SAPIENZA
UNIVERSITÀ DI ROMA

DEPARTMENT OF BASIC AND APPLIED SCIENCES FOR ENGINEERING

PHD THESIS IN MATHEMATICAL MODELS FOR ENGINEERING, ELECTROMAGNETISM AND NANOSCIENCES

Salvatore Fragapane

Regularity and asymptotics for p-Laplace type operators in fractal and pre-fractal domains

Advisor: Prof. Maria Agostina Vivaldi

Academic Year 2017-2018

Contents

Introduction

In this thesis we deal with double obstacle problems involving p-Laplace type operators in fractal and pre-fractal domains of $\mathbb{R}^2$.

The arising of these questions is almost obvious:

1) Why p-Laplacian?

2) Why fractal and pre-fractal domains?

The choice of p-Laplace type operators is motivated by the fact that they are able to describe and provide a model for many physical processes: quasi-Newtonian flow, power law material, non-linear filtration and diffusion (see [31] and the references therein). In particular, the origin of these operators has to be searched at the time when the engineers realized that the turbulent regime of fluids could not be described correctly through the linear Darcy law. More precisely, it must be traced back to the first one who used the power law into the continuity equation (see [6] and the references quoted there). Somehow, the use of p-Laplacian provides a mathematical formulation for the problems that are object of our investigation.

Also the choice of fractal and pre-fractal domains finds some motivations in the possibility of these objects to describe Nature in a new way: more precisely, they can provide a better description than the classical shapes of Geometry. In fact, these extraordinary objects can be powerful and useful tools to provide models for problems connected to various phenomena in many fields: Biology, Medicine, Engineering, etc. In particular, from a mathematical point of view, they are the best way to describe all the phenomena in which surface effects are prevalent on volume effects; furthermore, they are able to describe the irregular structure and the profile of many natural objects: coasts, pulmonary alveoli, Roman broccoli, etc (see [53] to visualize some examples).

We point out that the examples just mentioned can be described through pre-fractal sets with a self-similar structure.

It should not be forgotten that, among the various fields of application there is also Economy;

in fact, Mandelbrot used fractals to study certain behaviours of markets (see, for instance, [54]). Moreover, as further example of the use of domains with reentrant corner in the study of physical situations, we refer to [23]. Here, in order to predict the failure load for a structural member with a reentrant corner, the authors show how in these types of structures the fracture strength increases with the amplitude of the corner.

After providing some of the motivations that pushed us toward the choice of this setting and these operators, let us present the problems studied.
In the first part of this work, we will focus on double obstacle problems involving p-Laplace type operators in polygonal and non-convex domains in $\mathbb{R}^2$; in particular, our aim is the improvement of the regularity results and the approximation error estimates up to now known.
Let $p \in (2, \infty)$ and let us consider a conical domain Ω_ω (i.e. a polygonal domain with reentrant corners) and the following two obstacle problem:

$$\text{find } u \in \mathcal{K}, \quad a_p(u, v-u) - \int_{\Omega_\omega} f(v-u)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}, \tag{0.0.1}$$

with

$$a_p(u,v) = \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v \,\mathrm{d}x, \;\; k \in \mathbb{R},$$

and

$$\mathcal{K} = \{ v \in W_0^{1,p}(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega \},$$

where f, k, φ_1 and φ_2 are given.
It is known that, under suitable and natural assumptions, the existence and the uniqueness of a solution u to Problem (0.0.1) are guaranteed. Furthermore, about the regularity of the solution, in particular regarding the properties of the first order derivatives, in the literature are known, for instance, the studies of Li and Martio (see [49] and the references therein) and Lieberman (see [50] and the references quoted there).
Instead, to our knowledge, regarding second order derivatives, for $p > 2$, there are no L^2 regularity results concerning obstacle problems, even requiring stronger assumptions as the differentiability of the data and the smoothness of the boundary. The results recently obtained by Brasco and Santambrogio (see in [9]) and Mercuri, Riey and Sciunzi (see [58]), in particular, seem not to be applicable in the case of obstacle problems. As far as we know, up to now, for the solutions to obstacle problems, global regularity results in terms of Sobolev (or Besov)

spaces with smoothness index greater than 1, are only established for the case of $p = 2$ (see [18]).
The first improvement here presented (see Chapter 2) consists in establishing a regularity result (see Theorem 2.2.4) for the solution to obstacle Problem (0.0.1) in terms of weighted Sobolev spaces (involving second derivatives), where we take as weight the distance from the conical point, that is the point coincident with the vertex of the reentrant corner.
In the proof of our regularity result a crucial role is played by Lewy-Stampacchia inequality: first proved for super-harmonic functions (see [48]), subsequently it was extended to more general (linear) operators and more general obstacles by Mosco and Troianiello (see [61]) and for $T-$monotone operators (see [60]).
We want to underline that the importance of Theorem 2.2.4 is due to the fact that it represents a novelty not only for obstacle problems but in the case of the Dirichlet problem too. In fact, if on the one hand there is a very wide literature about the regularity in the Hölder classes for both the solution u and the gradient ∇u (see [45] and the references therein), on the other hand we have few investigations about the smoothness of the second derivatives in this type of irregular domains. About this subject we know only the contributions of Borsuk and Kondratiev (see [8]) and Cianchi and Maz'ya (see [27]).
Differently from the Dirichlet problem faced by the authors in [8], we require weaker assumptions and we prove a stronger regularity. Nevertheless, to state our results, we exploit some ideas of the studies just mentioned.
In particular, to obtain local estimates and estimates far away from the conical point, we follow some approach of [27], requiring only the membership of the data to $L^2(\Omega_\omega)$; furthermore, contrarily to the case studied by [27], our domains are not convex and they do not satisfy condition (2.12) in [27]. Whereas, to state estimates near the conical point, we follow the approach of Tolksdorf (see [70]) and Dobrowolski (see [33]).
Moreover, we prove the boundedness of the gradient far away from the conical point, using some results by Tolksdorf, by Cianchi and Mazy'a (see [26]) and by Barret and Liu (see [5]) (for the case of $k = 0$).

In addition to its innovation, the regularity result presented is interesting because it is useful also in order to face up the issue of numerical analysis. As known, indeed, regularity results play a crucial role to establish error estimates for the FEM approximations (see, for instance, [13]).

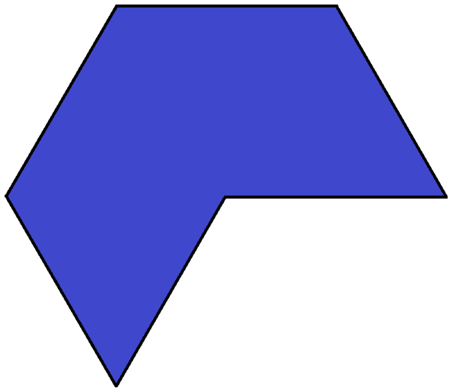

Figure 1: Simulation domain

In order to face the numerical approach of the solutions to obstacle problems in fractal domains, we have to consider the solutions to the corresponding obstacle problems in pre-fractal approximating domains, the related FEM-solutions and then evaluate the approximation error.

With this premises, to pursuit our goal (see Chapter 3), we apply Theorems 2.2.4 and 2.3.3. To state our sharp error estimate (see Theorem 3.2.1) for the FEM approximations we will follow the approach of P. Grisvard (see [38]); more precisely, we consider a suitable triangulation of the domains, adapted to the regularity of the solutions. We note that for $p = 2$ our Theorem 3.2.1 gives the sharp result of Grisvard (see Corollary 8.4.1.7 in [38]). Furthermore, we stress the fact that our Theorem 3.2.1 gives a better result than the one obtained in [22]; more precisely it gives a faster convergence than the one assured by estimate (5.63) in [22].

After stating this optimal estimate, we perform numerical simulations in order to apply and "confirm" the results obtained. To do this, according to the approach of Grisvard and following the suggestion of Raugel (see [65]), we consider a particular triangulation of the domain. In our simulations, the domain will be polygonal with only a reentrant corner with amplitude equal to $\frac{4\pi}{3}$ (see Figure 3).

In particular, we investigate the cases of $p = 2$ and $p > 2$ (fixed) underlying how the use of this particular mesh allows to obtain an improvement of the error estimate opposite to the case in which we use a "non-Grisvard" triangulation (i.e. an only regular and conformal mesh). Moreover, we point out that, since the value $1 - \gamma$ (see Figure 2.4 and Remark 2.2.2) decrease as p tends to infinity, then, when p increase, the adapted mesh will undergo a less significant change, becoming more like to a non-adapted mesh.

Up to this point, we have focused our efforts to improve previous regularity results and approximation error estimates.
Now, considering both the case of fractal domains and the n-th pre-fractal ones (choosing $k = 0$ as the value of the constant in the operators), we want to investigate the asymptotic behaviour of the solution with respect to p and n. Moreover, we briefly discuss the issue of the uniqueness (see Chapter 4).
The reason of this further step is the study of an optimal mass transport problem (as investigated in [56]). More precisely, there, the authors studied a double obstacle problem involving p-Laplacian in smooth domains, then by passing to the limit as $p \to \infty$ they obtained an answer to an optimal mass transport problem for the Euclidean distance (see [76] for notation and general results on mass transport theory).
We set the following problems.
Given $f \in L^1(\Omega_\alpha)$, let us consider this two obstacle problems on domains with fractal and pre-fractal boundary, respectively indicated with Ω_α and Ω_α^n.

$$\text{find } u \in \mathcal{K}_n, \quad \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla (v-u)\,\mathrm{d}x - \int_{\Omega_\alpha^n} f(v-u)\,\mathrm{d}x \geqslant 0 \ \ \forall v \in \mathcal{K}_n, \qquad (\mathcal{P}_{p,n})$$

$$\text{find } u \in \mathcal{K}, \quad \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla (v-u)\,\mathrm{d}x - \int_{\Omega_\alpha} f(v-u)\,\mathrm{d}x \geqslant 0 \ \ \forall v \in \mathcal{K}, \qquad (\mathcal{P}_p)$$

where

$$\mathcal{K}_n = \{ v \in W^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leqslant v \leqslant \varphi_{2,n} \text{ in } \Omega_\alpha^n \}$$

$$\mathcal{K} = \{ v \in W^{1,p}(\Omega_\alpha) : \varphi_1 \leqslant v \leqslant \varphi_2 \text{ in } \Omega_\alpha \}$$

with obstacles $\varphi_{1,n}$, $\varphi_{2,n}$, φ_1 and φ_2.
So, under suitable assumptions about the obstacles (see (4.1.3)), we are able to prove that, up to pass to a subsequence, the solutions to Problems $(\mathcal{P}_p)$ converge (uniformly) in $C(\bar{\Omega}_\alpha)$, as $p \to \infty$, to a solution to the following problem

$$\int_{\Omega_\alpha} u_\infty(x) f(x)\,\mathrm{d}x = \max \left\{ \int_{\Omega_\alpha} w(x) f(x)\,\mathrm{d}x : w \in \mathcal{K}^\infty \right\}, \qquad (\mathcal{P}_\infty)$$

where

$$\mathcal{K}^\infty = \{ u \in W^{1,\infty}(\Omega_\alpha) : \varphi_1 \leq u \leq \varphi_2 \text{ in } \Omega_\alpha, ||\nabla u||_{L^\infty(\Omega_\alpha)} \leq 1 \}$$

(see Theorem 4.1.1). In a totally analogous way, we can obtain a similar result (see Theorem

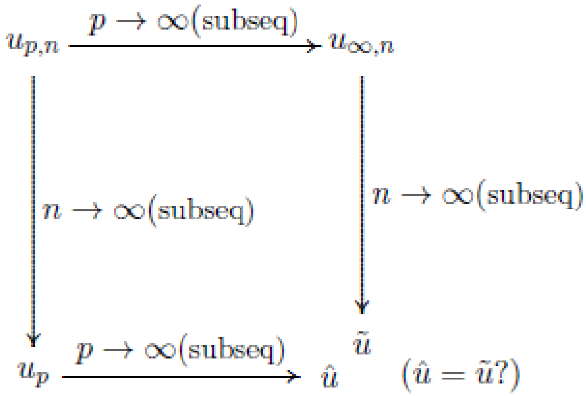

Figure 2: Summary of the asymptotics

4.1.2) for the pre-fractal case. In particular, we show that, up to pass to a subsequence, the solutions to Problems $(\mathcal{P}_{p,n})$ converge (uniformly) in $C(\bar{\Omega}^n_\alpha)$, as $p \to \infty$, to a solution to the problem

$$\int_{\Omega^n_\alpha} u_\infty(x) f(x)\,\mathrm{d}x = \max\left\{\int_{\Omega^n_\alpha} w(x) f(x)\,\mathrm{d}x : w \in \mathcal{K}^\infty_n\right\}, \tag{$\mathcal{P}_{\infty,n}$}$$

where

$$\mathcal{K}^\infty_n = \{u \in W^{1,\infty}(\Omega^n_\alpha) : \varphi_{1,n} \le u \le \varphi_{2,n} \text{ in } \Omega^n_\alpha, ||\nabla u||_{L^\infty(\Omega^n_\alpha)} \le 1\}.$$

We point out that to obtain the result that we present it is sufficient to require the membership of f to $L^1(\Omega_\alpha)$; however, in [56], the authors required that $f \in L^\infty(\Omega)$ in order to give sense to the interpretation as a transport problem.

Since we have the convergence, as $n \to \infty$, of the n-th pre-fractal curves to the fractal curve in the Hausdorff metric, the pre-fractal domains Ω^n_α converge to the fractal domain Ω_α. Hence, it make sense to study the asymptotic behavior of $(\mathcal{P}_{p,n})$ and $(\mathcal{P}_{\infty,n})$ for $n \to \infty$ (see [59] for the connections between convergence of convex sets and solutions of variational inequalities). In particular, making suitable assumptions about the obstacles, we have, as $n \to \infty$ and a fixed $p \in (2,\infty)$, the strong convergence in $W^{1,p}(\Omega_\alpha)$ of suitable extensions of the solutions to Problems $(\mathcal{P}_{p,n})$ to a solution to Problem $(\mathcal{P}_p)$. Similarly, as $n \to \infty$ and $p = \infty$, we obtain the $\star$-weakly convergence in $W^{1,\infty}(\Omega_\alpha)$ of suitable extensions of the solutions to Problem $(\mathcal{P}_{\infty,n})$ to a solution to Problem $(\mathcal{P}_\infty)$.

At this point, in a totally natural way, this question arises:

Do we obtain the same solution changing the order of the limits?

We note (see Figure 4), indeed, that passing to the limits, first for $p \to \infty$ and then for $n \to \infty$, we obtain a solution to Problem $(\mathcal{P}_\infty)$. Nevertheless, passing to the limits, first for $n \to \infty$ and then for $p \to \infty$, we find a solution to the same problem. Unfortunately, not having, in general, uniqueness results for the solutions to Problem $(\mathcal{P}_\infty)$, we cannot expect that the limit solutions coincide; that is, the answer to the previous question, in general, is negative. For this reason, the search of conditions which guarantee uniqueness takes a certain importance.
Eventually, we want to point out that condition (4.1.3) assures, in particular, the non-emptiness of the convex $\mathcal{K}^\infty$. This assumption can be substituted with the request that the convex is not empty in the proof of the existence of the solution (see Remark 4.1.1).

The organization of the thesis is the following.
In Chapter 1 we recall some basic results concerning the tools used throughout this work. In Chapter 2 we give the regularity result. In Chapter 3, starting from the result of the previous chapter, we state optimal estimates of the approximation error and perform numerical simulations. In Chapter 4 we study the asymptotic behaviour of the solutions, as well as stating some uniqueness results. Eventually, we conclude by briefly summarizing the results obtained, specifying some open problems and outlining some possibilities for future investigation.

Chapter 1

Preliminary concepts and tools

This first chapter is dedicated to recall those concepts and tools which will be constantly used during this work. Let us point out that the set Ω will be taken always as bounded domain of $\mathbb{R}^2$, where we will call domain an open, connected subset of $\mathbb{R}^2$. Clearly, the definitions can be given in more general cases.

1.1 Sobolev spaces

In this paragraph we recall definitions and some basic results about Sobolev spaces firstly with integer and fractional order and then with weight.

1.1.1 Sobolev spaces with integer order

Let Ω be a domain of $\mathbb{R}^2$ and let us consider $1 \leq p \leq \infty$, $m \in \mathbb{N} \cup \{0\}$ and a multi-index $\beta = (\beta_1, \beta_2)$, with $\beta_1, \beta_2 \in \mathbb{N} \cup \{0\}$.
In order to define the Sobolev spaces with integer order, we recall the definition of the Lebesgue spaces

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^p \,\mathrm{d}x < \infty\}, \text{ for } 1 \leq p < \infty,$$

and

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : \sup \mathrm{ess}_\Omega |u(x)| < \infty\},$$

which are Banach spaces with respect to the norms

$$||u||_{L^p(\Omega)} := ||u||_p = \Big(\int_\Omega |u(x)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}$$

$$||u||_{L^\infty(\Omega)} := ||u||_\infty = \sup \text{ess}_\Omega |u(x)|,$$

respectively (see Chapter 2 in [1] or Chapter 4 in [10] for proofs and details).

We recall that the elements of these spaces are equivalence classes of measurable functions, according to the equivalence relation to be equal almost everywhere (a.e.) in Ω.

Moreover, we recall that

$$Lip(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} : \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|} < \infty\}.$$

Now, we are able to define the Sobolev spaces $W^{m,p}(\Omega)$ as follow:

$$W^{0,p}(\Omega) = L^p(\Omega),$$

$$W^{m,p}(\Omega) = \{u \in W^{m-1,p}(\Omega) : D^\beta u \in L^p(\Omega), |\beta| = m\}, m \geq 1,$$

where $D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$ indicates the weak derivative of u, with $|\beta| = \beta_1 + \beta_2$ and $x = (x_1, x_2)$.

Moreover, we introduce the spaces

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) \ : \ u|_{\partial\Omega} = 0\},$$

as the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$, if $p < \infty$, and the space

$$W_0^{1,\infty}(\Omega) = \{u \in Lip(\overline{\Omega}) \ : \ u|_{\partial\Omega} = 0\}.$$

We endow these spaces, respectively, with the norms

$$||u||_{W^{m,p}(\Omega)} := ||u||_{m,p} = \Big(\sum_{|\beta|\leq m} ||D^\beta u||_p^p\Big)^{\frac{1}{p}}, \text{for } 1 \leq p < \infty.$$

$$||u||_{W^{1,\infty}(\Omega)} := ||u||_{1,\infty} = \max_{|\beta|\leq 1} ||D^\beta u||_\infty.$$

In the case of $p = 2$, we indicate them with the notation

$$H^m(\Omega) = W^{m,2}(\Omega) \text{ and } H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

For these spaces, the following results hold

Theorem 1.1.1. *The spaces* $(W^{m,p}(\Omega), ||\cdot||_{m,p})$ *are*

(1) Banach spaces for $1 \le p \le \infty$,

(2) separable for $1 \le p < \infty$,

(3) reflexive for $1 < p < \infty$.

We also recall that $f \in W^{m,p}_{loc}(\Omega)$ if $f \in W^{m,p}(A)$ for every open $A \subset\subset \Omega$, i.e. for every open A with compact closure in Ω.
Furthermore, we introduce the dual space of $W_0^{m,p}(\Omega)$ as

$$(W_0^{m,p}(\Omega))' = W^{-m,p'}(\Omega),$$

with $p \in [1,\infty)$ and p' the conjugate exponent of p.
Equipping these spaces with the norm

$$||u||_{W^{-m,p'}(\Omega)} := ||u||_{-m,p'} = \sup\{| < u, v > | \ : v \in W_0^{m,p}(\Omega), ||v||_{m,p} = 1\},$$

where $< \cdot, \cdot >$ is the pairing, we have Banach spaces (see, for instance, Chapter 3 in [1] or Chapter 9 in [10] for details about definitions and results).

Eventually, we want to point out that embedding results hold for the Sobolev spaces we have just considered (see, for instance, Chapters 5 and 6 in [1]). We propose to remind them after, when we will speak about fractional spaces.

1.1.2 Fractional Sobolev spaces

Now, let us consider an arbitrary, possibly non-smooth, domain $\Omega \subset \mathbb{R}^2$ (actually, in our case, it will be bounded). We want to introduce the fractional Sobolev spaces $W^{\sigma,p}(\Omega)$, for any $\sigma \in \mathbb{R}^+$ and for any $p \in [1,\infty)$.
Obviously, in the case of $\sigma = m$ integer we define $W^{\sigma,p}(\Omega) = W^{m,p}(\Omega)$, that is the Sobolev spaces already introduced. We now consider the case in which σ is not an integer.
Let us start fixing $\sigma \in (0,1)$. For any $p \in [1,\infty)$, we define

$$W^{\sigma,p}(\Omega) = \{u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{2}{p}+\sigma}} \in L^p(\Omega \times \Omega)\}.$$

Endowing this space with the norm

$$||u||_{W^{\sigma,p}(\Omega)} := ||u||_{\sigma,p} = \Big(||u||_p^p + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{2+\sigma p}}\,\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}},$$

it becomes a Banach space, intermediate between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

If $\sigma > 1$ (non-integer), we can write it as sum between an integer m and a number $s \in (0,1)$, that is $\sigma = m + s$. In this case, we define the space

$$W^{\sigma,p}(\Omega) = \{u \in W^{m,p}(\Omega) : D^\beta u \in W^{s,p}, \text{ with } \beta \text{ such that } |\beta| = m\}.$$

Introducing the norm

$$||u||_{W^{\sigma,p}(\Omega)} := ||u||_{\sigma,p} = \Big(||u||_{m,p}^p + \sum_{|\beta|=m} ||D^\beta u||_{s,p}^p\Big)^{\frac{1}{p}},$$

it becomes a Banach space. Let us define $W_0^{s,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm $||\cdot||_{\sigma,p}$. Even here, in the case of $p = 2$ we indicate them with the notation

$$H^\sigma(\Omega) = W^{\sigma,2}(\Omega) \text{ and } H_0^\sigma(\Omega) = W_0^{\sigma,2}(\Omega).$$

(see, for example, [30] or Chapter 7 in [1] or Chapter 4 in [29] for details)

Theorem 1.1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. The following facts hold*

i) if $\sigma p < 2$, then $W^{\sigma,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q \in [1, p^*[$, with $p^* = \frac{2p}{2-\sigma p}$;

ii) if $\sigma p = 2$, then $W^{\sigma,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q \in [1,\infty[$;

iii) if $\sigma p > 2$, then we have:

(a) if $\chi = \sigma - \frac{2}{p} \notin \mathbb{N}$, then $W^{\sigma,p}(\Omega)$ is compactly embedded in $C^{[\chi],\lambda}(\overline{\Omega})$, for every $\lambda < \chi - [\chi]$, where $[\cdot]$ here denotes the integer part,

(b) if $\chi = \sigma - \frac{2}{p} \in \mathbb{N}$, then $W^{\sigma,p}(\Omega)$ is compactly embedded in $C^{\chi-1,\lambda}(\overline{\Omega})$ for every $\lambda < 1$,

(see Theorem 4.58 in [29] for proofs and details).

To conclude, we specify some definitions used in the previous Theorem 1.1.2.

Definition 1.1.1. *Let Ω a domain in $\mathbb{R}^2$.*

(i) We define $C^k(\overline{\Omega})$ as the set of the functions having partial derivatives of order less or equal than k uniformly continuous on $\overline{\Omega}$.

(ii) We define $C^{k,\lambda}(\overline{\Omega})$*, for* $0 < \lambda \leq 1$*, as the subset of* $C^k(\Omega)$ *of the functions* v *such that*

$$\exists C > 0 : \forall \beta \text{ with } |\beta| = k \text{ we have } |D^\beta v(x) - D^\beta v(y)| \leq |x - y|^\lambda, \; \forall x, y \in \Omega.$$

These spaces become Banach spaces if we endow them with a suitable norm (see Section 1.3 in [29]).

Definition 1.1.2. *We say that a domain* $\Omega \in \mathbb{R}^2$ *is Lipschitz if for every* $x \in \partial\Omega$ *there exists a neighbourhood* $I(x)$ *in* $\mathbb{R}^2$ *and new orthogonal coordinates* $\{y_1, y_2\}$ *such that*
(i) $I(x)$ *is a rectangle in the new coordinates:*

$$I(x) = \{(y_1, y_2) \,|\, -a_i < y_i < a_i, \; i = 1, 2\};$$

(ii) There exists a Lipschitz function f *defined in* $(-a_1, a_1)$ *such that*

$$|f(y)| \leq \frac{a_2}{2} \text{ for every } y \in (-a_1, a_1),$$

$$\Omega \cap I(x) = \{(y_1, y_2) \in I(x) \,|\, y_2 < f(y_1)\},$$

$$\partial\Omega \cap I(x) = \{(y_1, y_2) \in I(x) \,|\, y_2 = f(y_1)\}$$

(see, for instance, Definition 1.2.1.1 in [38] for more details).

1.1.3 Weighted Sobolev spaces

With the aim of stating regularity results in the next chapter, we need to introduce a particular Sobolev space with weight (see Section 8.4.1 in [38]).
Let $\Omega = \Omega_\omega$ be a domain of $\mathbb{R}^2$ having a reentrant corner (that is a corner with amplitude $\omega > \pi$. Let us consider $\mu \in \mathbb{R}_0^+$ and $\rho : \Omega \to \mathbb{R}$ the distance function from the vertex of the reentrant corner.
We indicate with

$$H^{2,\mu}(\Omega)$$

the space of the functions u belonging to $H^1(\Omega)$ such that

$$D^\beta u \in L^{2,\mu}(\Omega), \text{ for all } \beta : |\beta| = 2,$$

where $L^{2,\mu}(\Omega)$ is the completion of the space $C^0(\Omega)$ with respect to the norm

$$||u||_{L^{2,\mu}(\Omega)} = \Big(\int_\Omega \rho^{2\mu} |u|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Endowing this space with the norm

$$||u||_{H^{2,\mu}(\Omega)} := ||u||_{2,\mu} = \Big(||u||_{1,2}^2 + \sum_{|\beta|=2} ||\rho^\mu D^\beta u||_2^2 \Big)^{\frac{1}{2}}, \qquad (1.1.1)$$

it becomes a Banach space (in particular a Hilbert space).
In the following, we will consider even the case of $\Omega = \Omega_\alpha^n$ (see Section 1.4 for the definition of Ω_α^n) in which we have more reentrant corners. In this situation, we consider as weight function the minimum distance from the set of the vertices of the reentrant corners.

Let us recall the following results.

Proposition 1.1.1. *The following fact holds*

$$\text{(i) } H^{2,\mu}(\Omega) \subset W^{2,p}(\Omega), \forall p \ : \ 1 < p < \frac{2}{\mu + 1}, \ \mu \in (0, 1)$$

(for instance, see Lemma 8.4.1.2 in [38] for the proof).

Proposition 1.1.2. *If $0 \le \sigma < 2 - \mu$, then the following embedding holds*

$$H^{2,\mu}(\Omega) \subset W^{\sigma,2}(\Omega),$$

with

$$||u||_{\sigma,2} \le C||u||_{2,\mu}.$$

(see, for instance, Proposition 3.2 in [68] for the proof).

1.2 Besov spaces

In this section we recall the definition and some basic results about Besov spaces (see, for instance, Section 4 in [67]). For more informations and details we refer to Chapter 2 in [72] and Chapter 6 in[7].
Let Ω be a domain in $\mathbb{R}^2$. Let us consider $\sigma \in (0, 1)$, $1 \le p \le \infty$ and $1 \le q \le \infty$. We define

$$B_{p,q}^\sigma(\Omega) := (W^{1,p}(\Omega), L^p(\Omega))_{\sigma,q},$$

$$B^{\sigma+1}_{p,q}(\Omega) := (W^{2,p}(\Omega), W^{1,p}(\Omega))_{\sigma,q} = \{u \in W^{1,p}(\Omega) : \nabla u \in B^{\sigma}_{p,q}(\Omega : \mathbb{R}^2)\}$$

with $(\cdot,\cdot)_{\sigma,q}$ the real interpolation functor.
Moreover, we have the following particular case

$$W^{\sigma,p}(\Omega) = B^{\sigma}_{p,p}(\Omega).$$

Now, for $\sigma \in (0,1)$ and $1 \le p \le \infty$, we consider the following semi-norm

$$|u|^p_{\sigma,p} = \sup_{h\in D\setminus\{0\}} \int_{\Omega_{|h|}} \Big|\frac{u(x+h)-u(x)}{|h|^\sigma}\Big|^p \,\mathrm{d}x,$$

where D is a set generating $\mathbb{R}^2$ and star-shaped with respect to 0 and $\Omega_{|h|}$ denotes the set of the points of Ω which have distance from the boundary greater that $|h|$. So, we can characterize the following space:

$$B^{\sigma}_{p,\infty} = \{u \in L^p(\Omega) \,:\, |u|^p_{\sigma,p} < \infty\}$$

Now, we give the following embedding results (see Theorem 6.2.4 in [7]):

Theorem 1.2.1. *Let us consider $p, q_1, q_2 \in [1,\infty]$ and $\sigma_1, \sigma_2 > 0$. Then*

(i) $B^{\sigma_1}_{p,q_1}(\Omega) \subset B^{\sigma_1}_{p,q_2}(\Omega)$, *with* $q_1 < q_2$.

(ii) $B^{\sigma_2}_{p,q}(\Omega) \subset B^{\sigma_1}_{p,q}(\Omega)$, *with* $\sigma_2 > \sigma_1$.

(iii) $B^{\sigma_1}_{p,1}(\Omega) \subset W^{\sigma_1,p}(\Omega) \subset B^{\sigma_1}_{p,\infty}(\Omega)$.

1.3 Hausdorff measure

In order to introduce fractal sets in the next section, we have to recall the Hausdorff measure (see, for instance, Chapter 2 in [35]).
Let $M \subseteq \mathbb{R}^d$.

Definition 1.3.1. *The family $\{V_i\}_{i\in I}$ is a δ-covering of M if*

$$M \subset \cup_{i\in I} V_i, \text{ with } 0 < diam_e V_i < \delta, \ \forall i,$$

with

$$diam_e M = \sup\{d(P,Q), \forall P, Q \in M\}.$$

Now, let us consider $s \geq 0$, for a fixed value of δ, we define

$$\mathcal{H}^s_\delta(M) = \inf\{\sum_{i\in I} k_s \Big(\frac{\text{diam}_e V_i}{2}\Big)^s \ : \ \{V_i\}_{i\in I}\ \delta - covering\},$$

where

$$k_s = \frac{s^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)},$$

with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ is the Gamma function.
For s and M as before, we define

$$\mathcal{H}^s(M) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(M) = \sup_{\delta>0} \mathcal{H}^s_\delta(M).$$

We call $\mathcal{H}^s$ the s-dimensional Hausdorff measure on $\mathbb{R}^d$.
It is possible to prove that the measure just defined is a Borel regular measure. Furthermore, it satisfies the following properties

Proposition 1.3.1. *Let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure on $\mathbb{R}^d$.*
(a) $\mathcal{H}^0$ is counting measure.
(b) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}^1$.
(c) $\mathcal{H}^s = 0$ on $\mathbb{R}^d$, $\forall s > d$.
(d) $\mathcal{H}^s(\lambda M) = \lambda^s \mathcal{H}^s(M)$, $\forall \lambda > 0$.
(e) $\mathcal{H}^s(\Psi(M)) = \alpha^s \mathcal{H}^s(M)$, with $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ contractive similarity having α as contraction factor.
(f) If $0 < \mathcal{H}^s(M) < \infty$, then $\mathcal{H}^q(M) = 0$, if $q > s$, and $\mathcal{H}^q(M) = \infty$, if $q < s$.

Now, we can give the following definition.

Definition 1.3.2. *The Hausdorff dimension of a set $M \subseteq \mathbb{R}^d$ is*

$$dim_{\mathcal{H}}(M) = \inf\{0 \leq s < \infty : \mathcal{H}^s(M) = 0\}.$$

1.4 Self-similar fractal sets: Koch curves

In the following, we will often use the words "bad domain" (which we will indicate with Ω_α^n or Ω_ω) to refer to irregular subsets of $\mathbb{R}^2$, for example sets which approximate the so-called Koch Islands (in the following indicated with Ω_α). These domains Ω_α are the limit, in the Hausdorff metric, of the domains Ω_α^n which are constructed starting from a regular polygon (triangle, square, pentagon, etc.) and replacing each side by the n-th pre-fractal of the Koch curve. For this reason, this paragraph focuses on how to obtain these particular sets.
In order to introduce them, let us recall some definitions and results (see Section 5 in [40]).

Let us consider $\Psi_\alpha = \{\psi_{1,\alpha}, \ldots, \psi_{N,\alpha}\}$, with $\psi_i : \mathbb{C} \to \mathbb{C}$, $i = 1, \ldots, N$, a set of contractive similarities having α^{-1} as contraction factor.

Definition 1.4.1. *(see [40])*
We say that K is self-similar with respect to Ψ_α if
(i) K is invariant with respect to Ψ_α, that is $K = \Psi_\alpha(K)$
(ii) $\mathcal{H}^s(K) > 0$, $\mathcal{H}^s(K_i \cap K_j) = 0$ for $i \neq j$, with $s = dim_{\mathcal{H}}(K)$ and $K_i = \psi_{i,\alpha}(K)$.

Definition 1.4.2. *(see [40])*
We say that Ψ_α satisfies the open set condition if there exists a non-empty bounded open set V such that
(i) $\cup_{i=1}^N \psi_{i,\alpha}(V) \subset V$,
(ii) $\psi_{i,\alpha}(V) \cap \psi_{j,\alpha}(V) = \emptyset$, if $i \neq j$.

Theorem 1.4.1. *Let Ψ_α be a set of contractive similarities satisfying the condition of definition (1.4.2). Then there exists a unique compact set $K \subset \mathbb{C}$ such that K is invariant on Ψ_α. Moreover the Hausdorff dimension of K is equal to $\frac{\ln N}{\ln \alpha}$.*

Now, let us examine the construction of these sets.
Let K^0 be the line segment of unit length that has as endpoints $A = (0,0)$ and $B = (1,0)$.
Furthermore, let us consider the family of 4 contractive similarities Ψ_α, by means
$\Psi_\alpha = \{\psi_{1,\alpha}, \ldots, \psi_{4,\alpha}\}$, with $\psi_{i,\alpha} : \mathbb{C} \to \mathbb{C}$, $i = 1, \ldots, 4$ and $2 < \alpha < 4$, defined in this way:

$$\psi_{1,\alpha}(z) = \frac{z}{\alpha}, \qquad \psi_{2,\alpha}(z) = \frac{z}{\alpha} e^{i\theta(\alpha)} + \frac{1}{\alpha},$$
$$\psi_{3,\alpha}(z) = \frac{z}{\alpha} e^{-i\theta(\alpha)} + \frac{1}{2} + i\sqrt{\frac{1}{\alpha} - \frac{1}{4}}, \qquad \psi_{4,\alpha}(z) = \frac{z-1}{\alpha} + 1,$$

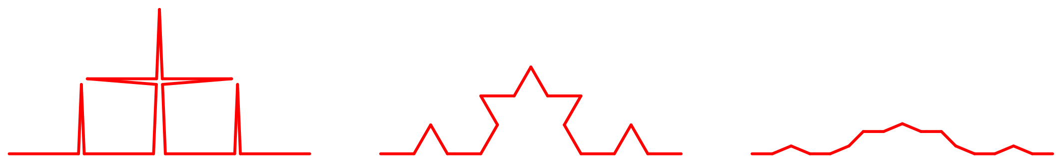

Figure 1.1: K_α^2, with $\alpha = 2.2$, $\alpha = 3$ and $\alpha = 3.8$, respectively

with

$$\theta(\alpha) = \arcsin\left(\frac{\sqrt{\alpha(4-\alpha)}}{2}\right). \tag{1.4.2}$$

So, for each $n \in \mathbb{N}$, we set

$$K_\alpha^1 = \bigcup_{i=1}^4 \psi_{i,\alpha}(K^0), \qquad K_\alpha^2 = \bigcup_{i=1}^4 \psi_{i,\alpha}(K_\alpha^1), \ldots, \qquad K_\alpha^n = \bigcup_{i=1}^4 \psi_{i,\alpha}(K_\alpha^{n-1}) = \bigcup_{i|n} \psi_{i|n,\alpha}(K^0),$$

where $\psi_{i|n,\alpha} = \psi_{i_1,\alpha} \circ \psi_{i_2,\alpha} \circ \cdots \circ \psi_{i_n,\alpha}$, for each integer $n > 0$, is the map associated with an arbitrary $n-$tuple of indices $i|n = (i_1, i_2, \ldots, i_n) \in \{1, \ldots, 4\}^n$ and it is the identity map in $\mathbb{R}^2$ for $n = 0$.

For each $n \in \mathbb{N}$, K_α^n is the so-called n-th pre-fractal Koch curve.

For $n \to \infty$ the n-th pre-fractal curves K_α^n converge to the fractal curve K_α in the Hausdorff metric. We recall the K_α is the unique closed bounded set in $\mathbb{R}^2$ which is invariant with respect to the family Ψ_α, that is,

$$K_\alpha = \cup_{i=1}^4 \psi_{i,\alpha}(K_\alpha).$$

Furthermore, there exists a unique Borel regular measure ν_α with $supp\, \nu_\alpha = K_\alpha$, invariant with respect to Ψ_α, which coincides with the normalized s-dimensional Hausdorff measure on K_α,

$$\nu_\alpha = (\mathcal{H}^s(K_\alpha))^{-1} \mathcal{H}^s|_{K_\alpha}, \tag{1.4.3}$$

where the Hausdorff dimension $s = \frac{\ln 4}{\ln \alpha}$ (see [40] for proofs and details).

To have an idea of pre-fractal curves we refer to Figures 1.1, 1.2 and 1.3; there, we can see the iterations for $n = 2$, $n = 3$ and $n = 4$, respectively, obtained by choosing different values for α.

In Figure 1.4, we can see a particular example of our domains Ω_α^n: the pre-fractal domains

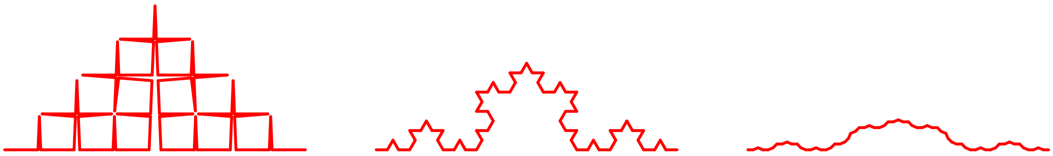

Figure 1.2: K^3_α, with $\alpha = 2.2$, $\alpha = 3$ and $\alpha = 3.8$, respectively

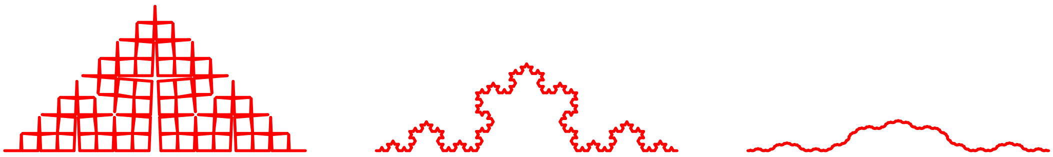

Figure 1.3: K^4_α, with $\alpha = 2.2$, $\alpha = 3$ and $\alpha = 3.8$, respectively

which approximate the snowflake. To realize them, we have chosen outward curves starting from a equilateral triangle and $\alpha = 3$. Instead, in Figure 1.5 we have an example of domain with fractal boundary obtained by choosing inward curves starting from a regular pentagon and $\alpha = \frac{3+\sqrt{5}}{2}$.

We observe that in both cases the pre-fractal domains Ω^n_α are polygonal, non convex and with an increasing number of sides which develop at the limit a fractal geometry.

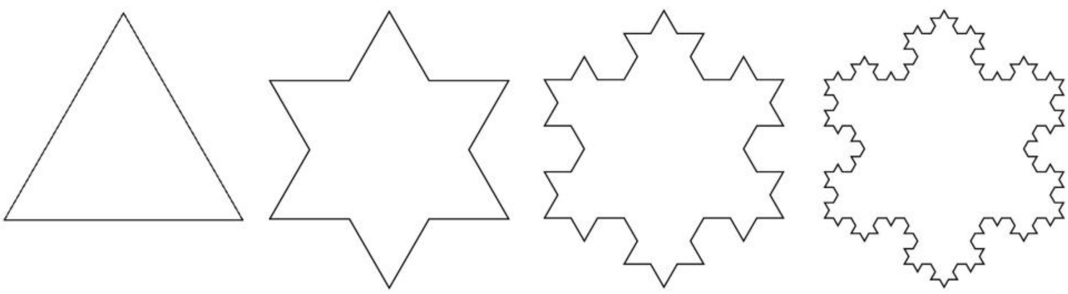

Figure 1.4: Ω^n_3, for $n = 0$, $n = 1$, $n = 2$ and $n = 3$, respectively

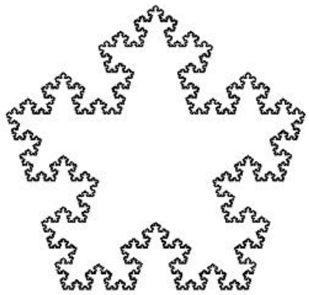

Figure 1.5: Example of bad domain

1.5 Extension operators

In this section we recall some definitions and results about extension operators which we will use in Chapter 4.

First, we recall a result due to McShane (see Theorem 1 in [57]) about Lipschitz extensions.

Theorem 1.5.1. *Let f be a real function defined in X subset of a metric space S satisfying the Lipschitz condition on X. Then, there exists a extension of f to S still satisfying the Lipschitz condition.*

Now, let us consider a fixed value of α and let $\Omega^n = \Omega^n_\alpha$ be the domains introduced before. The following result holds:

Theorem 1.5.2. *For any $n \in \mathbb{N}$, there exists a bounded linear extension operator*

$$Ext : W^{1,p}(\Omega^n) \to W^{1,p}(\mathbb{R}^2).$$

Furthermore, the norm of Ext on $W^{1,p}(\mathbb{R}^2)$ allows an upper bound independent from n, that is:

$$\exists C > 0 \ : ||Ext\, v||_{W^{1,p}(\mathbb{R}^2)} \leq C||v||_{W^{1,p}(\Omega^n)} \ ,$$

with C independent from n.

We refer to Section 5 in [20], Theorem 1 in [42] and the references quoted therein for proofs and details.

1.6 FEM approximation and Galerkin method

The Galerkin method is used to obtain approximate solutions of a differential problem deriving, for example, from the weak formulation of a partial differential equation. Clearly, besides to find these solutions, we want to evaluate the error made approximating the exact solution with the numerical one. We refer, for instance, to Chapters 5 and 6 in [64] for more informations about it.

According to the problem that we will have to face up, let us consider the following model problem (see Section 5.3 in [28]):

$$\text{find } u \in V \ :< Au, v >=< f, v >, \forall v \in V, \tag{1.6.4}$$

where V is a Banach space, V' is its dual, $f \in V'$ and $A : V \to V'$ is an operator.

If A has suitable properties, we can prove existence, uniqueness and approximation error estimates.

In order to approximate the solution u with this method, we replace the space V, which has infinite dimension, with a finite-dimensional subspace of V. We indicate this space with V_h and we construct it by using feasible triangulation of the domain Ω, if we choose as V_h the FEM-space. In this framework, h indicate the size of the triangulation.

In this way, we obtain the following problem:

$$\text{find } u_h \in V_h \ :< Au_h, v >=< f, v >, \forall v \in V_h. \tag{1.6.5}$$

This new problem is equivalent to a set of N_h equations with N_h unknowns, where N_h is the dimension of V_h; in particular:

· we set $u_h(x) = \sum_{i=1}^{N_h} \xi_i \psi_i(x)$;

· $\xi = (\xi_i)_{i=1,\dots N_h}$ is the vector of the unknowns;

·$\psi_i, i = 1, \dots N_h$ are the elements of a basis of the vector space V_h.

Obviously, we have a double goal:

(1) to solve this system through a minimal amount of calculations;

(2) to minimize the norm of the error $||u - u_h||$ in a suitable space.

In order to reach the objective (2), obtaining some estimates for the error, standard tools are the classical estimates (see, for instance, [5], [28], [24] and [36]), which will be recalled in the

Chapter 3, before stating ours. Furthermore, as we will see in the same chapter, the properties of the triangulations and the regularity of the solutions play a crucial role to establish the result we will present.
We remember the following results (see Theorem 5.1.1 and Theorem 5.2.1, respectively, in [64] for the proofs)

Proposition 1.6.1. ***(Lax-Milgram lemma)***
Let V be a (real) Hilbert space. If $< Au, v >$ is a continuous and coercive bilinear form and f is a linear continuous functional, then there exists a unique $u \in V$ solution to (1.6.4).
Moreover,

$$||u||_V \leq \frac{1}{\alpha}||f||_{V'}$$

with α the coercivity constant.

Proposition 1.6.2. ***(Céa's Lemma)***
Let us assume that $< Au, v >$ is a coercive and continuous bilinear form. Let u and u_h be the solutions to Problems (1.6.4) and (1.6.5), respectively. Then

$$||u - u_h||_V \leq \frac{M}{\alpha} \inf_{v \in V_h} ||u - v||_V, \tag{1.6.6}$$

with M and α the continuity and coercivity constants, respectively.

We conclude observing that the previous result can be generalized to the case in which $< Au, v >$ is not a bilinear form and the whole space V is replaced by a its convex subset; in particular, we will replace Problem (1.6.4) with Problem (2.1.1) and prove Lemma 3.2.2, as generalization of Proposition 1.6.2.

Chapter 2

Regularity results for p-Laplacian in non-convex polygons

This chapter has the aim to give a regularity result for the solutions of double obstacle problems involving p-Laplace type operators. The regularity is established in terms of weighted Sobolev spaces, where the weight function is chosen appropriately.
The innovation of this result is due to the fact that the smoothness of the second derivatives, here considered, is little investigated in such type of irregular domains.
The results presented in this chapter are contained in [16], recently published.
Before starting with the setting of our problem, we need to recall some definitions and notations (see, e.g., Section 1.8.1 in [73]).
Let a and b be two given functions, then

$$a \wedge b = \inf\{a, b\},$$

$$a \vee b = \sup\{a, b\}.$$

Definition 2.0.1. *We say that V is a linear lattice, if V is an ordered linear space such that $u \wedge v$ and $u \vee v$ exist for any u, $v \in V$.*

Definition 2.0.2. *Let V a Banach space. It is a Banach lattice if it is a linear lattice and, moreover, $||v^{\pm}||_V \leq ||v||_V$, $\forall v \in V$, where $v^+ = v \vee 0$ and $v^- = -(v \vee 0)$.*

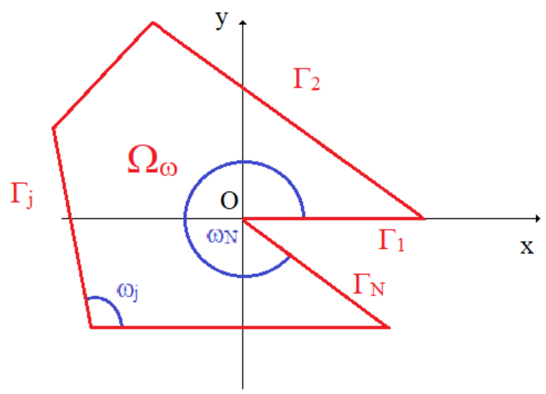

Figure 2.1: Example of domain Ω_ω

2.1 Setting and first results

Let Ω_ω be a polygonal domain in $\mathbb{R}^2$, where $\partial\Omega_\omega$ is the union of a finite number N of linear segments Γ_j numbered following the positive orientation. Let ω_j be the angle between Γ_j and Γ_{j+1} (where $\Gamma_{N+1} = \Gamma_1$), with $\omega_j < \pi$ for any $j < N$ and $\omega_N = \omega > \pi$. Without loss of generality, we can assume that the vertex of the angle ω_N is the origin and that Γ_1 lays in the positive abscissa axis (see Figure 2.1).

For $p \in (2, \infty)$ and a value $k \in \mathbb{R}$, we consider the following double obstacle problem:

$$\text{find } u \in \mathcal{K}, \quad a_p(u, v-u) - \int_{\Omega_\omega} f(v-u)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K} \tag{2.1.1}$$

where

$$a_p(u,v) = \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v \,\mathrm{d}x$$

and

$$\mathcal{K} = \{v \in W_0^{1,p}(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega\}.$$

Remark 2.1.1. *Since $\mathcal{K} \subset W^{1,p}(\Omega_\omega)$ is non-empty, closed (with respect to the metric in $W^{1,p}(\Omega_\omega)$) and convex and the functional $J_p(\cdot)$ on $\mathcal{K}$,*

$$J_p(v) = \frac{1}{p}\int_{\Omega_\omega} (k^2 + |\nabla v|^2)^{\frac{p}{2}} \,\mathrm{d}x - \int_{\Omega_\omega} fv\,\mathrm{d}x. \tag{2.1.2}$$

is convex, weakly lower semi-continuous and coercive in $\mathcal{K}$, then the variational problem

$$\min_{v\in\mathcal{K}} J_p(v)$$

has a minimizer u in $\mathcal{K}$, namely

$$J_p(u) = \min_{v\in\mathcal{K}} J_p(v).$$

Furthermore, see [73], we recall that u is a minimizer iff it is a solution to Problem (2.1.1).

For this problem, we can prove the following result.

Proposition 2.1.1. *Let*

$$\begin{cases} f \in W^{-1,p'}(\Omega_\omega), \quad \frac{1}{p}+\frac{1}{p'}=1, \ \varphi_i \in W^{1,p}(\Omega_\omega), \ i=1,2, \\ \varphi_1 \le \varphi_2 \ \textit{in} \ \Omega_\omega, \ \varphi_1 \le 0 \le \varphi_2 \ \textit{in} \ \partial\Omega_\omega. \end{cases} \tag{2.1.3}$$

Then, there exists a unique function u that solves Problem (2.1.1). *Moreover,*

$$||u||_{W^{1,p}(\Omega_\omega)} \le C\Big\{||f||_{W^{-1,p'}(\Omega_\omega)} + ||\varphi_1||_{W^{1,p}(\Omega_\omega)} + ||\varphi_2||_{W^{1,p}(\Omega_\omega)} + 1\Big\}, \tag{2.1.4}$$

with C depending from k.

Proof. The existence of the solution follows from Remark 2.1.1.
In order to prove the uniqueness, let us recall the following relation (see Lemma 2.1 in [52]). For any $p \ge 2$, $\delta \ge 0$ and $k \in \mathbb{R}$ there exists $c > 0$ such that, for all ξ, $\eta \in \mathbb{R}^2$

$$((k^2+|\xi|^2)^{\frac{p-2}{2}}\xi - (k^2+|\eta|^2)^{\frac{p-2}{2}}\eta, \xi-\eta)_{\mathbb{R}^2} \ge c|\xi-\eta|^{2+\delta}(|k|+|\xi|+|\eta|)^{p-2-\delta}. \tag{2.1.5}$$

Let us assume that there exist u_1 and u_2 solutions of (2.1.1) and substitute them in (2.1.1) taking first u_2 and then u_1 as test function. So, we have

$$\int_{\Omega_\omega} (k^2+|\nabla u_1|^2)^{\frac{p-2}{2}} \nabla u_1 \nabla(u_2-u_1)\,\mathrm{d}x - \int_{\Omega_\omega} f(u_2-u_1)\,\mathrm{d}x \ge 0 \tag{2.1.6}$$

and

$$-\int_{\Omega_\omega} (k^2+|\nabla u_2|^2)^{\frac{p-2}{2}} \nabla u_2 \nabla(u_2-u_1)\,\mathrm{d}x + \int_{\Omega_\omega} f(u_2-u_1)\,\mathrm{d}x \ge 0. \tag{2.1.7}$$

Summing (2.1.6) and (2.1.7), we obtain

$$\int_{\Omega_\omega} ((k^2 + |\nabla u_1|^2)^{\frac{p-2}{2}} \nabla u_1 - (k^2 + |\nabla u_2|^2)^{\frac{p-2}{2}} \nabla u_2, \nabla(u_1 - u_2))\, \mathrm{d}x \leq 0.$$

Hence, thanks to relation (2.1.5), choosing $\delta = p - 2$, we get

$$||\nabla(u_1 - u_2)|| \leq 0 \Longrightarrow u_1 = u_2.$$

Let us show that relation (2.1.4) hold.

$$||u||^p_{W^{1,p}(\Omega_\omega)} = ||u||^p_{L^p(\Omega_\omega)} + ||\nabla u||^p_{L^p(\Omega_\omega)},$$

hence, by Poincaré's inequality

$$||u||^p_{L^p(\Omega_\omega)} \leq C^* ||\nabla u||^p_{L^p(\Omega_\omega)},$$

we have

$$||u||^p_{W^{1,p}(\Omega_\omega)} \leq c||\nabla u||^p_{L^p(\Omega_\omega)}. \tag{2.1.8}$$

Now, we have to evaluate the norm at the right-hand side.
By relation (2.1.1), we have:

$$\int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v dx - \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 dx - \int_{\Omega_\omega} f(v - u) dx \geq 0, \forall v \in \mathcal{K}.$$

Now, for all $v \in \mathcal{K}$ and setting

$$B(u) := \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 dx,$$

we have

$$B(u) \leq \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v dx + \int_{\Omega_\omega} f(u - v)\, \mathrm{d}x.$$

Applying the Young's inequality with $p = p' = 2$ and by the definition of the norm in $W^{-1,p'}(\Omega_\omega)$, we obtain

$$B(u) \leq \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \Big(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} \frac{|\nabla v|^2}{2}\Big)\, \mathrm{d}x + ||f||_{W^{-1,p'}(\Omega_\omega)} ||u - v||_{L^p(\Omega_\omega)}, \forall \varepsilon > 0;$$

using the triangle inequality

$$B(u) \leq \frac{\varepsilon}{2}B(u)+\frac{1}{2\varepsilon}\int_{\Omega_\omega}(k^2+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla v|^2\mathrm{d}x+||f||_{W^{-1,p'}(\Omega_\omega)}||u||_{L^p(\Omega_\omega)}+||f||_{W^{-1,p'}(\Omega_\omega)}||v||_{L^p(\Omega_\omega)}.$$

Recalling the property

$$\forall r>0\ \exists S=S(r)>0:(a+b)^r\leq S(a^r+b^r),\ \forall a,b\geq 0$$

and using it in the the previous relation, we obtain:

$$B(u)\leq \frac{\varepsilon}{2}B(u)+\frac{S}{2\varepsilon}\int_{\Omega_\omega}|k|^{p-2}|\nabla v|^2\,\mathrm{d}x+\frac{S}{2\varepsilon}\int_{\Omega_\omega}|\nabla u|^{p-2}|\nabla v|^2\,\mathrm{d}x+$$

$$+||f||_{W^{-1,p'}(\Omega_\omega)}||u||_{L^p(\Omega_\omega)}+||f||_{W^{-1,p'}(\Omega_\omega)}||v||_{L^p(\Omega_\omega)}$$

Setting $c_\varepsilon=\frac{1}{\varepsilon}$ and applying to the third term in the right-hand side of the previous relation the Young's inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ and with exponents p and p' at the terms after, we have

$$B(u)\leq \frac{\varepsilon}{2}B(u)+\frac{c_\varepsilon S|k|^{p-2}}{2}\int_{\Omega_\omega}|\nabla v|^2\,\mathrm{d}x+\frac{c_\varepsilon S}{2}\Big[\frac{\varepsilon_1(p-2)}{p}||\nabla u||^p_{L^p(\Omega_\omega)}+\frac{2c_{\varepsilon_1}}{p}||\nabla v||^p_{L^p(\Omega_\omega)}\Big]+$$

$$\varepsilon_1\frac{||u||^p_{L^p(\Omega_\omega)}}{p}+c_{\varepsilon_1}\frac{||f||^{p'}_{W^{-1,p'}(\Omega_\omega)}}{p'}+\frac{||v||^p_{L^p(\Omega_\omega)}}{p}+\frac{||f||^{p'}_{W^{-1,p'}(\Omega_\omega)}}{p'},\forall\varepsilon>0.$$

So, by using Poincaré's inequality and again Young's inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$ on the second term of right-hand side, we obtain:

$$B(u)\leq \frac{\varepsilon}{2}B(u)+\frac{c_\varepsilon S|k|^{p-2}}{2}\Big[\frac{2}{p}\int_{\Omega_\omega}|\nabla v|^p\,\mathrm{d}x+\frac{p-2}{p}|\Omega_\omega|\Big]+\frac{c_\varepsilon S}{2}\Big[\frac{\varepsilon_1(p-2)}{p}||\nabla u||^p_{L^p(\Omega_\omega)}+$$

$$+\frac{2c_{\varepsilon_1}}{p}||\nabla v||^p_{L^p(\Omega_\omega)}\Big]+\frac{\varepsilon_1 C^*}{p}||\nabla u||^p_{L^p(\Omega_\omega)}+\frac{1+c_{\varepsilon_1}}{p'}||f||^{p'}_{W^{-1,p'}(\Omega_\omega)}+\frac{||v||^p_{L^p(\Omega_\omega)}}{p}.$$

Hence

$$B(u)\leq\frac{\varepsilon}{2}B(u)+\frac{1+c_{\varepsilon_1}}{p'}||f||^{p'}_{W^{-1,p'}(\Omega_\omega)}+\varepsilon_1C_1||\nabla u||^p_{L^p(\Omega_\omega)}+C_2||v||^p_{W^{1,p}(\Omega_\omega)}+C_3|k|^{p-2}[||\nabla v||^p_{L^p(\Omega_\omega)}+|\Omega_\omega|],$$

where $C_1=\frac{C^*}{p}+\frac{c_\varepsilon S(p-2)}{2p}$, $C_2=\max\{\frac{c_\varepsilon S c_{\varepsilon_1}}{p},\frac{1}{p}\}$ e $C_3=\max\{\frac{c_\varepsilon S}{p},\frac{c_\varepsilon S(p-2)}{2p}\}$.

Thus, we obtain

$$\left(1-\frac{\varepsilon}{2}\right)B(u)-\varepsilon_1 C_1||\nabla u||^p_{L^p(\Omega_\omega)} \leq \frac{1+c_{\varepsilon_1}}{p'}||f||^{p'}_{W^{-1,p'}(\Omega_\omega)}+C_2||v||^p_{W^{1,p}(\Omega_\omega)}+C_3|k|^{p-2}[||\nabla v||^p_{L^p(\Omega_\omega)}+|\Omega_\omega|].$$

By choosing appropriately ε and ε_1, we can find a constant C_4 such that

$$||\nabla u||^p_{L^p(\Omega_\omega)} \leq C_4\{||f||^{p'}_{W^{-1,p'}(\Omega_\omega)} + ||v||^p_{W^{1,p}(\Omega_\omega)} + |k|^{p-2}[||\nabla v||^p_{L^p(\Omega_\omega)} + |\Omega_\omega|]\}, \tag{2.1.9}$$

Now, let us consider this test function $v = (0 \wedge \varphi_2) \vee \varphi_1$, which belongs to the convex $\mathcal{K}$, and, since $\varphi_1 \leq \varphi_2$ in Ω_ω, we have

$$v = (0 \wedge \varphi_2) \vee \varphi_1 \begin{cases} \varphi_2, & \text{if } \varphi_2 \leq 0 \\ \varphi_1, & \text{if } \varphi_1 \geq 0 \\ 0, & \text{if } \varphi_2 > 0 \text{ and } \varphi_1 < 0 \end{cases} \tag{2.1.10}$$

Let us pose $\Omega_1 = \{x \in \Omega_\omega | \varphi_2 \leq 0\}$, $\Omega_2 = \{x \in \Omega_\omega | \varphi_2 > 0 \text{ and } \varphi_1 \geq 0\}$ and $\Omega_3 = \{x \in \Omega_\omega | \varphi_2 > 0 \text{ and } \varphi_1 < 0\}$.

With the previous choice of v and thanks to Theorem 1.56 in [73], we have:

$$||v||^p_{W^{1,p}(\Omega_\omega)} = \int_{\Omega_\omega} |v|^p \,\mathrm{d}x + \int_{\Omega_\omega} |\nabla v|^p \,\mathrm{d}x =$$

$$= \int_{\Omega_1} |\varphi_2|^p \,\mathrm{d}x + \int_{\Omega_2} |\varphi_1|^p \,\mathrm{d}x + \int_{\Omega_1} |\nabla \varphi_2|^p \,\mathrm{d}x + \int_{\Omega_2} |\nabla \varphi_1|^p \,\mathrm{d}x \leq$$

$$\leq ||\varphi_2||^p_{W^{1,p}(\Omega_\omega)} + ||\varphi_1||^p_{W^{1,p}(\Omega_\omega)}. \tag{2.1.11}$$

Finally, being $p' \leq p$ and choosing v like defined in (2.1.10), thanks to the relations (2.1.8), (2.1.11) and (2.1.9), we have

$$||u||^p_{W^{1,p}(\Omega_\omega)} \leq C\Big\{||f||^p_{W^{-1,p'}(\Omega_\omega)}+||\varphi_2||^p_{W^{1,p}(\Omega_\omega)}+||\varphi_1||^p_{W^{1,p}(\Omega_\omega)}+|k|^{p-2}[||\nabla\varphi_1||^p_{L^p(\Omega_\omega)}+||\nabla\varphi_2||^p_{L^p(\Omega_\omega)}+|\Omega_\omega|]\Big\},$$

which gives our estimate. □

In Figure 2.2, we see an one-dimensional example of test function v constructed according to relation (2.1.10) and choosing $\varphi_1 = \frac{1}{2} - |x-2|$, $\varphi_2 = |x| - 1$ and $\Omega = (-3, 3)$.

We point out that, from now onwards, we denote by C possibly different constants.

Let us recall the following definitions (see Section 4.1.1 in [73]).

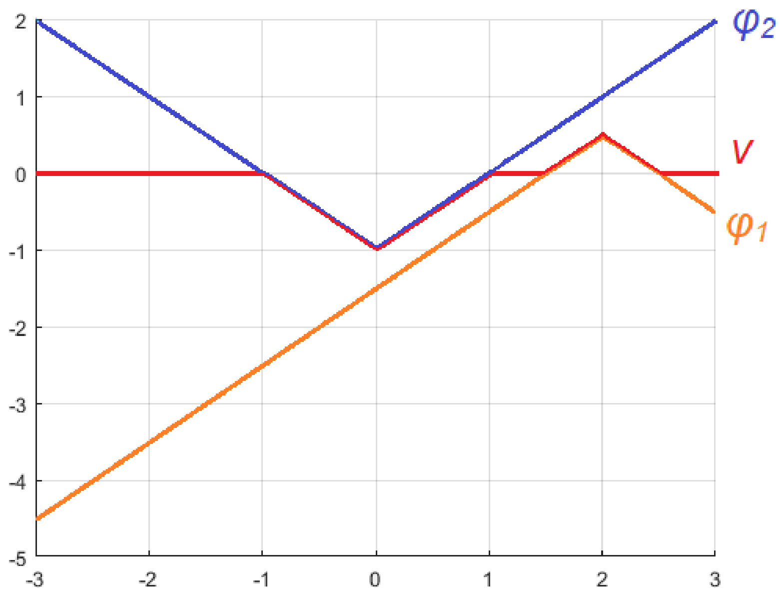

Figure 2.2: Example of test function v

Definition 2.1.1. *Let V be a Banach space and $J(\cdot)$ a functional.*
(a) $J(\cdot)$ is convex if

$$J(\lambda u + (1-\lambda)v) \leq \lambda J(u) + (1-\lambda)J(v) \text{ for } u, v \in V \text{ with } u \neq v, 0 \leq \lambda \leq 1.$$

(b) $J(\cdot)$ is coercive if

$$J(v) \longrightarrow \infty \text{ as } ||v||_V \longrightarrow \infty.$$

(c) $J(\cdot)$ is weakly lower semi-continuous if

$$J(v) \leq \liminf_{n\to\infty} J(v_n) \text{ when } v_n \longrightarrow v \text{ weakly in } V.$$

Moreover, we introduce the operator

$$A_p(u) = -div((k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u).$$

Finally, for the crucial role that it plays in order to obtain our regularity results, we recall the Lewy-Stampacchia inequality

Proposition 2.1.2. *We assume hypothesis (2.1.3) and*

$$f, A_p(\varphi_i) \in L^{p'}(\Omega_\omega),\ i = 1, 2,\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1.12}$$

Let u be solution of (2.1.1). Then

$$A_p(\varphi_2) \wedge f \le A_p(u) \le A_p(\varphi_1) \vee f \ \ in\ \ \Omega_\omega. \tag{2.1.13}$$

We observe that, actually, Lewy-Stampacchia inequality holds under weaker assumptions than the ones we have required (we refer to [48], [60] and [61] for more information about the initial proof and the following extensions).
Moreover, we have the following result.

Proposition 2.1.3. *We assume hypotheses (2.1.3) and (2.1.12). Then the solution u to Problem (2.1.1) is the solution of the Dirichlet problem*

$$\begin{cases} A_p(u) = f^* \ in\ \Omega_\omega \\ u = 0 \ in\ \partial\Omega_\omega \end{cases}, \tag{2.1.14}$$

where f^* belongs to the space $L^{p'}(\Omega_\omega)$ and

$$||f^*||_{L^{p'}(\Omega_\omega)} \le C\Big\{||f||_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_1)||_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_2)||_{L^{p'}(\Omega_\omega)}\Big\}. \tag{2.1.15}$$

Proof. The thesis follows from The Lewy-Stampacchia inequality and the uniqueness of the solution. □

Remark 2.1.2. *Since u, φ_1, φ_2 are continuous, we have*

$$f^* = \begin{cases} A(\varphi_2), & for\ x \in \Omega_\omega\ such\ that\ u = \varphi_2 \\ f, & for\ x \in \Omega_\omega\ such\ that\ \varphi_1 < u < \varphi_2 \\ A(\varphi_1), & for\ x \in \Omega_\omega\ such\ that\ u = \varphi_1 \end{cases}$$

Theorem 2.1.1. *We assume hypotheses (2.1.3) and (2.1.12). Let u be the solution to (2.1.1). Then u belongs to the Besov space $B^{1+1/p}_{p,\infty}(\Omega_\omega)$. Moreover,*

$$||u||_{B^{1+1/p}_{p,\infty}(\Omega_\omega)} \le C\Big\{1 + ||f||^{\frac{p'}{p}}_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_1)||^{\frac{p'}{p}}_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_2)||^{\frac{p'}{p}}_{L^{p'}(\Omega_\omega)}\Big\}. \tag{2.1.16}$$

Proof. We use Proposition 2.1.3, estimate (2.1.15) and Theorem 2 in [67]. □

We refer to [22] for the case $k \neq 0$.

Let us observe that putting in the previous theorem $p = 2$, we get, in particular, $u \in H^{3/2-\epsilon}(\Omega_\omega)$ in the Sobolev scale. It is important to underline that the previous result is, in a certain sense, the best possible as it holds for any value of $\omega \in (\pi, 2\pi)$ and as $\omega \to 2\pi$, the domain becomes "very bad".

So, it comes natural to ask ourselves: can we expect sharper regularity results considering a fixed value of ω?

Having in mind the by now classical results of Kondratiev (see [44]), we believe that the natural spaces to study the regularity in the case of non-convex polygons are the weighted Sobolev spaces, whose definition has been introduced in Chapter 1, choosing the distance from the reentrant corner as weight function.

In the next paragraph, we state our regularity results in terms of these spaces.

2.2 Regularity result

In this section, we state our results using the spaces above mentioned. Before stating and proving them, we need to state some preliminary theorems, which require weaker assumptions.

Theorem 2.2.1. *We assume hypotheses (2.1.3) and*

$$\begin{cases} k \neq 0 \\ f, A_p(\varphi_i) \in L^2_{loc}(\Omega_\omega),\ i = 1, 2. \end{cases} \tag{2.2.17}$$

Then the solution u to obstacle problem (2.1.1) in Ω_ω belongs to $H^2_{loc}(\Omega_\omega)$.

Proof. Thanks to inequality (2.1.13), assumption (2.2.17) and the lattice properties of $L^2_{loc}(\Omega_\omega)$, we have

$$||f^*||_{L^2_{loc}(\Omega_\omega)} \leq C\Big\{||f||_{L^2_{loc}(\Omega_\omega)} + ||A_p(\varphi_1)||_{L^2_{loc}(\Omega_\omega)} + ||A_p(\varphi_2)||_{L^2_{loc}(\Omega_\omega)}\Big\}. \tag{2.2.18}$$

Moreover, u solution of Problem (2.1.1) is the solution to the equation $A_p(u) = f^*$, with f^*

belonging to the space $L^2_{loc}(\Omega_\omega)$. Furthermore, we have that

$$\sup_{t>0} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = \lim_{t\to\infty} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = p-2 \tag{$*$}$$

and as $k \neq 0$

$$\inf_{t>0} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = \lim_{t\to 0^+} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = 0. \tag{$**$}$$

Since ($*$) and ($**$) (see assumption 2.2 in [27]) are satisfied, then using formula (5.11) in the proof of Theorem 2.1 in [27], we obtain

$$\begin{cases} \int_{B_R}(k^2+|\nabla u|^2)^{p-2}\sum_{|\beta|=2}|D^\beta u|^2dx \leq \\ \leq C\Big(||f||^2_{L^2(B_{2R})} + ||A_p(\varphi_1)||^2_{L^2(B_{2R})} + ||A_p(\varphi_2)||^2_{L^2(B_{2R})} + \frac{1}{R^4}(\int_{B_{2R}}(k^2+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|dx)^2\Big) \end{cases} \tag{2.2.19}$$

for any ball $B_{2R} \subset\subset \Omega_\omega$, with C independent of k.

Then, repeating steps 2 and 3 of the proof of Theorem 2.1 in [27] and by (2.1.4), we obtain that $u \in H^2_{loc}(\Omega_\omega)$. □

Now, we want to derive estimates far away from the origin.

With this aim, let us consider $x \in \partial\Omega_\omega \setminus \{O\}$ and define $\Omega_s(x) := B_s(x) \cap \bar{\Omega}_\omega$, for $s > 0$. Also, let $R \in (0, \frac{dist(x,O)}{4})$ be in such a way that $\Omega_{2R}(x) := B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Theorem 2.2.2. *We assume hypotheses (2.1.3) and*

$$f, A_p(\varphi_i) \in L^2(\Omega_\omega),\ i = 1, 2. \tag{2.2.20}$$

Then the solution u to double obstacle Problem (2.1.1) in Ω_ω satisfies

$$\begin{cases} \int_{\Omega_R(x)}(k^2+|\nabla u|^2)^{p-2}\sum_{|\beta|=2}|D^\beta u|^2dx \leq C\Big(||f||^2_{L^2(\Omega_{2R}(x))} + ||A_p(\varphi_1)||^2_{L^2(\Omega_{2R}(x))} + \\ +||A_p(\varphi_2)||^2_{L^2(\Omega_{2R}(x))} + \frac{1}{R^2}\int_{\Omega_{2R}(x)}(k^2+|\nabla u|^2)^{p-2}|\nabla u|^2dx\Big) \end{cases} \tag{2.2.21}$$

for any $x \in \partial\Omega_\omega \setminus \{O\}$ and $R \in (0, \frac{dist(x,O)}{4})$ such that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Proof. Thanks to the same argumentations of the previous proof, we obtain that u, solution to our problem, is the solution to the Dirichlet Problem (2.1.14), with f^* belonging to the space

$L^2(\Omega_\omega)$ and, moreover,

$$||f^*||_{L^2(\Omega_\omega)} \le C\Big\{||f||_{L^2(\Omega_\omega)} + ||A_p(\varphi_1)||_{L^2(\Omega_\omega)} + ||A_p(\varphi_2)||_{L^2(\Omega_\omega)}\Big\}. \qquad (2.2.22)$$

Using the terminology of [27], we want to underline the fact that far away from the origin the weak second fundamental form on $\partial\Omega_\omega$ is non-positive. So, we choose the cut function $\xi \in C_0^\infty(B_{2R}(x))$ with $\xi = 1$ in $B_R(x)$ and we go on as in step 1 of the proof of Theorem 2.4 in [27].

On $\Omega_\omega \cap \partial B_{2R}(x)$ we have $\xi = 0$ and on $\partial\Omega_\omega \cap B_{2R}(x)$ the Dirichlet condition holds, then we can ignore the boundary integrals (see (4.18) in [27]) and using the estimate (2.2.22), we obtain

$$\int_{\Omega_\omega} \xi^2(k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta|=2} |D^\beta u|^2 dx \le$$

$$\le C\left(||\xi^2 f||^2_{L^2(\Omega_\omega)} + ||\xi^2 A_p(\varphi_1)||^2_{L^2(\Omega_\omega)} + ||\xi^2 A_p(\varphi_2)||^2_{L^2(\Omega_\omega)} + \int_{\Omega_\omega} |\nabla\xi|^2(k^2 + |\nabla u|^2)^{p-2}|\nabla u|^2 dx\right) \qquad (2.2.23)$$

(see (4.74) in [27]).

Hence, repeating steps 2, 3 and 4 of the proof of Theorem 2.3 in [27], we get estimate (2.2.21), with the constant C independent from k. □

The following (final) preliminary theorem establishes estimates near the origin and it is valid for any $k \in \mathbb{R}$.

Before to state it, we want to specify that the parameter

$$\gamma = \gamma(p, \chi) = 1 + \frac{p(1-\chi)^2 + (1-\chi)\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)} \qquad (2.2.24)$$

(where $\chi = \frac{\omega}{\pi}$), is the least positive eigenvalue (with $\phi(\theta)$ the corresponding eigenfunction) of the problem

$$\begin{cases} \partial_\theta\{(\lambda^2\phi^2 + |\partial_\theta\phi|^2)^{\frac{p-2}{2}}\partial_\theta\phi\} + \lambda(\lambda(p-1) + 2 - p)(\lambda^2\phi^2 + |\partial_\theta\phi|^2)^{\frac{p-2}{2}}\phi = 0 \quad in \quad 0 < \theta < \omega, \\ \phi(0) = \phi(\omega) = 0 \end{cases} \qquad (2.2.25)$$

(see [70] and Theorem 8.12 and Remark 8.13 in [8]).

Theorem 2.2.3. *Assume hypotheses (2.1.3), (2.1.12) and*

$$A_p(\varphi_2) \wedge f \geq 0, \qquad A_p(\varphi_1) \vee f \leq C_1 r^{\lambda_0} \quad \textit{with } \lambda_0 > \gamma(p-1) - p \quad \textit{in} \quad \Omega_\omega \tag{2.2.26}$$

where γ *is defined in* (2.2.24). *Then the following estimates hold for the solution* u *to obstacle Problem (2.1.1) near the origin*

$$|u(x)| \leq Cr^\gamma, \quad |\nabla u(x)| \leq Cr^{\gamma-1}, |D^\beta u(x)| \leq Cr^{\gamma-2}, \quad |\beta| = 2. \tag{2.2.27}$$

Proof. Once again from the Lewy-Stampacchia inequality and assumption (2.2.26) we have that the solution u to our Problem (2.1.1) is the solution to the Dirichlet Problem (2.1.14), where the datum f^* has the following property

$$0 \leq f^* \leq C_1 r^{\lambda_0} \quad \text{with } \lambda_0 > \gamma(p-1) - p. \tag{2.2.28}$$

If $f^* = 0$, then the unique solution u to Problem (2.1.1) is identically zero and, clearly, estimates (2.2.27) are trivial; thus, we can suppose that $f^* \neq 0$.
In this case, using Theorem 3 and the following remarks in [33], we deduce that u admits the singular expansion

$$u(r,\theta) = C_2 r^\gamma \phi(\theta) + v(x), \tag{2.2.29}$$

where $C_2 > 0$ and

$$|v(x)| \leq C_3 r^{\gamma+\delta}, \ |\nabla v(x)| \leq C_3 r^{\gamma+\delta-1}, \ |D^\beta v(x)| \leq C_3 r^{\gamma+\delta-2}, \ |\beta| = 2. \tag{2.2.30}$$

We recall that γ is defined in (2.2.24) and $\phi(\theta)$ is the corresponding eigenfunction in problem (2.2.25), moreover the maximum $\delta > 0$ depends on γ and λ_0.
Finally, from (2.2.29) and (2.2.30), we obtain estimates (2.2.27). □

Now, finally, taking into account the previous results, we are able to state and prove our main result.

Theorem 2.2.4. *Assume hypotheses (2.1.3) and*

$$\begin{cases} k \neq 0 \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\omega),\ i = 1, 2, \\ A_p(\varphi_2) \wedge f \geq 0. \end{cases} \tag{2.2.31}$$

Then the solution u to double obstacle Problem (2.1.1) in Ω_ω belongs to the weighted Sobolev space

$$H^{2,\mu}(\Omega_\omega),\ \mu > 1 - \gamma \tag{2.2.32}$$

Moreover,

$$||u||_{H^{2,\mu}(\Omega_\omega)} \leq C\Big\{1 + ||f||_{L^\infty(\Omega_\omega)} + ||A_p(\varphi_1)||_{L^\infty(\Omega_\omega)} + ||A_p(\varphi_2)||_{L^\infty(\Omega_\omega)}\Big\}. \tag{2.2.33}$$

Proof. Since $k \neq 0$ and assumptions (2.1.3) and (2.2.31) imply that all the assumptions of Theorems 2.2.1 , 2.2.2 and 2.2.3 (with $\lambda_0 = 0$) are satisfied, then, combining all of the results stated, we obtain that the solution u to Problem (2.1.1) belongs to the space $H^{2,\mu}(\Omega_\omega)$ for any $\mu > 1 - \gamma$. Finally, estimate (2.2.33) follows from (2.1.4), (2.2.19), (2.2.21) and (2.2.27). □

The importance of this result is underlined in the following remarks.

Remark 2.2.1. *Considering the function $\gamma(p, \cdot)$ for any fixed value of $p > 2$, we note that it is decreasing with respect to χ; in particular, as $\chi \to 2$, it tends to $\frac{p-1}{p}$. Analogously, for any fixed value of $\chi < 2$, it is increasing with respect to p; in particular, as $p \to \infty$, it tends to 1. Consequently, μ has the opposite behaviour.*
In the particular case in which we choose $\omega = \frac{4\pi}{3}$ the expression for γ is

$$\gamma(p, \frac{4}{3}) = 1 + \frac{p - \sqrt{p^2 + 32p - 32}}{16(p-1)}.$$

Let us observe that, putting in the previous formula $p = 2$, we obtain $\gamma = \frac{3}{4}$ (hence $\mu = \frac{1}{4}$) according to the by now classical results of Kondratiev for equations (see, e.g., *[8]).*
Figures 2.3 and 2.4 show the behaviour of $\gamma(p, \frac{4}{3})$ for $2 < p < 10$ and $2 << p < 10.000$, respectively.

Remark 2.2.2. *We stress the fact that, also in the case of the Dirichlet problem with datum $f \in L^\infty$, the regularity result of Theorem 2.2.4 is not a consequence of Theorems 8.43, 8.44,*

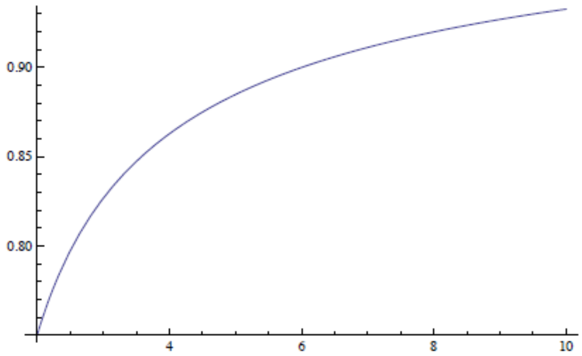

Figure 2.3: γ for $2 < p < 10$ and $\chi = \frac{4}{3}$

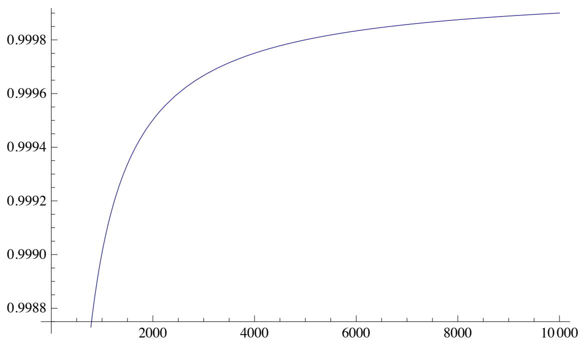

Figure 2.4: γ for $2 << p < 10.000$ and $\chi = \frac{4}{3}$

8.46 in [8] because we do not assume the differentiability of f and, moreover, for any $p > 2$, the exponent of the weight in [8] is greater than ours in (2.2.32). Indeed the exponent of the weight in formula (8.4.35) in [8] is required to be greater than $\frac{p}{2}(1-\gamma)$ (according to our notation) and it increases as $p \to \infty$, tending to $\frac{1}{2}$; instead μ in (2.2.32) *is required to be greater than $1 - \gamma$ and it decreases as $p \to \infty$, tending to* 0*.*

Remark 2.2.3. *We want to point out that this regularity result cannot be deduced from Theorems 2.4 in [27], since our boundary does not satisfy condition (2.12) in [27]. However, we use some ideas in [27] with the aim to state local estimates and estimates far away from the origin. We note that in this part only the membership of the data to $L^2(\Omega_\omega)$, and not their boundedness, is required (see Theorems 2.2.1 and 2.2.2).*

Remark 2.2.4. *For the properties of first order derivatives, we refer to [49] and [50] and the reference quoted there. Up to now, we have global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than* 1 *for the solutions to obstacles problems established only for the case $p = 2$ (see [18]).*
For $p > 2$, as far as we know, there are no second order L^2 regularity results concerning obstacle problems even assuming the differentiability of the data and the smoothness of the boundary; in particular, in the case of obstacle problems, recent results of L. Brasco and F. Santambrogio in [9] and C. Mercuri, G. Riey and B. Sciunzi in [58] do not seem to work.

2.3 Boundedness of the gradient far away from the origin.

This section is dedicated to the investigation of the boundedness of the gradient in L^∞ far away from the origin. We want to underline the fact that the following two theorems hold for any $k \in \mathbb{R}$.

Theorem 2.3.1. *We assume hypotheses (2.1.3) and*

$$f, A_p(\varphi_i) \in L^\infty(\Omega_\omega), \ \ i = 1, 2 \tag{2.3.34}$$

Then the solution u to double obstacle Problem (2.1.1) belongs to the Sobolev space $W^{1,\infty}_{loc}(\Omega_\omega)$.

Proof. Thanks to the Lewy-Stampacchia inequality, assumption (2.3.34) and proposition 2.1.3, we conclude that u, solution to Problem (2.1.1), is the solution to the Dirichlet Problem (2.1.14) with datum $f^* \in L^\infty(\Omega_\omega)$.

Hence the thesis follows from Theorem 1 of [71] (see also [51], [32] and [74]). $\square$

Theorem 2.3.2. *We assume hypotheses (2.1.3) and (2.3.34). Then the solution u to double obstacle Problem (2.1.1) belongs to the Sobolev space $W^{1,\infty}(\Omega_R(x))$ for any $x \in \partial\Omega_\omega \setminus \{O\}$ and $R \in (0, \frac{dist(x,O)}{4})$ such that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.*

Proof. Once again, using inequality (2.1.13), assumption (2.3.34) and Proposition (2.1.3), we derive that the solution u to double obstacle Problem (2.1.1) is the solution to the Dirichlet Problem (2.1.14) with datum $f^* \in L^\infty(\Omega_\omega)$.

Thus, we show that $u \in W^{1,\infty}(\Omega_R(x))$ proceeding as in Theorem 2.2 and Remark 2.7 of [26]. In particular, we replace Lemma 5.4 of [26] with a local version which exploits a cut-off function $\xi \in C_0^\infty(B_{2R}(x))$ with $\xi = 1$ in $B_R(x)$. Hence for smooth functions v vanishing on $\partial\Omega_\omega$, we obtain

$$C(k^2+t^2)^{\frac{p-2}{2}} t \int_{\{|\nabla v|=t\}} \xi^2 |\nabla|\nabla v|| d\mathcal{H}^1(x) \leq$$

$$\leq t \int_{\{|\nabla v|=t\}} \xi^2 |div((k^2+|\nabla v|^2)^{\frac{p-2}{2}} \nabla v)| d\mathcal{H}^1(x) +$$

$$+ \int_{\{|\nabla v|>t\}} \xi^2 \frac{1}{(k^2+|\nabla v|^2)^{\frac{p-2}{2}}} |div((k^2+|\nabla v|^2)^{\frac{p-2}{2}} \nabla v)|^2 \, dx +$$

$$+ C \int_{\{|\nabla v|>t\}} \xi^2 |\nabla v|^p \, dx,$$

thanks to the fact that the weak second fundamental form on $\partial\Omega_\omega \cap B_{2R}(x)$ is non-positive. $\square$

Now, for the case $k = 0$, we want to state a further property for the gradient which will be useful in the following. Similarly to what we have done before, for any $x \in \partial\Omega_\omega \setminus \{O\}$, we set $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$, with $R \in (0, \frac{dist(x,O)}{4})$ chosen in such a way that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Theorem 2.3.3. *We assume (2.1.3), (2.3.34) and*

$$k = 0, \;\; A_p(\varphi_2) \wedge f \geq c^* > 0. \tag{2.3.35}$$

We suppose that the solution u to double obstacle Problem (2.1.1) belongs to the space $W^{2,s}_{loc}(\Omega_\omega)$, and for any $x \in \partial\Omega_\omega \setminus \{O\}$ the restriction of u to the set $\Omega_{2R}(x)$ belongs to $W^{2,s}(\Omega_R(x))$, $s \in [1, 2]$.

Then for any $q \geq 1$, $p > 2$, we obtain

$$|\nabla u|^{-\frac{(p-t)q}{t-q}} \in L^1(\Omega_\omega) \tag{2.3.36}$$

with

$$t \geq \frac{q(p+(p-2)s)}{q+(p-2)s}.$$

Proof. With the same argumentations of the previous proof, the solution u to double obstacle Problem (2.1.1) is the solution to the Dirichlet Problem (2.1.14) with datum $f^* \in L^\infty(\Omega_\omega)$; also, by (2.3.35), we have that $f^* \geq c^* > 0$. In particular, we note that assumption (2.2.26) of Theorem 2.2.3 is satisfied with $\lambda_0 = 0$. Hence, from (2.2.29), we deduce that $|\nabla u|$ behaves like $r^{\gamma-1}$ in a neighborhood of O and then $|\nabla u|^{-1} \in L^\infty$ near the origin.
Far away from the origin, we apply Theorem 2.3.1 in order to obtain that u belongs to $W^{1,\infty}_{loc}(\Omega_\omega)$.
Let us consider a domain $G \subset\subset \Omega_\omega$. So $v \equiv |\nabla u| \in L^\infty(G)$ and, by assumption, $(v_1, v_2) \equiv \nabla u \in (W^{1,s}(G))^2$, then it follows that $v \in W^{1,s}(G)$ and $\nabla v = \frac{(v_1 \nabla v_1 + v_2 \nabla v_2)}{v}$.
Furthermore, we have that

$$f^* = -div(v^{p-2}\nabla u) = -\{v^{p-2}(v_{1x_1} + v_{2x_2}) + (p-2)v^{p-2}\frac{v_1 v_{x_1} + v_2 v_{x_2}}{v}\}.$$

Thus

$$c^* \leq f^* \leq M(x)|\nabla u|^{p-2} \qquad \text{a.e. in G} \tag{2.3.37}$$

with $M(x) \in L^s(G)$.
In particular, we obtain

$$\int_G |\nabla u|^{-\frac{(p-t)q}{t-q}} dx \leq C \int_G (M(x))^{\frac{(p-t)q}{(p-2)(t-q)}} \, dx$$

and if $t \geq \frac{q(p+(p-2)s)}{q+(p-2)s}$, then $\frac{(p-t)q}{(p-2)(t-q)} \leq s$.
At this point, we repeat the previous proof putting $\Omega_R(x)$ instead of G and using Theorem 2.3.2 instead of Theorem 2.3.1. □

Chapter 3

Numerical aspects for p-Laplacian in non-convex polygons

In this chapter, thanks to the regularity results obtained in the previous one, we will be able to prove optimal approximation error estimates of the FEM-solutions.
The theoretical results presented in this chapter are contained in [16].

3.1 Regularity results in pre-fractal sets

To face the numerical approach to the solutions to obstacle problems in fractal domains it is natural to consider the solutions of obstacle problems in pre-fractal approximating domains and the corresponding FEM-solutions and to evaluate the approximation error. We consider the pre-fractal Koch Islands Ω_α^n that are polygonal domains having as sides pre-fractal Koch curves. As we discussed in Chapter 1, we start by a regular polygon and we replace each side by a pre-fractal Koch curve (see Figures 3.1 and 3.2).

In Section 3 in [22] the authors, assuming some natural conditions, show as the solutions u_n to the obstacle problems in Ω_α^n converge to the fractal solution to the obstacle problem in the

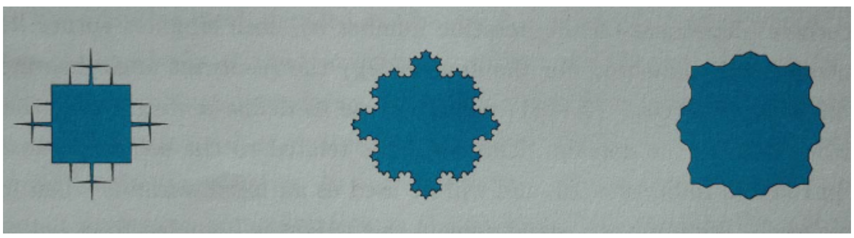

Figure 3.1: Ω_α^2, $\alpha = 2.1$, $\alpha = 3$ and $\alpha = 3.75$

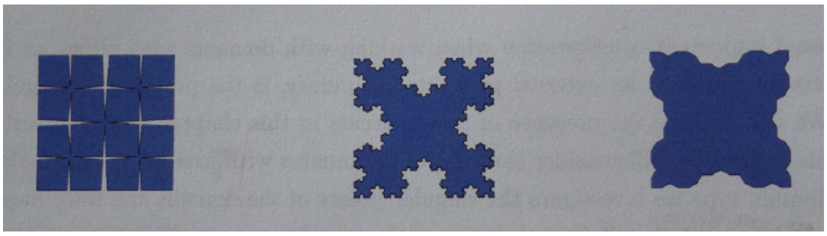

Figure 3.2: Ω_α^2, $\alpha = 2.1$, $\alpha = 3$ and $\alpha = 3.75$

Koch Island Ω_α.

For any (fixed) n, we consider the following double obstacle problem

$$\text{find } u_n \in \mathcal{K}_n, \quad a_p(u_n, v - u_n) - \int_{\Omega_\alpha^n} f(v - u_n)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}_n \tag{3.1.1}$$

where

$$a_p(u, v) = \int_{\Omega_\alpha^n} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v\,\mathrm{d}x, \; k \in \mathbb{R}$$

and

$$\mathcal{K}_n = \{v \in W_0^{1,p}(\Omega_\alpha^n) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\alpha^n\}.$$

Since n is fixed, the number of reentrant corners is fixed; hence, we can prove for the solution u_n to Problem (3.1.1) all the results obtained in the last section of the previous chapter with $\chi = \frac{\omega}{\pi}$, where we define

$$\omega = \begin{cases} \pi + \theta(\alpha) & \text{if the sides of the polygons are obtained by outward curves} \\ \pi + 2\theta(\alpha) & \text{if the sides of the polygons are obtained by inward curves.} \end{cases} \tag{3.1.2}$$

We recall that by $\theta(\alpha)$, defined in (1.4.2), we denote the opening of the rotation angle of the similarities involved in the construction of the Koch curve. Then $\chi \in (1, \frac{3}{2})$ in the case of outward curves or $\chi \in (1, 2)$ in the case of inward curves.

In this framework, we involved $H^{2,\mu}(\Omega_\alpha^n)$, the weighted Sobolev space introduced in the first chapter, which is a Hilbert space with respect to norm (1.1.1). As weight function $\rho = \rho_n(x)$, we chose the distance function from the set of the vertices of the reentrant corners of Ω_α^n.

In the previous chapter, we gave our regularity results in the domain Ω_ω. With the following theorems, we recall these results in the setting of pre-fractal Koch islands Ω_α^n.

Theorem 3.1.1. *We assume*

$$\begin{cases} \varphi_i \in W^{1,p}(\Omega_\alpha^n), \ i = 1, 2, \\ \varphi_1 \le \varphi_2 \ \ in \ \ \Omega_\alpha^n, \ \varphi_1 \le 0 \le \varphi_2 \ \ in \ \ \partial\Omega_\alpha^n \end{cases} \tag{3.1.3}$$

and

$$\begin{cases} k \neq 0 \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\alpha^n), \quad i = 1, 2, \\ A_p(\varphi_2) \wedge f \ge 0. \end{cases} \tag{3.1.4}$$

Then the solution u_n to obstacle Problem (3.1.1) belongs to the weighted Sobolev space

$$H^{2,\mu}(\Omega_\alpha^n), \ \mu > 1 - \gamma \tag{3.1.5}$$

where

$$\gamma = \gamma(p, \chi) = 1 + \frac{p(1-\chi)^2 + (1-\chi)\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)} \tag{3.1.6}$$

with $\chi = \frac{\omega}{\pi}$ and ω defined in (3.1.2).

Moreover,

$$||u_n||_{H^{2,\mu}(\Omega_\alpha^n)} \le C\Big\{1 + ||f||_{L^\infty(\Omega_\alpha^n)} + ||A_p(\varphi_1)||_{L^\infty(\Omega_\alpha^n)} + ||A_p(\varphi_2)||_{L^\infty(\Omega_\alpha^n)}\Big\}. \tag{3.1.7}$$

In the case of $k = 0$, an analogous of Theorem 3.1.1 holds.

Theorem 3.1.2. *We assume (3.1.3) and*

$$\begin{cases} k = 0 \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\alpha^n), \ \ i = 1, 2, \\ A_p(\varphi_2) \wedge f \ge c^* > 0. \end{cases} \tag{3.1.8}$$

If the solution u_n to obstacle Problem (3.1.1) belongs to the space $H^{2,\mu}(\Omega_\alpha^n)$, then for any $q \ge 1$, $p > 2$, we obtain

$$|\nabla u_n|^{-\frac{(p-t)q}{t-q}} \in L^1(\Omega_\alpha^n) \tag{3.1.9}$$

with

$$t \ge \frac{q(p + (p-2)2)}{q + (p-2)2}.$$

3.2 Optimal estimates

3.2.1 FEM spaces and solutions

In order to define the approximating solutions according to the Galerkin method, we introduce a triangulation of the domain Ω^n_α. Let T_h be a partitioning of the domain Ω^n_α into disjoint, open regular triangles τ, each side being bounded by h, that $\overline{\Omega}^n_\alpha = \bigcup_{\tau\in T_h} \bar{\tau}$. Associated with T_h, we consider the finite-dimensional spaces

$$S_h = \Big\{v \in C(\bar{\Omega}^n_\alpha) : v|_\tau \quad \text{is affine} \quad \forall \tau \in T_h\Big\} \text{ and } S_{h,0} = \Big\{v \in S_h : \;\; v = 0 \;\text{ on }\; \partial\Omega^n_\alpha\Big\}. \quad (3.2.10)$$

By π_h we denote the interpolation operator $\pi_h : C(\bar{\Omega}^n_\alpha) \to S_h$ such that $\pi_h v(P_i) = v(P_i)$ for any vertex P_i of the partitioning T_h.

We make some assumptions on the triangulation T_h. We give the following definition.

Definition 3.2.1. *The triangulation T_h of Ω^n_α is regular and conformal if*

- $\bar{\Omega}^n_\alpha = \bigcup\limits_{\tau\in T_h} \bar{\tau}$;
- $\tau \neq \emptyset, \forall \;\; \tau \in T_h$;
- $\tau_1 \cap \tau_2 = \emptyset, \quad \forall \;\; \tau_1, \tau_2 \in T_h : \quad \tau_1 \neq \tau_2$;
- $\bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset, \quad \tau_1 \neq \tau_2 \quad \Rightarrow \quad \bar{\tau}_1 \cap \bar{\tau}_2 =$ *edge or vertex*;
- $\exists \; \sigma > 0$ *such that* $\max_{\tau\in T_h} \frac{h_\tau}{\eta_\tau} \leq \sigma$.

where
$h_\tau = diam(\tau)$, $\eta_\tau = \sup\{diam(B) : \; B \text{ ball}, B \subset \tau \;\}$ *and* $h = \sup\{h_\tau\,,\, \tau \in T_h\}$ *denotes the size of the triangulation.*

Here, we recall the classical interpolation estimates (see [5], [28],[36] and [24])

$$||\nabla(v - \pi_h v)||_{L^q(\tau)} \leq ch \sum_{|\beta|=2} ||D^\beta v||_{L^q(\tau)}, \;\; q \geq 2;$$

$$||\nabla(v - \pi_h v)||_{L^q(\tau)} \leq c||\nabla v||_{L^q(\tau)}, \;\; q > 2;$$

$$||v - \pi_h v||_{L^p(\tau)} \leq ch^2 \sum_{|\beta|=2} ||D^\beta v||_{L^p(\tau)}, \;\; p \geq 2;$$

$$||v - \pi_h v||_{L^p(\tau)} \leq ch||\nabla v||_{L^p(\tau)}, \;\; p > 2. \quad (3.2.11)$$

We stress the fact that we are using piecewise-linear finite elements.

However, to obtain our new optimal estimates, we need to introduce the following definition.

Definition 3.2.2. *The family of triangulations T_h is* adapted *to the $H^{2,\mu}(\Omega_\alpha^n)-$regularity if the following conditions hold:*

- *the vertices of the polygonal curves $\partial\Omega_\alpha^n$ are nodes of the triangulations.*
- *the meshes are conformal and regular.*
- *there exists $\sigma^* > 0$ such that as $h \to 0$:*

$$\begin{cases} h_\tau \leq \sigma^* h^{\frac{1}{1-\mu}}, & \forall\tau \in T_h \text{ such that one of the vertices of } \tau \text{ belongs to } \mathcal{R}^n \\ h_\tau \leq \sigma^* h \cdot \inf_\tau \rho^\mu, & \forall\tau \in T_h \text{ with no vertex in } \mathcal{R}^n. \end{cases}$$

Here $\rho = \rho_n(x)$ denotes the distance of the point x from the set $\mathcal{R}^n$ of the vertices of the reentrant corners of Ω_α^n.

Now, we consider the two obstacle problem in the finite dimensional space $S_{h,0}$:

$$\text{find } u \in \mathcal{K}_h \,:\, a_p(u, v-u) - \int_{\Omega_\alpha^n} f(v-u)\,\mathrm{d}x \geqslant 0 \;\forall v \in \mathcal{K}_h \tag{3.2.12}$$

where

$$a_p(u,v) = \int_{\Omega_\alpha^n} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v \,\mathrm{d}x$$

and

$$\mathcal{K}_h = \{v \in S_{h,0} : \varphi_{1,h} \leqslant v \leqslant \varphi_{2,h} \text{ in } \Omega_\alpha^n\},$$

with $\varphi_{1,h} = \pi_h\varphi_1$ and $\varphi_{2,h} = \pi_h\varphi_2$.

Even for the FEM-problem, as for the initial one, we can state an existence and uniqueness result of the solution, with analogous estimates for the norm, whose proof is the same as the case previously discussed.

Proposition 3.2.1. *Let us assume hypothesis (3.1.3) and $f \in L^{p'}(\Omega)$. Then, there exists a unique function u_h that solves problem* (3.2.12)*. Moreover,*

$$||u_h||_{W^{1,p}(\Omega_\alpha^n)} \leq C\Big\{1 + ||f||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\varphi_1||_{W^{1,p}(\Omega_\alpha^n)} + ||\varphi_2||_{W^{1,p}(\Omega_\alpha^n)}\Big\} \tag{3.2.13}$$

Proof. With the same argumentations used in the proof of Proposition 2.1.1 we obtain existence and uniqueness of the solution.

Let us show that relation (3.2.13) holds.

Repeating the same passages of the proof of Proposition 2.1.1, we obtain

$$||\nabla u_h||^p_{L^p(\Omega^n_\alpha)} \leq C\{||f||^{p'}_{L^{p'}(\Omega^n_\alpha)} + ||v||^p_{W^{1,p}(\Omega^n_\alpha)} + |k|^{p-2}[||\nabla v||^p_{L^p(\Omega^n_\alpha)} + |\Omega^n_\alpha|]\}, \tag{3.2.14}$$

Now, let us consider this test function $v = (0 \wedge \varphi_{2,h}) \vee \varphi_{1,h}$, which belongs to the convex $\mathcal{K}_h$.

$$v = \begin{cases} \varphi_{2,h}, & \text{if } \varphi_{2,h} \leq 0 \\ \varphi_{1,h}, & \text{if } \varphi_{1,h} \geq 0 \\ 0, & \text{if } \varphi_{2,h} > 0 \text{ and } \varphi_{1,h} < 0 \end{cases} \tag{3.2.15}$$

Let us pose $\Omega_1 = \{x \in \Omega^n_\alpha | \varphi_{2,h} \leq 0\}$, $\Omega_2 = \{x \in \Omega^n_\alpha | \varphi_{2,h} > 0 \text{ and } \varphi_{1,h} \geq 0\}$ and $\Omega_3 = \{x \in \Omega^n_\alpha | \varphi_{2,h} > 0 \text{ and } \varphi_{1,h} < 0\}$.

With the previous choice of v and thanks to Theorem 1.56 in [73], we have:

$$||v||^p_{W^{1,p}(\Omega^n_\alpha)} = \int_{\Omega^n_\alpha} |v|^p \, dx + \int_{\Omega^n_\alpha} |\nabla v|^p \, dx =$$

$$= \int_{\Omega_1} |\varphi_{2,h}|^p \, dx + \int_{\Omega_2} |\varphi_{1,h}|^p \, dx + \int_{\Omega_1} |\nabla \varphi_{2,h}|^p \, dx + \int_{\Omega_2} |\nabla \varphi_{1,h}|^p \, dx \leq$$

(thanks to classical estimates (3.2.11) recalled before)

$$\leq C\Big(||\varphi_2||^p_{W^{1,p}(\Omega^n_\alpha)} + ||\varphi_1||^p_{W^{1,p}(\Omega^n_\alpha)}\Big). \tag{3.2.16}$$

Finally, choosing v like defined in (3.2.15) and thanks to relations (3.2.16), (3.2.14) and the Poincaré's inequality, we have

$$||u_h||^p_{W^{1,p}(\Omega^n_\alpha)} \leq C\Big\{||f||^{p'}_{L^{p'}(\Omega^n_\alpha)} + ||\varphi_2||^p_{W^{1,p}(\Omega^n_\alpha)} + ||\varphi_1||^p_{W^{1,p}(\Omega^n_\alpha)} + |k|^{p-2}[||\nabla \varphi_1||^p_{L^p(\Omega^n_\alpha)} + ||\nabla \varphi_2||^p_{L^p(\Omega^n_\alpha)} + |\Omega^n_\alpha|]\Big\},$$

which gives our estimate. □

As previously observed, the solution u_h to Problem (3.2.12) realizes the minimum on the convex $\mathcal{K}_h$ of the functional $J_p(\cdot)$, i. e.,

$$J_p(u) = \min_{v \in \mathcal{K}_h} J_p(v), \text{ where } J_p(v) = \frac{1}{p} \int_{\Omega^n_\alpha} (k^2 + |\nabla v|^2)^{\frac{p}{2}} \, dx - \int_{\Omega^n_\alpha} f v \, dx. \tag{3.2.17}$$

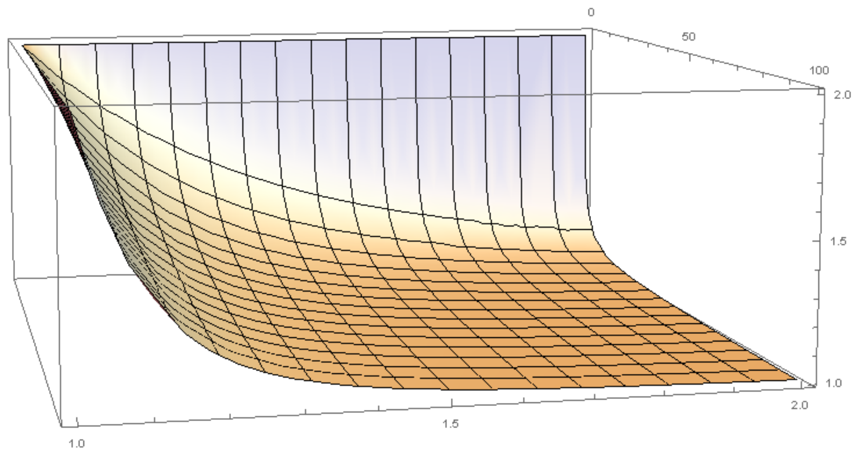

Figure 3.3: Surface plot of the behaviour of r with respect to χ and p

3.2.2 Approximation error estimates

Theorem 3.2.1. *Let $k \neq 0$ and let us denote by u_n and u_h the solutions to Problems (3.1.1) and (3.2.12), respectively. Let us assume hypotheses (3.1.3), (3.1.4) and*

$$\varphi_i \in H^{2,\mu}(\Omega_\alpha^n), \ i = 1, 2. \tag{3.2.18}$$

Let T_h be a triangulation of Ω_α^n adapted to the $H^{2,\mu}(\Omega_\alpha^n)-$regularity of the solution u_n. Then

$$||u_n - u_h||_{W^{1,t}(\Omega_\alpha^n)} \leq C\, h^{\frac{r}{t}} ||u_n||_{H^{2,\mu}(\Omega_\alpha^n)} \tag{3.2.19}$$

for any

$$r \in [1, \frac{2\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{\sqrt{p^2 - \chi(2-\chi)(p-2)^2} + (\chi - 1)(p-2)}), \quad t \in [2, p]. \tag{3.2.20}$$

(we recall again that we are using piecewise-linear finite elements).
Figure 3.3 shows the behaviour of r with respect to χ and p.

In order to prove Theorem 3.2.1, we need to introduce some tools.
First, for any $\sigma \in [0, p]$ we define the following semi-norm

$$|v|_{(p,\sigma)} = \Big(\int_{\Omega_\alpha^n} (|k| + |\nabla u_n| + |\nabla v|)^{p-\sigma} |\nabla v|^\sigma \, \mathrm{d}x \Big)^{\frac{1}{p}}. \tag{3.2.21}$$

Then, let us recall again that for any $p \geq 2$, $\delta \geq 0$ and $k \in \mathbb{R}$ there exist c_1, $c_2 > 0$ such that, for all ξ, $\eta \in \mathbb{R}^2$, the following relations hold (see Lemma 2.1 in [52]).

$$|(k^2+|\xi|^2)^{\frac{p-2}{2}}\xi-(k^2+|\eta|^2)^{\frac{p-2}{2}}\eta| \le c_1|\xi-\eta|^{1-\delta}(|k|+|\xi|+|\eta|)^{p-2+\delta} \tag{3.2.22}$$

and

$$((k^2+|\xi|^2)^{\frac{p-2}{2}}\xi-(k^2+|\eta|^2)^{\frac{p-2}{2}}\eta,\xi-\eta)_{\mathbb{R}^2} \ge c_2|\xi-\eta|^{2+\delta}(|k|+|\xi|+|\eta|)^{p-2-\delta}. \tag{3.2.23}$$

Eventually, we state the following Proposition which is a generalization of Proposition 1.6.2 (Céa's lemma).

Proposition 3.2.2. *Let us denote by u_n and u_h the solutions to Problems (3.1.1) and (3.2.12), respectively. Then, for any $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}_n$ we have*

$$|u_n-u_h|^p_{(p,t)} \le C\{|u_n-v_h|^p_{(p,r)}+||f-A_p(u_n)||_{L^2(\Omega^n_\alpha)}\big(||u_n-v_h||_{L^2(\Omega^n_\alpha)}+||v-u_h||_{L^2(\Omega^n_\alpha)}\big)\}, \tag{3.2.24}$$

where $r\in[1,2], t\in[2,p]$, and the constant C does not depend on h.

Proof. Let $v_h \in \mathcal{K}_h$. First, we observe that for the Lagrange's theorem

$$J_p(v_h)-J_p(u_n)=$$

$$=\int_0^1\Big(\int_{\Omega^n_\alpha}((k^2+|\nabla(u_n+s(v_h-u_n))|^2)^{\frac{p-2}{2}}\nabla(u_n+s(v_h-u_n)),\nabla(v_h-u_n))\mathrm{d}x\Big)ds-\int_{\Omega^n_\alpha}f(v_h-u_n)\mathrm{d}x=$$

$$=\int_0^1\frac{1}{s}\Big(a_p(u_n+s(v_h-u_n),s(v_h-u_n))-a_p(u_n,s(v_h-u_n))\Big)ds+a_p(u_n,v_h-u_n)-\int_{\Omega^n_\alpha}f(v_h-u_n)\mathrm{d}x=$$

$$=A(v_h)+a_p(u_n,v_h-u_n)-\int_{\Omega^n_\alpha}f(v_h-u_n)\,\mathrm{d}x$$

where we have put

$$A(v_h)=\int_0^1\Big(\int_{\Omega^n_\alpha}((k^2+|\nabla(u_n+s(v_h-u_n))|^2)^{\frac{p-2}{2}}\nabla(u_n+s(v_h-u_n))-(k^2+|\nabla u_n|^2)^{\frac{p-2}{2}}\nabla u_n,\nabla(v_h-u_n))\mathrm{d}x\Big)ds. \tag{3.2.25}$$

By choosing $\delta=t-2$ in (3.2.23) and using the following inequality

$$\frac{t}{2}(|a|+|b|)\le|a|+|a+tb|,\ \forall a,b\in\mathbb{R}\text{ and }t\in[0,1],$$

we obtain

$$|u_n - u_h|^p_{(p,t)} \leq cA(u_h) \tag{3.2.26}$$

where $A(\cdot)$ is defined in (3.2.25) and $c = \frac{2^{p-t}p}{c_2}$.

Now, since u_h is solution to Problem (3.2.17), we have

$$\begin{aligned} A(u_h) =& J_p(u_h) - J_p(u_n) - a_p(u_n, u_h - u_n) + \int_{\Omega^n_\alpha} f(u_h - u_n)\,\mathrm{d}x \leq \\ &\leq J_p(v_h) - J_p(u_n) - a_p(u_n, u_h - u_n) + \int_{\Omega^n_\alpha} f(u_h - u_n)\,\mathrm{d}x. \end{aligned} \tag{3.2.27}$$

Moreover

$$\begin{aligned} &J_p(v_h) - J_p(u_n) - a_p(u_n, u_h - u_n) + \int_{\Omega^n_\alpha} f(u_h - u_n)\,\mathrm{d}x = \\ &= A(v_h) + a_p(u_n, v_h - u_n) - \int_{\Omega^n_\alpha} f(v_h - u_n)\,\mathrm{d}x - a_p(u_n, u_h - u_n) + \int_{\Omega^n_\alpha} f(u_h - u_n)\,\mathrm{d}x = \\ &= A(v_h) + a_p(u_n, v_h - u_h) - \int_{\Omega^n_\alpha} f(v_h - u_h)\,\mathrm{d}x \end{aligned} \tag{3.2.28}$$

so, for any $v \in \mathcal{K}_n$, as u_n is solution to (3.1.1),

$$\begin{aligned} a_p(u_n, u_n - v + v_h + v - u_n - u_h) - \int_{\Omega^n_\alpha} f(u_n - v + v_h + v - u_n - u_h)\,\mathrm{d}x \leq \\ a_p(u_n, v_h - u_n) + a_p(u_n, v - u_h) - \int_{\Omega^n_\alpha} f(v_h - u_n)\,\mathrm{d}x - \int_{\Omega^n_\alpha} f(v - u_h)\,\mathrm{d}x. \end{aligned} \tag{3.2.29}$$

Hence, by (3.2.22) with $\delta = 2 - r$ and using the following inequality

$$|a| + |a + tb| \leq 2(|a| + |b|),\ \forall a, b \in \mathbb{R} \text{ and } t \in [0, 1],$$

we have

$$|A(v_h)| \leq c_1 \int_0^1 s^{r-1} \int_{\Omega^n_\alpha} \big(|k| + |\nabla(u_n + s(v_h - u_n))| + |\nabla u_n|\big)^{p-r} |\nabla(v_h - u_n)|^r\,\mathrm{d}x ds \leq c|v_h - u_n|^p_{(p,r)} \tag{3.2.30}$$

where $c = \frac{2^{p-r}c_1}{r}$.

Putting together estimates (3.2.26), (3.2.27), (3.2.28), (3.2.29), and (3.2.30) and taking into account the Levy-Stampacchia inequality, we obtain

$$|u_n - u_h|^p_{(p,t)} \leq C\{|u_n - v_h|^p_{(p,r)} + ||f - A_p(u_n)||_{L^2(\Omega^n_\alpha)}\big(||u_n - v_h||_{L^2(\Omega^n_\alpha)} + ||v - u_h||_{L^2(\Omega^n_\alpha)}\big)\}$$

where $r \in [1,2], t \in [2,p]$, and the constant C does not depend on h.

□

Proof. *(of Theorem 3.2.1)*

Let us evaluate the terms in the right-hand side in estimate (3.2.24) by choosing in an appropriate way the test functions $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}_n$. According to Theorem 3.1.1 the function u_n belongs to the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha^n)$ for any $\mu > 1-\gamma$ (see (4.1.18) and (3.1.6)). We choose $v_h = \pi_h u_n$ and by using the approximation estimates of Grisvard (see section 8.4.1 in [38]) we derive

$$||u_n - \pi_h u_n||_{L^2(\Omega_\alpha^n)} \leq Ch^2 ||u_n||_{H^{2,\mu}(\Omega_\alpha^n)}. \tag{3.2.31}$$

Then we choose $v = \varphi_2 \wedge (u_h \vee \varphi_1)$ and, as in Lemma 4.4 in [22], we have

$$||v - u_h||^2_{L^2(\Omega_\alpha^n)} \leq ||\pi_h \varphi_2 - \varphi_2||^2_{L^2(\Omega_\alpha^n)} + ||\pi_h \varphi_1 - \varphi_1||^2_{L^2(\Omega_\alpha^n)}.$$

Again using Grisvard estimates and assumption (3.2.18) we derive

$$||v - u_h||_{L^2(\Omega_\alpha^n)} \leq Ch^2. \tag{3.2.32}$$

We compare the seminorm $|u_n - u_h|_{W^{1,t}(\Omega_\alpha^n)}$ with $|u_n - u_h|^p_{(p,t)}$ (defined in (3.2.21)) and we obtain

$$|u_n - u_h|^t_{W^{1,t}(\Omega_\alpha^n)} \leq \frac{1}{|k|^{p-t}} |u_n - u_h|^p_{(p,t)}. \tag{3.2.33}$$

Now, we have to evaluate the term $|u_n - v_h|^p_{(p,r)}$, where $v_h = \pi_h u_n$.

By the embedding of weigthed Sobolev spaces in the fractional Sobolev spaces (see Proposition 1.1.2) we deduce that u_n belongs to the space $W^{\sigma_2,2}(\Omega_\alpha^n)$, for any $\sigma_2 < 1+\gamma$ and, taking into account the Sobolev embedding, we obtain

$$|\nabla u_n| \in L^{r*}(\Omega_\alpha^n) \text{ with } r^* = \frac{2}{2-\sigma_2}. \tag{3.2.34}$$

So, applying Hölder inequality with conjugate exponents $\frac{2}{r}$ and $\frac{2}{2-r}$ and using estimate (3.1.7) with $r = \frac{2(r^*-p)}{r^*-2}$, we have

$$|u_n - v_h|^p_{(p,r)} \leq C(r) |u_n - \pi_h u_n|^r_{W^{1,2}(\Omega_\alpha^n)}, \tag{3.2.35}$$

Thus, as $\sigma_2 < 1+\gamma$, r^* is defined in (3.2.34) and γ given in (3.1.6), we have to choose $r < p + \frac{2-p}{\gamma}$

and we obtain that $r < \frac{2\sqrt{p^2-\chi(2-\chi)(p-2)^2}}{\sqrt{p^2-\chi(2-\chi)(p-2)^2}+(\chi-1)(p-2)}$ (for the proof see the following Lemma 3.2.2). So, we use Theorem 8.4.1.6 in [38] to obtain

$$|u_n - \pi_h u_n|_{W^{1,2}(\Omega_\alpha^n)} \leq Ch. \tag{3.2.36}$$

Then, finally, taking into account estimates (3.2.24), (3.2.31), (3.2.32), (3.2.33),(3.2.35) and (3.2.36), we end the proof applying Poincaré inequality. □

Lemma 3.2.2. *For any $p \in [2, \infty)$ and $\chi \in (1, 2)$*

$$p + \frac{2-p}{\gamma} = \frac{2\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{\sqrt{p^2 - \chi(2-\chi)(p-2)^2} + (\chi-1)(p-2)}. \tag{3.2.37}$$

Proof. First, we show that the following relation holds

$$\gamma = 1 - \frac{2(\chi-1)}{\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} + p(\chi-1)}. \tag{3.2.38}$$

where γ is defined in relation (3.1.6). In fact:

$$1 - \frac{2(\chi-1)}{\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} + p(\chi-1)} =$$

$$= \frac{\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} + p(\chi-1) - 2(\chi-1)}{\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} + p(\chi-1)} =$$

$$= \frac{4\chi(2-\chi)(p-1) - 2(\chi-1)[\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} - p(\chi-1)]}{4\chi(2-\chi)(p-1)} =$$

$$= \frac{4\chi(2-\chi)(p-1) + 2p(\chi-1)^2 - 2(\chi-1)\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)}}{4\chi(2-\chi)(p-1)} =$$

$$= 1 + \frac{p(\chi-1)^2 + (1-\chi)\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)}}{2\chi(2-\chi)(p-1)},$$

which is our γ, if we note that

$$p^2 - \chi(2-\chi)(p-2)^2 = p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1). \tag{3.2.39}$$

Now, let us show that equality (3.2.37) holds

$$p + \frac{2-p}{\gamma} = \frac{p(\gamma-1)+2}{\gamma} =$$

$$= \frac{\frac{-2p(\chi-1)}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+p(\chi-1)}+2}{1-\frac{2(\chi-1)}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+p(\chi-1)}} =$$

$$\frac{2\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+(p-2)(\chi-1)}.$$

□

We note that in Theorem 3.2.1 we assume $k \neq 0$, if $k = 0$ the following (analogous) result holds. We point out that in the following theorem we require that the solution is $H^{2,\mu}$-regular because, unfortunately, until now we are not able to prove that it has this regularity. Actually, we show an example of solution belonging to the space $H^{2,\mu}$.

Example 3.2.1. *The function $u = \rho^\beta \sin(\beta\phi)$ (where ρ and θ are the polar coordinates), with $1 - \frac{2}{p} < \beta$ is the solution to the obstacle Problem (2.1.1) in the space $W^{1,p}(\Omega^*)$ with:*
(a) datum $f = C\rho^{(\beta-1)(p-1)-1}\sin(\beta\phi)$, where $C = \beta^{p-1}(1-\beta)(p-2)$;
(b) obstacles $\varphi_1 = 0, \varphi_2 = 1$;
(c) boundary datum $\Phi = \sin(\beta\phi)$;
(d) Ω^* the unit cone of opening $\frac{\pi}{\beta}$, that is $\Omega^* = \{(\rho,\phi) \in \mathbb{R}^2 \,:\, 0 \le \rho \le 1, 0 \le \phi \le \frac{\pi}{\beta}\}$.
Moreover, if we assume $\beta > 1 - \mu$, we obtain that $u \in H^{2,\mu}(\Omega^*)$.

Theorem 3.2.3. *Let us denote by u_n and u_h the solutions to Problems (3.1.1) and (3.2.12) respectively. Let us assume hypotheses (3.1.3), (3.1.8), (3.2.18) and that the solution u_n belongs to the space $H^{2,\mu}(\Omega^n_\alpha)$. Let T_h be a triangulation of Ω^n_α adapted to the $H^{2,\mu}(\Omega^n_\alpha)-$regularity of the solution u_n. Then*

$$||u_n - u_h||_{W^{1,q}(\Omega^n_\alpha)} \le C\, h^{\frac{r}{t}} ||u_n||_{H^{2,\mu}(\Omega^n_\alpha)} \tag{3.2.40}$$

for any

$$r \in [1, \frac{2\sqrt{p^2-\chi(2-\chi)(p-2)^2}}{\sqrt{p^2-\chi(2-\chi)(p-2)^2}+(\chi-1)(p-2)}) \tag{3.2.41}$$

$t \in [2,p]$, $q \in [1,t]$ and for $q < p$ we require $t \ge \frac{q(p+(p-2)2)}{q+(p-2)2}$.

Proof. We proceed as in the proof of Theorem 3.2.1: we replace estimate (3.2.33) by the following

$$|u_n - u_h|^t_{W^{1,q}(\Omega^n_\alpha)} \le |||\nabla u_n|^{-\frac{(p-t)q}{t-q}}||^{\frac{t-q}{q}}_{L^1(\Omega^n_\alpha)} \cdot \int_{\Omega^n_\alpha} |\nabla(u_n-u_h)|^t |\nabla u_n|^{p-t} dxdy \le C|u_n - u_h|^p_{(p,t)}. \tag{3.2.42}$$

Here we have applied Hölder inequality with conjugate exponents $\frac{t}{q}$ and $\frac{t}{t-q}$ and used estimate

(3.1.9). In fact, we have

$$|u_n - u_h|^t_{W^{1,q}(\Omega^n_\alpha)} = \Big(\int_{\Omega^n_\alpha} |\nabla(u_n - u_h)|^q \,\mathrm{d}x\Big)^{\frac{t}{q}} = \Big(\int_{\Omega^n_\alpha} |\nabla(u_n - u_h)|^q \frac{|\nabla u_n|^{\frac{(p-t)q}{t}}}{|\nabla u_n|^{\frac{(p-t)q}{t}}} \,\mathrm{d}x\Big)^{\frac{t}{q}} \le$$

$$\le \Big(\int_{\Omega^n_\alpha} |\nabla(u_n - u_h)|^t |\nabla u_n|^{p-t} \,\mathrm{d}x\Big)\Big(\int_{\Omega^n_\alpha} |\nabla u_n|^{-\frac{q(p-t)}{t-q}} \,\mathrm{d}x\Big)^{\frac{t-q}{q}} =$$

$$= || \, |\nabla u_n|^{-\frac{q(p-t)}{t-q}} \, ||^{\frac{t-q}{q}}_{L^1(\Omega^n_\alpha)} \int_{\Omega^n_\alpha} |\nabla(u_n - u_h)|^t |\nabla u_n|^{p-t} \,\mathrm{d}x \le C|u_n - u_h|^p_{(p,t)}$$

□

Remark 3.2.1. *From the previous proof we deduce that, for the linear case $p = 2$, Theorem 3.2.1 gives the "sharp" result of Grisvard (see Corollary 8.4.1.7 in [38]): in fact, we have $p = t = 2$ and, in particular, formula (3.2.35) holds true for $r = 2 = p$.*

Remark 3.2.2. *We note that Theorem 3.2.1 improves the results of [22]: in particular, estimate (3.2.19) gives a faster convergence than the convergence in estimate (5.63) in [22], that is.*

$$||u_n - u_h||_{W^{1,q}(\Omega^n_\alpha)} \le ch^\eta \{||u_n||^{\frac{2}{t}}_{W^{\sigma_2,2}(\Omega^n_\alpha)} + ||u_n||^{\frac{1}{t}}_{W^{\sigma,p}(\Omega^n_\alpha)} + ||\varphi_1||^{\frac{1}{t}}_{W^{\sigma,p}(\Omega^n_\alpha)} + ||\varphi_2||^{\frac{1}{t}}_{W^{\sigma,p}(\Omega^n_\alpha)}\},$$

where $\eta = \frac{\sigma}{t}$, with $\sigma = \sigma_2 - 1 + \frac{2}{p}$, $t \in [2, p]$, $q \in [1, t]$, $2 - \frac{1-\frac{2}{p}}{p-1} < \sigma_2 < 2$ and the constant c does not depend on h.
In fact the solution u_n belongs to the weighted Sobolev space $H^{2,\mu}(\Omega^n_\alpha)$ for any $\mu = \mu(p) > 1-\gamma$. This space is continuously embedded in the fractional Sobolev space $W^{\sigma_2,2}(\Omega^n_\alpha)$ for any $\sigma_2 < 2-\mu$ (see Proposition 1.1.2). Hence by the Sobolev embedding (see Theorem 1.1.2), for any $\sigma < \gamma+\frac{2}{p}$, $p \ge 2$, the fractional Sobolev space $W^{\sigma,p}(\Omega^n_\alpha)$ properly contains the weighted Sobolev space $H^{2,\mu}(\Omega^n_\alpha)$ for some $\mu = \mu(p) > 1 - \gamma$. Actually for every $p \ge 2$ the exponent r in (3.2.19) is strictly greater than $\gamma + \frac{2}{p}$, as follows from the following proposition.

Proposition 3.2.3. *For any choice of the parameters $p \in [2, +\infty)$ and $\chi \in (1, 2)$ we have*

$$\gamma + \frac{2}{p} < r \Longleftrightarrow \chi(2-\chi)(p-1)(p-2) + (\chi-1)\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} > 0, \quad (3.2.43)$$

where r is a value of (3.2.20) and γ is given by (3.2.38).

Proof. Let $p \in [2, +\infty)$, $\chi \in (1, 2)$.
Substituting the left-hand-side in (3.2.43) with the expressions of γ (3.2.38) and the upper

bound for r, we have:

$$1-\frac{2(\chi-1)}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+p(\chi-1)}+\frac{2}{p}<$$

$$<\frac{2\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+(\chi-1)(p-2)},$$

that is

$$\frac{-2(\chi-1)}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+p(\chi-1)}+\frac{2}{p}<$$

$$<\frac{2\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+(\chi-1)(p-2)}-1,$$

so

$$\frac{2\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}}{p(\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+p(\chi-1))}<\frac{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}-(\chi-1)(p-2)}{\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+(\chi-1)(p-2)}.$$

Then

$$2[p^2(\chi-1)^2+4\chi(2-\chi)(p-1)]+2(p-2)(\chi-1)\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}<$$

$$<p[p^2(\chi-1)^2+4\chi(2-\chi)(p-1)]-p(p-2)(\chi-1)\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}+$$

$$+p^2(\chi-1)\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}-p^2(\chi-1)^2(p-2).$$

Hence, moving all terms to the right-hand-side, we obtain

$$(p-2)[p^2(\chi-1)^2+4\chi(2-\chi)(p-1)]-p^2(\chi-1)^2(p-2)-$$

$$-[2(p-2)+p(p-2)-p^2](\chi-1)\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}>0,$$

and finally

$$4\chi(2-\chi)(p-1)(p-2)+4(\chi-1)\sqrt{p^2(\chi-1)^2+4\chi(2-\chi)(p-1)}>0$$

which, dividing by 4, is (3.2.43). □

Remark 3.2.3. *We note that the constant C in estimate (3.2.19) does not depend from n. However to deduce from (3.2.19) error estimates for the fractal solution we have to bound the* *norms* $||u_n||_{H^{2,\mu}(\Omega_\alpha^n)}$ *uniformly in n. Up to now, this type of results is only established for* $p = 2$ *(see [17] and [18]).*

3.3 Numerical simulations

In this last section, we want to show some numerical simulations which confirm the theoretical result obtained. We point out that the meshes have been constructed using Matlab. Instead all the simulations have been realized using FreeFem++. In particular, as solver we have used IPOPT. It is designed to perform optimization for both inequality and equality constrained problems. Actually non-linear inequalities are rearranged before the beginning of the optimization process. IPOPT is a smart Newton method for solving constrained optimization problems (see Section 8.3 in [39]).

First of all, in order to apply our result, we have to construct a triangulation satisfying the conditions of definition (3.2.2). Compared to the "usual" regular and conformal triangulation, this definition has an innovation: we have to satisfy these conditions:

there exists $\sigma^* > 0$ such that as $h \to 0$

$$h_\tau \leq \sigma^* h^{\frac{1}{1-\mu}}, \; \forall \tau \in T_h \text{ such that one of the vertices of } \tau \text{ belongs to } \mathcal{R}^n \tag{3.3.44}$$

$$h_\tau \; \leq \sigma^* h \cdot \inf_\tau \rho^\mu, \; \forall \tau \in T_h \text{ with no vertex in } \mathcal{R}^n. \tag{3.3.45}$$

What's the meaning of these assumptions?

The first one requires that the triangles which have a vertex in a reentrant corner have to be smaller than the ones in the usual regular and conformal triangulation. In others words, the triangulation closer to the reentrant corners must have a lower size than the ones far from the reentrant corner.

The second condition means that the triangles which do not have a vertex in a reentrant corner do not need to be much smaller that the ones used in the usual regular and conformal triangulation. In particular, the amount $\inf_\tau \rho^\mu$ means that the size increase gradually as we move away form the reentrant corner.

In this framework, μ establish how (much) the grid will be thick near the "bad" points. In Figure 3.4, we see different examples of triangulation made on a particular "bad domain" choosing different values for μ.

We want to specify two facts.

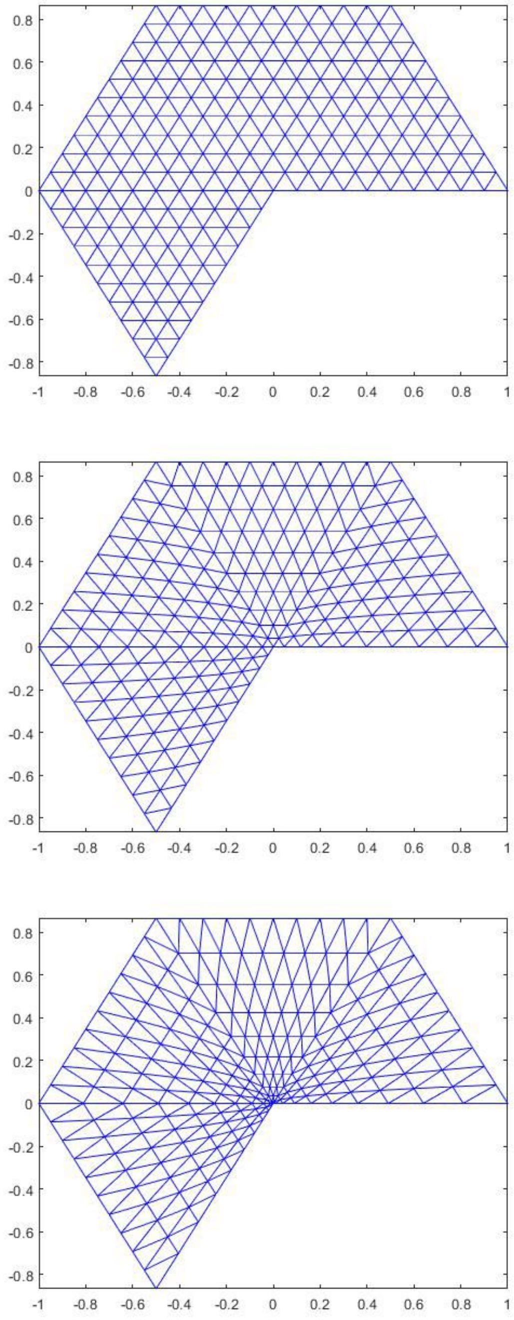

Figure 3.4: Triangulations in the cases of $\mu = 0$, $\mu = 0.25$ and $\mu = 0.5$, respectively

First, we know that the choice $\mu = 0$ is not feasible. In fact, since we require $\mu > 1 - \gamma$ and we have that $1-\gamma$ goes to 0, as $p \to \infty$, then we chose $\mu = \varepsilon > 0$. Nevertheless, this choice is useful in order to have a particular example of triangulation which is only regular and conformal (in fact the previous conditions collapse in only one) and compare it with the feasible cases.

The second fact is that all values of μ greater the $0.25 + \varepsilon$, with $\varepsilon > 0$ small, and smaller than 1 are feasible because the amount $1 - \gamma$ decreases, as $p \to \infty$, and it is equal to 0.25 when $p = 2$. Consequently, since $\mu > 1 - \gamma$, for p fixed, we have to choose $\mu = 1 - \gamma(p) + \varepsilon$, with $\varepsilon > 0$, in order to have an optimal value for μ.

Moreover, we want to observe that the triangulations here used (constructed according to Raugel's suggestion) are not the only possible. Indeed, see for instance [3], the authors propose another triangulation which satisfies Grisvard's conditions too, but is more appropriate for the type of the transmission problem faced and the geometry of the domain.

Before showing the numerical simulations results, we want to recall how Raugel (and Grisvard, see Section 8.4.1 in [38]) suggested to construct meshes which satisfy the conditions of the definition 3.2.2.

1) divide the domains in big (for instance, one half of the radius) triangles (see Figure 3.5);

2) divide the big triangles not having vertices in the reentrant corners in the usual way (i.e. with a only regular and conformal triangulation);

3) divide the big triangles which have a vertex in the reentrant corners according to this ratios

$$l \cdot \left(\frac{i}{n}\right)^{\frac{1}{1-\mu}}, \; i = 0, ..., n \text{ with } 0 < 1 - \gamma < \mu < 1 \text{ and } l = \textit{length of the side}$$

along the sides which have an endpoint in the reentrant vertex and the third side in n equal parts (see from Figure 3.6 to Figure 3.10).

In Figure 3.12 we see an example of regular and conformal mesh which does not satisfy Grisvard's conditions.

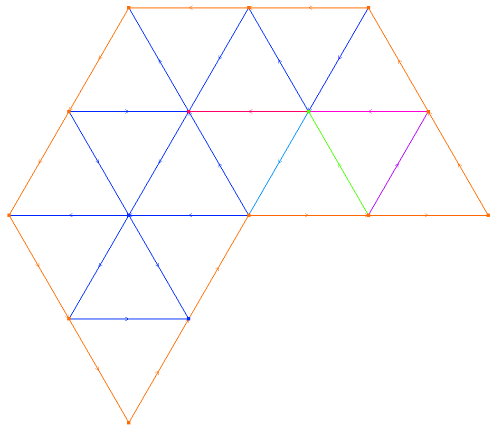

Figure 3.5: The domain divided in big triangles

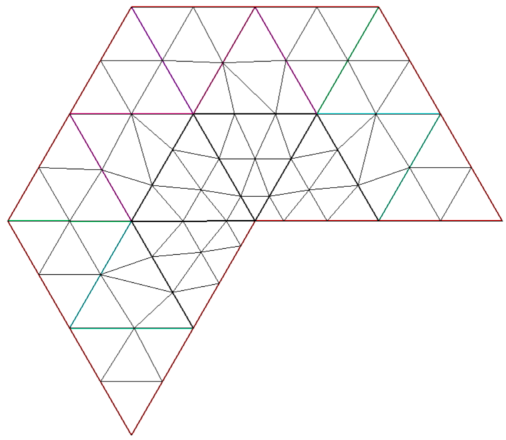

Figure 3.6: "Grisvard Mesh" with $h = 0,600921$ and $\mu = 0.2501$

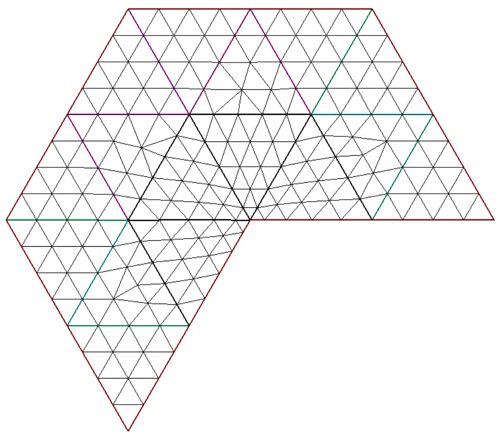

Figure 3.7: "Grisvard Mesh" with $h = 0,311832$ and $\mu = 0.2501$

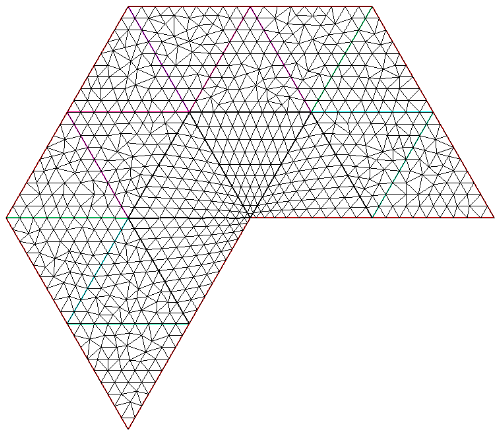

Figure 3.8: “Grisvard Mesh” with $h = 0,171633$ and $\mu = 0.2501$

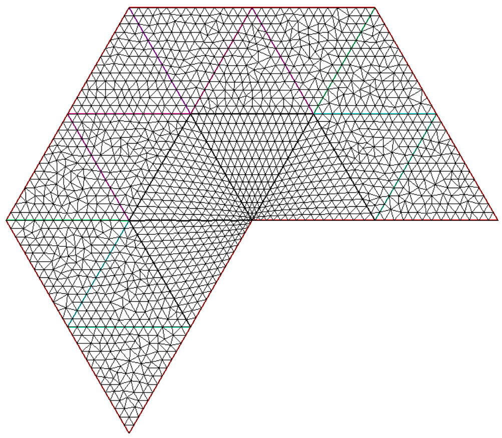

Figure 3.9: “Grisvard Mesh” with $h = 0,133225$ and $\mu = 0.2501$

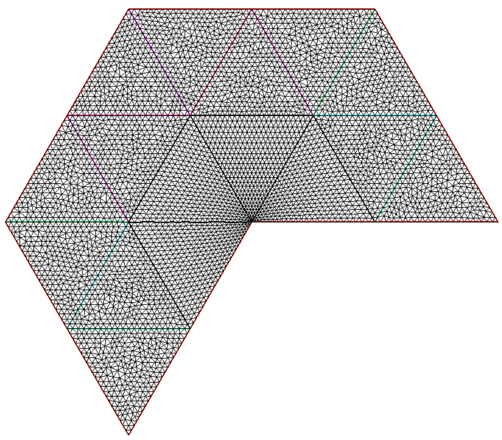

Figure 3.10: “Grisvard Mesh” with $h = 0,0787296$ and $\mu = 0.2501$

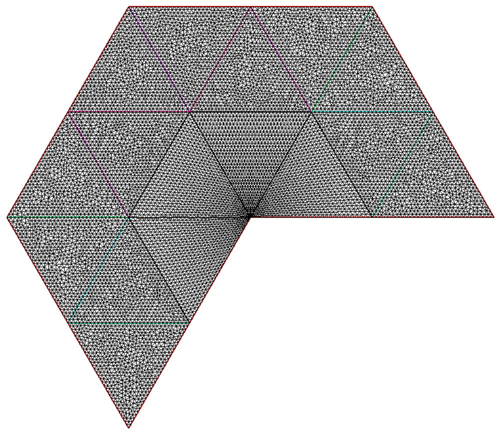

Figure 3.11: “Grisvard Mesh” with $h = 0,0512961$ and $\mu = 0.2501$

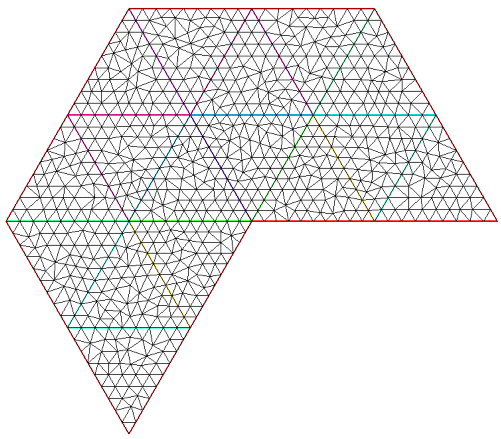

Figure 3.12: “Non-Grisvard Mesh” with $h = 0,171633$

3.3.1 Simulations in the case of $p = 2$

Now, we show some numerical simulations made using the adapted triangulation, constructed by choosing, for instance, $\mu = 0.2501$ (see from Figure 3.6 to 3.10) and taking

$$\varphi_1 = 0 \text{ and } \varphi_2 = 0.25$$

as obstacles and, as datum,

$$f = 5$$

and

$$f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}},$$

with, for $(x, y) \neq (0, 0)$,

$$a(x,y) = \begin{cases} \arctan(\frac{y}{x}), \text{ if } x > 0 \text{ and } y \geq 0 \\ \frac{\pi}{2}, \text{ if } x = 0 \text{ and } y > 0 \\ \arctan(\frac{y}{x}) + \pi, \text{ if } x < 0 \\ \frac{3\pi}{2}, \text{ if } x = 0 \text{ and } y < 0 \\ \arctan(\frac{y}{x}) + 2\pi, \text{ if } x > 0 \text{ and } y < 0 \end{cases},$$

For the last choice of f we were inspired by data used in the example 3.2.1.
As observed in Remark 3.2.1, we stress again the fact that in this case (linear), Theorem 3.2.1 gives the sharp result of Grisvard. In the following table we see the approximation errors and the approximation relative errors corresponding to the choice of meshes with different sizes, evaluated in the $W^{1,2}$-norm, comparing the meshes which satisfy and do not satisfy Grisvard's conditions, for the case $f = 5$

h	$ErrGris$	$ErrNonGris$	$ErrRelGris$	$ErrRelNonGris$
0,600964	1,01552	1,03247	0,675424	0,6867
0,311832	0,475222	0,480677	0,32445	0,328175
0,171633	0,213031	0,224887	0,146045	0,154173
0,133225	0,131189	0,138569	0,0899752	0,0950367
0,0787296	0,0624998	0,0665082	0,0428656	0,0456149
0,0512961	0,0291075	0,0362865	0,0199633	0,024887

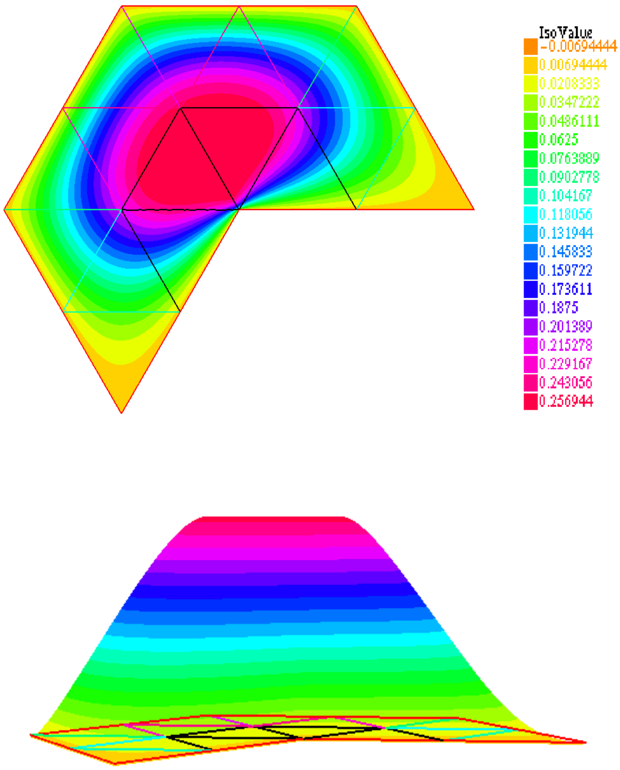

Figure 3.13: Reference solution for $p = 2$ and $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

Instead, in the following table we see the analogous table for the choice of $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

h	$ErrGris$	$ErrNonGris$	$ErrRelGris$	$ErrRelNonGris$
$0,600964$	$0,374391$	$0,413492$	$0,409424$	$0,452184$
$0,311832$	$0,254928$	$0,316685$	$0,278862$	$0,346416$
$0,171633$	$0,14431$	$0,203136$	$0,158462$	$0,223057$
$0,133225$	$0,0978755$	$0,156604$	$0,107398$	$0,17184$
$0,0787296$	$0,0615946$	$0,105964$	$0,0676602$	$0,116399$
$0,0512961$	$0,0308178$	$0,0859639$	$0,0338451$	$0,0944084$

Moreover, in Figure 3.13 we see the reference solution (obtained using a finer mesh with $h = 0,0376415$). Instead, in Figure 3.14 we see the errors corresponding to the meshes which satisfy and do not satisfy Grisvard's conditions, with $h = 0,133225$; in particular, it shows that the higher difference is concentrated near the reentrant corner.

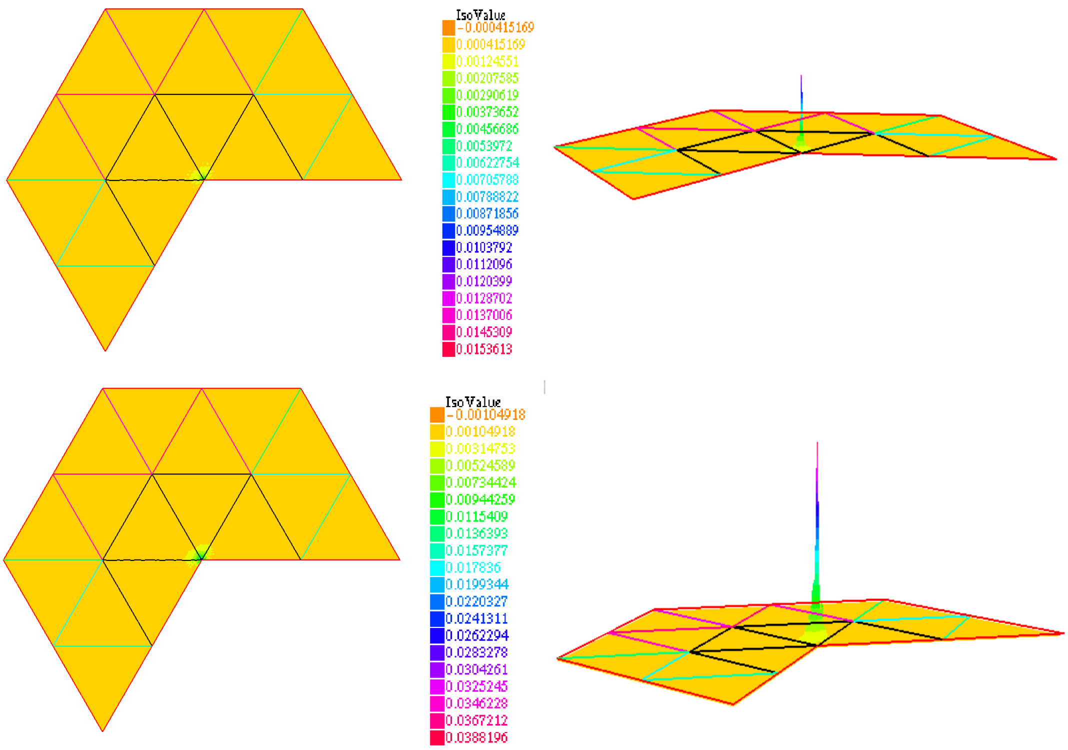

Figure 3.14: Errors for $p = 2$ and $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$ in "Grisvard" and "non-Grisvard Mesh", respectively, with size $h = 0,0512961$

3.3.2 Simulations in the case of $p > 2$

Now, we show other numerical simulations made by using the same triangulations, the same choices of f, $k = 1$ and the same obstacles we used before. In this simulations we have chosen $p = 8$.

In the first place, we show the FEM-solutions obtained using triangulations that satisfy and do not satisfy Grisvard's conditions (with size $h = 0,171633$) the reference solution (see Figure 3.15 and 3.16) and then the corresponding errors (see Figure 3.17), in the case of $f = 5$. Then, we show the analogous pictures obtained in the case of $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$ and meshes again with size $h = 0,171633$ (see Figures 3.18, 3.19 and 3.20).

As in the case of $p = 2$, in the following table we see the approximation relative errors corresponding to the choice of meshes with different sizes, evaluated in the $W^{1,2}$-norm, comparing the meshes which satisfy and do not satisfy Grisvard's conditions, for the case $f = 5$

h	$ErrGris$	$ErrNonGris$	$ErrRelGris$	$ErrRelNonGris$
$0,600964$	$0,353095$	$0,345724$	$0,283906$	$0,277979$
$0,311832$	$0,159531$	$0,159357$	$0,128252$	$0,128112$
$0,171633$	$0,0598404$	$0,0618406$	$0,0481258$	$0,0497344$

Instead, in the following table we see the table for the last choice of $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

h	$ErrGris$	$ErrNonGris$	$ErrRelGris$	$ErrRelNonGris$
$0,600964$	$0,148449$	$0,158407$	$0,200001$	$0,202639$
$0,311832$	$0,0668549$	$0,0726701$	$0,0900714$	$0,0979061$
$0,171633$	$0,0207721$	$0,0289916$	$0,0279857$	$0,0390595$

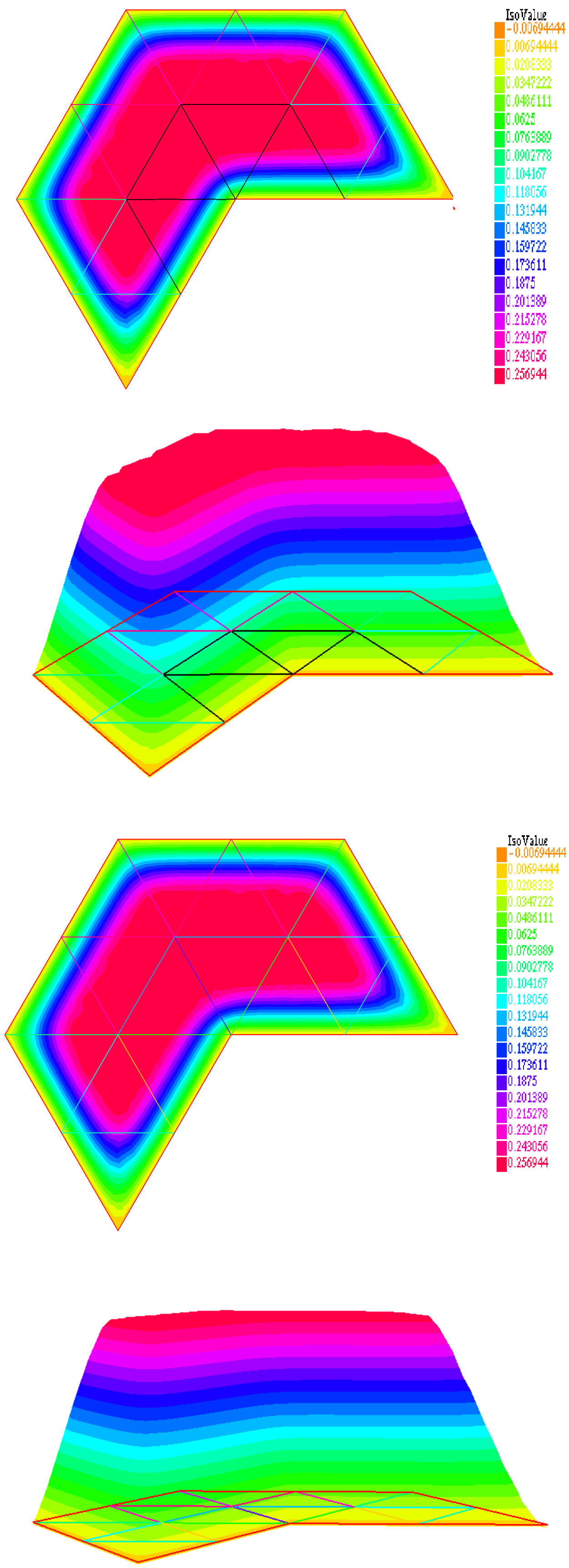

Figure 3.15: Solutions in “Grisvard” and “non-Grisvard” meshes, respectively, for $h = 0,171633$ and $f = 5$

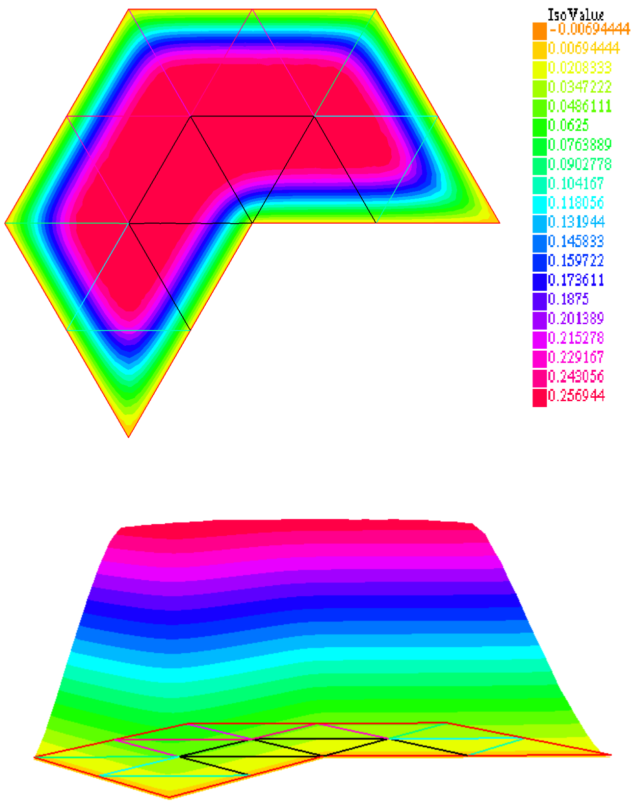

Figure 3.16: Reference solution in the case of $f = 5$

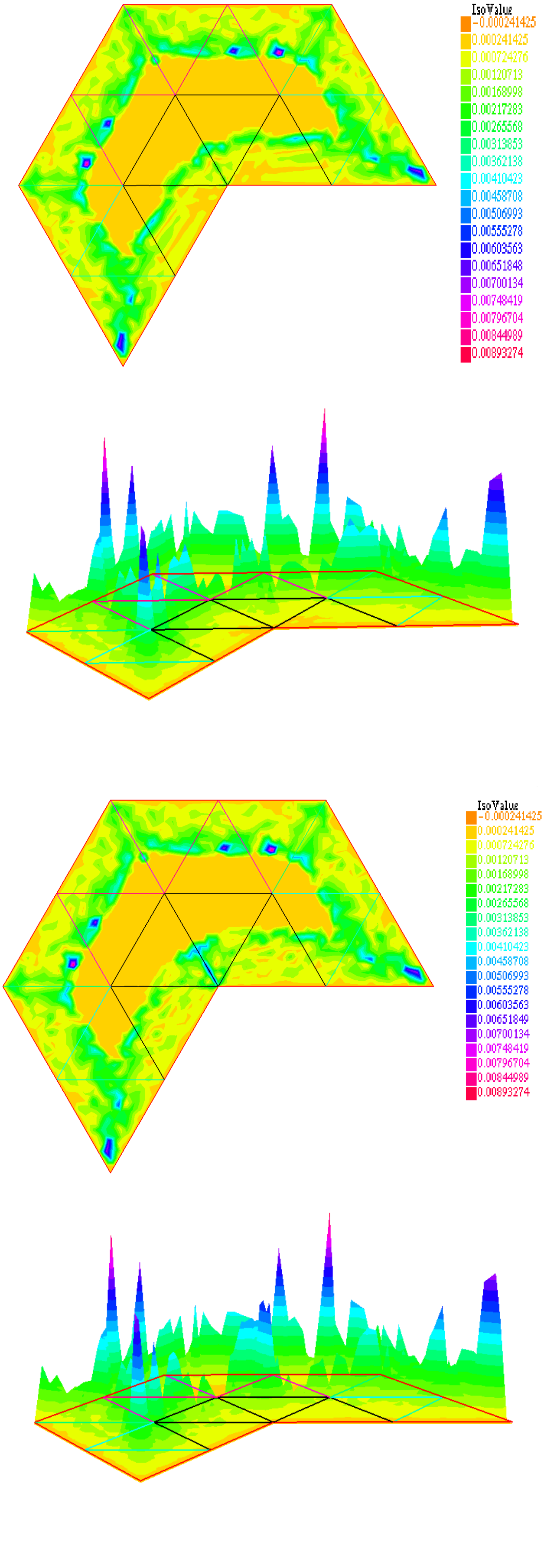

Figure 3.17: Errors in "Grisvard" and "non-Grisvard" meshes, respectively, with $h = 0,171633$ and $f = 5$

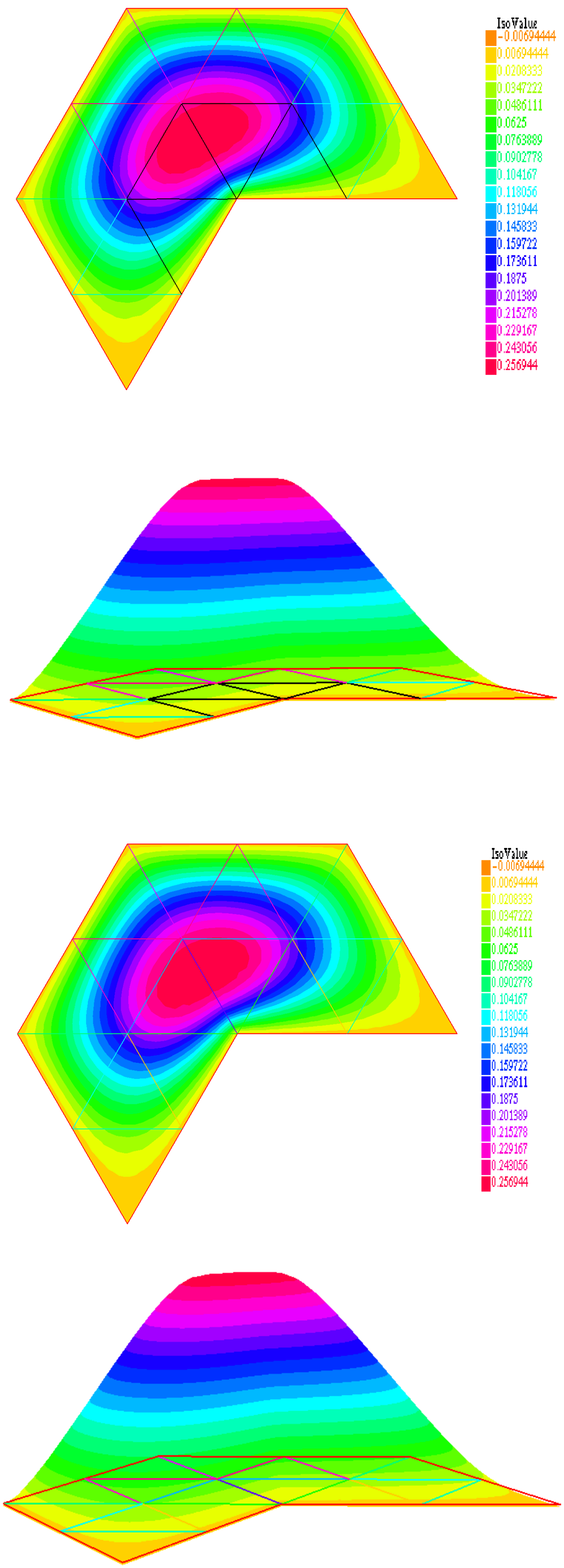

Figure 3.18: Solutions in "Grisvard" and "non-Grisvard" meshes, respectively, with $h = 0,171633$ and $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

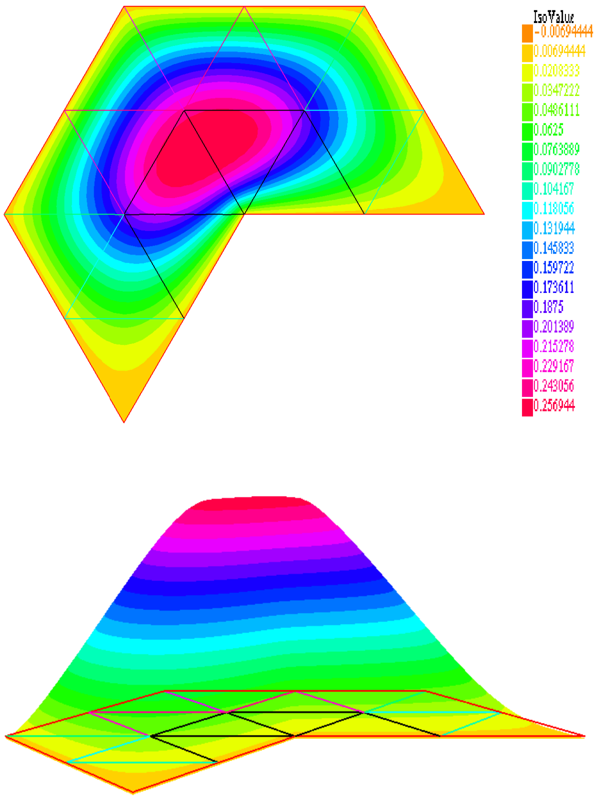

Figure 3.19: Reference solution in the case of $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

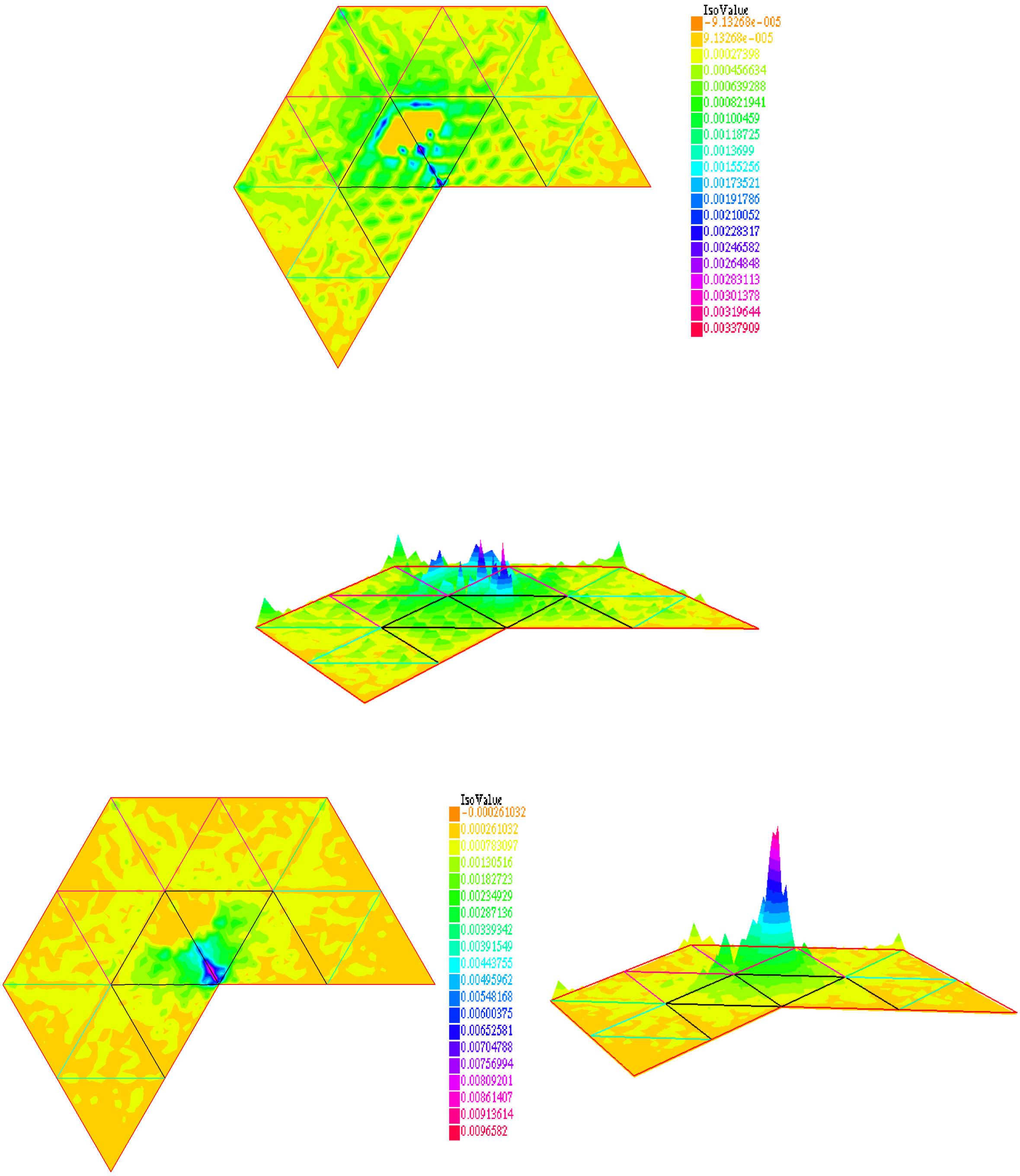

Figure 3.20: Errors in "Grisvard" and "non-Grisvard" meshes, respectively, with $h = 0,171633$ and $f = \frac{\sin(\frac{3}{4}a(x,y))}{(x^2+y^2)^{5/8}}$

Chapter 4

Asymptotic for quasilinear obstacle problems in bad domains

In this chapter we consider obstacle problems involving p-Laplacian, both in the Koch Island and in pre-fractal domains which approximate it (defined as we have introduced in Section 1.4 in Chapter 1). These domains are in $\mathbb{R}^2$ and we will indicate them with Ω_α and Ω_α^n, respectively in the fractal and pre-fractal case. As specified in Section 1.4, we recall that the pre-fractal domains Ω_α^n are constructed taking outward pre-fractal Koch curves.
The results presented in this chapter are contained in [15], recently published.
In particular, given $f \in L^1(\Omega_\alpha)$, we consider the following two obstacle problems:

$$\text{find } u \in \mathcal{K}_n, \quad \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla (v-u)\,\mathrm{d}x - \int_{\Omega_\alpha^n} f(v-u)\ \mathrm{d}x \geqslant 0 \ \ \forall v \in \mathcal{K}_n, \qquad (\mathcal{P}_{p,n})$$

and

$$\text{find } u \in \mathcal{K}, \quad \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla (v-u)\,\mathrm{d}x - \int_{\Omega_\alpha} f(v-u)\ \mathrm{d}x \geqslant 0 \ \ \forall v \in \mathcal{K}, \qquad (\mathcal{P}_{p})$$

where

$$\mathcal{K}_n = \{ v \in W^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leqslant v \leqslant \varphi_{2,n} \text{ in } \Omega_\alpha^n \}$$

$$\mathcal{K} = \{ v \in W^{1,p}(\Omega_\alpha) : \varphi_1 \leqslant v \leqslant \varphi_2 \text{ in } \Omega_\alpha \}$$

and $\varphi_{1,n}$, $\varphi_{2,n}$, φ_1, φ_2 the obstacles.
We study the asymptotic behaviour both for $p \to \infty$ and for $n \to \infty$ in order to give an answer to these questions:
· Keeping p fixed, do the solutions to Problems $(\mathcal{P}_{p,n})$ converge to the solution to Problem $(\mathcal{P}_p)$?

· Keeping n fixed, do the solutions to Problems $(\mathcal{P}_p)$ converge to the solution to some Problem $(\mathcal{P}_\infty)$?
· What is this problem?
· If the previous questions have positive answers, passing first to the limit in n and then to the limit in p can we expect to get the solution that would come out passing first to the limit in p and then to the limit in n? What can we say about uniqueness?

We want to point out that in this chapter we have considered also problems posed in $W^{1,p}(\Omega_\alpha)$ and $W^{1,p}(\Omega_\alpha^n)$ (following the approach in [56]), i.e. without Dirichlet conditions, and the corresponding limit problems, as $p \to \infty$, which are related to an optimal mass transport problem (see [56]). In particular, as we see in Section 4.3, in general these problems do not have unique solution. We state theorems which provide uniqueness under suitable assumptions.

4.1 Analysis for $p \longrightarrow \infty$

4.1.1 Asymptotic in the fractal case

Let Ω_α be the domain defined in Chapter 1 and let f be in $L^1(\Omega_\alpha)$.
We consider the following two obstacle problem in the domain Ω_α :

$$\text{find } u \in \mathcal{K}, \quad a_p(u, v-u) - \int_{\Omega_\alpha} f(v-u)\,\mathrm{d}x \geqslant 0 \;\; \forall v \in \mathcal{K}, \tag{4.1.1}$$

with

$$a_p(u,v) = \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla v\,\mathrm{d}x\,, \text{ with } p \in (2,\infty) \tag{4.1.2}$$

and

$$\mathcal{K} = \{v \in W^{1,p}(\Omega_\alpha) : \varphi_1 \leqslant v \leqslant \varphi_2 \text{ in } \Omega_\alpha\,\}.$$

We require that the obstacles satisfy the following assumption

$$\begin{cases} \varphi_i \in C(\bar{\Omega}_\alpha),\;\; i = 1,2, \\ \varphi_1(x) - \varphi_2(y) \leq |x-y|, \forall x, y \in \overline{\Omega}_\alpha. \end{cases} \tag{4.1.3}$$

This condition, trivially, implies that $\varphi_1(x) \leq \varphi_2(x), \forall x \in \overline{\Omega}_\alpha$. Moreover, we note the if we consider φ_2 satisfying Lipschitz condition with constant equal to 1, then condition (4.1.3) will hold for any continuous function $\varphi_1(x)$ such that $\varphi_1(x) \leq \varphi_2(x), \forall x \in \overline{\Omega}_\alpha$.

We observe that with this condition the interpretation of the limit problem as mass transport problem with courier makes sense. Indeed, when the cost due to the courier $(\varphi_1(z) - \varphi_2(w))$ is less or equal to the (euclidean) cost $|z - w|$, it is convenient to use the courier instead of taking the mass ourselves (see [56]).
The following couple of obstacle provides an one-dimensional example in which condition (4.1.3) is not satisfied

$$\varphi_1(x) = -2 \text{ and } \varphi_2(y) = 2 - y^2 \text{ in } \Omega = (-2, 2).$$

The following couple of obstacle provides an one-dimensional example in which condition (4.1.3) is satisfied

$$\varphi_1(x) = \frac{x^2}{2} \text{ and } \varphi_2(y) = \frac{y^2}{2} \text{ in } \Omega = (-1, 1).$$

Proposition 4.1.1. *Condition (4.1.3) assure that* $\mathcal{K} \neq \emptyset$ *and*

$$\mathcal{K}^\infty = \{u \in W^{1,\infty}(\Omega_\alpha) : \varphi_1 \leq u \leq \varphi_2 \text{ in } \Omega_\alpha, ||\nabla u||_{L^\infty(\Omega_\alpha)} \leq 1\} \neq \emptyset.$$

Proof. Let us define and consider the following function

$$w(x) = \max_{y \in \bar{\Omega}_\alpha} \{\varphi_1(y) - |x - y|\};$$

our aim is to show that $w \in \mathcal{K}^\infty$.
We show that

$$|w(x) - w(y)| \leq |x - y|, \forall x, y \in \bar{\Omega}_\alpha, \text{i. e. } w \in Lip(\overline{\Omega}) \text{ with constant } L = 1 \tag{4.1.4}$$

(this condition implies that $w \in W^{1,p}(\Omega_\alpha)$).
Indeed, thanks to the properties of modulus, we have

$$|x - z| \leq |x - y| + |y - z| \Longrightarrow -|x - z| \geq -|x - y| - |y - z|.$$

Adding $\varphi_1(z)$ in both sides and passing to the maximum, we obtain

$$\max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |x - z|\} \geq \max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |x - y| - |y - z|\},$$

i.e.

$$\max_{z\in\bar{\Omega}_\alpha}\{\varphi_1(z)-|x-z|\}\geq \max_{z\in\bar{\Omega}_\alpha}\{\varphi_1(z)-|y-z|\}-|x-y|,$$

then

$$w(x)\geq w(y)-|x-y|.$$

Analogously, exchanging the role of x and y, we have

$$w(x)-w(y)\leq |x-y|,$$

and then (4.1.4). We stress the fact that the bound of the gradient correspond to this property.

Clearly, choosing $y=x$, we obtain

$$w(x)=\max_{y\in\bar{\Omega}_\alpha}\{\varphi_1(y)-|x-y|\}\geq\varphi_1(x), \qquad \forall x\in\overline{\Omega}_\alpha.$$

Finally, from (4.1.3) we have

$$\varphi_1(y)-|x-y|\leq\varphi_2(x), \forall x,y\in\overline{\Omega}_\alpha$$

and taking the maximum of the left-hand side we have that

$$w(x)\leq\varphi_2(x), \forall x\in\overline{\Omega}_\alpha.$$

□

As we have observed in Chapter 2, Problems (4.1.1) is equivalent to the following minimum problem

$$\min_{u\in\mathcal{K}} J_p(u), \tag{4.1.5}$$

with

$$J_p(v)=\frac{1}{p}a_p(v,v)-\int_{\Omega_\alpha} fv\,\mathrm{d}x. \tag{4.1.6}$$

Moreover, like we have observed in Remark (2.1.1), it has a minimizer u_p in $\mathcal{K}$. In general, this minimizer is not unique. We will discuss about uniqueness in the following.
Now, we perform the asymptotic analysis in the fractal case for $p\longrightarrow\infty$ as in [56] (see also [4]).

Theorem 4.1.1. *Let $f \in L^1(\Omega_\alpha)$. Assume that φ_1 and φ_2 verify (4.1.3). Then, for any $p \in (2, \infty)$, a minimizer u_p of Problem (4.1.5) exists. The family of the minimizers $\{u_p\}_p$ is pre-compact in $C(\bar{\Omega}_\alpha)$. In particular, for any sequence $p_k \to \infty$ there is a subsequence p_{kj} such that $u_{p_{kj}} \longrightarrow u_\infty$ weakly in $W^{1,m}(\Omega_\alpha)$ ($\forall m > 2$), u_∞ being a maximizer of the following variational problem*

$$\int_{\Omega_\alpha} u_\infty(x) f(x)\,\mathrm{d}x = \max\left\{\int_{\Omega_\alpha} w(x) f(x)\,\mathrm{d}x : w \in \mathcal{K}^\infty\right\}, \tag{4.1.7}$$

The following proof of this theorem is structured as follow. First, we will show the existence of the minimizers u_p and the non-emptiness of $\mathcal{K}^\infty$. Then, we will prove the pre-compactness of $\{u_p\}_p$ (showing that all the assumption of Ascoli-Arzelà compactness criterion are satisfied). Eventually, we will prove that the limit function u_∞ belongs to $\mathcal{K}^\infty$ and satisfies Problem (4.1.7).

Proof. Assuming that φ_1 and φ_2 verify (4.1.3), we have that $\mathcal{K} \neq \emptyset$, then (see Remark 2.1.1) a minimizer u_p exists. Furthermore, assuming (4.1.3) we also have $\mathcal{K}^\infty \neq \emptyset$ (see Proposition 4.1.1).

For any function $w \in \mathcal{K}^\infty$, we have

$$-\int_{\Omega_\alpha} f u_p\,\mathrm{d}x \le \frac{1}{p}\int_{\Omega_\alpha} |\nabla u_p|^p\,\mathrm{d}x - \int_{\Omega_\alpha} f u_p\,\mathrm{d}x \le$$

$$\le \frac{1}{p}\int_{\Omega_\alpha} |\nabla w|^p\,\mathrm{d}x - \int_{\Omega_\alpha} f w\,\mathrm{d}x \le \frac{|\Omega_\alpha|}{p} - \int_{\Omega_\alpha} f w\,\mathrm{d}x \tag{4.1.8}$$

(where $|\cdot| = \mathrm{meas}(\cdot)$).

Moreover, by the fact that $u_p \in \mathcal{K}$, we have, naturally,

$$\min_{\bar{\Omega}_\alpha} \varphi_1 \le \varphi_1 \le u_p \le \varphi_2 \le \max_{\bar{\Omega}_\alpha} \varphi_2,$$

then

$$||u_p||_{L^\infty(\Omega_\alpha)} \le C, \tag{4.1.9}$$

with C independent from p.

Now, from (4.1.8), applying Hölder's inequality and taking into account (4.1.9), we get

$$\frac{1}{p}\int_{\Omega_\alpha} |\nabla u_p|^p\,\mathrm{d}x \le \frac{|\Omega_\alpha|}{p} + \int_{\Omega_\alpha} f u_p\,\mathrm{d}x - \int_{\Omega_\alpha} f w\,\mathrm{d}x \le$$

$$\le \frac{|\Omega_\alpha|}{p} + ||f||_{L^1(\Omega_\alpha)} ||u_p||_{L^\infty(\Omega_\alpha)} - \int_{\Omega_\alpha} fw \, \mathrm{d}x \le$$

$$\le \frac{|\Omega_\alpha|}{p} + ||f||_{L^1(\Omega_\alpha)} ||u_p||_{L^\infty(\Omega_\alpha)} - \int_{\Omega_\alpha} f^+ \varphi_1 \, \mathrm{d}x + \int_{\Omega_\alpha} f^- \varphi_2 \, \mathrm{d}x \le C,$$

from where we have

$$||\nabla u_p||^p_{L^p(\Omega_\alpha)} \le pC, \forall p > 2, \tag{4.1.10}$$

with C independent from p.

From (4.1.9) and (4.1.10), we deduce that $\{u_p\}_{p>2}$ is bounded in $W^{1,p}(\Omega_\alpha)$.

In particular (4.1.10), we have that

$$||\nabla u_p||_{L^p(\Omega_\alpha)} \le (pC)^{\frac{1}{p}}.$$

So, considering $m_0 > 2$ and $p > m_0$, we have

$$||\nabla u_p||^{m_0}_{L^{m_0}(\Omega_\alpha)} \le ||\nabla u_p||^{m_0}_{L^p(\Omega_\alpha)} |\Omega_\alpha|^{1-\frac{m_0}{p}} \le (pC)^{\frac{m_0}{p}} |\Omega_\alpha|^{1-\frac{m_0}{p}}$$

Hence

$$\limsup_{p\to\infty} ||\nabla u_p||_{L^{m_0}(\Omega_\alpha)} \le |\Omega_\alpha|^{\frac{1}{m_0}}. \tag{4.1.11}$$

Thanks to (4.1.11), by Theorem 1.1.2, we have

$$|u_p(x) - u_p(y)| \le C^*(m_0) |x-y|^{1-\frac{2}{m_0}}. \tag{4.1.12}$$

Thanks to the Ascoli-Arzelà compactness criterion, by using (4.1.9) and (4.1.12), we can extract a subsequence of the previous one, that we indicate again with $\{u_p\}$, such that, for $p \longrightarrow \infty$, we obtain

$$u_p \longrightarrow u_\infty \text{ uniformly in } \bar{\Omega}_\alpha. \tag{4.1.13}$$

Furthermore, thanks to (4.1.9) and (4.1.11), for all $m > 2$, we have

$$||u_p||_{W^{1,m}(\Omega_\alpha)} \le C$$

(with C independent from p).

Hence, there exists a subsequence that we denote with u_{p_k}, such that, for $k \longrightarrow \infty$,

$$u_{p_k} \longrightarrow u_m \text{ weakly in } W^{1,m}(\Omega_\alpha).$$

So, by (4.1.13), we deduce that $u_m = u_\infty$ and then the whole sequence $\{u_p\}$ converges to u_∞. Then

$$||\nabla u_\infty||_{L^m(\Omega_\alpha)} \leq \liminf_{p\to\infty} ||\nabla u_p||_{L^m(\Omega_\alpha)},$$

hence, passing to the limit as $m \to \infty$ (see Chapter 6 in [66]), we have

$$||\nabla u_\infty||_{L^\infty(\Omega_\alpha)} = \lim_{m\to\infty} ||\nabla u_\infty||_{L^m(\Omega_\alpha)} \leq \lim_{m\to\infty} \liminf_{p\to\infty} ||\nabla u_p||_{L^m(\Omega_\alpha)}.$$

Thus, since

$$\liminf_{p\to\infty} ||\nabla u_p||_{L^m(\Omega_\alpha)} \leq \liminf_{p\to\infty} (pC)^{\frac{1}{p}} |\Omega_\alpha|^{\frac{1}{m}-\frac{1}{p}},$$

namely

$$\liminf_{p\to\infty} ||\nabla u_p||_{L^m(\Omega_\alpha)} \leq |\Omega_\alpha|^{\frac{1}{m}},$$

we obtain

$$\lim_{m\to\infty} ||\nabla u_\infty||_{L^m(\Omega_\alpha)} \leq \lim_{m\to\infty} |\Omega_\alpha|^{\frac{1}{m}} = 1,$$

then we have that $u_\infty \in \mathcal{K}^\infty$.

Now, passing to the limit in (4.1.8), we have

$$\lim_{p\to\infty} \left(- \int_{\Omega_\alpha} f u_p \, \mathrm{d}x \right) \leq \lim_{p\to\infty} \left(\frac{|\Omega_\alpha|}{p} - \int_{\Omega_\alpha} f w \, \mathrm{d}x \right),$$

so

$$- \int_{\Omega_\alpha} f u_\infty \, \mathrm{d}x \leq - \int_{\Omega_\alpha} f w \, \mathrm{d}x,$$

that is

$$\int_{\Omega_\alpha} f u_\infty \, \mathrm{d}x \geq \int_{\Omega_\alpha} f w \, \mathrm{d}x.$$

Hence, finally, we obtain

$$\int_{\Omega_\alpha} u_\infty(x) f(x) \, \mathrm{d}x = \max \left\{ \int_{\Omega_\alpha} w(x) f(x) \, \mathrm{d}x : w \in \mathcal{K}^\infty \right\},$$

which was our thesis. □

4.1.2 Asymptotic in the pre-fractal case

Let us consider the analogous two obstacle problems in the pre-fractal approximating domains Ω_α^n :

$$\text{find } u \in \mathcal{K}_n, \quad a_{p,n}(u, v-u) - \int_{\Omega_\alpha^n} f(v-u)\,\mathrm{d}x \geqslant 0 \ \ \forall v \in \mathcal{K}_n, \tag{4.1.14}$$

with

$$a_{p,n}(u,v) = \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla v \,\mathrm{d}x$$

and

$$\mathcal{K}_n = \{v \in W^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leqslant v \leqslant \varphi_{2,n} \text{ in } \Omega_\alpha^n\},$$

We require that the obstacles satisfy the following assumption

$$\begin{cases} \varphi_{i,n} \in C(\bar{\Omega}_\alpha^n), \ i = 1, 2, \\ \varphi_{1,n}(x) - \varphi_{2,n}(y) \leq |x-y|, \forall x, y \in \bar{\Omega}_\alpha^n. \end{cases} \tag{4.1.15}$$

As in the fractal case, this assumption implies that $\mathcal{K}_n \neq \emptyset$. So, with the same arguments of Remark 2.1.1, the variational problem

$$\min_{v \in \mathcal{K}_n} J_{p,n}(v), \tag{4.1.16}$$

with

$$J_{p,n}(v) = \frac{1}{p} a_{p,n}(v,v) - \int_{\Omega_\alpha^n} fv \,\mathrm{d}x, \tag{4.1.17}$$

has a minimizer $u_{p,n}$ in $\mathcal{K}_n$, namely

$$J_{p,n}(u_{p,n}) = \min_{v \in \mathcal{K}_n} J_{p,n}(v). \tag{4.1.18}$$

Even in this case, an analogous theorem to the one of the case of fractal domain holds; the proof is the same.

Theorem 4.1.2. *Let $f \in L^1(\Omega_\alpha^n)$. Assume that $\varphi_{1,n}$ and $\varphi_{2,n}$ verify (4.1.15). Then, a mimimizer $u_{p,n}$ of Problem (4.1.16) exists. The family of the minimizers $\{u_{p,n}\}_p$ is pre-compact in $C(\bar{\Omega}_\alpha^n)$. In particular, for any sequence $p_k \to \infty$ there is a subsequence p_{k_j} such that $u_{p_{k_j},n} \longrightarrow u_{\infty,n}$ weakly in $W^{1,m}(\Omega_\alpha^n), \forall m > 2$, $u_{\infty,n}$ being a maximizer of the following varia-*

tional problem

$$\int_{\Omega_\alpha^n} u_{\infty,n}(x)f(x)\,\mathrm{d}x = \max\left\{\int_{\Omega_\alpha^n} w(x)f(x)\,\mathrm{d}x : w \in \mathcal{K}_n^\infty\right\}, \tag{4.1.19}$$

where $\mathcal{K}_n^\infty = \{u \in W^{1,\infty}(\Omega_\alpha^n) : \varphi_{1,n} \le u \le \varphi_{2,n} \text{ in } \Omega_\alpha^n, ||\nabla u||_{L^\infty(\Omega_\alpha^n)} \le 1\}$.

Remark 4.1.1. *To state Theorem 4.1.1, and the analogous Theorem 4.1.2 for the pre-fractal case, we have made and used the assumptions*

$$\varphi_1(x) - \varphi_2(y) \le |x-y|, \forall x, y \in \bar{\Omega}_\alpha$$

and

$$\varphi_{1,n}(x) - \varphi_{2,n}(y) \le |x-y|, \forall x, y \in \bar{\Omega}_\alpha^n$$

in order to have that the sets $\mathcal{K}$ *and* $\mathcal{K}_n$ *are not empty.*
Actually, if we require directly that

$$\mathcal{K}_0^\infty = \{v \in W_0^{1,\infty}(\Omega_\alpha) : \varphi_1 \le v \le \varphi_2 \text{ in } \Omega_\alpha,\ ||\nabla v||_{L^\infty(\Omega_\alpha)} \le 1\} \neq \emptyset \tag{4.1.20}$$

and

$$\mathcal{K}_{0,n}^\infty = \{v \in W_0^{1,\infty}(\Omega_\alpha^n) : \varphi_{1,n} \le v \le \varphi_{2,n} \text{ in } \Omega_\alpha^n,\ ||\nabla v||_{L^\infty(\Omega_\alpha^n)} \le 1\} \neq \emptyset, \tag{4.1.21}$$

then the previous assumptions can be removed and the proofs are still valid; so Theorems 4.1.1 and 4.1.2 still hold.

4.2 Analysis for $n \longrightarrow \infty$

In this section we keep p fixed and send n to infinity.

4.2.1 Asymptotic for $p \in (2, \infty)$ fixed

Thanks to Theorem 1.5.2 we know that there exists a bounded linear extension operator

$$Ext : W^{1,p}(\Omega_\alpha^n) \to W^{1,p}(\mathbb{R}^2),$$

whose norm is independent from n, that is,

$$||Ext\, v_n||_{W^{1,p}(\mathbb{R}^2)} \leqslant C||v_n||_{W^{1,p}(\Omega_\alpha^n)}, \tag{4.2.22}$$

with C independent from n.
We set

$$\hat{u}_{p,n} = (Ext\, u_{p,n})|_{\Omega_\alpha}, \tag{4.2.23}$$

with $u_{p,n}$ solution to Problem (4.1.14), that is the restriction to Ω_α of the extension $\hat{u}_{p,n} = (Ext\, u_{p,n})$.

Theorem 4.2.1. *Let $f \in L^1(\Omega_\alpha)$,*

$$\begin{cases} \varphi_i \in W^{1,p}(\Omega_\alpha), \ i = 1, 2, \\ \varphi_1 \leqslant \varphi_2 \ in \ \Omega_\alpha \end{cases} \tag{4.2.24}$$

$$\begin{cases} \varphi_{i,n} \in W^{1,p}(\Omega_\alpha), \ i = 1, 2, \\ \varphi_{1,n} \leqslant \varphi_{2,n} \ in \ \Omega_\alpha^n \end{cases} \tag{4.2.25}$$

and

$$\varphi_{i,n} \to \varphi_i \ in \ W^{1,p}(\Omega_\alpha), \ i = 1, 2. \tag{4.2.26}$$

Then, there exists a subsequence of functions $\hat{u}_{p,n}$ defined in (4.2.23) such that $\hat{u}_{p,n}$ strongly converges as $n \to \infty$ in $W^{1,p}(\Omega_\alpha)$ to a solution to Problem (4.1.1).

Proof. First of all, we observe that Problem (4.1.14) admits a solution because condition (4.2.25) guarantees that the convex $\mathcal{K}_n$ is non-empty. Analogously, Problem (4.1.1) admits a solution as condition (4.2.24) assures that the convex $\mathcal{K}$ is non-empty.
Following the same steps of Theorem 4.1.1, we obtain that

$$||u_{p,n}||_{L^\infty(\Omega_\alpha^n)} \leq C, \tag{4.2.27}$$

and

$$||\nabla u_{p,n}||^p_{L^p(\Omega_\alpha^n)} \leq pC, \forall p > 2, \tag{4.2.28}$$

with C independent from n.

By (4.2.27) and (4.2.28), we have that $\{u_{p,n}\}_{p>2}$ is bounded in $W^{1,p}(\Omega^n_\alpha)$.
As

$$\hat{u}_{p,n} = (Ext\, u_{p,n})|_{\Omega_\alpha},$$

with $u_{p,n}$ solution of Problem (4.1.14), we have

$$||\hat{u}_{p,n}||_{W^{1,p}(\mathbb{R}^2)} \leqslant C||u_{p,n}||_{W^{1,p}(\Omega^n_\alpha)}, \tag{4.2.29}$$

with C independent from n.
Hence there exists $\hat{u} \in W^{1,p}(\Omega_\alpha)$ and a subsequence of $\hat{u}_{p,n}$, denoted by $\hat{u}_{p,n}$ again, weakly converging to $\hat{u}$ in $W^{1,p}(\Omega_\alpha)$.
Remembering that solutions $u_{p,n}$ to Problems (4.1.14) realize the minimum on $\mathcal{K}_n$ of the functional $J_{p,n}(\cdot)$ (see (4.1.17)), we will show that

$$J_p(\hat{u}) = \min_{v\in\mathcal{K}} J_p(v), \tag{4.2.30}$$

where $J_p(\cdot)$ is the functional defined in (4.1.6).
In fact, since $\hat{u}_{p,n}$ weakly converges to $\hat{u}$ in $W^{1,p}(\Omega_\alpha)$ (because we have outward curves), for all fixed $m \in \mathbb{N}$, we obtain that

$$\liminf_{n\to\infty} \int_{\Omega^n_\alpha} |\nabla u_{p,n}|^p \,\mathrm{d}x \geq \liminf_{n\to\infty} \int_{\Omega^m_\alpha} |\nabla \hat{u}_{p,n}|^p \,\mathrm{d}x \geq \int_{\Omega^m_\alpha} |\nabla \hat{u}|^p \,\mathrm{d}x. \tag{4.2.31}$$

Then, passing to the limit for $m \to \infty$, we have

$$J_p(\hat{u}) \leq \liminf_{n\to\infty} J_{p,n}(u_{p,n}) \leq \liminf_{n\to\infty} \min_{v\in\mathcal{K}_n} J_{p,n}(v). \tag{4.2.32}$$

Furthermore, given u_p solution to Problem (4.1.1), we are able to construct a sequence of functions $w_n \in \mathcal{K}_n$ strongly converging to u_p in $W^{1,p}(\Omega_\alpha)$ by putting

$$w_n = \varphi_{2,n} \wedge (u_p \vee \varphi_{1,n})$$

In particular, we have

$$w_n = u_p + (\varphi_{1,n} - u_p)^+ - (u_p + (\varphi_{1,n} - u_p)^+ - \varphi_{2,n})^+$$

and

$$\liminf_{n\to\infty} \min_{v\in\mathcal{K}_n} J_{p,n}(v) \le \liminf_{n\to\infty} J_{p,n}(w_n) = J_p(u_p). \tag{4.2.33}$$

(see Theorem 1.56 in [73]). Then, from (4.2.32) and (4.2.33), we obtain (4.2.30).

Now, we want to prove that we have strong convergence in $W^{1,p}(\Omega_\alpha)$.

Using the following estimate (see, for instance, Lemma 2.1 in [5])

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} \ge c_1|\xi - \eta|^p \qquad c_1 > 0, \tag{4.2.34}$$

and choosing $v_n = (\hat{u} \vee \varphi_{1,n}) \wedge \varphi_{2,n}$ (clearly $v_n \in \mathcal{K}_n$ and $v_n \to \hat{u}$ in $W^{1,p}(\Omega_\alpha)$, see Theorem 1.56 in [73]), $\forall m \in \mathbb{N}$, $n \ge m$, we obtain that

$$c_1 \int_{\Omega_\alpha^m} |\nabla(\hat{u}_{p,n} - \hat{u})|^p \,\mathrm{d}x \le c_1 \int_{\Omega_\alpha^n} |\nabla(\hat{u}_{p,n} - \hat{u})|^p \,\mathrm{d}x \le$$

$$\le \int_{\Omega_\alpha^n} \{|\nabla u_{p,n}|^{p-2}\nabla u_{p,n} - |\nabla\hat{u}|^{p-2}\nabla\hat{u}\}\nabla(u_{p,n} - \hat{u}) \,\mathrm{d}x =$$

$$= a_{p,n}(u_{p,n}, u_{p,n} - v_n) + a_{p,n}(u_{p,n}, v_n - \hat{u}) + \int_{\Omega_\alpha^n} |\nabla\hat{u}|^{p-2}\nabla\hat{u}\nabla(\hat{u} - u_{p,n}) \,\mathrm{d}x \le$$

$$\le \int_{\Omega_\alpha^n} f(u_{p,n} - v_n) \,\mathrm{d}x + a_{p,n}(u_{p,n}, v_n - \hat{u}) + \int_{\Omega_\alpha^n} |\nabla\hat{u}|^{p-2}\nabla\hat{u}\nabla(\hat{u} - u_{p,n}) \,\mathrm{d}x.$$

From this chain of inequalities, considering the first and the last member and passing to lim sup as $n \to \infty$, we have

$$c_1 \limsup_{n\to\infty} \int_{\Omega_\alpha^m} |\nabla(\hat{u}_{p,n} - \hat{u})|^p \,\mathrm{d}x \le \limsup_{n\to\infty} \left\{ \int_{\Omega_\alpha^n} f(u_{p,n} - v_n) \,\mathrm{d}x + a_{p,n}(u_{p,n}, v_n - \hat{u}) \right\} +$$

$$+ \lim_{n\to\infty} \left\{ \int_{\Omega_\alpha^n} |\nabla\hat{u}|^{p-2}\nabla\hat{u}\nabla(\hat{u} - u_{p,n}) \,\mathrm{d}x \right\} = 0.$$

In fact:

(i) the third term goes to zero because $\hat{u}_{p,n}$ weakly convergence to $\hat{u}$ in $W^{1,p}(\Omega_\alpha)$;

(ii) the first term tends to zero, since

$$\int_{\Omega_\alpha^n} f(u_{p,n} - v_n) \,\mathrm{d}x = \int_{\Omega_\alpha} f(\hat{u}_{p,n} - v_n) \,\mathrm{d}x - \int_{\Omega_\alpha \setminus \Omega_\alpha^n} f(\hat{u}_{p,n} - v_n) \,\mathrm{d}x \longrightarrow 0,$$

as $n \to \infty$, $\hat{u}_{p,n}$ and v_n converge to $\hat{u}$ in $C(\overline{\Omega}_\alpha)$ and $|\Omega_\alpha \setminus \Omega_\alpha^n| \to 0$;

(iii) the second term goes to zero since v_n strongly converges to $\hat{u}$ in $W^{1,p}(\Omega_\alpha)$.

Finally, passing to the limit for $m \to \infty$, we conclude our proof. $\square$

Remark 4.2.1. *We observe that an analogous of Theorem 4.2.1, can be stated even in the case in which solutions vanish on the boundary. We refer to Proposition 3.3 in [22], see also Section 4.3*

4.2.2 Asymptotic for $p = \infty$

Finally, we perform the asymptotic analysis for $p = \infty$ and $n \to \infty$.
Let $u_{\infty,n}$ be a maximizer of Problem (4.1.19).
In order to obtain a extension preserving the same properties, we put

$$\tilde{u}_{\infty,n}(x) = \max_{y \in \overline{\Omega}_\alpha^n} \{u_{\infty,n}(y) - |x - y|\}, \tag{4.2.35}$$

for any $x \in \overline{\Omega}_\alpha$ (see [57]). So, we have

$$||\nabla \tilde{u}_{\infty,n}||_{L^\infty(\Omega_\alpha)} \leq 1. \tag{4.2.36}$$

Theorem 4.2.2. *Let* $f \in L^1(\Omega_\alpha)$. *Suppose that* (4.1.3) *holds,*

$$\begin{cases} \varphi_{i,n} \in C(\overline{\Omega}_\alpha), \;\; i = 1, 2, \\ \varphi_{1,n}(x) - \varphi_{2,n}(y) \leq |x - y|, \forall x, y \in \overline{\Omega}_\alpha, \end{cases} \tag{4.2.37}$$

and

$$\varphi_{i,n} \longrightarrow \varphi_i \qquad \textit{in } C(\overline{\Omega}_\alpha), \;\; \textit{for } \; i = 1, 2. \tag{4.2.38}$$

Then, there exists a subsequence of functions $\tilde{u}_{\infty,n}$ defined in (4.2.35) such that $\tilde{u}_{\infty,n}$ $\star$-weakly converges as $n \to \infty$ in $W^{1,\infty}(\Omega_\alpha)$ to a maximizer u_∞ of Problem (4.1.7).

Proof. Let $u_{\infty,n}$ be a maximizer of Problem (4.1.19). Thanks to (4.2.36), (4.2.37), and (4.2.38) we have that there exists $\tilde{v} \in W^{1,\infty}(\Omega_\alpha)$ and a subsequence of $\tilde{u}_{\infty,n}$, that we denote with $\tilde{u}_{\infty,n}$ too, which $\star$-weakly converges to $\tilde{v}$ in $W^{1,\infty}(\Omega_\alpha)$. Hence

$$||\nabla \tilde{v}||_{L^\infty(\Omega_\alpha)} \leq 1.$$

Thus, for any $w \in \mathcal{K}^\infty$, we can construct a function $w_n \in \mathcal{K}_n^\infty$ such that

$$\lim_{n\to\infty} \int_{\Omega_\alpha^n} f w_n \,\mathrm{d}x = \int_{\Omega_\alpha} f w \,\mathrm{d}x.$$

First of all, we set

$$\varphi_{1,n}^*(x) = \max_{y\in\overline{\Omega}_\alpha} \{\varphi_{1,n}(y) - |x-y|\}, \tag{4.2.39}$$

$$\varphi_{2,n}^*(x) = \min_{y\in\overline{\Omega}_\alpha} \{\varphi_{2,n}(y) + |x-y|\}, \tag{4.2.40}$$

$$\varphi_1^*(x) = \max_{y\in\overline{\Omega}_\alpha} \{\varphi_1(y) - |x-y|\}, \tag{4.2.41}$$

$$\varphi_2^*(x) = \min_{y\in\overline{\Omega}_\alpha} \{\varphi_2(y) + |x-y|\}, \tag{4.2.42}$$

for any $x \in \overline{\Omega}_\alpha$ (according to [57]).

Clearly, for any $x \in \overline{\Omega}_\alpha$, we have

$$\varphi_{1,n}(x) \le \varphi_{1,n}^*(x) \le \varphi_{2,n}^*(x) \le \varphi_{2,n}(x), \tag{4.2.43}$$

$$||\nabla\varphi_{1,n}^*||_{L^\infty(\Omega_\alpha)} \le 1, \qquad ||\nabla\varphi_{2,n}^*||_{L^\infty(\Omega_\alpha)} \le 1. \tag{4.2.44}$$

Now, for any $w \in \mathcal{K}^\infty$, we can define the function w_n setting $w_n = \varphi_{2,n}^* \wedge (w \vee \varphi_{1,n}^*)$. Thanks to (4.2.43) and (4.2.44) we deduce that $w_n \in \mathcal{K}_n^\infty$. Moreover, from the assumption (4.2.38), we obtain that $w_n \to \varphi_2^* \wedge (w \vee \varphi_1^*)$ in $L^\infty(\Omega_\alpha)$. Hence, since $w \in \mathcal{K}^\infty$, from the fact that $\varphi_1 \le w \le \varphi_2$, we have that $\varphi_1^* \le w \le \varphi_2^*$ and then $w_n \to w$ in $L^\infty(\Omega_\alpha)$. In particular, we have

$$\lim_{n\to\infty} \int_{\Omega_\alpha^n} f w_n \,\mathrm{d}x = \int_{\Omega_\alpha} f w \,\mathrm{d}x.$$

Finally, from

$$\int_{\Omega_\alpha^n} f u_{\infty,n} \,\mathrm{d}x \ge \int_{\Omega_\alpha^n} f w_n \,\mathrm{d}x,$$

passing to the limit, we have

$$\int_{\Omega_\alpha} f \tilde{v} \,\mathrm{d}x \ge \int_{\Omega_\alpha} f w \,\mathrm{d}x,$$

for any $w \in \mathcal{K}^\infty$.

This shows that $\tilde{v}$ is a maximizer of (4.1.7). □

4.3 The issue of the uniqueness

Until now, we have focused our attention only on the analysis of the asymptotic behaviour. We have discussed nothing about uniqueness.

Looking at the example 3.6 (case $0 < k < 1$) in [56] we can see that, in general, in the case $p = \infty$, there is not uniqueness of the solution to Problem (4.1.7).

In the case of an obstacle problem, for $p \in (2, \infty)$, we can obtain uniqueness results both in fractal and in pre-fractal domain making, for instance, the following assumption:

$$\int_{\Omega_\alpha} f \, \mathrm{d}x \neq 0. \tag{4.3.45}$$

In the following theorem, concerning the fractal case, we will see how condition (4.3.45) implies uniqueness of the solution for the corresponding two obstacle problem. Similarly, we can prove the same result for the pre-fractal case.

Theorem 4.3.1. *Let $a_p(u, v)$ defined as in (4.1.2). Let us assume that (4.1.3) and (4.3.45) hold. Then we have a unique solution to Problem (4.1.1).*

Proof. Since we have already shown the existence of solutions, let us focus only on the uniqueness.

Let us assume condition (4.3.45) holds. In particular, we suppose that $\int_{\Omega_\alpha} f \mathrm{d}x < 0$ (analogously we can prove the case in which $\int_{\Omega_\alpha} f \, \mathrm{d}x > 0$).

We show the uniqueness proceeding by contradiction.

Let u_1 and u_2 be two solutions to (4.1.1). Taking them, first u_1 and then u_2, as test function in (4.1.1), we obtain

$$a_p(u_1, u_1 - u_2) \leq \int_{\Omega_\alpha} f(u_1 - u_2) \, \mathrm{d}x, \tag{4.3.46}$$

$$a_p(u_2, u_2 - u_1) \leq \int_{\Omega_\alpha} f(u_2 - u_1) \, \mathrm{d}x. \tag{4.3.47}$$

Then, by (4.2.34), we have

$$c_1 ||\nabla(u_1 - u_2)||^p_{L^p(\Omega_\alpha)} \leq a_p(u_1, u_1 - u_2) - a_p(u_2, u_1 - u_2) \leq 0, \qquad c_1 > 0,$$

so

$$||\nabla(u_1 - u_2)||_{L^p(\Omega_\alpha)} = 0$$

and then

$$u_1 = u_2 + c, \qquad c \in \mathbb{R}. \tag{4.3.48}$$

Hence, by (4.3.46) and (4.3.48) we have that

$$0 = a_p(u_1, c) \leq c \int_{\Omega_\alpha} f \, \mathrm{d}x$$

and so $c \leq 0$, while by (4.3.47) and (4.3.48) we have that

$$0 = a_p(u_2, -c) \leq -c \int_{\Omega_\alpha} f \, \mathrm{d}x$$

and so $c \geq 0$.
Then we deduce that $c = 0$, hence $u_1 = u_2$.

□

Another case in which we have uniqueness is the one of homogeneous Dirichlet boundary conditions (see [22] for two obstacle problem with Dirichlet boundary condition in fractal and pre-fractal domains). In particular, as we did before, if we substitute the second request of the assumptions (4.2.37) and (4.1.3) with the requests $\mathcal{K}_{0,n}^\infty \neq \emptyset$ and $\mathcal{K}_0^\infty \neq \emptyset$, respectively, we can prove an analogous result about the uniqueness of the solution.

Theorem 4.3.2. *Let $a_p(u, v)$ defined as in (4.1.2). Let us assume $\mathcal{K}_0 = \{u \in \mathcal{K} \, : \, u|_{\partial\Omega_\alpha} = 0\} \neq \emptyset$. Then we have a unique solution to Problem (4.1.1) with $\mathcal{K}_0$ in place of $\mathcal{K}$.*

Proof. We have already proved the existence of solutions for this case. Let us focus on the uniqueness. Even here we proceed by contradiction.
Let u_1 and u_2 be two solutions to (4.1.1). Repeating the same passages of the previous proof, we find again that

$$u_1 = u_2 + c, \qquad c \in \mathbb{R}.$$

Then, having u_1 and u_2 to be zero on the boundary, we deduce that $c = 0$. □

Either in the pre-fractal case or in the fractal one, as consequences of uniqueness, we can deduce that the whole sequence $\hat{u}_{p,n}$ of Theorem 4.2.1 converges to the solution to Problem (4.1.1), as $n \to \infty$.

We summarise the results obtained showing the following scheme

$$\begin{array}{ccc} u_{p,n} & \xrightarrow{p \to \infty(\text{subseq})} & u_{\infty,n} \\ \Big\downarrow {\scriptstyle n \to \infty(\text{subseq})} & & \Big\downarrow {\scriptstyle n \to \infty(\text{subseq})} \\ & & \tilde{u}_\infty \\ u_p & \xrightarrow{p \to \infty(\text{subseq})} & \hat{u}_\infty \quad (\hat{u}_\infty = \tilde{u}_\infty ?) \end{array}$$

We can observe that, passing to the limit first for $n \to \infty$ (see Theorem 4.2.1) and then for $p \to \infty$ (see Theorem 4.1.1), the sequence $\hat{u}_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution $\hat{u}$ to Problem (4.1.7) (writing "subseq" we indicate the convergence along subsequences).
Moreover, passing to the limit first for $p \to \infty$ (see Theorem 4.1.2) and then for $n \to \infty$ (see Theorem 4.2.2), the sequence $u_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution $\tilde{u}$ to the same problem.
Clearly, the question is whether we can expect that $\tilde{u} = \hat{u}$.
Since, until now, we do not have uniqueness result, then the answer is not obvious. This makes the search for conditions which guarantee uniqueness of the solution interesting and still open (see [2], [41], [63] and the references therein).

Conclusions and future developments

The problems faced in this work have provided answers to some questions of open problems. In particular, the following results are new:

1) a regularity result for obstacle Problem (2.1.1), which take into account the second derivatives, involving suitable weighted Sobolev spaces;

2) optimal approximation error estimates, exploiting the regularity result and following the approach of P. Grisvard;

3) the convergence of the solutions to Problems $(\mathcal{P}_{p,n})$ and $(\mathcal{P}_p)$ towards the solutions to Problems (4.1.19) and (4.1.7), respectively, as $p \to \infty$; the convergence of the solutions to Problems $(\mathcal{P}_{p,n})$ and (4.1.19) towards the solutions to Problems $(\mathcal{P}_p)$ and (4.1.7), respectively, as $n \to \infty$.

4) uniqueness results for the solutions to Problems $(\mathcal{P}_{p,n})$ and $(\mathcal{P}_p)$, in the case of $p \in (2, \infty)$ fixed.

We have left some open problems. In particular, we highlight the following ones:

1) finding uniform estimates for the norm in $H^{2,\mu}(\Omega_\alpha^n)$ of the solution to Problem (3.1.1) in the case of $p > 2$ (we only have the answer in the case $p = 2$);

2) finding conditions that guarantee the uniqueness of the solution to Problems (4.1.7) and (4.1.19);

3) the numerical solutions to the corresponding numerical approximating problems of Problems $(\mathcal{P}_{\infty,n})$.

Acknowledgements

I want and have to thank many people

Prof Maria Agostina Vivaldi, for having guided me during my doctorate and my work.

Prof Raffaela Capitanelli, for her help, suggestions and the work together.

Prof Finzi and Prof Carlini, for helping me and for their availability.

My referees, Prof Gianazza and Prof Moiola, for all precious suggestions.

My family and my aunts, for their support.

My love Angelo, to be always with me.

My dear friend Christian, for his constant support in these years.

My dear friend Giuseppe C., for reading and correcting my English.

My dear friends Giuseppe and Sarah, for supporting me, always.

My dear friend Giuseppe M., for helping me in hard times.

All member of Berti-Baldacelli family, because they welcomed me like a son.

My mates of Room 24, 26 and 27.

A posteriori, the fact that I did not have a scholarship because it allowed me to learn many thing.

Thanks a lot to all my friend that are not named explicitly here, but that have supported me in every moment.

Bibliography

[1] R. A. Adams, *Sobolev spaces.* Academic Press, 1975.

[2] G. Aronsson, M. G. Crandall, P. Juutinen, *A tour of the theory of absolutely minimizing functions.* Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 4, 439–505.

[3] P. Bagnerini, P. Buffa, E. Vacca, *Mesh generation and numerical analysis of a Galerkin method for highly conductive prefractal layers.* Appl. Numer. Math. 65 (2013), 63-78.

[4] T. Bhattacharya, E. DiBenedetto, J. Manfredi, *Limits as* $p \to +\infty$ *of* $\Delta_p u_p = f$ *and related extremal problems.* Some topics in nonlinear PDEs (Turin, 1989). Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, 15–68 (1991).

[5] J. W. Barrett, W. B. Liu, *Finite element approximation of the p-Laplacian.* Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 16, pp. 1-17.

[6] J. Benedikt, P. Girg, L. Kotrla, P. Takáč, *Origin of the p-Laplacian and A. Missbach.* Math. Comp. 61 (1993), no. 204, 523-537.

[7] J. Bergh, J. Löfstrom, *Interpolation Spaces. An Introduction.* Springer-Verlag, Berlin, 1976.

[8] M. Borsuk, V. Kondratiev, *Elliptic Boundary value problems of Second Order in Piecewise Smooth Domains.* Noth-Holland Mathematical Library Eds., 2006.

[9] L. Brasco, F. Santambrogio, *A sharp estimate à la Calderón-Zygmund for the p-Laplacian.* Commun. Contemp. Math. 20 (2018), no. 3, 1750030, 24 pp.

[10] H. Brezis, *Analyse fonctionnelle. Theorie et applications.* Masson, Paris, 1983.

[11] F. Brezzi, *Finite element methods for nonlinear problems. A survey on the theoretical and numerical trends in nonlinear analysis* I (Proc. Third Sem. Funct. Anal. Appl., Bari, 1978). Confer. Sem. Mat. Univ. Bari No. 158-162 (1979), 47-68.

[12] F. Brezzi, G. Gilardi, *Fundamentals of P.D.E. for numerical analysis*, in Finite Element Handbook, (H. Kardestuncer Ed.), McGraw-Hill Book Co., New York, 1987.

[13] F. Brezzi, W. W. Hager, P. Raviart, *Error estimates for the finite element solution of variational inequalities.* II. Mixed methods. Numer. Math. 28 (1977), no. 4, 431- 443.

[14] F. Camilli, R. Capitanelli, M.A. Vivaldi, *Absolutely minimizing Lipschitz extensions and infinity harmonic functions on the Sierpinski gasket.* Nonlinear Anal. 163 (2017), 71–85.

[15] R. Capitanelli, S. Fragapane, *Asymptotics for quasilinear obstacle problems in bad domains.* Discrete Contin. Dyn. Syst. Ser. S 12 (2019), no. 1, 43–56.

[16] R. Capitanelli, S. Fragapane, M. A. Vivaldi, *Regularity results for p-Laplacians in pre-fractal domains.* Advances in Nonlinear Analysis. DOI: 10.1515/anona-2017-0248

[17] R. Capitanelli, M. A. Vivaldi, *Uniform weighted estimates on pre-fractal domains.* Discrete Contin. Dyn. Syst. Ser. B 19 (2014), no. 7, 1969-1985.

[18] R. Capitanelli, M. A. Vivaldi, *Weighted estimates on fractal domains.* Mathematika 61 (2015), no. 2, 370-384.

[19] R. Capitanelli, M. A. Vivaldi, *Reinforcement problems for variational inequalities on fractal sets.* Calc. Var. Partial Differential Equations 54, (2015), no. 3, 2751-2783.

[20] R. Capitanelli, *Asymptotics for mixed Dirichlet-Robin problems in irregular domains*, J. Math. Anal. Appl. **362** (2010), no. 2, 450–459.

[21] R. Capitanelli, M. A. Vivaldi, *Dynamical Quasi-Filling Fractal Layers.* SIAM J. Math. Anal. **48** (2016), no. 6, 3931-3961.

[22] R. Capitanelli, M. A. Vivaldi, *FEM for quasilinear obstacle problems in bad domains.* ESAIM Math. Model. Numer. Anal. 51 (2017), no. 6, 2465–2485.

[23] A. Carpinteri, N. Pugno, *Fracture instability and limit strength condition in structures with re-entrant corners.* Engineering Fracture Mechanics 72 (2005) 1254–1267.

[24] J. Céa, *Approximation variationnelle des problèmes aux limites.* Studies in Mathematics and its Applications. **4** Ann. Inst. Fourier **14** (1964).

[25] M. Cefalo, M. R. Lancia, H. Liang, *Heat-flow problems across fractal mixtures: regularity results of the solutions and numerical approximation.* Differential Integral Equations 26 (2013), no. 9-10, 1027-1054.

[26] A. Cianchi, V. Maz'ya, *Global boundedness of the gradient for a class of nonlinear elliptic systems.* Arch. Ration. Mech. Anal. 212 (2014), no. 1, 129–177.

[27] A. Cianchi, V. Maz'ya, *Second-order two-sided estimates in nonlinear elliptic problems.* Arch. Ration. Mech. Anal. 229 (2018), no. 2, 569–599.

[28] P. G . Ciarlet, *The Finite Element Method for Elliptic Problems.* Studies in Mathematics and its Applications. 4 North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[29] F. Demengel, G. Demengel, *Functional Spaces for the Theory of Elliptic Partial Differential Equations.* Springer-Verlag London, 2012.

[30] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[31] J. I. Diaz, *Nonlinear partial differential equations and free boundaries.* Vol. I. Elliptic equations. Research Notes in Mathematics. **106**. Pitman, Boston, MA, 1985.

[32] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal. 7 (1983), no. 8, 827–850.

[33] M. Dobrowolski, *On quasilinear elliptic equations in domains with conical boundary points.* J. Reine Angew. Math. 394, (1989),186–195.

[34] L. C. Evans, W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem.* Mem. Amer. Math. Soc. **137** no. 653 (1999).

[35] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions, Revised Edition.* CRC Press, 2015.

[36] R. S. Falk, *Approximation of an elliptic boundary value problem with unilateral constraints.* Rev. Française Automat. Informat. Recherche Opérationnelle RAIRO Analyse Numérique **9** (1975).

[37] D. S. Grebenkov, M. Filoche, B. Sapoval, *Mathematical basis for a general theory of Laplacian transport towards irregular interfaces.* Phys. Rev. E **73** (2006), 021103.

[38] P. Grisvard, *Elliptic problems in nonsmooth domains.* Monographs and Studies in Mathematics, 24. Pitman, Boston, MA, 1985.

[39] F. Hecht, *FreeFem++ Third Edition, Version 3.56-2.* Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris. http://www.freefem.org/ff++

[40] J. E. Hutchinson, *Fractals and selfsimilarity.* Indiana Univ. Math. J. **30** (1981) no. 5, 713-747.

[41] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient.* Arch. Rational Mech. Anal. **123** (1993), no. 1, 51–74.

[42] P.W. Jones, *Quasiconformal mapping and extendability of functions in Sobolev spaces.* Acta Math. 147 (1981) 71–88.

[43] A. Jonsson, H. Wallin, *Function spaces on subsets of* R^n, Mathematical Reports Vol. 2, Part 1, 1984.

[44] V. A. Kondratiev, *Boundary value problems for elliptic equations in domains with conical or angular points.* Tr. Mosk. Mat. Obs. 16 (1967), 209-292.

[45] T. Kuusi, G. Mingione, *Guide to nonlinear potential estimates.* Bull. Math. Sci. 4 (1) (2014), 1-82.

[46] M. R. Lancia, M. Cefalo, G. Dell'Acqua, *Numerical approximation of transmission problems across Koch-type highly conductive layers.* Appl. Math. Comput. **218** (2012), no. 9, 5453-5473.

[47] M. R. Lancia, M. Cefalo, *An optimal mesh generation algorithm for domains with Koch type boundaries.* Math. Comput. Simulation **106** (2014), 133-162.

[48] H. Lewy, G. Stampacchia, *On the smoothness of superharmonics which solve a minimum problem.* J. Analyse Math. 23, (1970), 227-236.

[49] G. Li, O. Martio, *Stability and higher integrability of derivatives of solutions in double obstacle problems.* J. Math. Anal. Appl. 272 (2002), no. 1. 19-29.

[50] G. M. Lieberman, *Regularity of solutions to some degenerate double obstacle problems.* Indiana Univ. Math. J. 40 (1991), no. 3, 1009-1028.

[51] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations.* Nonlinear analysis, T.M.A. 12, (1988), 1203–1219.

[52] W. B. Liu, J. W. Barrett, *Quasi-norm error bounds for the finite element approximation of some degenerate quasilinear elliptic equations and variational inequalities.* RAIRO – Modélisation mathématique et analyse numérique, tome 28, no 6 (1994), p. 725-744.

[53] B. B. Mandelbrot, *The Fractal Geometry of Nature.* W. H. Freeman Co, 1982.

[54] B. B. Mandelbrot, *Fractals and Scaling in Finance.* Springer, 1997.

[55] V. G. Maz'ya, S. V. Poborchi, *Differentiable functions on bad domains.* World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[56] J. M. Mazón, J. D. Rossi, J. Toledo, *Mass transport problems for the Euclidean distance obtained as limits of p-Laplacian type problems with obstacles.* Journal of Differential Equations **256** (2014), 3208-3244.

[57] E. J. McShane, *Extension of range of functions.* Bull. Amer. Math. Soc. **40** (1934), no. 12, 837–842.

[58] C. Mercuri, G. Riey, B. Sciunzi, *A regularity result for the p-Laplacian near uniform ellipticity.* SIAM J. Math. Anal. 48 (2016), no. 3, 2059–2075.

[59] U. Mosco, *Convergence of convex sets and solutions of variational inequalities.* Adv. Math. **3** (1969) 510–585.

[60] U. Mosco, *Implicit Variational Problems and Quasi Variational Inequalities.* Lecture notes in Mathematics, 543, Springer-Verlag, Bruxells (1976).

[61] U. Mosco, G. M. Troianiello, *On the smoothness of solutions of unilateral Dirichlet problem.* Boll. Un. Mat. Ital. (4) 8 (1973), 57-67.

[62] U. Mosco, M. A.Vivaldi, *Layered fractal fibers and potentials.* J. Math. Pures Appl. (9) **103** (2015), no. 5, 1198-1227.

[63] Y. Peres, O. Schramm, S. Sheffield, D. B. Wilson, *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc. **22** (2009), no. 1, 167–210.

[64] A. Quarteroni, A. Valli, *Numerical Approximation of Partial Differential Equations.* Springer-Verlag Berlin Heidelberg 1994.

[65] G. Raugel, *Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone.* C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 18, A791–A794.

[66] H. L. Royden, *Real Analysis.* Second edition. Macmillan Publishing Company, New York, 1968.

[67] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains.* J. Funct. Anal. 152 (1998), no. 1, 176-201.

[68] V. A. Solonnikov, M. A.Vivaldi, *Mixed type, nonlinear systems in polygonal domains.* Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 24 (2013), no. 1, 39-81.

[69] E. Tadmor, *A Review of Numerical Methods for Nonlinear Partial Differential Equation.* BULLETIN (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY Volume 49, Number 4, October 2012, Pages 507–554.

[70] P. Tolksdorf, *On the Dirichlet problem for quasilinear elliptic equations in domains with conical boundary points.* Comm. in Partial Differential Equations 8, (1983), no. 7, 773–817.

[71] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations.* J. Differential Equations 51 (1984), no. 1, 126–150.

[72] H. Triebel, *Theory of Function Spaces II.* Monographs in Mathematics Vol.84, Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[73] G. M. Troianiello, *Elliptic Differential Equations and Obstacle Problems.* The University Series in Mathematics. Plenum Press, New York, 1987.

[74] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems.* Acta. Math. 138 (1977), 219-240.

[75] E. Vacca, *Galerkin approximation for highly conductive layers.* Ph.D. thesis Dipartimento di Metodi e Modelli Matematici per l'Ingegneria, Univ. degli Studi di Roma "La Sapienza", 2005.

[76] C. Villani, *Optimal transport. Old and new.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338. Springer-Verlag, Berlin, 2009.

[77] R. Wasyk, *Numerical solution of a transmission problem with a prefractal interface.* Ph.D. thesis Department of Mathematical Sciences, Worcester Polytechinique Institute, 2007,