

HOMOGENIZATION OF A MODIFIED BIDOMAIN MODEL INVOLVING IMPERFECT TRANSMISSION

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ABSTRACT. We study, by means of the periodic unfolding technique, the homogenization of a modified bidomain model, which describes the propagation of the action potential in the cardiac electrophysiology. Such a model, allowing the presence of pathological zones in the heart, involves various geometries and non-standard transmission conditions on the interface between the healthy and the damaged part of the cardiac muscle.

KEYWORDS: Homogenization, time-periodic unfolding, bidomain models, imperfect transmission.

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1. INTRODUCTION

In the last years, the mathematical modeling of the electrical activity of the heart was a topic of major interest in biomedical research. A better understanding of the complex bioelectrical processes involved in the activity of the heart is a key issue in order to find new drugs and diagnostic techniques, being well-known that a huge part of the heart diseases is produced by some disorders of its electrical activity.

One of the most well-known mathematical models in cardiac electrophysiology is the so-called *bidomain model* (see, e.g., [24, 37, 38] and the references therein; see, also, the references quoted in [25, Introduction]). In this model, at a macroscopic scale, the electric activity of the heart is governed by a system of two degenerate reaction-diffusion partial differential equations for the averaged intra-cellular and, respectively, extra-cellular electric potentials, along with the transmembrane potential, coupled in a nonlinear manner to ordinary differential equations describing the dynamics of the

ion channels. In such a model, the cardiac tissue is represented, at a macroscopic scale, despite its discrete cellular structure, as the superposition of two continuous media, called the *intra-cellular* and, respectively, the *extra-cellular domain*, coexisting at each point of the heart tissue and connected through a distributed continuous cellular membrane.

Several ionic models are considered in the literature for describing the cellular membrane dynamics, starting with the famous Hodgkin-Huxley formalism and continuing with more and more complex models (see, for instance, [25, 28, 30, 37]). The well-posedness of the bidomain model has been studied, for different nonlinear ionic models and by using different techniques, by several authors (see, for instance, [13, 18, 41, 42, 45]).

The bidomain model can be obtained from a corresponding appropriate microscopic one by homogenization techniques (see, among others, [7, 8, 11, 18, 25, 32, 38, 41, 44]). Such a model is widely recognized as being the standard model used in cardiac electrophysiology for describing the propagation of the action potential in a perfectly healthy cardiac tissue, but it is no longer valid in pathological situations, in which the heart contains electrically passive zones of fibrotic tissue, collagen or fat, as observed for instance in scars, inflammations, ischemic or rheumatic heart diseases, etc. Thus, it is important to find a suitable mathematical model that accounts for the presence of pathological zones in the heart. Such a model was proposed in [14, 16, 26, 27, 46]; it takes into account the presence in the cardiac tissue of damaged zones, called *diffusive inclusions* and assumed to be passive electrical conductors.

In the above mentioned papers, at the mesoscopic scale, the heart tissue is considered to be a periodic structure obtained by inserting in a healthy tissue a set of periodically distributed diffusive inclusions. From the mathematical point of view, we have a bidomain system coupled with a diffusion equation. More precisely, the model consists of a degenerate reaction-diffusion system of partial differential equations modeling the intra-cellular and, respectively, the extra-cellular electric potentials of the healthy cardiac tissue, coupled with an elliptic equation for the passive regions and with an ordinary differential equation describing the cellular membrane dynamics. A similar model arises also in coupling the torso to the heart (see, e.g., [14, 17, 45]). The above model is, indeed, a *mesoscopic one*, the diffusive inclusions being considered at an *intermediate scale* in between the cardiac cell scale and the heart tissue scale.

The mentioned modifications assume a perfect electrical coupling between the healthy part of the heart and the damaged tissue. More general coupling conditions were proposed in [17] and investigated only through numerical simulations in [15, 46], in order to take into account the possible capacitive and resistive effects of the surface of the diffusive inclusions. Up to our knowledge, the well-posedness of such a problem was addressed for the first time in [10]. Here, we rigorously investigate, in more general geometries than the ones considered in [27], the homogenization of the problem with non-standard interface conditions, this being, in fact, the main novelty of our paper. To achieve our goal, we use the periodic homogenization unfolding technique. The limit problems highly depend on the scaling of the imperfect transmission across the membrane and on the geometry of the domain. The influence of the diffusive zone is captured in the limit in several different ways. In particular, for some special scalings

and geometries, we obtain a bidomain system with memory effects (see Theorem 4.6 and Remark 4.7) or a kind of tridomain model (see Theorem 4.16 and Remark 4.18). We point out again that our model generalizes the modified bidomain one with diffusive inclusions and perfect transmission conditions considered in [27, 28], the original model being recovered by suitably rearranging the parameters appearing in equation (2.20).

The paper is organized as follows: Section 2 is devoted to the geometrical and functional setting and to the introduction of the microscopic problem. In Section 3, we introduce and recall the main properties of the time-dependent unfolding operator. In Section 4, we state and prove our main homogenization results.

2. THE MICROSCOPIC PROBLEM

2.1. Geometrical setting. The typical periodic geometrical setting is displayed in Figure 1 and Figure 2. Here we give, for the sake of clarity, its detailed formal definition. Let $N \geq 3$. Let $\Omega \subset \mathbb{R}^N$ be an open connected bounded subset of \mathbb{R}^N and introduce a periodic open subset E of \mathbb{R}^N , such that $E + z = E$ for all $z \in \mathbb{Z}^N$. We assume that Ω and E are of class \mathcal{C}^∞ , though this assumption can be weakened. We employ the notation $Y = (0, 1)^N$ and $E^D = E \cap Y$, $E^B = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. We assume that E^B is connected, while E^D may be connected or not. Moreover, we stipulate that $|\Gamma \cap \partial Y|_{N-1} = 0$.

Let $\varepsilon \in (0, 1)$ be a small positive parameter, related to the characteristic dimension of the microstructure and which takes values in a sequence of strictly positive numbers tending to zero. We define $\Omega^{D,\varepsilon} = \Omega \cap \varepsilon E$, $\Omega^{B,\varepsilon} = \Omega \setminus \overline{\varepsilon E}$, so that $\Omega = \Omega^{D,\varepsilon} \cup \Omega^{B,\varepsilon} \cup \Gamma^\varepsilon$, where $\Omega^{D,\varepsilon}$ and $\Omega^{B,\varepsilon}$ are two disjoint open subsets of Ω and $\Gamma^\varepsilon = \partial \Omega^{D,\varepsilon} \cap \Omega = \partial \Omega^{B,\varepsilon} \cap \Omega$. From the biological point of view, Ω represents the region occupied by the cardiac tissue, $\Omega^{B,\varepsilon}$ [respectively, $\Omega^{D,\varepsilon}$] corresponds to the bidomain phase [respectively, the damaged part], while Γ^ε is the interface between these two regions; in fact, these definitions are slightly modified below. We assume also that $\Omega^{B,\varepsilon}$ is connected at each step $\varepsilon > 0$, while $\Omega^{D,\varepsilon}$ will be connected or disconnected. Indeed, we will consider two different cases: in the first one (to which we will refer as the *connected/disconnected case*, see Figure 1), we will assume that $\Gamma \cap \partial Y = \emptyset$. We also stipulate that all the cells which intersect $\partial \Omega$ do not contain any inclusion, so that, for all $\varepsilon > 0$, $\partial \Omega \cap \partial \Omega^{D,\varepsilon} \neq \emptyset$ and, moreover, $\text{dist}(\Gamma^\varepsilon, \partial \Omega) \geq c\varepsilon$, where c is a suitable strictly positive constant.

In the second case (to which we will refer as the *connected/connected case*, see Figure 2) we will assume that E^D , E^B , $\Omega^{D,\varepsilon}$ and $\Omega^{B,\varepsilon}$ are connected and, without loss of generality, that they have Lipschitz continuous boundary. In this last case, we have that, for all $\varepsilon > 0$, both $\partial \Omega \cap \partial \Omega^{B,\varepsilon} \neq \emptyset$ and $\partial \Omega \cap \partial \Omega^{D,\varepsilon} \neq \emptyset$. Moreover, we assume that our geometry satisfies all the assumptions stated in [33, Section 3.2.1].

Finally, let ν denote the normal unit vector to Γ pointing into E^B , extended by periodicity to the whole of \mathbb{R}^N , so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denotes the normal unit vector to Γ^ε pointing into $\Omega^{B,\varepsilon}$.

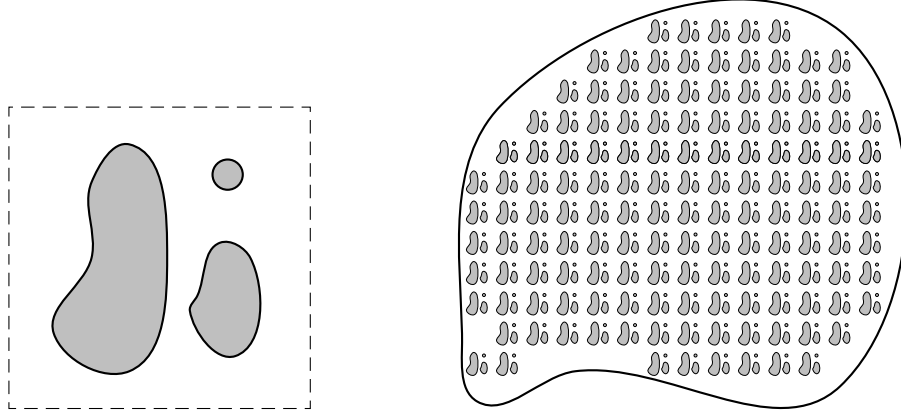


FIGURE 1. *Left:* the periodic cell Y . E^D is the shaded region and E^B is the white region. *Right:* the region Ω .

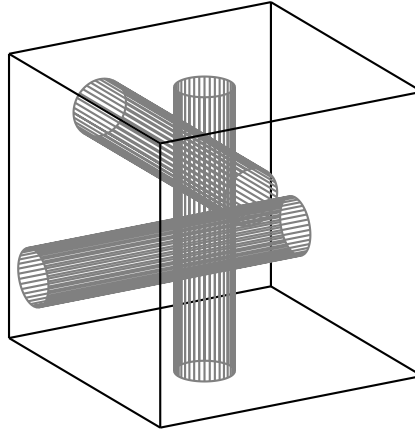


FIGURE 2. The periodic cell Y . E^D is the shaded region and E^B is the white region.

In the following, by γ we shall denote a strictly positive constant, which may depend on the geometry and on the other parameters of the problem; γ may vary from line to line. Moreover, if $\mathcal{G} \subset \mathbb{R}^N$ is an open set and $T > 0$, we set $\mathcal{G}_T = \mathcal{G} \times (0, T)$.

2.2. Functional spaces. Following [10], we consider the functional spaces

$$\begin{aligned} H_{null}^1(\Omega^{B,\varepsilon}) &:= \{w \in H^1(\Omega^{B,\varepsilon}) : w = 0 \text{ on } \partial\Omega^{B,\varepsilon} \cap \partial\Omega, \text{ in the sense of traces}\}; \\ H_{null}^1(\Omega^{D,\varepsilon}) &:= \{w \in H^1(\Omega^{D,\varepsilon}) : w = 0 \text{ on } \partial\Omega^{D,\varepsilon} \cap \partial\Omega, \text{ in the sense of traces}\}. \end{aligned} \quad (2.1)$$

Notice that in the connected/disconnected case $H_{null}^1(\Omega^{D,\varepsilon}) = H^1(\Omega^{D,\varepsilon})$.

We also define the space

$$\mathcal{X}_{0\varepsilon}^1(\Omega) := \{w : \Omega \rightarrow \mathbb{R} : w|_{\Omega^{B,\varepsilon}} \in H_{null}^1(\Omega^{B,\varepsilon}), w|_{\Omega^{D,\varepsilon}} \in H_{null}^1(\Omega^{D,\varepsilon})\}, \quad (2.2)$$

endowed with the norm

$$\|w\|_{\mathcal{X}_{0\varepsilon}^1(\Omega)}^2 := \|\nabla w\|_{L^2(\Omega^{B,\varepsilon})}^2 + \|w\|_{H^1(\Omega^{D,\varepsilon})}^2. \quad (2.3)$$

By our assumptions, we have that $\partial\Omega^{B,\varepsilon} \cap \partial\Omega$ is always non-empty, while $\partial\Omega^{D,\varepsilon}$ can intersect or not the boundary of Ω , depending on the geometry. We recall that, for $w \in \mathcal{X}_{0\varepsilon}^1(\Omega)$, the following Poincaré inequality holds (see [5, Proposition 7.1 and Remark 7.1]):

$$\|w\|_{L^2(\Omega)}^2 \leq \gamma \left(\|\nabla w\|_{L^2(\Omega^{B,\varepsilon})}^2 + \|\nabla w\|_{L^2(\Omega^{D,\varepsilon})}^2 + \varepsilon \| [w] \|_{L^2(\Gamma^\varepsilon)}^2 \right), \quad (2.4)$$

where $[w] = w|_{\Omega^{B,\varepsilon}} - w|_{\Omega^{D,\varepsilon}}$ and the constant γ is independent of ε . We point out that the last term is not necessary in the connected/connected case. Therefore, an equivalent norm on $\mathcal{X}_{0\varepsilon}^1(\Omega)$ is given by

$$\|w\|_{\mathcal{X}_{0\varepsilon}^1(\Omega)}^2 \sim \|\nabla w\|_{L^2(\Omega)}^2 + \| [w] \|_{L^2(\Gamma^\varepsilon)}^2; \quad (2.5)$$

again, the last term can be dropped in the connected/connected case.

2.3. Position of the problem. Let α, β be strictly positive constants and $\sigma_1^B, \sigma_2^B, \sigma^D$ be Y -periodic bounded and symmetric matrices such that there exist $\gamma_0, \tilde{\gamma}_0 > 0$ with

$$\begin{aligned} \gamma_0 |\zeta|^2 &\leq \sigma_1^B(y) \zeta \cdot \zeta \leq \tilde{\gamma}_0 |\zeta|^2, & \text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } y \in Y; \\ \gamma_0 |\zeta|^2 &\leq \sigma_2^B(y) \zeta \cdot \zeta \leq \tilde{\gamma}_0 |\zeta|^2, & \text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } y \in Y; \\ \gamma_0 |\zeta|^2 &\leq \sigma^D(y) \zeta \cdot \zeta \leq \tilde{\gamma}_0 |\zeta|^2, & \text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } y \in Y. \end{aligned} \quad (2.6)$$

Moreover, set $\sigma_1^{B,\varepsilon}(x) = \sigma_1^B(\varepsilon^{-1}x)$, $\sigma_2^{B,\varepsilon}(x) = \sigma_2^B(\varepsilon^{-1}x)$, $\sigma^{D,\varepsilon}(x) = \sigma^D(\varepsilon^{-1}x)$ for a.e. $x \in \Omega$.

As in [10], let us consider a locally Lipschitz continuous function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $g(p, 1) \geq 0$ and $g(p, 0) \leq 0$. The example we have in mind here is a function of the form

$$g(p, q) = a(p)(q - 1) + b(p)q, \quad (2.7)$$

where $a, b : \mathbb{R} \rightarrow \mathbb{R}$ are positive, bounded and Lipschitz functions. Notice that the form of g in (2.7) is classical in this framework (see, for instance, [45]) and that g is Lipschitz continuous with respect to p and affine with respect to q . Let $I_{\text{ion}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$I_{\text{ion}}(p, q) = h_1(p) + h_2(p)q, \quad (2.8)$$

where h_1, h_2 are Lipschitz continuous functions and h_2 is bounded. Let $w_o \in L^\infty(\Omega)$, with $0 \leq w_o(x) \leq 1$ a.e. in Ω and $p^\varepsilon \in L^2(\Omega_T^{B,\varepsilon})$. Consider the gating equation

$$\partial_t \tilde{w}_{p^\varepsilon}^\varepsilon + g(p^\varepsilon, \tilde{w}_{p^\varepsilon}^\varepsilon) = 0, \quad \text{in } \Omega_T^{B,\varepsilon}; \quad (2.9)$$

$$\tilde{w}_{p^\varepsilon}^\varepsilon(x, 0) = w_o(x), \quad \text{in } \Omega^{B,\varepsilon}. \quad (2.10)$$

Notice that, by classical results, the previous problem admits a unique solution $\tilde{w}_p^\varepsilon \in H^1(0, T; L^\infty(\Omega^{B,\varepsilon}))$ and, from our assumptions, $0 \leq \tilde{w}_{p^\varepsilon}^\varepsilon(x, t) \leq 1$ a.e. in $\Omega_T^{B,\varepsilon}$, since

$0 \leq w_o(x) \leq 1$ a.e. in $\Omega^{B,\varepsilon}$ (see [34]). Moreover, we can write

$$\tilde{w}_{p^\varepsilon}^\varepsilon(x, t) = w_o(x) + \int_0^t g(p^\varepsilon, \tilde{w}_{p^\varepsilon}^\varepsilon) d\tau, \quad \text{for a.e. } x \in \Omega. \quad (2.11)$$

From the previous assumptions, we can prove that I_{ion} is a uniformly globally Lipschitz continuous function, i.e. there exists a strictly positive constant γ , independent of ε , such that

$$\|I_{\text{ion}}(p_1^\varepsilon, \tilde{w}_{p_1^\varepsilon}^\varepsilon) - I_{\text{ion}}(p_2^\varepsilon, \tilde{w}_{p_2^\varepsilon}^\varepsilon)\|_{L^2(\Omega_T^{B,\varepsilon})} \leq \gamma \|p_1^\varepsilon - p_2^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})}, \quad (2.12)$$

due to the uniform Lipschitz dependence of $\tilde{w}_{p^\varepsilon}^\varepsilon$ on p^ε and to the bound $0 \leq \tilde{w}_{p^\varepsilon}^\varepsilon(x, t) \leq 1$ a.e. in $\Omega_T^{B,\varepsilon}$.

Remark 2.1. Different examples of functions I_{ion} and g are considered in the literature. We consider here a Hodgkin-Huxley type model (see (2.7)–(2.8)), as in [25, 45]. However, we point out that the results obtained in this paper are also valid for a regularized version of the Mitchell-Schaeffer model proposed in [27] (see, also, [26, 28]). For this last model, the ionic current I_{ion} satisfies (2.12) and

$$I_{\text{ion}}(0, \tilde{w}_0(x, t)) = 0, \quad \text{a.e. in } \Omega_T^{B,\varepsilon}, \quad (2.13)$$

the function g is supposed to be an affine function with respect to q and smooth with respect to p . More precisely, assuming p given, the ionic current $I_{\text{ion}}(p, q)$ is defined as being

$$I_{\text{ion}}(p, q) = \frac{1}{\tau_{\text{in}}} q p^2 (p - 1) e^{-(p/p_{th})^2} - \frac{1}{\tau_{\text{out}}} p (1 + r_{\text{max}} e^{-(p_{th}/p)^2})$$

and the function g is given by

$$g(p, q) = \left(\frac{1}{\tau_{\text{cl}}} + \frac{\tau_{\text{cl}} - \tau_{\text{op}}}{\tau_{\text{cl}} \tau_{\text{op}}} q_\infty(p) \right) (q - q_\infty(p)),$$

with

$$q_\infty(p) = 1 - e^{-(p_{\text{gate}}/p)^2}.$$

Here, all the model parameters are supposed to be positive constants (see, for the interpretation of these constants, [27, 28, 36]). Further, it is assumed that

$$0 < \tau_{\text{op}} < \tau_{\text{cl}} \quad p_{th} \gg p_{\text{gate}} \quad \text{and} \quad r_{\text{max}} \gg 1.$$

Under the same assumptions we made for the initial data, one can prove that, also for the Mitchell-Schaeffer model, the ionic function verifies condition (2.12); moreover, $\tilde{w}(t, \cdot), \partial_t \tilde{w}(t, \cdot) \in L^\infty(\Omega^B)$ and $0 < \tilde{w}(t, x) \leq 1$ (see [28, Lemma 6 and Proposition 17] or [27, Proposition 1]). \square

We give here a complete formulation of the problem we shall address in this paper (the operators div and ∇ act only with respect to the space variable x).

Let $\ell \geq -1$. Assume that $f_1, f_2 \in L^2(0, T; H_0^1(\Omega))$, $\bar{v}_0 \in L^2(\Omega)$ and, for every $\varepsilon > 0$, let $s_{0\varepsilon} \in L^2(\Gamma^\varepsilon)$ be such that

$$\frac{1}{\varepsilon^\ell} \int_{\Gamma^\varepsilon} s_{0\varepsilon}^2 d\sigma \leq \gamma, \quad (2.14)$$

for a suitable $\gamma > 0$, independent of ε .

Let us consider the problem for $u_1^{B,\varepsilon}, u_2^{B,\varepsilon} \in L^2(0, T; H_{null}^1(\Omega^{B,\varepsilon}))$, $u^{D,\varepsilon} \in L^2(0, T; H_{null}^1(\Omega^{D,\varepsilon}))$ and $\tilde{w}^\varepsilon \in H^1(0, T; L^\infty(\Omega^{B,\varepsilon}))$ given by

$$\frac{\partial}{\partial t}(u_1^{B,\varepsilon} - u_2^{B,\varepsilon}) - \operatorname{div}(\sigma_1^{B,\varepsilon} \nabla u_1^{B,\varepsilon}) + I_{ion}(u_1^{B,\varepsilon} - u_2^{B,\varepsilon}, \tilde{w}^\varepsilon) = f_1, \quad \text{in } \Omega_T^{B,\varepsilon}; \quad (2.15)$$

$$\frac{\partial}{\partial t}(u_1^{B,\varepsilon} - u_2^{B,\varepsilon}) + \operatorname{div}(\sigma_2^{B,\varepsilon} \nabla u_2^{B,\varepsilon}) + I_{ion}(u_1^{B,\varepsilon} - u_2^{B,\varepsilon}, \tilde{w}^\varepsilon) = f_2, \quad \text{in } \Omega_T^{B,\varepsilon}; \quad (2.16)$$

$$- \operatorname{div}(\sigma^{D,\varepsilon} \nabla u^{D,\varepsilon}) = 0, \quad \text{in } \Omega_T^{D,\varepsilon}; \quad (2.17)$$

$$\sigma_1^{B,\varepsilon} \nabla u_1^{B,\varepsilon} \cdot \nu_\varepsilon = 0, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.18)$$

$$\sigma_2^{B,\varepsilon} \nabla u_2^{B,\varepsilon} \cdot \nu_\varepsilon - \sigma^{D,\varepsilon} \nabla u^{D,\varepsilon} \cdot \nu_\varepsilon = 0, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.19)$$

$$\frac{\alpha}{\varepsilon^\ell} \frac{\partial}{\partial t}(u_2^{B,\varepsilon} - u^{D,\varepsilon}) + \frac{\beta}{\varepsilon^\ell}(u_2^{B,\varepsilon} - u^{D,\varepsilon}) = \sigma_2^{B,\varepsilon} \nabla u_2^{B,\varepsilon} \cdot \nu_\varepsilon, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.20)$$

$$u_1^{B,\varepsilon}(x, t), u_2^{B,\varepsilon}(x, t), u^{D,\varepsilon}(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T); \quad (2.21)$$

$$u_1^{B,\varepsilon}(x, 0) - u_2^{B,\varepsilon}(x, 0) = \bar{v}_0(x), \quad \text{in } \Omega^{B,\varepsilon}; \quad (2.22)$$

$$u_2^{B,\varepsilon}(x, 0) - u^{D,\varepsilon}(x, 0) = s_{0\varepsilon}(x), \quad \text{on } \Gamma^\varepsilon, \quad (2.23)$$

where \tilde{w}^ε is the solution of the gating problem (2.9)-(2.10), with $p^\varepsilon = u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$.

Remark 2.2 (Biological interpretation). The previous system of equations represents the coupling of a standard bidomain model in $\Omega^{B,\varepsilon}$, for the intra and the extra-cellular potentials $u_1^{B,\varepsilon}$ and $u_2^{B,\varepsilon}$ of the healthy zone, with a Poisson equation in the diffusive part $\Omega^{D,\varepsilon}$, for the electrical potential $u^{D,\varepsilon}$ of the damaged zone. The function $u_2^{B,\varepsilon} - u_1^{B,\varepsilon}$ is the so-called transmembrane potential. The sources f_1 and f_2 are the internal and the external current stimulus, respectively. The coefficients σ_1^B, σ_2^B and σ^D are the conductivities of the two healthy phases and of the damaged one, respectively, while α and β are given parameters related to the capacitive and the resistive behaviour of the interface Γ^ε . We point out that for the intra-cellular potential $u_1^{B,\varepsilon}$ we assume no flux condition on Γ^ε (see (2.18)), while the extra-cellular potential $u_2^{B,\varepsilon}$ is coupled with the electrical potential $u^{D,\varepsilon}$ of the damaged zone through non-standard imperfect transmission conditions (see (2.19) and (2.20)). Our system is completed with suitable initial and boundary conditions. The variable \tilde{w}^ε , called the *gating variable*, describes the ionic transport through the cell membrane. The terms g and I_{ion} are nonlinear functions, modeling the membrane ionic currents.

For simplicity, we consider only one gating variable, but our results hold true also for the case in which the gating variable is vector valued. \square

Notice that, by setting $v^\varepsilon = p^\varepsilon = u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$, $u^\varepsilon = u_2^{B,\varepsilon}$ a.e. in $\Omega_T^{B,\varepsilon}$, $u^\varepsilon = u^{D,\varepsilon}$ a.e. in $\Omega_T^{D,\varepsilon}$, and denoting by $[\cdot]$ the jump across Γ^ε of the quantity in the square brackets, i.e., $[u^\varepsilon] = u_2^{B,\varepsilon} - u^{D,\varepsilon}$ and $[\sigma^\varepsilon \nabla u^\varepsilon \cdot \nu_\varepsilon] = (\sigma_2^{B,\varepsilon} \nabla u_2^{B,\varepsilon} - \sigma^{D,\varepsilon} \nabla u^{D,\varepsilon}) \cdot \nu_\varepsilon$, the previous

system can be written in the more convenient form

$$\frac{\partial v^\varepsilon}{\partial t} - \operatorname{div}(\sigma_1^{B,\varepsilon} \nabla v^\varepsilon) + I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon) = f_1 + \operatorname{div}(\sigma_1^{B,\varepsilon} \nabla u^\varepsilon), \quad \text{in } \Omega_T^{B,\varepsilon}; \quad (2.24)$$

$$- \operatorname{div}((\sigma_1^{B,\varepsilon} + \sigma_2^{B,\varepsilon}) \nabla u^\varepsilon) = f_1 - f_2 + \operatorname{div}(\sigma_1^{B,\varepsilon} \nabla v^\varepsilon), \quad \text{in } \Omega_T^{B,\varepsilon}; \quad (2.25)$$

$$- \operatorname{div}(\sigma^{D,\varepsilon} \nabla u^\varepsilon) = 0, \quad \text{in } \Omega_T^{D,\varepsilon}; \quad (2.26)$$

$$\sigma_1^{B,\varepsilon} \nabla(v^\varepsilon + u^\varepsilon) \cdot \nu_\varepsilon = 0, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.27)$$

$$[\sigma^\varepsilon \nabla u^\varepsilon \cdot \nu_\varepsilon] = 0, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.28)$$

$$\frac{\alpha}{\varepsilon^\ell} \frac{\partial}{\partial t} [u^\varepsilon] + \frac{\beta}{\varepsilon^\ell} [u^\varepsilon] = \sigma_2^{B,\varepsilon} \nabla u^\varepsilon \cdot \nu_\varepsilon, \quad \text{on } \Gamma_T^\varepsilon; \quad (2.29)$$

$$v^\varepsilon, u^\varepsilon = 0, \quad \text{on } \partial\Omega \times (0, T); \quad (2.30)$$

$$v^\varepsilon(x, 0) = \bar{v}_0(x), \quad \text{on } \Omega^{B,\varepsilon}; \quad (2.31)$$

$$[u^\varepsilon](x, 0) = s_{0\varepsilon}(x), \quad \text{on } \Gamma^\varepsilon, \quad (2.32)$$

complemented with the gating problem (2.9)–(2.10), where again $u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$ is replaced by v^ε . Clearly, $v^\varepsilon \in L^2(0, T; H_{null}^1(\Omega^{B,\varepsilon}))$ and $u^\varepsilon \in L^2(0, T; \mathcal{X}_{0\varepsilon}^1(\Omega))$. We stress again that, by (2.12), the composed function $I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon)$ is a Lipschitz function with respect to v^ε .

The weak formulation of the previous problem is given by

$$\begin{aligned} & - \int_{\Omega_T^{B,\varepsilon}} v^\varepsilon \partial_t \varphi_B \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi_B \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi_B \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon) \varphi_B \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} (\sigma_1^{B,\varepsilon} + \sigma_2^{B,\varepsilon}) \nabla u^\varepsilon \cdot \nabla \varphi_D^1 \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi_D^1 \, dx \, dt + \int_{\Omega_T^{D,\varepsilon}} \sigma^{D,\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi_D^2 \, dx \, dt \\ & - \frac{\alpha}{\varepsilon^\ell} \int_{\Gamma_T^\varepsilon} [u^\varepsilon] \partial_t [\varphi_D] \, d\sigma \, dt + \frac{\beta}{\varepsilon^\ell} \int_{\Gamma_T^\varepsilon} [u^\varepsilon] [\varphi_D] \, d\sigma \, dt \\ & = \int_{\Omega_T^{B,\varepsilon}} f_1 \varphi_B \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} (f_1 - f_2) \varphi_D^1 \, dx \, dt \\ & + \int_{\Omega^{B,\varepsilon}} \bar{v}_0 \varphi_B(0) \, dx + \frac{\alpha}{\varepsilon^\ell} \int_{\Gamma^\varepsilon} s_{0\varepsilon} [\varphi_D](0) \, d\sigma, \quad (2.33) \end{aligned}$$

for every $\varphi_B \in L^2(0, T; H_{null}^1(\Omega^{B,\varepsilon})) \cap H^1(0, T; L^2(\Omega^{B,\varepsilon}))$, $\varphi_D^1 \in L^2(0, T; H_{null}^1(\Omega^{B,\varepsilon}))$, $\varphi_D^2 \in L^2(0, T; H_{null}^1(\Omega^{D,\varepsilon}))$, and $[\varphi_D] \in H^1(0, T; L^2(\Gamma^\varepsilon))$, with $\varphi_B(T) = 0$ and $[\varphi_D](T) = 0$. Here, $[\varphi_D] = \varphi_D^1 - \varphi_D^2$ on Γ^ε and (2.33) shall be complemented with the gating problem.

For any $\varepsilon > 0$ fixed, by [10, Theorem 3.6], it follows that the system (2.24)–(2.30), complemented with the gating problem (2.9)–(2.10), admits a unique solution $v^\varepsilon \in L^2(0, T; H_{null}^1(\Omega^{B,\varepsilon}))$, $u^\varepsilon \in L^2(0, T; \mathcal{X}_{0\varepsilon}^1(\Omega))$ and $\tilde{w}^\varepsilon \in H^1(0, T; L^\infty(\Omega^{B,\varepsilon}))$, such that $v^\varepsilon \in \mathcal{C}^0([0, T]; L^2(\Omega^{B,\varepsilon}))$, $[u^\varepsilon] \in \mathcal{C}^0([0, T]; L^2(\Gamma^\varepsilon))$, at least when $\sigma_1^{B,\varepsilon}, \sigma_2^{B,\varepsilon}, \sigma^{D,\varepsilon}$ are scalar coefficients or special matrices as in [18, Lemma 1] and [35, Formula (1)] (see, also, [14] and [28]).

Moreover, by a standard regularization procedure, multiplying (2.24) by v^ε , (2.25) and (2.26) by u^ε , adding the three equations, integrating by parts, using (2.27)–(2.32), moving the integral containing I_{ion} to the right-hand side, using (2.7)–(2.12) and Hölder and Gronwall inequalities, as in [10, inequality (2.38)], we get the following energy estimate:

$$\begin{aligned} & \sup_{t \in (0, T)} \int_{\Omega^{B,\varepsilon}} (v^\varepsilon)^2(x, t) dx + \int_{\Omega_T^{B,\varepsilon}} |\nabla v^\varepsilon + \nabla u^\varepsilon|^2 dx dt + \int_{\Omega_T^{B,\varepsilon}} |\nabla u^\varepsilon|^2 dx dt \\ & + \int_{\Omega_T^{D,\varepsilon}} |\nabla u^\varepsilon|^2 dx dt + \sup_{t \in (0, T)} \frac{1}{\varepsilon^\ell} \int_{\Gamma^\varepsilon} [u^\varepsilon]^2(x, t) d\sigma + \frac{1}{\varepsilon^\ell} \int_{\Gamma_T^\varepsilon} [u^\varepsilon]^2 d\sigma dt \\ & \leq \gamma \left(\|f_1\|_{L^2(\Omega_T)}^2 + \|f_2\|_{L^2(\Omega_T)}^2 + \|\bar{v}_0\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^\ell} \|s_{0\varepsilon}\|_{L^2(\Gamma^\varepsilon)}^2 + 1 \right), \end{aligned} \quad (2.34)$$

where γ is independent of ε . Notice that, by (2.34), it follows also that

$$\int_{\Omega_T^{B,\varepsilon}} |\nabla v^\varepsilon|^2 dx dt \leq \gamma \left(\|f_1\|_{L^2(\Omega_T)}^2 + \|f_2\|_{L^2(\Omega_T)}^2 + \|\bar{v}_0\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^\ell} \|s_{0\varepsilon}\|_{L^2(\Gamma^\varepsilon)}^2 + 1 \right). \quad (2.35)$$

Finally, taking into account condition (2.14), it follows that the right-hand side in (2.34) and (2.35) is uniformly bounded with respect to ε . Therefore, recalling that, both in the connected/connected case and in the connected/disconnected one, the trace of v^ε and u^ε on $(\partial\Omega \cap \partial\Omega^{B,\varepsilon}) \times (0, T)$ is null and using the Poincaré inequality (2.4), we get

$$\begin{aligned} & \|v^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} + \|\nabla v^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} \leq \gamma; \\ & \|u^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} + \|\nabla u^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} \leq \gamma; \\ & \|u^\varepsilon\|_{L^2(\Omega_T^{D,\varepsilon})} + \|\nabla u^\varepsilon\|_{L^2(\Omega_T^{D,\varepsilon})} \leq \gamma; \\ & \sup_{t \in (0, T)} \frac{1}{\varepsilon^\ell} \int_{\Gamma_T^\varepsilon} [u^\varepsilon]^2 d\sigma \leq \gamma. \end{aligned} \quad (2.36)$$

3. TIME-DEPENDING UNFOLDING OPERATOR

A space-time version of the unfolding operator in a more general framework, in which also a time-microscale is actually present, has been introduced in [3] and [4], to which we also refer for a survey on this topic. However, in the present case, the time variable does not play any special role and can be treated essentially as a parameter; hence,

most of the properties of this operator can be proven as in [19, 21, 22] and are therefore omitted.

For $\xi \in \Xi_\varepsilon$, we define

$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + Y) \subset \Omega\}$$

and set

$$\widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \quad \Lambda_T^\varepsilon = \widehat{\Omega}_\varepsilon \times (0, T).$$

Denoting by $[r]$ the integer part and by $\{r\}$ the fractional part of $r \in \mathbb{R}$, we define for $x \in \mathbb{R}^N$

$$\left[\frac{x}{\varepsilon} \right]_Y = \left(\left[\frac{x_1}{\varepsilon} \right], \dots, \left[\frac{x_N}{\varepsilon} \right] \right) \quad \text{and} \quad \left\{ \frac{x}{\varepsilon} \right\}_Y = \left(\left\{ \frac{x_1}{\varepsilon} \right\}, \dots, \left\{ \frac{x_N}{\varepsilon} \right\} \right),$$

so that

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right).$$

Then, we introduce the space cell containing x as $Y_\varepsilon(x) = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + Y \right)$.

Definition 3.1. For w Lebesgue-measurable on Ω_T , the (time-dependent) periodic unfolding operator \mathcal{T}_ε is defined as

$$\mathcal{T}_\varepsilon(w)(x, t, y) = \begin{cases} w \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y, t \right), & (x, t, y) \in \Lambda_T^\varepsilon \times Y, \\ 0, & \text{otherwise.} \end{cases}$$

For w Lebesgue-measurable on Γ_T^ε , the (time-dependent) boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as

$$\mathcal{T}_\varepsilon^b(w)(x, t, y) = \begin{cases} w \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y, t \right), & (x, t, y) \in \Lambda_T^\varepsilon \times \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

□

Clearly, for w_1, w_2 , as in Definition 3.1,

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1) \mathcal{T}_\varepsilon(w_2) \tag{3.1}$$

and the same property holds for the boundary unfolding operator. Note that $\mathcal{T}_\varepsilon^b(w)$ is the trace of the unfolding operator on $\Lambda_T^\varepsilon \times \Gamma$, when both the operators are defined.

We need also an average operator in space.

Definition 3.2. Let w be integrable in Ω_T . The local (time-dependent) space average operator is defined by

$$\mathcal{M}^\varepsilon(w)(x, t) = \begin{cases} \frac{1}{\varepsilon^N} \int_{Y_\varepsilon(x)} w(\zeta, t) \, d\zeta, & \text{if } (x, t) \in \Lambda_T^\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

□

Remark 3.3. From the above definitions, it follows that

$$\mathcal{M}^\varepsilon(w)(x, t) = \int_Y \mathcal{T}_\varepsilon(w)(x, t, y) \, dy = \mathcal{M}_Y(\mathcal{T}_\varepsilon(w))(x, t), \quad (3.3)$$

where $\mathcal{M}_Y(\mathcal{T}_\varepsilon(w))$ denotes the mean average of $\mathcal{T}_\varepsilon(w)$ over Y .

More in general, given an open set $\mathcal{G} \subset \mathbb{R}^N$, we denote by $\mathcal{M}_\mathcal{G}(w)$ the mean average of w over \mathcal{G} . \square

We collect here some properties of the operators defined above.

Proposition 3.4. *The operator $\mathcal{T}_\varepsilon : L^2(\Omega_T) \rightarrow L^2(\Omega_T \times Y)$, given by Definition 3.1, is linear and continuous and satisfies the following estimates:*

$$\|\mathcal{T}_\varepsilon(w)\|_{L^2(\Omega_T \times Y)} \leq \|w\|_{L^2(\Omega_T)} \quad (3.4)$$

and

$$\left| \int_{\Omega_T} w \, dx \, dt - \int_{\Omega_T} \int_Y \mathcal{T}_\varepsilon(w) \, dy \, dx \, dt \right| \leq \int_{\Omega_T \setminus \Lambda_T^\varepsilon} |w| \, dx \, dt, \quad (3.5)$$

for every $w \in L^2(\Omega_T)$.

Proposition 3.5. *Let $\{w_\varepsilon\}$ be a sequence of functions in $L^2(\Omega_T)$.*

If $w_\varepsilon \rightarrow w$ strongly in $L^2(\Omega_T)$ as $\varepsilon \rightarrow 0$, then

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightarrow w, \quad \text{strongly in } L^2(\Omega_T \times Y). \quad (3.6)$$

If $\{w_\varepsilon\}$ is a bounded sequence of functions in $L^2(\Omega_T)$, then, up to a subsequence,

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}, \quad \text{weakly in } L^2(\Omega_T \times Y) \quad (3.7)$$

and

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}), \quad \text{weakly in } L^2(\Omega_T). \quad (3.8)$$

Remark 3.6. In particular, if $w \in L^2(\Omega_T)$, we get that $\mathcal{T}_\varepsilon(w) \rightarrow w$, for $\varepsilon \rightarrow 0$, strongly in $L^2(\Omega_T \times Y)$. Moreover, $\mathcal{M}^\varepsilon(w) \rightarrow w$ strongly in $L^2(\Omega_T)$. \square

Remark 3.7. Actually, the only classes for which the strong convergence of the unfolding $\mathcal{T}_\varepsilon(w_\varepsilon)$ is known to hold, even without strong convergence of w_ε , are sums of the following cases: $w_\varepsilon(x, t) = f_1(x, t)f_2(\varepsilon^{-1}x)$, $w_\varepsilon(x, t) = w(x, \varepsilon^{-1}x, t)$ with $w \in L^2(Y; \mathcal{C}(\overline{\Omega_T}))$ or $w \in L^2(\Omega_T; \mathcal{C}(\overline{Y}))$. In all such cases, $\mathcal{T}_\varepsilon(w_\varepsilon) \rightarrow w$ strongly in $L^2(\Omega_T \times Y)$ (see [2, 20, 21] and [4, Remark 2.9]). \square

Proposition 3.8. *The operator $\mathcal{T}_\varepsilon^b : L^2(\Gamma_T^\varepsilon) \rightarrow L^2(\Omega_T \times \Gamma)$ is linear and continuous. In addition, we have*

$$\|\mathcal{T}_\varepsilon^b(w)\|_{L^2(\Omega_T \times \Gamma)} \leq \sqrt{\varepsilon} \|w\|_{L^2(\Gamma_T^\varepsilon)} \quad (3.9)$$

and

$$\left| \int_{\Gamma_T^\varepsilon} w \, d\sigma \, dt - \frac{1}{\varepsilon} \int_{\Omega_T} \int_\Gamma \mathcal{T}_\varepsilon^b(w) \, d\sigma \, dx \, dt \right| \leq \int_{\Gamma_T^\varepsilon \setminus \Lambda_T^\varepsilon} |w| \, d\sigma \, d\tau. \quad (3.10)$$

Notice that, in the connected/disconnected case, the last integral in (3.10) is identically null, since by the assumptions made in Subsection 2.1 we have that $\Gamma_T^\varepsilon \setminus \Lambda_T^\varepsilon = \emptyset$. On the contrary, in the connected/connected case, it remains bounded, but not infinitesimal. Indeed,

$$|\Gamma_T^\varepsilon \setminus \Lambda_T^\varepsilon| \approx \frac{|\Omega \setminus \Lambda_T^\varepsilon|}{\varepsilon^N} \varepsilon^{N-1} \quad \text{and} \quad |\Omega \setminus \Lambda_T^\varepsilon| \approx |\partial\Omega| \varepsilon.$$

However, in the homogenization process, the function w in (3.10) will be replaced with $[u^\varepsilon]\varphi$, where u^ε is the function appearing in problem (2.24)–(2.32) and $\varphi \in \mathcal{C}^1([0, T]; \mathcal{C}_0^1(\Omega))$, so that

$$\int_{\Gamma_T^\varepsilon} [u^\varepsilon]\varphi \, d\sigma \, dt = \frac{1}{\varepsilon} \iint_{\Omega_T \Gamma} \mathcal{T}_\varepsilon^b([u^\varepsilon]) \mathcal{T}_\varepsilon^b(\varphi) \, d\sigma \, dx \, dt + O(\varepsilon) \quad \text{for } \varepsilon \rightarrow 0. \quad (3.11)$$

By [12, Propositions 4.10 and 4.11 and Corollary 4.12], we get the following result.

Proposition 3.9. *Assume that $w_\varepsilon \rightharpoonup w$ weakly in $L^2(0, T; H_0^1(\Omega))$. Then,*

$$\mathcal{T}_\varepsilon^b(w_\varepsilon) \rightharpoonup w, \quad \text{weakly in } L^2(\Omega_T \times \Gamma).$$

Let w be a function belonging to $\mathcal{C}(\overline{\Omega}_T)$. Then, as $\varepsilon \rightarrow 0$,

$$\mathcal{T}_\varepsilon^b(w) \rightarrow w, \quad \text{strongly in } L^2(\Omega_T \times \Gamma). \quad (3.12)$$

Let w be a function belonging to $L^2(0, T; H^1(\Omega))$. Then, as $\varepsilon \rightarrow 0$,

$$\mathcal{T}_\varepsilon^b(w) \rightarrow w, \quad \text{strongly in } L^2(\Omega_T \times \Gamma). \quad (3.13)$$

Proposition 3.10. *Let $\phi : Y \rightarrow \mathbb{R}$ be a function extended by Y -periodicity to the whole of \mathbb{R}^N and define the sequence*

$$\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N. \quad (3.14)$$

If ϕ is measurable on Y , then

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x, y) = \begin{cases} \phi(y), & (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0, & \text{otherwise.} \end{cases} \quad (3.15)$$

Moreover, if $\phi \in L^2(Y)$, as $\varepsilon \rightarrow 0$,

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \rightarrow \phi, \quad \text{strongly in } L^2(\Omega \times Y); \quad (3.16)$$

if $\phi \in H^1(Y)$, as $\varepsilon \rightarrow 0$,

$$\nabla_y(\mathcal{T}_\varepsilon(\phi^\varepsilon)) \rightarrow \nabla_y \phi, \quad \text{strongly in } L^2(\Omega \times Y). \quad (3.17)$$

We note that

$$\nabla_y[\mathcal{T}_\varepsilon(w) - \mathcal{M}^\varepsilon(w)] = \nabla_y \mathcal{T}_\varepsilon(w) = \varepsilon \mathcal{T}_\varepsilon(\nabla w). \quad (3.18)$$

Proposition 3.11. *Let $w \in L^2(\Omega_T)$. Then,*

$$\frac{1}{\varepsilon} [\mathcal{T}_\varepsilon(w) - \mathcal{M}^\varepsilon(w)] \rightarrow y^c \cdot \nabla w, \quad \text{strongly in } L^2(\Omega_T \times Y), \quad (3.19)$$

where

$$y^c = \left(y_1 - \frac{1}{2}, y_2 - \frac{1}{2}, \dots, y_N - \frac{1}{2} \right).$$

Let $\{w_\varepsilon\}$ be a sequence converging weakly to w in $L^2(0, T; H_0^1(\Omega))$. Then, up to a subsequence, there exists $\hat{w} = \hat{w}(x, t, y) \in L^2(\Omega_T; H_\#^1(Y))$, with $\mathcal{M}_Y(\hat{w}) = 0$, such that, as $\varepsilon \rightarrow 0$,

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \hat{w}, \quad \text{weakly in } L^2(\Omega_T \times Y), \quad (3.20)$$

$$\frac{1}{\varepsilon} [\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}^\varepsilon(w_\varepsilon)] \rightharpoonup y^c \cdot \nabla w + \hat{w}, \quad \text{weakly in } L^2(\Omega_T; H_\#^1(Y)). \quad (3.21)$$

For later use, we set

$$\begin{aligned} \mathcal{X}_\#^1(Y) &:= \{\hat{w} = (\hat{w}^B, \hat{w}^D) : \hat{w}^B = \hat{w}|_{E^B} \in H^1(E^B), \\ &\quad \hat{w}^D = \hat{w}|_{E^D} \in H^1(E^D), \hat{w} \text{ is } Y\text{-periodic}\}. \end{aligned} \quad (3.22)$$

Proposition 3.12. *Let $\{w_\varepsilon\} \subset L^2(0, T; \mathcal{X}_{0\varepsilon}^1(\Omega))$ and assume that we are in the connected/connected geometry. Assume that there exists $\gamma > 0$ (independent of ε) such that*

$$\int_{\Omega_T} |w_\varepsilon|^2 dx dt + \int_{\Omega_T} |\nabla w_\varepsilon|^2 dx dt \leq \gamma, \quad \forall \varepsilon > 0. \quad (3.23)$$

Then, there exists $w \in L^2(\Omega_T; \mathcal{X}_\#^1(Y))$, whose restrictions to E^D and E^B satisfy

$$\begin{aligned} w|_{E^B}(x, t, y) &=: w^B(x, t) \in L^2(0, T; H^1(\Omega)), \quad \text{for a.e. } y \in E^B, \\ w|_{E^D}(x, t, y) &=: w^D(x, t) \in L^2(0, T; H^1(\Omega)), \quad \text{for a.e. } y \in E^D, \end{aligned}$$

and there exists $\hat{w} \in L^2(\Omega; \mathcal{X}_\#^1(Y)/\mathbb{R})$ such that, up to subsequence, as $\varepsilon \rightarrow 0$, we have

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} w_\varepsilon) \rightharpoonup \chi_{E^D} w^D, \quad \mathcal{T}_\varepsilon(\chi_{\Omega^{B,\varepsilon}} w_\varepsilon) \rightharpoonup \chi_{E^B} w^B, \quad \text{weakly in } L^2(\Omega_T \times Y); \quad (3.24)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} \nabla w_\varepsilon) \rightharpoonup \chi_{E^D} (\nabla w^D + \nabla_y \hat{w}^D), \quad \text{weakly in } L^2(\Omega_T \times Y); \quad (3.25)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{B,\varepsilon}} \nabla w_\varepsilon) \rightharpoonup \chi_{E^B} (\nabla w^B + \nabla_y \hat{w}^B), \quad \text{weakly in } L^2(\Omega_T \times Y), \quad (3.26)$$

where, for $\mathcal{O} \subseteq \mathbb{R}^N$, $\chi_{\mathcal{O}}$ denotes the characteristic function of \mathcal{O} . Moreover, we have also

$$\varepsilon \int_{\Gamma_T^\varepsilon} [w_\varepsilon]^2 d\sigma dt \leq 2\varepsilon \int_{\Gamma_T^\varepsilon} (|w_\varepsilon^B|^2 + |w_\varepsilon^D|^2) d\sigma dt \leq \gamma, \quad \forall \varepsilon > 0, \quad (3.27)$$

with γ independent of ε , and

$$\mathcal{T}_\varepsilon^b([w_\varepsilon]) \rightharpoonup [w], \quad \text{weakly in } L^2(\Omega_T \times \Gamma), \quad (3.28)$$

where $[w_\varepsilon] = w_\varepsilon^B - w_\varepsilon^D$ and $[w] = w^B - w^D$ and we have denoted by $w_\varepsilon^D, w_\varepsilon^B$ the trace on Γ^ε of w_ε from $\Omega^{D,\varepsilon}$ and $\Omega^{B,\varepsilon}$, respectively. Moreover, on Γ , we have also identified w^D, w^B with their traces.

Proof. The convergences (3.24)–(3.26) follow by [19, Theorem 2.13] applied in $\Omega^{D,\varepsilon}$ and $\Omega^{B,\varepsilon}$, separately. Inequality (3.27) is a consequence of the standard trace inequality together with a rescaling argument. Finally, (3.28) follows from the fact that

$$\mathcal{T}_\varepsilon^b(w_\varepsilon^D) \rightharpoonup w^D, \quad \mathcal{T}_\varepsilon^b(w_\varepsilon^B) \rightharpoonup w^B, \quad \text{weakly in } L^2(\Omega_T \times \Gamma). \quad (3.29)$$

Indeed, by (3.27), we obtain that there exists $W \in L^2(\Omega_T \times \Gamma)$ such that, up to a subsequence, $\mathcal{T}_\varepsilon^b(w_\varepsilon^D) \rightharpoonup W$ weakly in $L^2(\Omega_T \times \Gamma)$. Moreover, by Gauss-Green Theorem and (3.24)–(3.26), recalling that $\nabla_y \mathcal{T}_\varepsilon(w_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla w_\varepsilon)$, we get

$$\begin{aligned} \int_{\Omega_T} \int_{\Gamma} W \varphi \cdot \nu_i \, d\sigma \, dx \, dt &\leftarrow \int_{\Omega_T} \int_{\Gamma^\varepsilon} \mathcal{T}_\varepsilon^b(w_\varepsilon^D) \varphi \nu_i \, d\sigma \, dx \, dt = \int_{\Omega_T} \int_{E^D} \frac{\partial}{\partial y_i} (\mathcal{T}_\varepsilon(w_\varepsilon) \varphi) \, dy \, dx \, dt \\ &= \varepsilon \int_{\Omega_T} \int_{E^D} \mathcal{T}_\varepsilon(\partial_i w_\varepsilon) \varphi \, dy \, dx \, dt + \int_{\Omega_T} \int_{E^D} \mathcal{T}_\varepsilon(w_\varepsilon) \frac{\partial \varphi}{\partial y_i} \, dy \, dx \, dt \\ &\rightarrow \int_{\Omega_T} \int_{E^D} w^D \frac{\partial \varphi}{\partial y_i} \, dy \, dx \, dt = \int_{\Omega_T} \int_{E^D} \frac{\partial}{\partial y_i} (w^D \varphi) \, dy \, dx \, dt = \int_{\Omega_T} \int_{\Gamma} w^D \varphi \cdot \nu_i \, d\sigma \, dx \, dt, \end{aligned}$$

for any $\varphi \in \mathcal{C}^1(\overline{\Omega_T \times Y})$ with $\text{supp}(\varphi) \subset\subset \Omega_T \times Y$ and for $i = 1, \dots, N$, where $\nu = (\nu_1, \dots, \nu_N)$ is the unit normal vector pointing into E^B . This implies that $W = w^D$ on Γ . Clearly, the same procedure can be applied to w_ε^B and w^B . \square

Remark 3.13. Notice that in the connected/disconnected geometry, the result stated in Proposition 3.12 is still true, up to the fact that, now, w^D belongs only to the space $L^2(\Omega_T)$ and, consequently, (3.25) is replaced by

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} \nabla w_\varepsilon) \rightharpoonup \chi_{E^D} (\nabla w^B + \nabla_y \hat{w}^D), \quad \text{weakly in } L^2(\Omega_T \times Y). \quad (3.30)$$

Indeed, by [29, Theorem 4.3], we obtain

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} \nabla w_\varepsilon) \rightharpoonup \chi_{E^D} \nabla_y \hat{v}^D, \quad \text{weakly in } L^2(\Omega_T \times Y),$$

for a suitable $\hat{v}^D \in L^2(\Omega_T; H^1(E^D))$, and, by [29, Remark 4.4], we can identify $\nabla_y \hat{v}^D = \nabla w^B + \nabla_y \hat{w}^D$. Moreover, it is worthwhile to remark that, in the connected/disconnected geometry, the space $H^1(E^D)$ coincides with $H_\#^1(E^D)$. \square

4. HOMOGENIZATION

In what follows, we extend v^ε to the whole of Ω (still denoting the extension by v^ε), maintaining its $L^2(0, T; H^1(\Omega))$ -norm uniformly bounded, as it can be done following [23, 43], in the connected/disconnected case, and [1, Theorem 2.1] and [40, Lemma 1], in the connected/connected case. We also extend \tilde{w}^ε to the whole of Ω (still keeping the notation \tilde{w}^ε), simply by taking $\tilde{w}^\varepsilon = 0$ in $\Omega^{D,\varepsilon}$.

Our goal in this section is to describe the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the triplet $(v^\varepsilon, u^\varepsilon, \tilde{w}^\varepsilon)$ given by the system (2.24)–(2.32). To this aim, we state the following compactness result.

Lemma 4.1. *Suppose that $\alpha, \beta, \sigma_1^{B,\varepsilon}, \sigma_2^{B,\varepsilon}, \sigma^{D,\varepsilon}, f_1, f_2, \bar{v}_0, \tilde{w}_0, s_{0\varepsilon}$ satisfy the assumptions stated in Subsection 2.3. For every $\varepsilon > 0$, let $(v^\varepsilon, u^\varepsilon, \tilde{w}^\varepsilon)$ be the unique solution of the system (2.24)–(2.32), complemented with the gating problem (2.9)–(2.10). Then, up to a subsequence, still denoted by ε , there exist $v \in L^2(0, T; H_0^1(\Omega))$, $\hat{v} \in L^2(\Omega_T; H_\#^1(E^B))$ with $\mathcal{M}_{E^B}(\hat{v}) = 0$, $u \in L^2(\Omega_T)$, and $w \in L^2(\Omega_T)$ such that*

$$v^\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(0, T; H^1(\Omega)); \quad (4.1)$$

$$\mathcal{T}_\varepsilon(v^\varepsilon) \rightharpoonup v \quad \text{weakly in } L^2(\Omega_T \times Y); \quad (4.2)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{B,\varepsilon}} \nabla v^\varepsilon) \rightharpoonup \nabla v + \nabla_y \hat{v} \quad \text{weakly in } L^2(\Omega_T \times E^B). \quad (4.3)$$

Moreover,

$$v^\varepsilon \rightarrow v \quad \text{strongly in } L^2(\Omega_T); \quad (4.4)$$

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega_T); \quad (4.5)$$

$$\tilde{w}^\varepsilon \rightharpoonup w \quad \text{weakly in } L^2(\Omega_T). \quad (4.6)$$

Proof. Assertions (4.1) and (4.5) are direct consequence of the estimate (2.36), while assertion (4.2) follows by [9, Theorem 2.11]. On the other hand, assertion (4.3) follows from Proposition 3.11, while (4.6) is a consequence of the fact that $0 \leq \tilde{w}^\varepsilon(x, t) \leq 1$ a.e. in Ω_T , uniformly with respect to ε . Finally, (4.4) follows from the next Proposition 4.2.

To achieve the thesis, it remains to prove that the trace of v on $\partial\Omega$ is null. In the connected/disconnected case, this is a direct consequence of the extension technique, while in the connected/connected one, it is due to [33, Theorems 3.5 and 3.6], thanks to our geometrical assumptions. \square

Proposition 4.2. *Under the assumptions of Lemma 4.1, we have that $v^\varepsilon \rightarrow v$ strongly in $L^2(\Omega_T)$.*

Proof. Following the ideas in [32, Lemma 3.10], let us consider the temporal translation $v_{\Delta t}^\varepsilon(t) = v^\varepsilon(t + \Delta t)$ and $u_{\Delta t}^\varepsilon(t) = u^\varepsilon(t + \Delta t)$ of v^ε and u^ε , respectively. Clearly, $v_{\Delta t}^\varepsilon$ and $u_{\Delta t}^\varepsilon$ satisfy the system (2.24)–(2.30) in $(0, T - \Delta t)$, with initial conditions $v_{\Delta t}^\varepsilon(0) = v^\varepsilon(\Delta t)$ and $u_{\Delta t}^\varepsilon(0) = u^\varepsilon(\Delta t)$. We subtract the original equations from the corresponding ones satisfied by the temporal translated functions and set $\hat{v}^\varepsilon(t) = v^\varepsilon(t + \Delta t) - v^\varepsilon(t)$ and $\hat{u}^\varepsilon(t) = u^\varepsilon(t + \Delta t) - u^\varepsilon(t)$ (the same notation will be adopted for all the other quantities). Thus, taking into account only equations (2.24) and (2.27) and using as test function $\varphi_B(t) = -\int_t^{t+\Delta t} v^\varepsilon(s) ds$, we obtain

$$\begin{aligned} & \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} (\hat{v}^\varepsilon)^2 dx dt = \\ & \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla \hat{v}^\varepsilon \cdot \left(\int_t^{t+\Delta t} \nabla v^\varepsilon(s) ds \right) dx dt + \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla \hat{u}^\varepsilon \cdot \left(\int_t^{t+\Delta t} \nabla v^\varepsilon(s) ds \right) dx dt \end{aligned} \quad (4.7)$$

$$\begin{aligned}
& + \int_{\Omega^{B,\varepsilon}} (v^\varepsilon(T) - v^\varepsilon(T - \Delta t)) \left(\int_{T-\Delta t}^T v^\varepsilon(s) \, ds \right) \, dx \\
& \quad - \int_{\Omega^{B,\varepsilon}} (v^\varepsilon(\Delta t) - \bar{v}_0) \left(\int_0^{\Delta t} v^\varepsilon(s) \, ds \right) \, dx \\
& \quad + \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} (I_{ion}(v_{\Delta t}^\varepsilon, \tilde{w}_{\Delta t}^\varepsilon) - I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon)) \left(\int_t^{t+\Delta t} v^\varepsilon(s) \, ds \right) \, dx \, dt \\
& \quad - \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} \hat{f}_1 \left(\int_t^{t+\Delta t} v^\varepsilon(s) \, ds \right) \, dx \, dt = \sum_{k=1}^6 I_k.
\end{aligned}$$

Clearly, (4.7) shall be complemented with the gating problems for $\tilde{w}^\varepsilon(t)$ and $\tilde{w}^\varepsilon(t + \Delta t)$.

Taking into account (2.36) and using Hölder inequality, we get

$$\begin{aligned}
I_1 &= \int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla \hat{v}^\varepsilon \cdot \left(\int_t^{t+\Delta t} \nabla v^\varepsilon(s) \, ds \right) \, dx \, dt \\
&\leq \gamma \left(\int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} |\nabla v^\varepsilon|^2 \, dx \, dt \right)^{1/2} \left\| \int_t^{t+\Delta t} \nabla v^\varepsilon(s) \, ds \right\|_{L^2(\Omega^{B,\varepsilon} \times (0, T-\Delta t))} \\
&\leq \gamma \|\nabla v^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})}^2 \sqrt{\Delta t} \leq \gamma \sqrt{\Delta t}.
\end{aligned} \tag{4.8}$$

Similar computations lead to

$$\begin{aligned}
I_2 &\leq \gamma \|\nabla u^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} \|\nabla v^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} \sqrt{\Delta t} \leq \gamma \sqrt{\Delta t}, \\
I_3 &\leq \gamma \sqrt{\Delta t} \sup_{t \in (0, T)} \int_{\Omega^{B,\varepsilon}} (v^\varepsilon)^2(x, t) \, dx \leq \gamma \sqrt{\Delta t}, \\
I_4 &\leq \gamma \sqrt{\Delta t} \sup_{t \in (0, T)} \left(\int_{\Omega^{B,\varepsilon}} (v^\varepsilon)^2(x, t) \, dx + \|\bar{v}_0\|_{L^2(\Omega)} \left(\int_{\Omega^{B,\varepsilon}} (v^\varepsilon)^2(x, t) \, dx \right)^{1/2} \right) \leq \gamma \sqrt{\Delta t}, \\
I_6 &\leq \gamma \|\nabla \hat{f}_1\|_{L^2(\Omega_T^{B,\varepsilon})} \|\nabla v^\varepsilon\|_{L^2(\Omega_T^{B,\varepsilon})} \sqrt{\Delta t} \leq \gamma \sqrt{\Delta t}.
\end{aligned}$$

Finally,

$$\begin{aligned}
I_5 &\leq \gamma \|I_{ion}(v_{\Delta t}^\varepsilon, \tilde{w}_{\Delta t}^\varepsilon) - I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon)\|_{L^2(\Omega \times (T-\Delta t))} \|v^\varepsilon\|_{L^2(\Omega_T)} \sqrt{\Delta t} \\
&\leq \gamma \|\hat{v}^\varepsilon\|_{L^2(\Omega \times (T-\Delta t))} \|v^\varepsilon\|_{L^2(\Omega_T)} \sqrt{\Delta t} \leq \gamma \|v^\varepsilon\|_{L^2(\Omega_T)}^2 \sqrt{\Delta t},
\end{aligned}$$

where, in the second inequality, we used (2.12). Collecting all the previous estimates, from (4.7) we obtain

$$\int_0^{T-\Delta t} \int_{\Omega^{B,\varepsilon}} |v^\varepsilon(x, t + \Delta t) - v^\varepsilon(x, t)|^2 dx dt \leq \gamma \sqrt{\Delta t}. \quad (4.9)$$

Therefore, taking into account the energy estimate (2.36), by (4.9) we can infer that $v^\varepsilon \rightarrow v$ strongly in $L^2(\Omega_T)$. \square

Lemma 4.3. *Under the assumptions of Lemma 4.1, we have that, up to a subsequence, still denoted by ε , there exist $u^B \in L^2(0, T; H_0^1(\Omega))$, $u^D \in L^2(\Omega_T)$ and $\hat{u} = (\hat{u}^B, \hat{u}^D) \in L^2(\Omega_T; \mathcal{X}_\#^1(Y))$ with $\mathcal{M}_Y(\hat{u}) = 0$, such that*

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{B,\varepsilon}} \nabla u^\varepsilon) \rightharpoonup \nabla u^B + \nabla_y \hat{u}^B, \quad \text{weakly in } L^2(\Omega_T \times E^B). \quad (4.10)$$

Moreover,

- in the connected/connected case, $u^D \in L^2(0, T; H_0^1(\Omega))$ and

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} \nabla u^\varepsilon) \rightharpoonup \nabla u^D + \nabla_y \hat{u}^D, \quad \text{weakly in } L^2(\Omega_T \times E^D); \quad (4.11)$$

- in the connected/disconnected case,

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{D,\varepsilon}} \nabla u^\varepsilon) \rightharpoonup \nabla u^B + \nabla_y \hat{u}^D, \quad \text{weakly in } L^2(\Omega_T \times E^D); \quad (4.12)$$

- for $\ell > -1$ and in both geometries, we have

$$\mathcal{T}_\varepsilon^b([u^\varepsilon]) \rightarrow 0 \quad \text{weakly in } L^2(\Omega_T \times \Gamma), \quad (4.13)$$

so that $u^B = u^D =: u \in L^2(0, T; H_0^1(\Omega))$.

Proof. By (2.36), it follows that Proposition 3.12 and Remark 3.13 hold. Therefore, assertions (4.10), (4.11) and (4.12) are direct consequence of (3.25), (3.26) and (3.30). Moreover, in the connected/connected case, the traces of u^B and u^D are zero on $\partial\Omega \times (0, T)$. Indeed, we can identify $u^\varepsilon|_{\Omega^{B,\varepsilon}} = u_2^{B,\varepsilon}$ and $u^\varepsilon|_{\Omega^{D,\varepsilon}} = u_2^{D,\varepsilon}$ (as already done for v^ε and \tilde{w}^ε) with their extensions and then apply [33, Theorems 3.5 and 3.6]. In the connected/disconnected case, we have only to identify $u^\varepsilon|_{\Omega^{B,\varepsilon}} = u_2^{B,\varepsilon}$ with its extension from outside, so that $u_2^{B,\varepsilon} \in L^2(0, T; H_0^1(\Omega))$. Moreover, by the energy estimate (2.36) and the properties of the extension, we obtain also that $u_2^{B,\varepsilon} \rightharpoonup u^B$ weakly in $L^2(0, T; H_0^1(\Omega))$.

Finally, in order to prove (4.13), recalling that $\ell > -1$ and taking into account (2.36), we obtain that

$$\frac{1}{\varepsilon^\ell} \int_{\Gamma_T^\varepsilon} [u^\varepsilon]^2(t) d\sigma dt \leq \gamma,$$

with γ independent of ε . Therefore, (3.9) implies that

$$\int_{\Omega_T \times \Gamma} \mathcal{T}_\varepsilon^b([u^\varepsilon]^2) d\sigma dt \leq \gamma \varepsilon^{1+\ell} \rightarrow 0.$$

Hence, $\mathcal{T}_\varepsilon^b([u^\varepsilon]) \rightarrow 0$ strongly in $L^2(\Omega_T \times \Gamma)$. Taking into account that, by (3.28), $\mathcal{T}_\varepsilon^b([u^\varepsilon]) \rightharpoonup [u]$ weakly in $L^2(\Omega_T \times \Gamma)$, we get $[u] = 0$, i.e. $u \in L^2(0, T; H_0^1(\Omega))$. \square

We now have to distinguish the different scalings.

4.1. The scaling $\ell = 1$. In addition to what stated in Lemmas 4.1 and 4.3, we can also state the following result.

Lemma 4.4. *Under the assumptions of Lemma 4.1, we have that, up to a subsequence, still denoted by ε ,*

$$\mathcal{T}_\varepsilon^b(\varepsilon^{-1}[u^\varepsilon]) \rightharpoonup [\hat{u}] \quad \text{weakly in } L^2(\Omega_T \times \Gamma). \quad (4.14)$$

Proof. Assertion (4.14) is a consequence of the result in [39, Theorem 3] and [29, Theorem 4.3 and Remark 4.4], once we redefine, according to the notation in that paper,

$$\hat{u}^B = \hat{u}^1 + \xi_\Gamma - m, \quad \hat{u}^D = \hat{u}^2 - m,$$

where $\hat{u}^1 \in L^2(\Omega_T; H_\#^1(E^B))$, $\hat{u}^2 \in L^2(\Omega_T; H_\#^1(E^D))$, $\xi_\Gamma, m \in L^2(\Omega_T)$, with $\hat{u}^1, \hat{u}^2, \xi_\Gamma$ given in [39, Theorem 3] and $m = \mathcal{M}_{E^B}(\hat{u}^1) + |E^B|\xi_\Gamma + \mathcal{M}_{E^D}(\hat{u}^2)$. \square

Theorem 4.5. *Let $\alpha, \beta, \sigma_1^{B,\varepsilon}, \sigma_2^{B,\varepsilon}, \sigma^{D,\varepsilon}, f_1, f_2, \bar{v}_0, w_o$ and $s_{0\varepsilon}$ be as in Subsection 2.3. Assume also that $\mathcal{T}_\varepsilon^b(\varepsilon^{-1}s_{0\varepsilon}) \rightharpoonup s_1$ weakly in $L^2(\Omega \times \Gamma)$. For every $\varepsilon > 0$, let $(v^\varepsilon, u^\varepsilon, \tilde{w}^\varepsilon)$ be the unique solution of the system (2.24)–(2.32), complemented with the gating problem (2.9)–(2.10). Then, there exist $v, u \in L^2(0, T; H_0^1(\Omega))$, $\hat{v} \in L^2(\Omega_T; H_\#^1(E^B))$ with $\mathcal{M}_{E^B}(\hat{v}) = 0$, $\hat{u} \in L^2(\Omega_T; \mathcal{X}_\#^1(Y))$ with $\mathcal{M}_Y(\hat{u}) = 0$, and $w \in L^2(\Omega_T)$, such that $v^\varepsilon \rightharpoonup v$, $u^\varepsilon \rightharpoonup u$, $\tilde{w}^\varepsilon \rightharpoonup w$ in the sense of Lemmas 4.1, 4.3 and 4.4. Moreover, $v, \hat{v}, u, \hat{u}, w$ are the unique solutions of the two-scale homogenized system given by*

$$\begin{aligned} |E^B|v_t - \operatorname{div} \left(\int_{E^B} \sigma_1^B(\nabla(v+u) + \nabla_y(\hat{v} + \hat{u}^B)) \, dy \right) \\ + |E^B|I_{ion}(v, w) = |E^B|f_1, \end{aligned} \quad \text{in } \Omega_T; \quad (4.15)$$

$$\begin{aligned} - \operatorname{div} \left(\int_{E^B} \{(\sigma_1^B + \sigma_2^B)(\nabla u + \nabla_y \hat{u}^B) + \sigma_1^B(\nabla v + \nabla_y \hat{v})\} \, dy \right) \\ - \operatorname{div} \left(\int_{E^D} \sigma^D(\nabla u + \nabla \hat{u}^D) \, dy \right) = |E^B|(f_1 - f_2), \end{aligned} \quad \text{in } \Omega_T; \quad (4.16)$$

$$- \operatorname{div}_y(\sigma_1^B \nabla(v+u) + \sigma_1^B \nabla_y(\hat{v} + \hat{u}^B)) = 0, \quad \text{in } \Omega_T \times E^B; \quad (4.17)$$

$$- \operatorname{div}_y(\sigma_2^B(\nabla u + \nabla_y \hat{u}^B)) = 0, \quad \text{in } \Omega_T \times E^B; \quad (4.18)$$

$$- \operatorname{div}_y(\sigma^D(\nabla u + \nabla_y \hat{u}^D)) = 0, \quad \text{in } \Omega_T \times E^D; \quad (4.19)$$

$$\sigma_1^B \nabla(v+u) + \sigma_1^B \nabla_y(\hat{v} + \hat{u}^B) \cdot \nu = 0, \quad \text{on } \Omega_T \times \Gamma; \quad (4.20)$$

$$[\sigma(\nabla u + \nabla_y \hat{u}) \cdot \nu] = 0, \quad \text{on } \Omega_T \times \Gamma; \quad (4.21)$$

$$\alpha[\hat{u}]_t + \beta[\hat{u}] = \sigma_2^B(\nabla u + \nabla_y \hat{u}^B) \cdot \nu, \quad \text{on } \Omega_T \times \Gamma; \quad (4.22)$$

$$v(x, 0) = \bar{v}_0, \quad \text{in } \Omega; \quad (4.23)$$

$$[\hat{u}](x, y, 0) = s_1, \quad \text{in } \Omega \times \Gamma; \quad (4.24)$$

$$v, u = 0, \quad \text{on } \partial\Omega, \quad (4.25)$$

and

$$\partial_t w + g(v, w) = 0, \quad \text{in } \Omega_T; \quad (4.26)$$

$$w(x, 0) = w_o(x), \quad \text{in } \Omega. \quad (4.27)$$

Here, σ is the matrix which coincides with σ_2^B in E^B and with σ^D in E^D .

Proof. In the weak formulation (2.33), let us take, as test functions,

$$\varphi_B = \phi_B(x, t) + \varepsilon \psi_B(x, t, x/\varepsilon) \quad \text{and} \quad \varphi_D = \phi_D(x, t) + \varepsilon \psi_D(x, t, x/\varepsilon),$$

where $\phi_B, \phi_D \in \mathcal{C}^1(\overline{\Omega}_T)$, with compact support in Ω , for every $t \in [0, T]$, and such that $\phi_B(x, T) = \phi_D(x, T) = 0$, for every $x \in \overline{\Omega}$, $\psi_B \in \mathcal{C}^1(\overline{\Omega}_T; \mathcal{C}_\#^1(\overline{E}^B))$, with compact support in Ω , for every $(t, y) \in [0, T] \times \overline{Y}$, and such that $\psi_B(x, T, y) = 0$, for every $(x, y) \in \overline{\Omega} \times \overline{Y}$, $\psi_D \in \mathcal{C}^1(\Omega_T; \mathcal{X}_\#^1(Y))$, with compact support in $\overline{\Omega}$, for every $(t, y) \in [0, T] \times \overline{Y}$, and such that $[\psi_D(x, T, y)] = 0$, for every $(x, y) \in \overline{\Omega} \times \overline{Y}$. Then, we obtain

$$\begin{aligned} & - \int_{\Omega_T^{B,\varepsilon}} v^\varepsilon (\partial_t \phi_B + \varepsilon \partial_t \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot (\nabla \phi_B + \varepsilon \nabla_x \psi_B + \nabla_y \psi_B) \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla u^\varepsilon \cdot (\nabla \phi_B + \varepsilon \nabla_x \psi_B + \nabla_y \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon)(\phi_B + \varepsilon \psi_B) \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} (\sigma_1^{B,\varepsilon} + \sigma_2^{B,\varepsilon}) \nabla u^\varepsilon \cdot (\nabla \phi_D + \varepsilon \nabla_x \psi_D^1 + \nabla_y \psi_D^1) \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot (\nabla \phi_D + \varepsilon \nabla_x \psi_D^1 + \nabla_y \psi_D^1) \, dx \, dt + \int_{\Omega_T^{D,\varepsilon}} \sigma^{D,\varepsilon} \nabla u^\varepsilon \cdot (\nabla \phi_D + \varepsilon \nabla_x \psi_D^2 + \nabla_y \psi_D^2) \, dx \, dt \\ & - \alpha \varepsilon \int_{\Gamma_T^\varepsilon} \frac{[u^\varepsilon]}{\varepsilon} \partial_t [\psi_D] \, d\sigma \, dt + \beta \varepsilon \int_{\Gamma_T^\varepsilon} \frac{[u^\varepsilon]}{\varepsilon} [\psi_D] \, d\sigma \, dt \\ & = \int_{\Omega_T^{B,\varepsilon}} f_1(\phi_B + \varepsilon \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} (f_1 - f_2)(\phi_D + \varepsilon \psi_D^1) \, dx \, dt \\ & + \int_{\Omega_T^{B,\varepsilon}} \bar{v}_0(\phi_B(0) + \varepsilon \psi_B(0)) \, dx + \alpha \varepsilon \int_{\Gamma^\varepsilon} \frac{s_{0\varepsilon}}{\varepsilon} [\psi_D](0) \, d\sigma. \quad (4.28) \end{aligned}$$

Unfolding and passing to the limit, we arrive at

$$\begin{aligned}
& -|E^B| \int_{\Omega_T} v \partial_t \phi_B \, dx \, dt + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla v + \nabla_y \hat{v}) \cdot (\nabla \phi_B + \nabla_y \psi_B) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla u + \nabla_y \hat{u}^B) \cdot (\nabla \phi_B + \nabla_y \psi_B) \, dy \, dx \, dt + |E^B| \int_{\Omega_T} I_{ion}(v, w) \phi_B \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} (\sigma_1^B + \sigma_2^B) (\nabla u + \nabla_y \hat{u}^B) \cdot (\nabla \phi_D + \nabla_y \psi_D^1) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla v + \nabla_y \hat{v}) \cdot (\nabla \phi_D + \nabla_y \psi_D^1) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^D} \sigma^D (\nabla u + \nabla_y \hat{u}^D) \cdot (\nabla \phi_D + \nabla_y \psi_D^2) \, dy \, dx \, dt \\
& - \alpha \int_{\Omega_T} \int_{\Gamma} [\hat{u}] \partial_t [\psi_D] \, d\sigma \, dx \, dt + \beta \int_{\Omega_T} \int_{\Gamma} [\hat{u}] [\psi_D] \, d\sigma \, dx \, dt \\
& = |E^B| \int_{\Omega_T} f_1 \phi_B \, dx \, dt + |E^B| \int_{\Omega_T} (f_1 - f_2) \phi_D \, dx \, dt \\
& + |E^B| \int_{\Omega} \bar{v}_0 \phi_B(0) \, dx + \alpha \int_{\Omega} \int_{\Gamma} s_1 [\psi_D](0) \, d\sigma \, dx, \quad (4.29)
\end{aligned}$$

where we have used Lemmas 4.1, 4.3 and 4.4. In order to get the strong formulation (4.15)–(4.25), we localize (4.29), taking first $\psi_B = \phi_D = \psi_D^1 = \psi_D^2 = 0$ and then $\phi_B = \psi_B = \psi_D^1 = \psi_D^2 = 0$, so that we arrive at (4.15), (4.16) and (4.23). Moreover, we take $\phi_B = \phi_D = \psi_D^1 = \psi_D^2 = 0$, which gives (4.17) and (4.20). In the next step, we take first $\phi_B = \phi_D = \psi_B = \psi_D^2 = 0$ and ψ_D^1 with compact support in E^B and then $\phi_B = \phi_D = \psi_B = \psi_D^1 = 0$ and ψ_D^2 with compact support in E^D , in order to obtain

$$-\operatorname{div}_y((\sigma_1^B + \sigma_2^B)(\nabla u + \nabla_y \hat{u}^B)) - \operatorname{div}_y(\sigma_1^B(\nabla v + \nabla \hat{v})) = 0, \quad \text{in } \Omega_T \times E^B; \quad (4.30)$$

and (4.19), respectively. Clearly, subtracting (4.17) from (4.30), we get also (4.18). In the last step, we let $\phi_B = \phi_D = \psi_B = 0$ and we take advantage of the equations

previously found, obtaining

$$\begin{aligned}
& \alpha \int_{\Omega} \int_{\Gamma} s_1[\psi_D](0) \, d\sigma \, dx = - \int_{\Omega_T} \int_{\Gamma} (\sigma_1^B + \sigma_2^B)(\nabla u + \nabla_y \hat{u}^B) \cdot \nu \psi_D^1 \, d\sigma \, dx \, dt \\
& - \int_{\Omega_T} \int_{\Gamma} \sigma_1^B(\nabla v + \nabla_y \hat{v}) \cdot \nu \psi_D^1 \, d\sigma \, dx \, dt + \int_{\Omega_T} \int_{\Gamma} \sigma^D(\nabla u + \nabla_y \hat{u}^D) \cdot \nu \psi_D^2 \, d\sigma \, dx \, dt \\
& - \alpha \int_{\Omega_T} \int_{\Gamma} [\hat{u}] \partial_t [\psi_D] \, d\sigma \, dx \, dt + \beta \int_{\Omega_t} \int_{\Gamma} [\hat{u}] [\psi_D] \, d\sigma \, dx \, dt = \\
& - \int_{\Omega_T} \int_{\Gamma} \sigma_2^B(\nabla u + \nabla_y \hat{u}^B) \cdot \nu \psi_D^1 \, d\sigma \, dx \, dt + \int_{\Omega_T} \int_{\Gamma} \sigma^D(\nabla u + \nabla_y \hat{u}^D) \cdot \nu \psi_D^2 \, d\sigma \, dx \, dt \\
& - \alpha \int_{\Omega_T} \int_{\Gamma} [\hat{u}] \partial_t [\psi_D] \, d\sigma \, dx \, dt + \beta \int_{\Omega_t} \int_{\Gamma} [\hat{u}] [\psi_D] \, d\sigma \, dx \, dt = \\
& - \int_{\Omega_T} \int_{\Gamma} [\sigma(\nabla u + \nabla_y \hat{u}) \cdot \nu] \psi_D^2 \, d\sigma \, dx \, dt - \int_{\Omega_T} \int_{\Gamma} \sigma_2^B(\nabla u + \nabla_y \hat{u}^B) \cdot \nu [\psi_D] \, d\sigma \, dx \, dt \\
& - \alpha \int_{\Omega_T} \int_{\Gamma} [\hat{u}] \partial_t [\psi_D] \, d\sigma \, dx \, dt + \beta \int_{\Omega_T} \int_{\Gamma} [\hat{u}] [\psi_D] \, d\sigma \, dx \, dt,
\end{aligned}$$

where, in the second equality, we have taken into account (4.20). Therefore, if we take $[\psi] = 0$, it follows (4.21), while, when $[\psi] \neq 0$, we get (4.22) and (4.24). The boundary condition (4.25) is a direct consequence of the fact that $v, u \in L^2(0, T; H_0^1(\Omega))$.

Finally, the limit gating problem (4.26)–(4.27) follows from (4.4), (4.6) and (2.7), once we pass to the limit in (2.11), written for $p^\varepsilon = u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$, similarly as done in [25, Proposition 4.7] and [27, Section 5.3] (see, also, [31, Lemma 2.5]).

In order to conclude the proof, it remains to guarantee that the two-scale homogenized system (4.15)–(4.25) admits a unique solution, but this is a direct consequence of the linearity of the system jointly with the Lipschitz continuity of I_{ion} . Therefore, the whole sequence, and not only a subsequence, converges. \square

Theorem 4.6. *The two-scale system (4.15)–(4.25) can be rewritten as the single-scale degenerate parabolic system given by*

$$\begin{aligned}
& \partial_t v - \operatorname{div} (A^* \nabla (v + u)) + I_{ion}(v, w) = f_1, \quad \text{in } \Omega_T; \\
& - \operatorname{div} (A^* \nabla (v + u)) - \operatorname{div} \left(A^{hom} \nabla u + \int_0^t \tilde{A}(t - \tau) \nabla u(\tau) \, d\tau \right) \\
& \quad = \mathcal{F} + (f_1 - f_2), \quad \text{in } \Omega_T,
\end{aligned} \tag{4.31}$$

complemented with the initial and the boundary conditions (4.23), (4.25) and the gating problem (4.26)–(4.27). Here, the matrices A^* , A^{hom} , \tilde{A} are defined in (4.39) and \mathcal{F} is defined in (4.40). Moreover, the matrices A^* and A^{hom} are symmetric and positive definite and \tilde{A} is symmetric.

Proof. Taking into account (4.17) and (4.20), we can factorize

$$(\hat{v} + \hat{u}^B)(x, y, t) = -\zeta(y) \cdot \nabla(v + u)(x, t), \quad (4.32)$$

where the cell functions $\zeta = (\zeta^1, \dots, \zeta^N)$, with $\zeta^j \in H_{\#}^1(E^B)$ and $\mathcal{M}_{E^B}(\zeta^j) = 0$, are the solutions of the cell problem

$$\begin{aligned} -\operatorname{div}_y(\sigma_1^B \nabla_y(y^j - \zeta^j)) &= 0, & \text{in } E^B; \\ \sigma_1^B \nabla_y(y^j - \zeta^j) \cdot \nu &= 0, & \text{on } \Gamma. \end{aligned} \quad (4.33)$$

Moreover, following [5, Section 3] and taking into account (4.18)–(4.19), (4.21)–(4.22) and (4.24), we can factorize

$$\hat{u}(x, t, y) = -\chi_0(y) \cdot \nabla u(x, t) - \int_0^t \chi_1(y, t - \tau) \cdot \nabla u(x, \tau) \, d\tau + \mathcal{T}(s_1(x, \cdot))(t, y), \quad (4.34)$$

where we need two families of cell functions $\chi_0 = (\chi_0^1, \dots, \chi_0^N)$, with $\chi_0^j \in H_{\#}^1(Y)$ and $\mathcal{M}_Y(\chi_0^j) = 0$, and $(\chi_1^1, \dots, \chi_1^N)$, with $\chi_1^j \in \mathcal{X}_{\#}^1(Y)$ and $\mathcal{M}_Y(\chi_1^j) = 0$. More precisely, for $j = 1, \dots, N$, χ_0^j satisfies the cell problem

$$\begin{aligned} -\operatorname{div}_y(\sigma_2^B \nabla_y(y^j - \chi_0^j)) &= 0, & \text{in } E^B; \\ -\operatorname{div}_y(\sigma^D \nabla_y(y^j - \chi_0^j)) &= 0, & \text{in } E^D; \\ [\sigma \nabla_y(y^j - \chi_0^j) \cdot \nu] &= 0, & \text{on } \Gamma, \end{aligned} \quad (4.35)$$

which can be simply rewritten as

$$-\operatorname{div}_y(\sigma \nabla_y(y^j - \chi_0^j)) = 0, \quad \text{in } Y. \quad (4.36)$$

In turn, for $j = 1, \dots, N$, χ_1^j satisfies the cell problem

$$\begin{aligned} -\operatorname{div}_y(\sigma_2^B \nabla_y \chi_1^j) &= 0, & \text{in } E^B \times (0, T); \\ -\operatorname{div}_y(\sigma^D \nabla_y \chi_1^j) &= 0, & \text{in } E^D \times (0, T); \\ [\sigma \nabla_y \chi_1^j \cdot \nu] &= 0, & \text{on } \Gamma \times (0, T); \\ \alpha \partial_t[\chi_1^j] + \beta[\chi_1^j] &= \sigma_2^B \nabla_y \chi_1^j \cdot \nu, & \text{on } \Gamma \times (0, T); \\ \alpha[\chi_1^j](0) &= \sigma_2^B \nabla_y(\chi_0^j - y^j) \cdot \nu, & \text{on } \Gamma. \end{aligned} \quad (4.37)$$

Finally, $\mathcal{T}(s_1) \in L^2(\Omega_T; \mathcal{X}_{\#}^1(Y))$, for a.e. $x \in \Omega$, is defined as the solution of the problem

$$\begin{aligned} -\operatorname{div}_y(\sigma_2^B \nabla_y \mathcal{T}(s_1)) &= 0, & \text{in } E^B \times (0, T); \\ -\operatorname{div}_y(\sigma^D \nabla_y \mathcal{T}(s_1)) &= 0, & \text{in } E^D \times (0, T); \\ [\sigma \nabla_y \mathcal{T}(s_1) \cdot \nu] &= 0, & \text{on } \Gamma \times (0, T); \\ \alpha \partial_t[\mathcal{T}(s_1)] + \beta[\mathcal{T}(s_1)] &= \sigma_2^B \nabla_y \mathcal{T}(s_1) \cdot \nu, & \text{on } \Gamma \times (0, T); \\ \alpha[\mathcal{T}(s_1)](0) &= s_1, & \text{on } \Gamma, \end{aligned} \quad (4.38)$$

with the additional condition $\mathcal{M}_Y(\mathcal{T}(s_1)) = 0$, a.e. in Ω_T .

Notice that well-posedness for (4.33) and (4.36) is a classical problem, while systems (4.37) and (4.38) admit a unique solution by [6, Theorem 6 and Remark 7].

Inserting (4.32) and (4.34) in (4.15) and (4.16), we get the single-scale homogenized degenerate parabolic system (4.31), where the matrices A^* , A^{hom} and \tilde{A} are defined as

$$\begin{aligned}
A^* &= \frac{1}{|E^B|} \int_{E^B} \sigma_1^B \nabla_y(y - \zeta) \, dy = \frac{1}{|E^B|} \int_{E^B} (\nabla_y(y - \zeta))^T \sigma_1^B \nabla_y(y - \zeta) \, dy, \\
A^{hom} &= \frac{1}{|E^B|} \left(\int_{E^B} \sigma_2^B \nabla_y(y - \chi_0) \, dy + \int_{E^D} \sigma^D \nabla_y(y - \chi_0) \, dy \right) \\
&= \frac{1}{|E^B|} \int_Y \sigma \nabla_y(y - \chi_0) \, dy = \frac{1}{|E^B|} \int_Y (\nabla_y(y - \chi_0))^T \sigma \nabla_y(y - \chi_0) \, dy, \\
\tilde{A}(t) &= -\frac{1}{|E^B|} \int_Y \sigma \nabla_y \chi_1(y, t) \, dy,
\end{aligned} \tag{4.39}$$

and

$$\mathcal{F} = \frac{1}{|E^B|} \operatorname{div} \left(\int_Y \sigma \nabla_y \mathcal{T}(s_1) \, dy \right). \tag{4.40}$$

Clearly, A^* and A^{hom} are symmetric and their positive definiteness is a standard matter. Regarding \tilde{A} , we notice that, in the case of σ constant in E^D and E^B (with possibly two different constants), using Gauss-Green formula, it can be written as

$$\tilde{A}(t) = \frac{1}{|E^B|} \int_{\Gamma} [\sigma \chi_1(y, t)] \otimes \nu \, d\sigma,$$

whose symmetry has been proved in [5, Corollary 4.1]. However, still using the ideas in [5, Section 4], we can prove that \tilde{A} is symmetric also for a non piecewise constant matrix σ , satisfying (2.6). Indeed, by [5, Lemma 4.1] (applied to $s_1 = \sigma_2^B \nabla_y(\chi_0^j - y^j) \cdot \nu$ and $s_2 = \sigma_2^B \nabla_y(\chi_0^h - y^h) \cdot \nu$), it follows

$$\int_{\Gamma} [\chi_1^j](t) [\chi_1^h](0) \, d\sigma = \int_{\Gamma} [\chi_1^j](0) [\chi_1^h](t) \, d\sigma. \tag{4.41}$$

Moreover, recalling the initial condition in (4.37), we also have

$$\alpha \int_{\Gamma} [\chi_1^j](t) [\chi_1^h](0) \, d\sigma = \int_{\Gamma} [\chi_1^j](t) \sigma_2^B \nabla_y(\chi_0^h - y^h) \cdot \nu \, d\sigma. \tag{4.42}$$

Now, let us take χ_1^j as test function for the cell equation (4.36) (written for χ_0^h) and χ_0^h as test function for the cell problem (4.37). We get

$$\begin{aligned} \int_Y \sigma \nabla_y (\chi_0^h - y^h) \nabla_y \chi_1^j \, dy &= - \int_\Gamma \sigma_2^B \nabla_y (\chi_0^h - y^h) \cdot \nu [\chi_1^j] \, d\sigma, \\ \int_Y \sigma \nabla_y \chi_1^j \nabla_y \chi_0^h \, dy &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \tilde{A}_{hj}(t) &= - \frac{1}{|E^B|} \int_Y \sigma \mathbf{e}^h \nabla_y \chi_1^j(y, t) \, dy = - \frac{1}{|E^B|} \int_\Gamma \sigma_2^B \nabla_y (\chi_0^h - y^h) \cdot \nu [\chi_1^j](t) \, d\sigma \\ &= - \frac{\alpha}{|E^B|} \int_\Gamma [\chi_1^j](t) [\chi_1^h](0) \, d\sigma, \end{aligned}$$

where, in the last equality, we have used (4.42). Reasoning as above, we arrive also to

$$\tilde{A}_{jh}(t) = - \frac{1}{|E^B|} \int_Y \sigma \mathbf{e}^j \nabla_y \chi_1^h(y, t) \, dy = - \frac{\alpha}{|E^B|} \int_\Gamma [\chi_1^h](t) [\chi_1^j](0) \, d\sigma.$$

Finally, the symmetry is proven taken into account (4.41). \square

Remark 4.7. Notice that the limit problem (4.31) leads to a bidomain model with memory effects. Indeed, let us denote by u^B and u^D , respectively, the limits of the functions $u_1^{B,\varepsilon}$ and $u^{D,\varepsilon}$, appearing in the system (2.15)–(2.23). Recalling that $v^\varepsilon = u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$, $u^\varepsilon|_{\Omega_T^{B,\varepsilon}} = u_2^{B,\varepsilon}$ and $u^\varepsilon|_{\Omega_T^{D,\varepsilon}} = u^{D,\varepsilon}$, and taking into account (4.13), we can replace $v = u^B - u^D$ and $u = u^D$ in (4.31), thus obtaining

$$\begin{aligned} \partial_t(u^B - u^D) - \operatorname{div}(A^* \nabla u^B) + I_{ion}(u^B - u^D, w) &= f_1, \quad \text{in } \Omega_T; \\ - \operatorname{div}(A^* \nabla u^B) - \operatorname{div} \left(A^{hom} \nabla u^D + \int_0^t \tilde{A}(t - \tau) \nabla u^D(\tau) \, d\tau \right) & \\ &= \mathcal{F} + (f_1 - f_2), \quad \text{in } \Omega_T. \end{aligned} \tag{4.43}$$

\square

Remark 4.8. In the case $\ell > 1$, by unfolding the last inequality in the energy estimate (2.36), we obtain

$$\mathcal{T}_\varepsilon^b(\varepsilon^{-1}[u^\varepsilon]) \rightharpoonup 0 \quad \text{strongly in } L^2(\Omega_T \times \Gamma).$$

In particular, this implies that there is no jump in the corrector \hat{u} , and thus one can check that the limit problem is standard. \square

4.2. The scaling $\ell \in (-1, 1)$. We recall that Lemmas 4.1 and 4.3 are still in force.

Theorem 4.9. *Assume that $\alpha, \beta, \sigma_1^{B,\varepsilon}, \sigma_2^{B,\varepsilon}, \sigma^{D,\varepsilon}, f_1, f_2, \bar{v}_0, w_o$ and $s_{0\varepsilon}$ are as in Subsection 2.3. For every $\varepsilon > 0$, let $(v^\varepsilon, u^\varepsilon, \tilde{w}^\varepsilon)$ be the unique solution of the system (2.24)–(2.32), complemented with the gating problem (2.9)–(2.10). Then, there exist $v, u \in L^2(0, T; H_0^1(\Omega))$, $\hat{v} \in L^2(\Omega_T; H_\#^1(E^B))$ with $\mathcal{M}_{EB}(\hat{v}) = 0$, $\hat{u} \in L^2(\Omega_T; \mathcal{X}_\#^1(Y))$ with $\mathcal{M}_{EB}(\hat{u}^B) = 0 = \mathcal{M}_{ED}(\hat{u}^D)$, and $w \in L^2(\Omega_T)$, such that $v^\varepsilon \rightharpoonup v$, $u^\varepsilon \rightharpoonup u$, $\tilde{w}^\varepsilon \rightharpoonup w$ in the sense of Lemmas 4.1 and 4.3. Moreover, $v, \hat{v}, u, \hat{u}, w$ are the unique solutions of two-scale homogenized system given by (4.15)–(4.20), with (4.21) and (4.22) replaced with*

$$\sigma_2^B(\nabla u + \nabla_y \hat{u}^B) \cdot \nu = 0, \text{ on } \Omega_T \times \Gamma; \quad (4.44)$$

$$\sigma^D(\nabla u + \nabla_y \hat{u}^D) \cdot \nu = 0, \text{ on } \Omega_T \times \Gamma, \quad (4.45)$$

complemented with the initial-boundary conditions (4.23), (4.25) and the gating problem (4.26)–(4.27).

Proof. The proof can be carried out as in the case of Theorem 4.5. The main difference is that, now, the last integral in (4.28) is replaced by

$$\frac{\alpha}{\varepsilon^\ell} \varepsilon \int_{\Gamma^\varepsilon} s_{0\varepsilon} [\psi^D](0) \, d\sigma$$

and the fifth line of (4.28) is replaced by

$$-\frac{\alpha}{\varepsilon^\ell} \varepsilon \int_{\Gamma^\varepsilon} [u^\varepsilon] \partial_t [\psi^D] \, d\sigma + \frac{\beta}{\varepsilon^\ell} \varepsilon \int_{\Gamma^\varepsilon} [u^\varepsilon] [\psi^D] \, d\sigma. \quad (4.46)$$

However, taking into account that (2.14) can be rewritten in the form

$$\alpha \int_{\Omega \times \Gamma} \left(\mathcal{T}_\varepsilon \left(\frac{s_{0\varepsilon}}{\varepsilon^{\frac{\ell+1}{2}}} \right) \right)^2 \, d\sigma = \alpha \varepsilon \int_{\Gamma^\varepsilon} \left(\frac{s_{0\varepsilon}}{\varepsilon^{\frac{\ell+1}{2}}} \right)^2 \, d\sigma = \frac{\alpha}{\varepsilon^\ell} \int_{\Gamma^\varepsilon} s_{0\varepsilon}^2 \leq \gamma,$$

it follows

$$\frac{\alpha}{\varepsilon^\ell} \varepsilon \int_{\Gamma^\varepsilon} s_{0\varepsilon} [\psi^D](0) \, d\sigma = \alpha \varepsilon^{\frac{1-\ell}{2}} \int_{\Omega \times \Gamma} \mathcal{T}_\varepsilon \left(\frac{s_{0\varepsilon}}{\varepsilon^{\frac{\ell+1}{2}}} \right) \mathcal{T}_\varepsilon ([\psi^D])(0) \, d\sigma \leq \gamma \varepsilon^{\frac{1-\ell}{2}} \rightarrow 0, \quad (4.47)$$

and, thanks to (2.36), similar computations lead to the result that also the integrals in (4.46) tend to zero, for $\varepsilon \rightarrow 0$. Hence, passing to the limit in (4.28), taking into account the previous facts and, finally, localizing, we get the thesis. \square

Theorem 4.10. *The two-scale system (4.15)–(4.20), (4.44) and (4.45) can be rewritten as the single-scale degenerate parabolic system given by*

$$\begin{aligned} \partial_t v - \operatorname{div}(A^* \nabla(v + u)) + I_{ion}(v, w) &= f_1, & \text{in } \Omega_T; \\ -\operatorname{div}(A^* \nabla(v + u)) - \operatorname{div}(\hat{A}^{hom} \nabla u) &= f_1 - f_2, & \text{in } \Omega_T, \end{aligned} \quad (4.48)$$

complemented with the initial and the boundary conditions (4.23), (4.25) and the gating problem (4.26)–(4.27), where the matrix A^* is given in (4.39) and $\hat{A}^{hom} = A_B^{hom} + A_D^{hom}$ is defined in (4.51) and (4.52).

Proof. As in the case $\ell = 1$, taking into account (4.17) and (4.20), we can factorize $\hat{v} + \hat{u}^B$ as in (4.32), where the cell functions $\zeta = (\zeta^1, \dots, \zeta^N)$, with $\zeta^j \in H_{\#}^1(E^B)$ and $\mathcal{M}_{E^B}(\zeta^j) = 0$, are the solutions of the cell problem (4.33). Moreover, taking into account (4.18), (4.19), (4.44) and (4.45), we can factorize

$$\hat{u}(x, t, y) = -\hat{\chi}_0(y) \cdot \nabla u(x, t), \quad (4.49)$$

where the cell functions $\hat{\chi}_0 = (\hat{\chi}_0^1, \dots, \hat{\chi}_0^N)$, with $\hat{\chi}_0^j = (\hat{\chi}_0^{B,j}, \hat{\chi}_0^{D,j}) \in \mathcal{X}_{\#}^1(Y)$ and $\mathcal{M}_{E^B}(\hat{\chi}_0^{B,j}) = 0 = \mathcal{M}_{E^D}(\hat{\chi}_0^{D,j})$, for $j = 1, \dots, N$, satisfy the cell problem

$$\begin{aligned} -\operatorname{div}_y(\sigma_2^B \nabla_y(y^j - \hat{\chi}_0^{B,j})) &= 0, & \text{in } E^B; \\ \sigma_2^B \nabla_y(y^j - \hat{\chi}_0^{B,j}) \cdot \nu &= 0, & \text{on } \Gamma; \\ -\operatorname{div}_y(\sigma^D \nabla_y(y^j - \hat{\chi}_0^{D,j})) &= 0, & \text{in } E^D; \\ \sigma^D \nabla_y(y^j - \hat{\chi}_0^{D,j}) \cdot \nu &= 0, & \text{on } \Gamma, \end{aligned} \quad (4.50)$$

which are two independent Neuman problems. Inserting these factorizations in (4.15) and (4.16), we get the single-scale homogenized degenerate parabolic system (4.48), where the matrix A^* coincides with the one defined in (4.39), while $\hat{A}^{hom} = A_B^{hom} + A_D^{hom}$ is given by

$$A_B^{hom} = \frac{1}{|E^B|} \int_{E^B} \sigma_2^B \nabla_y(y - \hat{\chi}_0^B) dy = \frac{1}{|E^B|} \int_{E^B} (\nabla_y(y - \hat{\chi}_0^B))^T \sigma_2^B \nabla_y(y - \hat{\chi}_0^B) dy, \quad (4.51)$$

$$A_D^{hom} = \frac{1}{|E^B|} \int_{E^D} \sigma^D \nabla_y(y - \hat{\chi}_0^D) dy = \frac{1}{|E^B|} \int_{E^D} (\nabla_y(y - \hat{\chi}_0^D))^T \sigma^D \nabla_y(y - \hat{\chi}_0^D) dy. \quad (4.52)$$

Clearly, A_B^{hom} and A_D^{hom} are symmetric and their positive definiteness is a standard matter. \square

Remark 4.11. As in Remark 4.7, let us denote by u^B and u^D , respectively, the limits of the functions $u_1^{B,\varepsilon}$ and $u^{D,\varepsilon}$, appearing in the system (2.15)–(2.23), so that, replacing $v = u^B - u^D$ and $u = u^D$ in (4.48), we obtain

$$\begin{aligned} \partial_t(u^B - u^D) - \operatorname{div}(A^* \nabla u^B) + I_{ion}(u^B - u^D, w) &= f_1, & \text{in } \Omega_T; \\ -\operatorname{div}(A^* \nabla u^B) - \operatorname{div}(\hat{A}^{hom} \nabla u^D) &= f_1 - f_2, & \text{in } \Omega_T. \end{aligned} \quad (4.53)$$

\square

Remark 4.12. In the connected/disconnected case, $\chi_0^{D,j}(y) = y^j$, up to an additive constant, so that $A_D^{hom} = 0$. Therefore, $A^{hom} = A_B^{hom}$ and the limit problem is affected only by the physical properties of the phase E^B . \square

4.3. The scaling $\ell = -1$. In addition to what stated in Lemmas 4.1 and 4.3, we can also state the following result.

Lemma 4.13. *Under the assumptions of Lemma 4.1, we have that, up to a subsequence, still denoted by ε ,*

$$\mathcal{T}_{\varepsilon}^b([u^{\varepsilon}]) \rightharpoonup [u] \quad \text{weakly in } L^2(\Omega_T \times \Gamma). \quad (4.54)$$

Here, with a little abuse of notation, $[u] = u^B - u^D$.

Proof. Assertion (4.54) is a consequence of (2.36) and (3.28) in Proposition 3.12. \square

Theorem 4.14. *Assume to be in the connected/connected geometry. Let $\alpha, \beta, \sigma_1^{B,\varepsilon}, \sigma_2^{B,\varepsilon}, \sigma^{D,\varepsilon}, f_1, f_2, \bar{v}_0, W_o$ and $s_{0\varepsilon}$ be as in Subsection 2.3 and assume that $\mathcal{T}_\varepsilon^b(s_{0\varepsilon}) \rightharpoonup \bar{s}_1$ weakly in $L^2(\Omega \times \Gamma)$. For every $\varepsilon > 0$, let $(v^\varepsilon, u^\varepsilon, \tilde{w}^\varepsilon)$ be the unique solution of the system (2.24)–(2.32), complemented with the gating problem (2.9)–(2.10). Then, there exist $v, u^B, u^D \in L^2(0, T; H_0^1(\Omega))$, $\hat{v} \in L^2(\Omega_T; H_\#^1(E^B))$ with $\mathcal{M}_{E^B}(\hat{v}) = 0$, $\hat{u} \in L^2(\Omega_T; \mathcal{X}_\#^1(Y))$ with $\mathcal{M}_{E^B}(\hat{u}^B) = 0 = \mathcal{M}_{E^D}(\hat{u}^D)$, and $w \in L^2(\Omega_T)$, such that $v^\varepsilon \rightharpoonup v$, $u^\varepsilon \chi_{\Omega_T^{B,\varepsilon}} \rightharpoonup u^B$, $u^\varepsilon \chi_{\Omega_T^{D,\varepsilon}} \rightharpoonup u^D$, $\tilde{w}^\varepsilon \rightharpoonup w$ in the sense of Lemmas 4.1, 4.3 and 4.4. Moreover, $v, \hat{v}, u^B, u^D, \hat{u}, w$ are the unique solutions of two-scale homogenized system given by*

$$\begin{aligned} |E^B|v_t - \operatorname{div} \left(\int_{E^B} \sigma_1^B (\nabla(v+u) + \nabla_y(\hat{v} + \hat{u}^B)) \, dy \right) \\ + |E^B|I_{ion}(v, w) = |E^B|f_1, \end{aligned} \quad \text{in } \Omega_T; \quad (4.55)$$

$$\begin{aligned} \alpha|\Gamma|\partial_t[u] + \beta|\Gamma|[u] - \operatorname{div} \left(\int_{E^B} (\sigma_1^B + \sigma_2^B)(\nabla u^B + \nabla_y \hat{u}^B) \, dy \right) \\ - \operatorname{div} \left(\int_{E^B} \sigma_1^B (\nabla v^B + \nabla_y \hat{v}^B) \, dy \right) = |E^B|(f_1 - f_2), \end{aligned} \quad \text{in } \Omega_T; \quad (4.56)$$

$$\alpha|\Gamma|\partial_t[u] + \beta|\Gamma|[u] + \operatorname{div} \left(\int_{E^D} \sigma^D (\nabla u^D + \nabla_y \hat{u}^D) \, dy \right) = 0, \quad \text{in } \Omega_T; \quad (4.57)$$

$$-\operatorname{div}_y(\sigma_1^B \nabla(v+u^B) + \sigma_1^B \nabla_y(\hat{v} + \hat{u}^B)) = 0, \quad \text{in } \Omega_T \times E^B; \quad (4.58)$$

$$-\operatorname{div}_y(\sigma_2^B (\nabla u^B + \nabla_y \hat{u}^B)) = 0, \quad \text{in } \Omega_T \times E^B; \quad (4.59)$$

$$-\operatorname{div}_y(\sigma^D (\nabla u^D + \nabla_y \hat{u}^D)) = 0, \quad \text{in } \Omega_T \times E^D; \quad (4.60)$$

$$\sigma_1^B \nabla(v+u^B) + \sigma_1^B \nabla_y(\hat{v} + \hat{u}^B) \cdot \nu = 0, \quad \text{on } \Omega_T \times \Gamma; \quad (4.61)$$

$$\sigma_2^B (\nabla u^B + \nabla_y \hat{u}^B) \cdot \nu = 0, \quad \text{on } \Omega_T \times \Gamma; \quad (4.62)$$

$$\sigma^D (\nabla u^D + \nabla_y \hat{u}^D) \cdot \nu = 0, \quad \text{on } \Omega_T \times \Gamma; \quad (4.63)$$

$$v(x, 0) = \bar{v}_0, \quad \text{in } \Omega; \quad (4.64)$$

$$[u](x, 0) = \frac{1}{|\Gamma|} \int_\Gamma \bar{s}_1 \, d\sigma, \quad \text{in } \Omega; \quad (4.65)$$

$$v, u^B, u^D = 0, \quad \text{on } \partial\Omega, \quad (4.66)$$

complemented with the gating problem (4.26)–(4.27).

Remark 4.15. In the connected/disconnected case, $u^D \in L^2(\Omega_T)$ and equations (4.55), (4.56), (4.58), (4.59), (4.61), (4.62), (4.64), (4.65) are still in force, with v, u^B having

null trace on $\partial\Omega \times (0, T)$. However, as we will see in Remark 4.17 below, we will find that equation (4.57) becomes

$$\alpha \partial_t[u] + \beta[u] = 0, \quad \text{in } \Omega_T, \quad (4.67)$$

(4.60) and (4.63) disappear, and equation (4.56) simplifies to

$$\begin{aligned} & -\operatorname{div} \left(\int_{E^B} (\sigma_1^B + \sigma_2^B) (\nabla u^B + \nabla_y \hat{u}^B) \, dy \right) \\ & -\operatorname{div} \left(\int_{E^B} \sigma_1^B (\nabla v^B + \nabla_y \hat{v}^B) \, dy \right) = |E^B| (f_1 - f_2), \quad \text{in } \Omega_T. \end{aligned} \quad (4.68)$$

Therefore, the function u^D can be explicitly determined in terms of u^B and \bar{s}_1 , i.e.

$$u^D(x, t) = u^B(x, t) - \left(\frac{1}{|T|} \int_T \bar{s}_1(x, y) \, d\sigma(y) \right) e^{-\beta t/\alpha}.$$

In particular, the damaged zone affects the macroscopic model only through the physical properties (α, β) of the boundary of such a zone, while σ^D has no influence in the homogenized limit. \square

Proof of Theorem 4.14. In the weak formulation (2.33), let us take, as test functions,

$$\varphi_B = \phi_B(x, t) + \varepsilon \psi_B(x, t, x/\varepsilon) \quad \text{and} \quad \varphi_D = \phi_D^i(x, t) + \varepsilon \psi_D^i(x, t, x/\varepsilon), \quad i = 1, 2,$$

where $\phi_B, \phi_D^1, \phi_D^2 \in \mathcal{C}^1(\overline{\Omega}_T)$, with compact support in Ω , for every $t \in [0, T]$, and such that $\phi_B(x, T) = \phi_D(x, T) = 0$, for every $x \in \overline{\Omega}$, $\psi_B \in \mathcal{C}^1(\overline{\Omega}_T; \mathcal{C}_\#^1(\overline{E^B}))$, with compact support in Ω , for every $(t, y) \in [0, T] \times \overline{Y}$, and such that $\psi_B(x, T, y) = 0$, for every $(x, y) \in \overline{\Omega} \times \overline{Y}$, $\psi_D = (\psi_D^1, \psi_D^2) \in \mathcal{C}^1(\Omega_T; \mathcal{X}_\#^1(Y))$, with compact support in $\overline{\Omega}$, for every $(t, y) \in [0, T] \times \overline{Y}$, and such that $[\psi_D(x, T, y)] = 0$, for every $(x, y) \in \overline{\Omega} \times \overline{Y}$.

Then, we obtain

$$\begin{aligned}
& - \int_{\Omega_T^{B,\varepsilon}} v^\varepsilon (\partial_t \phi_B + \varepsilon \partial_t \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot (\nabla \phi_B + \varepsilon \nabla_x \psi_B + \nabla_y \psi_B) \, dx \, dt \\
& + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla u^\varepsilon \cdot (\nabla \phi_B + \varepsilon \nabla_x \psi_B + \nabla_y \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} I_{ion}(v^\varepsilon, \tilde{w}^\varepsilon) (\phi_B + \varepsilon \psi_B) \, dx \, dt \\
& + \int_{\Omega_T^{B,\varepsilon}} (\sigma_1^{B,\varepsilon} + \sigma_2^{B,\varepsilon}) \nabla u^\varepsilon \cdot (\nabla \phi_D^1 + \varepsilon \nabla_x \psi_D^1 + \nabla_y \psi_D^1) \, dx \, dt \\
& + \int_{\Omega_T^{B,\varepsilon}} \sigma_1^{B,\varepsilon} \nabla v^\varepsilon \cdot (\nabla \phi_D^1 + \varepsilon \nabla_x \psi_D^1 + \nabla_y \psi_D^1) \, dx \, dt + \int_{\Omega_T^{D,\varepsilon}} \sigma^{D,\varepsilon} \nabla u^\varepsilon \cdot (\nabla \phi_D^2 + \varepsilon \nabla_x \psi_D^2 + \nabla_y \psi_D^2) \, dx \, dt \\
& - \alpha \varepsilon \int_{\Gamma_T^\varepsilon} [u^\varepsilon] \partial_t [\phi^D + \varepsilon \psi_D] \, d\sigma \, dt + \beta \varepsilon \int_{\Gamma_T^\varepsilon} [u^\varepsilon] [\phi^D + \varepsilon \psi_D] \, d\sigma \, dt \\
& = \int_{\Omega_T^{B,\varepsilon}} f_1(\phi_B + \varepsilon \psi_B) \, dx \, dt + \int_{\Omega_T^{B,\varepsilon}} (f_1 - f_2)(\phi_D^1 + \varepsilon \psi_D^1) \, dx \, dt \\
& + \int_{\Omega_T^{B,\varepsilon}} \bar{v}_0(\phi_B(0) + \varepsilon \psi_B(0)) \, dx + \alpha \varepsilon \int_{\Gamma^\varepsilon} \frac{s_{0\varepsilon}}{\varepsilon} [\psi_D](0) \, d\sigma, \quad (4.69)
\end{aligned}$$

where, with a little abuse of notation, we denote by $[\phi^D] = \phi_D^1 - \phi_D^2$. Unfolding and passing to the limit, we arrive at

$$\begin{aligned}
& - |E^B| \int_{\Omega_T} v \partial_t \phi_B \, dx \, dt + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla v + \nabla_y \hat{v}) \cdot (\nabla \phi_B + \nabla_y \psi_B) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla u^B + \nabla_y \hat{u}^B) \cdot (\nabla \phi_B + \nabla_y \psi_B) \, dy \, dx \, dt + |E^B| \int_{\Omega_T} I_{ion}(v, w) \phi_B \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} (\sigma_1^B + \sigma_2^B) (\nabla u^B + \nabla_y \hat{u}^B) \cdot (\nabla \phi_D^1 + \nabla_y \psi_D^1) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^B} \sigma_1^B (\nabla v + \nabla_y \hat{v}) \cdot (\nabla \phi_D^1 + \nabla_y \psi_D^1) \, dy \, dx \, dt \\
& + \int_{\Omega_T} \int_{E^D} \sigma^D (\nabla u^D + \nabla_y \hat{u}^D) \cdot (\nabla \phi_D^2 + \nabla_y \psi_D^2) \, dy \, dx \, dt
\end{aligned}$$

$$\begin{aligned}
& -\alpha \int_{\Omega_T} \int_{\Gamma} [u] \partial_t [\phi_D] \, d\sigma \, dx \, dt + \beta \int_{\Omega_T} \int_{\Gamma} [u] [\phi_D] \, d\sigma \, dx \, dt \\
& = |E^B| \int_{\Omega_T} f_1 \phi_B \, dx \, dt + |E^B| \int_{\Omega_T} (f_1 - f_2) \phi_D^1 \, dx \, dt \\
& \quad + |E^B| \int_{\Omega} \bar{v}_0 \phi_B(0) \, dx + \alpha \int_{\Omega} \int_{\Gamma} \bar{s}_1 [\phi_D](0) \, d\sigma \, dx, \quad (4.70)
\end{aligned}$$

where we have used Lemmas 4.1, 4.3 and 4.13. In order to get the strong formulation (4.55)–(4.66), we localize (4.70), taking first $\psi_B = \phi_D^1 = \phi_D^2 = \psi_D^1 = \psi_D^2 = 0$, then $\phi_B = \phi_D^2 = \psi_B = \psi_D^1 = \psi_D^2 = 0$ and finally $\phi_B = \phi_D^1 = \psi_B = \psi_D^1 = \psi_D^2 = 0$, so that we arrive at (4.55)–(4.57) and (4.64), (4.65). Moreover, we take $\phi_B = \phi_D^1 = \phi_D^2 = \psi_D^1 = \psi_D^2 = 0$, which gives (4.58) and (4.61). In the next step, we take first $\phi_B = \phi_D^1 = \phi_D^2 = \psi_B = \psi_D^2 = 0$ and then $\phi_B = \phi_D^1 = \phi_D^2 = \psi_B = \psi_D^1 = 0$, in order to obtain (4.59), (4.60) and (4.62), (4.63). The boundary condition (4.66) is a direct consequence of the fact that $v, u_D^1, u_D^2 \in L^2(0, T; H_0^1(\Omega))$.

Finally, the limit gating problem (4.26)–(4.27) and the uniqueness for the two-scale homogenized system are obtained as in the proof of Theorem 4.5. \square

Theorem 4.16. *Assume to be in the connected/connected geometry. Then, the two-scale system (4.55)–(4.66) can be rewritten as the single-scale degenerate parabolic system given by*

$$\begin{aligned}
& \partial_t v - \operatorname{div} (A^* \nabla (v + u^B)) + I_{ion}(v, w) = f_1, \quad \text{in } \Omega_T; \\
& -\operatorname{div} (A^* \nabla (u^B + v)) - \operatorname{div} (A_B^{hom} \nabla u^B + A_D^{hom} \nabla u^D) = f_1 - f_2, \quad \text{in } \Omega_T; \quad (4.71) \\
& \frac{\alpha |\Gamma|}{|E^B|} \partial_t [u] + \frac{\beta |\Gamma|}{|E^B|} [u] + \operatorname{div} (A_D^{hom} \nabla u^D) = 0, \quad \text{in } \Omega_T,
\end{aligned}$$

complemented with the initial and the boundary conditions (4.64)–(4.66) and the gating problem (4.26)–(4.27), where the matrix A^* is defined in (4.39) and A_B^{hom}, A_D^{hom} are defined in (4.51) and (4.52), respectively.

Proof. As in Subsection 4.1, thanks to (4.58) and (4.61), we can factorize

$$(\hat{v} + \hat{u}^B)(x, y, t) = -\zeta(y) \cdot \nabla (v + u^B)(x, t), \quad (4.72)$$

where the cell functions $\zeta = (\zeta^1, \dots, \zeta^N)$, with $\zeta^j \in H_{\#}^1(E^B)$ and $\mathcal{M}_{E^B}(\zeta^j) = 0$, are the solutions of the cell problem (4.33). Moreover, taking into account (4.59), (4.62) and (4.60), (4.63), we can factorize

$$\hat{u}^B(x, t, y) = -\hat{\chi}_0^B(y) \cdot \nabla u^B(x, t), \quad \hat{u}^D(x, t, y) = -\hat{\chi}_0^D(y) \cdot \nabla u^D(x, t), \quad (4.73)$$

where, for $j = 1, \dots, N$, $\hat{\chi}_0^{B,j} \in H_{\#}^1(E^B)$, $\hat{\chi}_0^{D,j} \in H_{\#}^1(E^D)$, $\mathcal{M}_{E^B}(\hat{\chi}_0^{B,j}) = 0 = \mathcal{M}_{E^D}(\hat{\chi}_0^{D,j})$ are the solutions of (4.50). Inserting (4.72) and (4.73) in (4.55)–(4.57), we get the single-scale homogenized degenerate parabolic system (4.71), where the matrices $A^*, A_B^{hom}, A_D^{hom}$ are defined in (4.39), (4.51) and (4.52), respectively. \square

Remark 4.17. In the connected/disconnected geometry, the system (4.55)–(4.65), with u^D (which now is only an $L^2(\Omega_T)$ -function) replaced by u^B in (4.57), (4.60)

and (4.63), and v, u^B having null trace on the boundary $\partial\Omega \times (0, T)$, is still in force. Then, in (4.73), the factorization of \hat{u}^D is replaced by $\hat{u}^D = -\hat{\chi}_0^D \cdot \nabla u^B$, with $\hat{\chi}_0^D$ as above. However, as in Subsection 4.2, we obtain that $\hat{\chi}_0^{D,j}(y) = y^j$, up to an additive constant. This implies $\nabla u^B + \nabla_y \hat{u}^D = 0$ and, hence, (4.57) is replaced by (4.67) and equations (4.60) and (4.63), actually, disappear. Moreover, equation (4.56) simplifies in equation (4.68) and, finally, the matrix $A_D^{hom} = 0$. Therefore, the single-scale degenerate parabolic system (4.71) becomes

$$\begin{aligned} \partial_t v - \operatorname{div} (A^* \nabla (v + u^B)) + I_{ion}(v, w) &= f_1, & \text{in } \Omega_T; \\ -\operatorname{div} (A^* \nabla (u^B + v)) - \operatorname{div} (A_B^{hom} \nabla u^B) &= f_1 - f_2, & \text{in } \Omega_T; \\ \alpha \partial_t [u] + \beta [u] &= 0, & \text{in } \Omega_T. \end{aligned} \quad (4.74)$$

□

Remark 4.18. Notice that the limit problem (4.71), in the connected/connected case, and the limit problem (4.74), in the connected/disconnected case, both lead to a kind of tridomain model. Indeed, similarly as in Remark 4.7, let us denote by u_1^B, u_2^B and u^D , respectively, the limits of the functions $u_1^{B,\varepsilon}, u_2^{B,\varepsilon}$ and $u^{D,\varepsilon}$, appearing in the system (2.15)–(2.23). Recalling that $v^\varepsilon = u_1^{B,\varepsilon} - u_2^{B,\varepsilon}$, $u^\varepsilon|_{\Omega_T^{B,\varepsilon}} = u_2^{B,\varepsilon}$ and $u^\varepsilon|_{\Omega_T^{D,\varepsilon}} = u^{D,\varepsilon}$, we can replace $v = u_1^B - u_2^B$ and $u^B = u_2^B$ in (4.71), thus obtaining

$$\begin{aligned} \partial_t (u_1^B - u_2^B) - \operatorname{div} (A^* \nabla (u_1^B)) + I_{ion}(u_1^B - u_2^B, w) &= f_1, & \text{in } \Omega_T; \\ -\operatorname{div} (A^* \nabla u_1^B) - \operatorname{div} (A_B^{hom} \nabla u_2^B + A_D^{hom} \nabla u^D) &= f_1 - f_2, & \text{in } \Omega_T; \\ \frac{\alpha |I|}{|EB|} \partial_t (u_2^B - u^D) + \frac{\beta |I|}{|EB|} (u_2^B - u^D) + \operatorname{div} (A_D^{hom} \nabla u^D) &= 0, & \text{in } \Omega_T. \end{aligned}$$

Analogously, (4.74) becomes

$$\begin{aligned} \partial_t (u_1^B - u_2^B) - \operatorname{div} (A^* \nabla (u_1^B)) + I_{ion}(u_1^B - u_2^B, w) &= f_1, & \text{in } \Omega_T; \\ -\operatorname{div} (A^* \nabla u_1^B) - \operatorname{div} (A_B^{hom} \nabla u_2^B) &= f_1 - f_2, & \text{in } \Omega_T; \\ \frac{\alpha |I|}{|EB|} \partial_t (u_2^B - u^D) + \frac{\beta |I|}{|EB|} (u_2^B - u^D) &= 0, & \text{in } \Omega_T. \end{aligned}$$

□

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