solution of the problem Eq. (7) at different values of time; in the  
 right panel the norm ( $H^1$ ,  $L^2$ , and  $L^\infty$ ) of the difference between the solution  
 at time  $t$  and  $t - \Delta t$ . Different cases will correspond to different choices of the  
 125 initial profile.

In this section the function  $W$  is chosen as

$$W(u) = \frac{1}{2}u^2(1 - u^2), \quad (33)$$

so that the stationary solution Eq. (10) is

$$\bar{u}(x) = \frac{e^x}{1 + e^x}. \quad (34)$$

The stiffness function will be

$$\sigma(u) = A + B \sin(2\pi\nu u) \quad (35)$$

with  $A, B, \nu \geq 0$ ; note that for  $\nu > 1/2$  we shall choose  $B < A$  so that the  
 condition  $\sigma(u) > 0$  is safely satisfied. In the following picture we shall report  
 results for the case  $\nu = 5/2$ ; other possible values of  $\nu$  has been considered and  
 similar results have been found.

130 The type I initial condition will be a continuous function obtained by sub-  
 stituting the  $\bar{u}$  profile in the interval  $[a, b]$  with an arc of parabola with vertex  
 in  $c$ , with  $a < c < b$ . The type II initial condition will be a continuous function  
 obtained by adding to  $\bar{u}$  the function  $D \sin[2\pi\mu(x - d)]$  in the interval  $[d, \infty)$ ,  
 with  $0 < D \leq 1/4$ ,  $d \geq 1$ , and  $\mu > 0$ .

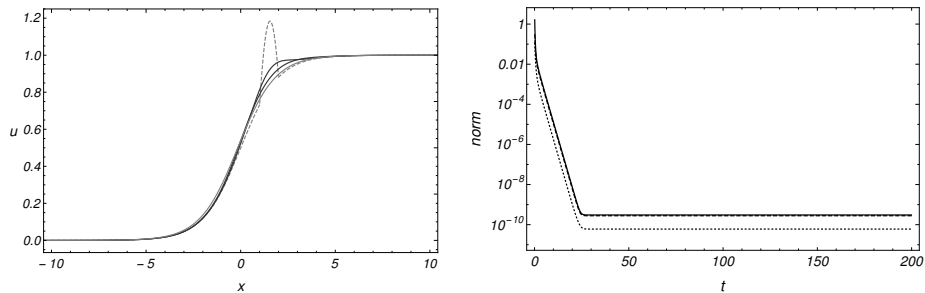


Figure 2: Solution of problem Eq. (7) for the initial condition of type I with  $a = 1$ ,  $b = 2$ , and  $c = 1.55$  and stiffness Eq. (35) with  $A = 1$ ,  $B = 1/2$ , and  $\nu = 5/2$ . Left panel: profile  $u(x, t)$  at times  $t = 0$  (dashed gray), 5 (solid black), 10 (solid black), and the stationary infinite time limit (solid gray). Right panel: semi-logarithmic plot of  $H^1$  norm (solid line),  $L^2$  norm (dashed line),  $L^\infty$  norm (dotted line) of the difference between the solution at time  $t$  and  $t - \Delta t$ .

135 Eq. (7) is solved via the finite element method powered with the Newton–  
 Raphson algorithm over a finite domain. The boundary conditions are of Neu-  
 mann type. The spatial discretization of the domain is based on quadratic shape  
 functions with one hundred elements. As time discretization we use the back-  
 ward Euler scheme with time step  $\Delta t = 1$ . The algorithm is stopped when the  
 140  $L^2$  norm of the difference between the current profile and the one computed one  
 time step before is smaller than  $10^{-4}$ .

In Figure 2 an initial condition of type I is used (see the caption for more de-  
 tails) and, as already remarked, for the stiffness Eq. (35) we chose the frequency  
 $\nu = 5/2$ . The solution of the equation converges to a solution in  $\mathcal{M}$  slightly  
 145 different from  $\bar{u}$  which, we recall, has been perturbed to construct the initial  
 condition. The behavior of the norm as a function of time is compatible with  
 an exponential decay and show three different regimes. An initial quick drop  
 is followed by a linear decrease and, finally, a long constant behavior onsets.  
 As shown in Figure 3, where we report the profile corresponding to the times  
 150 where the steepness of the norm changes abruptly, during the first part of the  
 evolution the parabolic bump is removed, whereas in the second part the profile



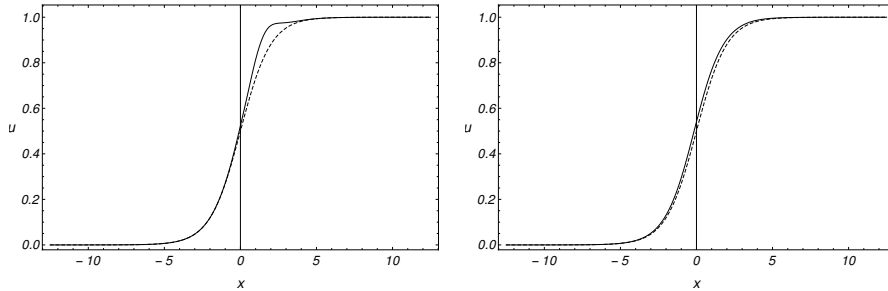


Figure 3: Solid lines represent the profiles at times  $t = 5$  (left) and  $t = 25$  (right) of the evolution described in Figure 2. The dashed line is the stationary profile already reported in Figure 2.

is slowly deformed to reach the stationary shape.

In Figure 4 an initial condition of type II is used (see the caption for more details) and for the stiffness Eq. (35) we chose, again, the frequency  $\nu = 5/2$ .  
 155 The solution of the equation quickly converges to a monotone interface profile with constant plateau lower than one. Then, the solution starts to increase the value of the plateau slowly approaching the correct limiting value one. The behavior of the norm as a function of time is compatible with an exponential decay, also in this case. The three regimes already noted in Figure 2 can be  
 160 observed and explained as we did above. Indeed, as illustrated in Figure 5 in the first part of the evolution, corresponding to the sudden drop of the norms, the ripples are removed and then the profile linearly converges to the stationary profile.

Finally, we note that our simulations have been performed on a finite interval,  
 165 whereas the theoretical discussion of the previous sections refer to the evolution on the real infinite line. In Figure 6 we, thus, compare results obtained in simulations referring to different sizes of the space interval. Since, the stationary profiles obtained for the three different cases coincide (see the left panel), we can infer that our simulations capture, indeed, the infinite volume behavior.  
 170 Moreover, we also note that by suitably shifting the exact profile, as shown in the right panel, we obtain a perfect match with the stationary profiles obtained

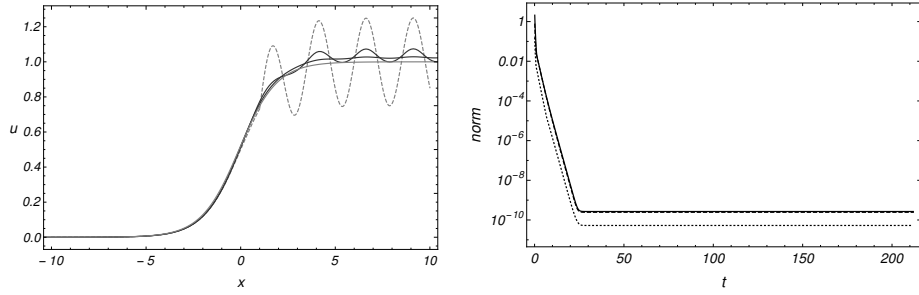


Figure 4: Solution of problem (7) for the initial condition of type II with  $d = 1$ ,  $D = 1/4$ , and  $\mu = 0.4$  and stiffness Eq. (35) with  $A = 1$ ,  $B = 1/2$ , and  $\nu = 5/2$ . Left panel: profile  $u(x, t)$  at times  $t = 0$  (dashed gray), 5 (solid black), 10 (solid black), and the stationary infinite time limit (solid gray). Right panel: semi-logarithmic plot of  $H^1$  norm (solid line),  $L^2$  norm (dashed line),  $L^\infty$  norm (dotted line) of the difference between the solution at time  $t$  and  $t - \Delta t$ .

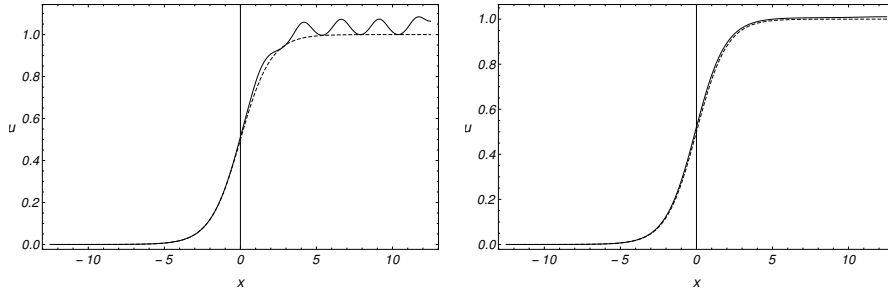


Figure 5: Solid lines represent the profiles at times  $t = 5$  (left) and  $t = 25$  (right) of the evolution described in Figure 4. The dashed line is the stationary profile already reported in Figure 4.

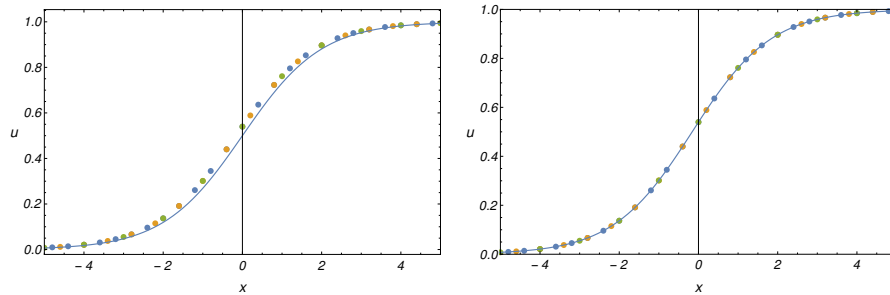


Figure 6: On the left, exact solution (34) and stationary profiles obtained for the simulation discussed in Figure 2 on the intervals  $[-10, 10]$  (yellow dots),  $[-12.5, 12.5]$  (blue dots), and  $[-15, 15]$  (green dots). On the right: the same, but the exact solution has been shifted by the amount 0.15881 for a perfect match.

on the finite space interval.

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