Metric representations of a preference ordering

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Abstract

We prove that when we decompose the expected utility function inside of an m-dimensional metric space we refer to a preference ordering based on the notion of distance. We prove that when we deal with a scale of measurable utilities we refer to a preference ordering based on the notion of distance. A contingent consumption plan is studied inside of an m-dimensional metric space because utility and probability are both subjective. The right closed structure in order to deal with utility and probability is a metric space in which we study coherent decisions under uncertainty having as their goal the maximization of the prevision of the utility associated with a contingent consumption plan.

Keywords: collinearity, monetary risk, expected utility function, random process, direct and orthogonal sum, contingent consumption plan.

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1 Introduction

When we speak about a univariate random quantity denoted by X we mean that it admits two or more than two possible values. They are real numbers. One and only one of them will be true a posteriori. Every possible value of X is connected with a single random event contained in X. We consider incompatible and exhaustive random events whose number is finite ((Nunke and Savage, 1952)). The set of all possible values of X is denoted by

$$\{x^1, x^2, \ldots, x^m\}, \tag{1.1}$$

where we consider $x^1 < x^2 < \ldots < x^m$ without loss of generality. These possible values are monetary values within this context, so X is a random gain meant in an algebraic sense. This means that a loss is a negative gain. It follows that all possible values of X or some of them could be negative. We consider the relationship "final wealth = initial wealth + return of investment", so all possible values of X represent different and possible outcomes of an investment. We suppose that the final wealth is fully consumed, so we also identify a contingent consumption plan in this way. It specifies what it will be consumed in each different outcome of the random process under consideration. We speak about a random process because we do not know a priori which is the quantity belonging to $\{x^1, x^2, \ldots, x^m\}$ that will be consumed a posteriori ((Manzini and Mariotti, 2014)). It is associated with the single random event which occurs. To be in doubt between $x^1, x^2, \ldots, x^m$ involves that it is appropriate to assign to them subjective probabilities denoted by $p_1, p_2, \ldots, p_m$ before knowing which is the true value of X occurring. We deal with a distribution of probability in this way. We denote it by

$$[(x^1, p_1), (x^2, p_2), \ldots, (x^m, p_m)], \tag{1.2}$$

where we have $x^1 < x^2 < \ldots < x^m$ as well as $p_1 + p_2 + \ldots + p_m = 1$. Every distribution of probability is always a coherent expression of the attitude of the individual under consideration with respect to uncertainty about random events ((de Finetti, 1972a)). It does not depend on one or more than one parameter like a pre-established distribution ((de Finetti, 1972b)). This means that one engages oneself in saying all what it is of interest about the specific case that is considered. Thus, one does not prefer to race on ahead occupying oneself with not real problems characterized by infinite and repeatable cases. On the other hand, it is known that the nature of probability is unitary in all fields ((Pfanzagl, 1967)). Nevertheless, information used in order to make a coherent evaluation of probability related to a given set of events can be different with respect to its external aspects. This implies that the criteria for the evaluation of probability are different ((Anscombe and Aumann, 1963)). However, they lead to an evaluation which is always subjective. This is because an equiprobable judgment is itself subjective. Such a judgment intrinsically characterizes symmetric probability. Concerning frequentist probability, it makes sense that each individual relates probability back to observed frequency only when he specifies the meaning and conditions of this thing. Symmetric probabilities as well as frequentist probabilities are only elements of judgment evaluated by each individual on the basis of his own judgment. Subjective probability results from this necessary judgment. On the other hand, it is not excluded that subjective probability coincides with symmetric probability. It is not even excluded that subjective probability coincides with frequentist probability.

2 A decomposition of the possible outcomes of a contingent consumption plan

Random events characterizing X are expressed by points in the space of X. Such a space is a linear space. In particular, they are points belonging to an m-dimensional linear space equipped with a Euclidean metric. We denote it by E^m. This is because we consider m possible values of X coinciding with x^i, $i = 1, \ldots, m$. They are different from one another because we deal with a partition of incompatible and exhaustive events ((Siniscalchi, 2009)). Given an orthonormal basis of E^m denoted by $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, we are able to consider m oriented straight lines of E^m which are measured in the same unit of length. They are pairwise orthogonal. The point where they meet is the origin of E^m. It is the zero vector of E^m ((von Neumann, 1936)). Each possible value of X coinciding with x^i, $i = 1, \ldots, m$, belongs to one of these m straight lines of E^m. We do not consider particular m-tuples of real numbers belonging to every straight line of E^m but we consider only real numbers connected with each of them. This thing results from a particular geometric property that we are going to use. Each straight line of E^m represents the whole of the space of alternatives whose number is infinite with respect to one of m alternatives of X ((Tversky and Kahneman, 1974)). Each point on a straight line of E^m corresponds to a single and possible alternative of X and vice versa. Concerning one of m alternatives of X we observe that information and knowledge of a given individual at a given instant permit him of not to excluding a real number only ((deGroot, 1962)). It remains possible for him because it is not either true or false ((Coletti, Scozzafava and Vantaggi, 2015)). Having said that, we observe that X is geometrically identified with the components of an m-dimensional vector of E^m. We write

$$\mathbf{x} = x^i \mathbf{e}_i, \tag{2.1}$$

with $\mathbf{x} \in E^m$, because we use the Einstein summation convention. It follows that we are able to consider the following

Proposition 2.1. *Let $X = \{x^1, x^2, \ldots, x^m\}$ be a contingent consumption plan, where we have $x^1 < x^2 < \ldots < x^m$ without loss of generality. If $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, is an orthonormal basis of E^m then $\mathbf{x} = x^i \mathbf{e}_i \in E^m$ is a direct and orthogonal sum of m vectors belonging to m one-dimensional subspaces of E^m.* □

Proof. We show that each contravariant component of $\mathbf{x} \in E^m$ can be viewed as an m-dimensional vector of E^m. It is denoted by ${}_{(i)}\mathbf{x}$, $i = 1, \ldots, m$. This vector and the corresponding vector of the orthonormal basis of E^m denoted by $\mathbf{e}_i$, $i = 1, \ldots, m$, are collinear. This is because there exists a real number denoted by α such that it turns out to be

$$_{(i)}\mathbf{x} = \alpha\mathbf{e}_i, \tag{2.2}$$

where we have $i = 1,\dots,m$. With regard to (2.2) we observe that α takes any value in $\mathbb{R}$. If we have $i = 1$ in (2.2) then the vector $\mathbf{e}_1$ identifies a straight line having a given direction in E^m. When α takes a value in $\mathbb{R}$ we note that α identifies an m-dimensional vector lying on the same straight line established by $\mathbf{e}_1$ in E^m. We therefore say that this m-dimensional vector and $\mathbf{e}_1$ are collinear. The same thing goes when α takes all values in $\mathbb{R}$. In particular, we observe that α always takes a value coinciding with the first contravariant component of $\mathbf{x}$. We note that only the first component of $\mathbf{x}$ in (2.1) is not equal to 0. All other components of it are equal to 0. We therefore write

$$_{(1)}\mathbf{x} = x^1\mathbf{e}_1. \tag{2.3}$$

If we have $i = 2$ in (2.2) then the vector $\mathbf{e}_2$ identifies a straight line having a given direction in E^m. This direction is orthogonal to the one of $\mathbf{e}_1$. When α takes a value in $\mathbb{R}$ we note that α identifies an m-dimensional vector lying on the same straight line established by $\mathbf{e}_2$ in E^m. We therefore say that this m-dimensional vector and $\mathbf{e}_2$ are collinear. The same thing goes when α takes all values in $\mathbb{R}$. In particular, we observe that α always takes a value coinciding with the second contravariant component of $\mathbf{x}$. We note that only the second component of $\mathbf{x}$ in (2.1) is not equal to 0. All other components of it are equal to 0. We therefore write

$$_{(2)}\mathbf{x} = x^2\mathbf{e}_2. \tag{2.4}$$

If we have $i = m$ in (2.2) then the vector $\mathbf{e}_m$ identifies a straight line having a given direction in E^m. This direction is orthogonal to the one of $\mathbf{e}_1$. It is also orthogonal to the one of $\mathbf{e}_2$ and so on until you get to $\mathbf{e}_{m-1}$. When α takes a value in $\mathbb{R}$ we note that α identifies an m-dimensional vector lying on the same straight line established by $\mathbf{e}_m$ in E^m. We therefore say that this m-dimensional vector and $\mathbf{e}_m$ are collinear. The same thing goes when α takes all values in $\mathbb{R}$. In particular, we observe that α always takes a value coinciding with the m-th contravariant component of $\mathbf{x}$. We note that only the m-th component of $\mathbf{x}$ in (2.1) is not equal to 0. All other components of it are equal to 0. We write

$$_{(m)}\mathbf{x} = x^m\mathbf{e}_m. \tag{2.5}$$

Each random event geometrically coincides with a straight line belonging to E^m. Each straight line of E^m identifies an one-dimensional subspace of E^m. The direct sum of m

one-dimensional subspaces of E^m coincides with E^m itself, so we write

$$E^m_{(1)} \oplus \ldots \oplus E^m_{(m)} = E^m, \tag{2.6}$$

where each $E^m_{(i)}$, $i = 1, \ldots, m$, denotes the i-th one-dimensional subspace of E^m. We note that this direct sum is also orthogonal. Also, it turns out to be

$$\dim E^m_{(1)} + \ldots + \dim E^m_{(m)} = \dim E^m, \tag{2.7}$$

where we have $\dim E^m = m$. After taking (2.6) into account we write

$$_{(1)}\mathbf{x} + \ldots + _{(m)}\mathbf{x} = \mathbf{x}, \tag{2.8}$$

where each $_{(i)}\mathbf{x}$ is an element of $E^m_{(i)}$, $i = 1, \ldots, m$, while $\mathbf{x}$ is an element of E^m. The contravariant components of $_{(i)}\mathbf{x}$ are given by

$$_{(i)}\mathbf{x} = {}_{(i)}x^i \delta^j_i, \tag{2.9}$$

where we have $i = 1, \ldots, m$. We note that δ^j_i denotes the Kronecker delta. If it turns out to be $i = j$ then we have $\delta^j_i = 1$. If it turns out to be $i \neq j$ then we have $\delta^j_i = 0$. We note that (2.9) is characterized by the Einstein summation convention. Thus, we are able to write

$$_{(i)}\mathbf{x} = \begin{pmatrix} _{(i)}x^1 \delta^1_1 + {}_{(i)}x^2 \delta^1_2 + \ldots + {}_{(i)}x^m \delta^1_m \\ _{(i)}x^1 \delta^2_1 + {}_{(i)}x^2 \delta^2_2 + \ldots + {}_{(i)}x^m \delta^2_m \\ \vdots \\ _{(i)}x^1 \delta^m_1 + {}_{(i)}x^2 \delta^m_2 + \ldots + {}_{(i)}x^m \delta^m_m \end{pmatrix}, \tag{2.10}$$

where we have $i = 1, \ldots, m$. □

We say that we have $x^1 < x^2 < \ldots < x^m$ without loss of generality because we could indifferently choose any ordered m-tuple of straight lines of E^m ((Mattsson and Weibull, 2002)). All these straight lines of E^m are the axes of the coordinate system under consideration. A single random event denoted by E_i, $i = 1, \ldots, m$, is a particular random quantity ((Gilio and Sanfilippo, 2014)). It admits a posteriori only two values coinciding with two different numbers, 1 and 0 ((Coletti, Petturiti and Vantaggi, 2016b)).

3 A distribution of probability embedded in a linear space provided with a metric on it

Probability is always defined inside of the domain of events ((de Finetti, 1982b)). We have realized that all random events contained in X are embedded in E^m ((de Finetti, 1980)). Probability meant as a mass is then defined inside of a metric space. The probability of an event viewed as a well-determined proposition is conceptually contained in the prevision or expected value or mathematical expectation of a random quantity ((de Finetti, 1981)). The notion of prevision of a random quantity is a unique notion ((Berti, Regazzini and Rigo, 2001)). It is called probability in the case of events. Hence, the same symbol $\mathbf{P}$ is used in order to denote both the prevision of a random quantity and the probability of an event ((Good, 1962)). Anyway, we deal with m masses denoted by $p_1, p_2, \ldots, p_m$ such that we write $p_1 + p_2 + \ldots + p_m = 1$ ((Piccinato, 1986)). They are located on m components denoted by $x^1, x^2, \ldots, x^m$ of m vectors denoted by ${}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}, \ldots, {}_{(m)}\mathbf{x}$ of E^m. We consider a distribution of probability on $\mathbb{R}$ inside of E^m in this way. This is because $x^1, x^2, \ldots, x^m$ are real numbers. We have evidently $\{x^1\} \in \mathbb{R}$, with ${}_{(1)}\mathbf{x} = x^1 \mathbf{e}_1 \in E^m_{(1)}, \ldots, \{x^m\} \in \mathbb{R}$, with ${}_{(m)}\mathbf{x} = x^m \mathbf{e}_m \in E^m_{(m)}$. After writing

$$\mathbf{w} = x^1 |E_1| \mathbf{e}_1 + x^2 |E_2| \mathbf{e}_2 + \ldots + x^m |E_m| \mathbf{e}_m, \tag{3.1}$$

with $\mathbf{w} \in E^m$, where $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, is an orthonormal basis of E^m, it turns out to be

$$X = x^1 |E_1| + x^2 |E_2| + \ldots + x^m |E_m|, \tag{3.2}$$

where we have

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases} \tag{3.3}$$

for every $i = 1, \ldots, m$. We consider m elementary events of a finite partition of incompatible and exhaustive events. They are denoted by $E_1, E_2, \ldots, E_m$. We observe that X is an identity function such that it is possible to write

$$id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}, \tag{3.4}$$

where $\mathbb{R}$ is a linear space over itself and it is of dimension 1. We say that X is a linear operator whose canonical expression coincides with (3.2). We say that X is an isometry. It follows that each single event could uniquely be identified with infinite numbers, so we could also write $\{x^1 + a, x^2 + a, \ldots, x^m + a\}$, where $a \in \mathbb{R}$ is an arbitrary constant. This means that we consider infinite translations in this way. We consider different quantities from a geometric viewpoint. They are nevertheless the same quantity from a randomness viewpoint because events and probabilities associated with them do not change. On the other hand, if two or more than two propositions can express the same event contained in X then two or more than two real numbers can identify it. Hence, we consider a different closed structure in this way. Such a structure is not a σ-algebra but it is a linear subspace over $\mathbb{R}$. We deal with m subspaces of dimension 1 because every event contained in X belongs to one of them according to (3.1). On the other hand, a univariate random quantity $X = \{x^1, x^2, \ldots, x^m\}$ viewed as an m-dimensional vector of E^m is an element of a set of univariate random quantities denoted by ${}_{(1)}S$. All these quantities are viewed as m-dimensional vectors of E^m from a geometric viewpoint. We note that it turns out to be

$$ {}_{(1)}S \subset E^m, \tag{3.5} $$

where ${}_{(1)}S$ is an m-dimensional linear space contained in E^m. This is because the sum of two vectors belonging to ${}_{(1)}S$ must be a vector whose components are all different. Thus, it belongs to ${}_{(1)}S$ in this way. We say that it belongs to ${}_{(1)}S$ if and only if its components are all different. The same thing goes when we consider the multiplication of a vector of ${}_{(1)}S$ by a real number that is different from zero. Hence, we say that ${}_{(1)}S$ is closed with respect to the sum of two vectors of it and the multiplication of a vector of it by a real number that is different from zero. We consider a closed structure coinciding with an m-dimensional linear space contained in E^m in this way. We note that E^m can also be viewed as an affine space over itself. Each element of E^m is firstly an m-dimensional vector viewed as an ordered list of m real numbers. Nevertheless, each element of E^m can also be viewed as a point of an affine space, where the zero vector of E^m is the origin of it. Thus, the zero vector of E^m characterizes an affine frame of E^m when it is viewed as an affine space. An affine frame of E^m viewed as an affine space consists of a point coinciding with the zero vector of E^m and an orthonormal basis of E^m. We are able to consider a point of an affine space having m coordinates or a vector of a linear space having m components. We choose a contravariant notation with respect to the components of $\mathbf{x}$. It is consequently possible to write

$$ \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix}. \tag{3.6} $$

We choose a covariant notation with respect to the components of $\mathbf{p} \in E^m$. It is therefore possible to write

$$ \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}, \tag{3.7} $$

where p_i represents a subjective probability assigned to x^i, $i = 1, \ldots, m$, by a given individual according to his degree of belief in the occurrence of x^i ((de Finetti, 1982a)). To say what probability is does not matter when one establishes the axioms that probability follows. A σ-algebra is therefore the field over which probability is defined. If this happens then we speak about the axiomatic probability theory ((Berti, Pratelli and Rigo, 2015)). We do not refer to it within this context. It is too much for what it is of our interest. Our methodological approach is however faithful to axioms of probability. We consequently note that it turns out to be $\sum_{i=1}^{m} p_i = 1$. We therefore consider a coherent evaluation of the probabilities associated with every single event. It is finitely additive. Different individuals whose state of knowledge is hypothetically identical may choose different p_i whose sum is equal to 1 ((Coletti, Petturiti and Vantaggi, 2016a)). Indeed, each of them may subjectively give greater attention to certain circumstances than to others ((Coletti, Petturiti and Vantaggi, 2014)). In any case, if we write

$$(\mathbf{x}, \mathbf{p}) \subset E^m \tag{3.8}$$

then we identify a distribution of probability embedded in a linear space provided with a metric on it ((Pompilj, 1957)). Such a distribution can always vary from individual to individual ((de Finetti, 1964)). Moreover, it can also vary with respect to the state of information of a given individual ((de Finetti, 1989)). We have to note a very important point: we should exactly speak about components of $\mathbf{x}$ and $\mathbf{p}$ having upper and lower indices because we deal with an orthonormal basis of E^m. This means that the covariant components of every m-dimensional vector of E^m coincide with the contravariant ones. We use these terms because we distinguish what it is objective from what it is subjective in this way.

4 A decomposition of the expected utility function

The expected utility function expressed by

$$\mathbf{P}(U) = U(x^1, \ldots, x^m) = u(x^i)p_i = u(x^1)p_1 + \ldots + u(x^m)p_m \tag{4.1}$$

is always characterized by $p_1, p_2, \ldots, p_m$ ((Schoemaker, 1982)). After decomposing X into m real numbers belonging to m straight lines of E^m we note that it is possible to assign a further real number to every element x^i, $i = 1, \ldots, m$, of X ((Wold, Shackle and Savage, 1952)). It is denoted by $u(x^i)$, $i = 1, \ldots, m$, where each x^i is a consumption bundle connected with E_i, $i = 1, \ldots, m$. Given $x^1 < x^2 < \ldots < x^m$, we observe that it turns out to be $u(x^1) < u(x^2) < \ldots < u(x^m)$ because one assumes that more is better when there is no satiation ((Maccheroni, Marinacci and Rustichini, 2006)). We are not speaking about bads but we are speaking about goods ((Manzini and Mariotti, 2007)). All these numbers represent a way of describing subjective preferences for which changes of origin (and unit of measurement) are inessential ((Gul and Pesendorfer, 2006)). Each $u(x^i)$, $i = 1, \ldots, m$, is a point of the corresponding straight line of E^m. It represents a distance from the number 0. We consider the 1-norm distance between $u(x^i)$, $i = 1, \ldots, m$, and 0. We say that (4.1) is then a weighted average of distances. We consider the following

Proposition 4.1. *Let $X = \{x^1, x^2, \ldots, x^m\}$ be a contingent consumption plan, where we have $x^1 < x^2 < \ldots < x^m$ without loss of generality. Let $u(x^1) + a < u(x^2) + a < \ldots < u(x^m + a)$ be a preference ordering, where $a \in \mathbb{R}$ is an arbitrary constant. If $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, is an orthonormal basis of E^m then $\{[u(x^i) + a]p_i\}\mathbf{e}_i = \mathbf{y} \in E^m$ is a direct and orthogonal sum of m vectors belonging to m one-dimensional subspaces of E^m.* □

Proof. Given any one-dimensional subspace of E^m, the collinear vectors related to $U(x^1, \ldots, x^m)$ are two. We have

$$\mathbf{e}_1 = 1 \cdot \mathbf{e}_1 \tag{4.2}$$

as well as

$$u({}_{(1)}\mathbf{x}) = [u(x^1)]\mathbf{e}_1 \tag{4.3}$$

with regard to the first one-dimensional subspace of E^m. A same probability denoted by p_1 is associated with $u(x^1)$ even when $u(x^1)$ varies. In general, a same probability denoted by p_1 is associated with $u(x^1)$ when we consider $u(x^1) + a$, where $a \in \mathbb{R}$ is an arbitrary constant. We note that $u(x^1)$ as well as $u(x^1) + a$ are distances from the number 0. We identify different m-dimensional vectors on a same straight line in E^m in this way. The direction of this straight line is established by $\mathbf{e}_1$. All these collinear vectors lying on the straight line established by $\mathbf{e}_1$ represent the same event from a randomness viewpoint on condition that the starting inequalities given by $u(x^1) < u(x^2) < \ldots < u(x^m)$, where we have $a = 0$, continue to be valid in the form expressed by $u(x^1) + a < u(x^2) + a < \ldots < u(x^m) + a$, where we have $a \neq 0$. We evidently consider a positive monotonic transformation in this way. This same event is then realized when the true value of X to be verified a posteriori coincides with the lowest possible value of it. Conversely, we write

$$\mathbf{e}_m = 1 \cdot \mathbf{e}_m \tag{4.4}$$

as well as

$$u({}_{(m)}\mathbf{x}) = [u(x^m)]\mathbf{e}_m \tag{4.5}$$

with regard to the m-th one-dimensional subspace of E^m. A same probability denoted by p_m is associated with $u(x^m)$ even when $u(x^m)$ varies. In general, a same probability denoted by p_m is associated with $u(x^m)$ when we consider $u(x^m)+a$, where $a \in \mathbb{R}$ is an arbitrary constant. We note that $u(x^m)$ as well as $u(x^m)+a$ are distances from the number 0. We identify different m-dimensional vectors on a same straight line in E^m in this way. The direction of this straight line is established by $\mathbf{e}_m$. All these collinear vectors lying on the straight line established by $\mathbf{e}_m$ represent the same event from a randomness viewpoint on condition that the starting inequalities given by $u(x^1) < u(x^2) < \ldots < u(x^m)$, where we have $a = 0$, continue to be valid in the form expressed by $u(x^1) + a < u(x^2) + a < \ldots < u(x^m) + a$, where we have $a \neq 0$. We evidently consider a positive monotonic transformation in this way. This same event is then realized when the true value of X to be verified a posteriori coincides with the highest possible value of it. The same thing evidently goes when we consider all other one-dimensional subspaces of E^m ((Debreu, 1960)). Given any one-dimensional subspace of E^m established by a straight line in E^m, we are able to consider different scalars related to this straight line of E^m. They coincide with the contravariant components of m-dimensional collinear vectors with respect to one of the basis vectors. If a varies in $\mathbb{R}$ then there are infinite possible positive monotonic transformations that can theoretically be considered. We say that there are infinite possible preference orderings that can theoretically be considered. It is then possible to move along every straight line of E^m in order to consider them. We write

$$u({}_{(1)}\mathbf{x}) + \ldots + u({}_{(m)}\mathbf{x}) = u(\mathbf{x}), \tag{4.6}$$

where each $u({}_{(i)}\mathbf{x})$ is an element of $E^m_{(i)}$, $i = 1, \ldots, m$, while $u(\mathbf{x})$ is an element of E^m. It turns out to be

$$u(\mathbf{x}) = u(x^1)\mathbf{e}_1 + \ldots + u(x^m)\mathbf{e}_m, \tag{4.7}$$

where we have

$$u(\mathbf{x}) = \begin{pmatrix} u(x^1) \\ u(x^2) \\ \vdots \\ u(x^m) \end{pmatrix} \tag{4.8}$$

as well as

$$(u(\mathbf{x}), \mathbf{p}) \subset E^m. \tag{4.9}$$

It follows that the expected utility function always coincides with the direct sum of m vectors related to m incompatible and exhaustive events ((Tversky, 1975)). Such a direct sum is also orthogonal. These m vectors belong to m one-dimensional subspaces of

E^m. An m-dimensional vector belonging to E^m is uniquely obtained by means of a linear combination of m basis vectors. We denote it by $\mathbf{y}$. The contravariant components of this m-dimensional vector are m scalars whose sum coincides with the expected utility function. We write

$$\{[u(x^1)+a]p_1\}\mathbf{e}_1+\ldots+\{[u(x^m)+a]p_m\}\mathbf{e}_m=\mathbf{y}, \tag{4.10}$$

where we have $\mathbf{y}\in E^m$. Each of these m scalars is obtained by multiplying one of the m probabilities related to m incompatible and exhaustive events by the contravariant component of the corresponding m-dimensional collinear vector belonging to one of the m one-dimensional subspaces of E^m. □

The expected utility function has an additive structure ((Friedman and Savage, 1952)). This means that the choices that a given individual makes when a random event occurs are independent from the choices that he makes when another random event occurs, where one and only one random event occurs ((Machina, 1982)). This independence assumption is entirely caught by the linear independence of the basis vectors.

5 Comparison between two contingent consumption plans inside of a metric space

If we refer to the geometric property of collinearity then the expected utility function is given by

$$\mathbf{P}(U)=U(x^1,\ldots,x^m)=\bar{\mathbf{x}}=\sum_{i=1}^{m}u({}_{(i)}\mathbf{x})p_i, \tag{5.1}$$

where we have $0\leq p_i=\mathbf{P}(E_i)\leq 1$, $i=1,\ldots,m$, and $\sum_{i=1}^{m}p_i=1$ because we consider a coherent evaluation of the probabilities connected with the set of events expressed by $\{E_1,\ldots,E_m\}$ ((Koopman, 1940)). We consider the same events contained in X. We consider the same probabilities assigned to $x^1,x^2,\ldots,x^m$ ((Kip Viscusi, 1985)). We note that all events contained in X are characterized by different numbers ((Marschak, 1959)). They coincide with $\{u(x^1),\ldots,u(x^m)\}$, where we have

$$u({}_{(i)}\mathbf{x})=[u(x^i)]\mathbf{e}_i \tag{5.2}$$

for every $i=1,\ldots,m$. We note that the Einstein summation convention does not hold with regard to (5.2). We say that (5.1) is an m-dimensional vector belonging to E^m whose contravariant components are all equal. We consider its distance from the zero vector of E^m. We note that the i-th contravariant component of $\bar{\mathbf{x}}$ is given by

$$\bar{x}^i=u({}_{(i)}x^i\delta_i^j)p_i, \tag{5.3}$$

where we have $i = 1, \ldots, m$. We observe that it turns out to be $u(0) = 0$. Each contravariant component of $\bar{\mathbf{x}}$ is then obtained by means of a linear combination. This linear combination is characterized by (5.3). We observe that the Einstein summation convention holds with regard to (5.3). Each contravariant component of $\bar{\mathbf{x}}$ is therefore established by m groups of numbers where every group of numbers consists of m numbers that are added. When we consider the first contravariant component of $\bar{\mathbf{x}}$ we note that only the first element of the first group having m elements as summands is not equal to 0. All other elements of the first group having m elements as summands are equal to 0. When we consider the first contravariant component of $\bar{\mathbf{x}}$ we note that only the second element of the second group having m elements as summands is not equal to 0. All other elements of the second group having m elements as summands are equal to 0. When we consider the first contravariant component of $\bar{\mathbf{x}}$ we note that only the m-th element of the m-th group having m elements as summands is not equal to 0. All other elements of the m-th group having m elements as summands are equal to 0. The same thing goes when we consider all other contravariant components of $\bar{\mathbf{x}}$. We note that it turns out to be

$$\mathbf{P}(X) = \bar{\bar{\mathbf{x}}} = \sum_{i=1}^{m} {}_{(i)}\mathbf{x} p_i, \tag{5.4}$$

where all components of $\bar{\bar{\mathbf{x}}} \in E^m$ are equal. We consider its distance from the zero vector of E^m. We write

$$\bar{\bar{x}}^i = {}_{(i)}x^i \delta_i^j p_i \tag{5.5}$$

for every $i = 1, \ldots, m$. A univariate random quantity representing all possible values of an investment denoted by A is expressed by X_A. A univariate random quantity representing all possible values of an investment denoted by B is expressed by X_B ((Markowitz, 1952)). We decompose $X_A = \{x_A^1, x_A^2, \ldots, x_A^m\}$, with $x_A^1 < x_A^2 < \ldots < x_A^m$, and $X_B = \{x_B^1, x_B^2, \ldots, x_B^m\}$, with $x_B^1 < x_B^2 < \ldots < x_B^m$, inside of E^m. We assign a number to every consumption bundle given by x_A^i, $i = 1, \ldots, m$, and x_B^i, $i = 1, \ldots, m$. It is denoted by $u(x_A^i)$, $i = 1, \ldots, m$, and $u(x_B^i)$, $i = 1, \ldots, m$. We associate a probability denoted by p_i^A with every $u(x_A^i)$, $i = 1, \ldots, m$, where we have $p_1^A + \ldots + p_m^A = 1$. These probabilities are the same of the ones associated with $x_A^1, x_A^2, \ldots, x_A^m$ because we deal with the same events. We associate a probability denoted by p_i^B with every $u(x_B^i)$, $i = 1, \ldots, m$, where we have $p_1^B + \ldots + p_m^B = 1$. These probabilities are the same of the ones associated with $x_B^1, x_B^2, \ldots, x_B^m$ because we deal with the same events. It is therefore possible to calculate $\mathbf{P}(U_A) = U(x_A^1, \ldots, x_A^m) = \bar{\mathbf{x}}_A$ as well as $\mathbf{P}(U_B) = U(x_B^1, \ldots, x_B^m) = \bar{\mathbf{x}}_B$. It is also possible to calculate $\mathbf{P}(X_A) = \bar{\bar{\mathbf{x}}}_A$ as well as $\mathbf{P}(X_B) = \bar{\bar{\mathbf{x}}}_B$. We always consider their distances from the zero vector of E^m. For instance, we observe that if the components of $\bar{\mathbf{x}}_A$ and $\bar{\mathbf{x}}_B$ are positive then we strictly prefer $\bar{\mathbf{x}}_A$ when it is more distant from the origin than $\bar{\mathbf{x}}_B$. We mathematically write

$$\|\bar{\mathbf{x}}_A\| > \|\bar{\mathbf{x}}_B\|, \tag{5.6}$$

where $\|\bar{\mathbf{x}}_A\|$ is the norm of $\bar{\mathbf{x}}_A$ while $\|\bar{\mathbf{x}}_B\|$ is the norm of $\bar{\mathbf{x}}_B$. The same thing goes when we consider $\bar{\bar{\mathbf{x}}}_A$ and $\bar{\bar{\mathbf{x}}}_B$. In general, one establishes if it turns out to be

$$U(x_A^1, \ldots, x_A^m) > U(x_B^1, \ldots, x_B^m) \iff \{x_A^1, \ldots, x_A^m\} \succ \{x_B^1, \ldots, x_B^m\} \tag{5.7}$$

or

$$U(x_B^1, \ldots, x_B^m) > U(x_A^1, \ldots, x_A^m) \iff \{x_B^1, \ldots, x_B^m\} \succ \{x_A^1, \ldots, x_A^m\} \tag{5.8}$$

or

$$U(x_A^1, \ldots, x_A^m) = U(x_B^1, \ldots, x_B^m) \iff \{x_A^1, \ldots, x_A^m\} \sim \{x_B^1, \ldots, x_B^m\}. \tag{5.9}$$

6 A geometric and analytical condition of coherence based on the notion of distance

After decomposing the expected utility function inside of E^m we say that an individual coherently behaves in the face of risk when there exists an m-dimensional vector of E^m uniquely obtained by means of a linear combination of m basis vectors such that it turns out to be

$$\mathbf{z} = [u(x^1)p_1]\mathbf{e}_1 + \ldots + [u(x^m)p_m]\mathbf{e}_m, \tag{6.1}$$

with $\mathbf{z} \in E^m$, where $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, is an orthonormal basis of E^m. The sum of the real coefficients of this linear combination coincides with the expected utility function given by $u(x^i)p_i \in \mathbb{R}$, $i = 1, \ldots, m$. We propose a geometric condition of coherence based on the notion of distance because each $u(x^i)$, $i = 1, \ldots, m$, represents a distance from the number 0 inside of E^m. We say that the right of getting x^1 associated with E_1, x^2 associated with E_2, ..., x^m associated with E_m, whose probabilities are expressed by $p_1, p_2, \ldots, p_m$, is equal to the weighted average of distances given by $u(x^i)p_i$, $i = 1, \ldots, m$. This is because the underlying events are incompatible and exhaustive ((Mc Fadden, 1974)). We then say that $\mathbf{z}$ additively behaves and we sum its components for this reason. We propose a more general condition of coherence because we consider a preference ordering expressed by

$$u(x^1) < u(x^2) < \ldots < u(x^m). \tag{6.2}$$

It is compatible with any analytical utility function denoted by $u(x)$ which we have decomposed inside of E^m. It is a continuous and strictly increasing utility function. It could indifferently be a concave or convex or linear utility function. Given (6.2), it is also possible to consider infinite positive monotonic transformations for which we observe the same preference ordering. On the other hand, it is appropriate to propose a more general condition of coherence because the attitude in the face of risk of an individual could unexpectedly change. It depends on his temperament and his current mood. Moreover, it is also influenced by the value of his estate denoted by F. It is a random quantity. This means that if the true value of F is unexpectedly great or low then his attitude in the face of risk may alter. In general, there are transactions for which an individual is usually neutral with respect to risk. Monetary value and utility consequently coincide with respect to these transactions. If the true value of F is unexpectedly low then the same individual could become risk averse. In all cases he will then prefer the certain alternative to the uncertain one ((Nau, 2006)). Conversely, there are transactions for which an individual is usually averse with respect to risk ((Slovic, Fischhoff and Lichtenstein, 1977)). If the true value of F is unexpectedly great then he could become risk neutral. In all these cases we note that if $u(x)$ has wrongly been chosen by a given individual then another utility function has coherently to be chosen by him ((Marschak, 1950)). Having said that, we say that the criteria of coherent decisions under uncertainty are all those consisting of the consideration of infinite preference orderings compatible with any analytical utility function denoted by $u(x)$. Also, they are all those consisting of the choice of any coherent evaluation of the probabilities associated with every single random case denoted by $E_1, E_2, \ldots, E_m$. One of these analytical utility functions must coherently be chosen ((Johnson and Payne, 1985)). Moreover, the criteria of coherent decisions under uncertainty are all those by means of which one fixes as one's goal the maximization of the prevision of the utility associated with a contingent consumption plan ((MacCrimmon, 1968)). We have to note a very important point: we get out of E^m in order to put back together $u(x)$, where $u(x)$ is a real-valued function having a real variable. We consequently consider an ordered pair of perpendicular axes, a single unit of length for both axes and an orientation for each axis. The point where they meet coincides with the origin for both. Each axis is then nothing but a real number line, where every point of it corresponds to a real number and every real number to a point. We therefore consider $x^1, x^2, \ldots, x^m$ together with their masses denoted by $p_1, p_2, \ldots, p_m$ onto the x-axis. We consider $u(x^1), u(x^2), \ldots, u(x^m)$ together with their masses denoted by $p_1, p_2, \ldots, p_m$ onto the y-axis. We note that our geometric condition of coherence is compatible with

$$u(x) = u(x^i)p_i, \tag{6.3}$$

where we have $i = 1, \ldots, m$. We observe that (6.3) means that the right of getting x^1 associated with E_1, x^2 associated with E_2, ..., x^m associated with E_m, whose probabilities are expressed by $p_1, p_2, \ldots, p_m$, is equal to the right of getting the certainty equivalent expressed by x. We say that (6.3) is an analytical condition of coherence. There evidently exists an analytical utility function denoted by $u(x)$ which additively behaves according to (6.3). Its increments (onto the y-axis) between A and x as well as between x and B are equal for a given individual when and only when he is indifferent (onto the x-axis) between the choice of x, which is the certainty equivalent, and the choice of purchasing a lottery ticket connected with two random events. They are A = "the ticket is not drawn" and B = "the ticket is drawn", where A and B have equal probabilities. We have $\mathbf{P}(A+B) = \mathbf{P}(A) + \mathbf{P}(B) = 1$ because A and B are incompatible and exhaustive events. This means that it turns out to be $\mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{2}$. We have to note another very important point: we refer to a scale of measurable utilities based on the notion of distance. By considering the inverse $u^{-1}(y)$ of $u(x)$ we obtain

$$x = u^{-1}\{u(x^i)p_i\}. \tag{6.4}$$

If x is less than the prevision or mathematical expectation of X given by $\mathbf{P}(X) = x^1 p_1 + \ldots + x^m p_m$ then we deal with a risk-averse individual whose utility function denoted by $u(x)$ is a concave function. We do not consider degenerate cases. We observe equal levels of utility (equal distances meant as 1-norm distances) onto the y-axis in passing from A to x and from x to B onto the x-axis. When we pass from A to x and from x to B we observe 1-norm distances between different points onto the x-axis. If x is greater than the prevision or mathematical expectation of X given by $\mathbf{P}(X) = x^1 p_1 + \ldots + x^m p_m$ then we deal with a risk-loving individual whose utility function denoted by $u(x)$ is a convex function. We do not consider degenerate cases. We observe equal levels of utility (equal distances meant as 1-norm distances) onto the y-axis in passing from A to x and from x to B onto the x-axis. When we pass from A to x and from x to B we observe 1-norm distances between different points onto the x-axis. If x is equal to the prevision or mathematical expectation of X given by $\mathbf{P}(X) = x^1 p_1 + \ldots + x^m p_m$ then we deal with a risk-neutral individual whose utility function denoted by $u(x)$ is a linear function. We observe equal levels of utility (equal distances meant as 1-norm distances) onto the y-axis in passing from A to x and from x to B onto the x-axis. When we pass from A to x and from x to B we observe 1-norm distances between different points onto the x-axis. We always divide a more or less spacious interval into two indifferent increments. They are 1-norm distances onto the x-axis. It is unimportant the finite number of possible values of X contained in it. We always observe equal levels of utility (equal distances meant as 1-norm distances) onto the y-axis.

7 Conclusions

We have decomposed the expected utility function inside of E^m. We have considered a geometric and unified approach to an integrated formulation of decision theory in its two subjective components: utility and probability. A univariate random quantity representing a contingent consumption plan has been studied inside of E^m because utility and probability are both subjective. We have considered distributions of probability embedded in a metric space. We have replaced a closed structure with another one: we have replaced a σ-algebra with a linear space over $\mathbb{R}$. We have studied coherent decisions under uncertainty having as their goal the maximization of the prevision of the utility associated with a contingent consumption plan. We have studied the criterion of the mathematical expectation when it is applied to utility and monetary values. When it is applied to monetary values we have observed that among decisions under uncertainty leading to different random gains an individual chooses that random gain having the highest prevision or mathematical expectation. When it is applied to the notion of utility we have considered the independence assumption as an implicit condition of coherence. This assumption is entirely caught by the linear independence of the vectors of an orthonormal basis of E^m. In any case, we have considered distances of m-dimensional vectors from the zero vector of E^m in order to study the criterion of the mathematical expectation applied to monetary values and utility. Each individual coherently chooses an analytical utility function denoted by $u(x)$ with respect to his attitude in the face of risk as well as he coherently chooses his probabilities associated with every single random case. He does not choose what it is necessary in order to be coherent. This thing cannot arbitrarily be chosen. We have therefore proposed a geometric condition of coherence compatible with all possible attitudes in the face of risk of an individual because we have measured utility inside of E^m by using the notion of distance of $u(x^1),\ldots,u(x^m)$ from the number 0. We have consequently decomposed $u(x)$ inside of E^m in this way. We have proved that if we use an analytical utility function denoted by $u(x)$ in order to measure utility then we continue to refer to the notion of distance. A decomposition of the expected utility function inside of E^m is therefore well-founded.

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