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Diversity Combining Type I-Hybrid ARQ protocol over *m*-ary Asymmetric Varshamov channels[☆,☆☆]

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ABSTRACT

Errors in multilevel flash memories and multilevel optical systems are asymmetric in nature because of the underlying device physics. These errors can be modeled using the m-ary asymmetric channel. In this work, a general $m(\geq 2)$-ary Asymmetric Varshamov (AV) channel for these systems is introduced. Also, a Divesting Combining Type-I Hybrid ARQ protocol using t-Asymmetric Error Correcting/All Asymmetric Error Detecting (t-AEC/AAED) codes is presented and its throughput performance is studied.

1. Introduction

There are two main schemes used in overcoming errors in communication systems: Automatic Repeat Request (ARQ) protocol and Forward Error Correction (FEC) [1–3]. In ARQ protocol, Error Detecting (ED) codes are used and upon receiving a data word, the receiver checks for any error in it. If it detects some errors, then it requests the sender to retransmit the same data. The transmitter then sends the same word again to the receiver. This process continues till no error is detected. At that time, the receiver sends a Positive Acknowledge signal (ACK) to the transmitter. Then, the transmitter sends the next word to the receiver [1–3].

In FEC, the transmitter continuously sends the data words encoded with some t-Error Correcting (t-EC) code. If there are t or fewer errors in the received word, the decoder at the receiver side corrects the errors [1,2].

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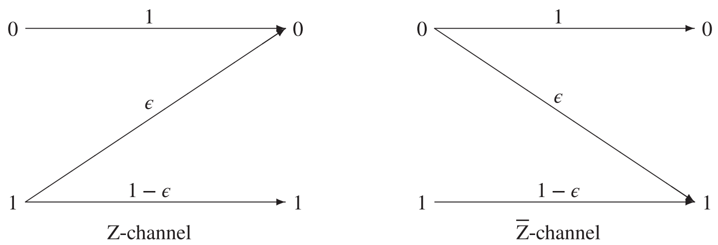


Fig. 1. Z and $\overline{Z}$ channels.

A combination of ARQ and FEC is called Hybrid ARQ (H-ARQ). In Type-I Hybrid ARQ (T_1-H-ARQ), the system uses t-Error Correcting and d-Error Detecting (t-EC/d-ED) code, $d \geq t$, to encode the data word. Assume that the received word contains t_1 errors. If $t_1 \leq t$, the decoder corrects t_1 errors and recovers the data. On the other hand, if the receiver is not able to correct the errors but detects those errors (i. e., $t < t_1 \leq d$ errors), then it sends a Negative Acknowledge signal (NAK) to the sender requesting the retransmission of the same code word. The sender then resends this code word [1–3].

ARQ can also be classified as Plain ARQ (P-ARQ) and Diversity Combining ARQ (DC-ARQ) protocols [3,4]. In the P-ARQ protocol, the receiver at the ith iteration, $i = 1, 2, \dots,$ makes a decision to send NAK or ACK signal to the transmitter purely based on the word it received in iteration number i. The words received in the previous iterations are discarded. But, in the case of DC-ARQ protocol, the receiver makes a decision by combining all the words received so far.

Errors in communication systems are of many types. In general, they are classified as symmetric, asymmetric, and unidirectional [2,5]. In symmetric type, $1 \to 0$ and $0 \to 1$ errors can simultaneously occur in any data word. Binary Symmetric Chanel (BSC) is used to model this type if the probabilities of $1 \to 0$ and $0 \to 1$ errors are equal.

When only $1 \to 0$ errors are possible in the received code word, the errors are of asymmetric type. The Z-channel is used to model these errors. Errors in many practical systems can be modeled using the $Z-$channel. For example, in optical communication, photon may fade or decay, and cannot be produced during the transmissions. Such systems can use the Z-channel model when representing 1 and 0 as the existence and non-existence of photons respectively [3,5,6]. Also, when only $0 \to 1$ errors are possible in the received code word, the $\overline{Z}$ channel can model such errors. Fig. 1 shows the Z and $\overline{Z}$ channels.

In unidirectional errors, both of $1 \to 0$ and $0 \to 1$ errors can occur with two conditions, first, these errors cannot simultaneously occur in a code word and, second, the type of error may vary from one word to another word [5,7–9]. Some errors that can occur in interconnection networks, ROM, RAM, etc., are of unidirectional type [3,5].

The Varshamov Error (VE) model is a generalization of the Z-channel for the m-ary alphabet

$$\mathbb{Z}_m \stackrel{\text{def}}{=} \{0, 1, \dots, m-1\},$$

$m \geq 2$. When $A = a_1 a_2 \dots a_n \in \mathbb{Z}_m^n$ is transmitted, the number of errors in the received word $B = b_1 b_2 \dots b_n \in \mathbb{N}^n$, can be measured using the L_1 (also called Manhattan) distance as:

$$d_{L_1}(A, B) \stackrel{\text{def}}{=} \sum_{i=1}^{n} |a_i - b_i|;$$

where $|\cdot|$ is the absolute value of a number and this represents the error magnitude [10–12].

When the number of errors is measured according to the Hamming distance (i. e., $D_H(A, B)$ is the number of positions in which A and B differ), this is referred to as the Hamming error model. Traditional coding methods, specially those based on the Hamming error model, are not efficient to be applied for asymmetric errors [13]. So, some efficient codes were designed to deal with asymmetric errors, mainly in flash memories [14–19,19].

The VE-model is applied particularly for $m > 2$, when the m-ary transmission channel has the error probability:

$$Pr(\ b \text{ is received} \mid a \text{ is transmitted }) \simeq c_{a,b} \cdot \epsilon^{|a-b|} \tag{1}$$

where $\epsilon \in [0, 1] \subseteq \mathbb{R}^+$ and $c_{a,b} : \Omega \to \mathbb{N}$ are random variables for all $a, b \in \mathbb{Z}_m \subseteq \mathbb{R}$ with $a \neq b$ [12]. In [11,20–22], some more theoretical concepts of L_1 distance EC codes are presented. In this paper, we focus on the case of $c_{a,b} = 0$ in the last equation (i. e., the asymmetric type) for $a < b$. Thus, the $(t_+ = 0, t_- = t)$-EC/$(d_+ = 0, d_- = +\infty)$-ED codes presented in [11,20,23] for the L_1 distance are applied to recover t asymmetric errors and detect all asymmetric errors. Hence, these codes are suited to achieve error-free communication in the T_1-H-ARQ scheme.

Some practical systems, whose error behavior fits with this model, are multilevel optical systems [24] and Multilevel Flash Memories (MLFM) [11,13,14]. In MLFM, each cell stores $\log_2 r$ bits since it is programmed into one of r voltage levels. Errors can occur in these memories mostly in only one direction [14]. Noticeable examples of a Varshamov Z-channels are those such that:

$$c_{a,b} = \binom{a}{b}, \quad \text{for all symbols } a, b \in \mathbb{Z}_m; \tag{2}$$

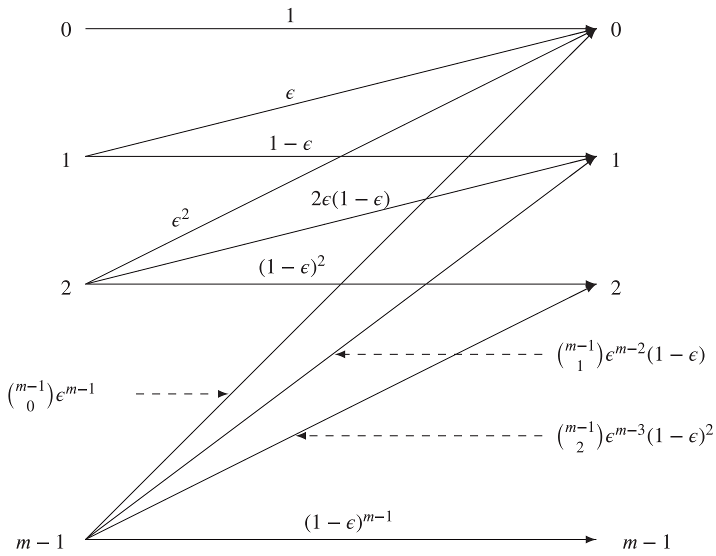


Fig. 2. The m-ary Z-channel which fits the VE-model.

where $\binom{n}{k}=0$, for all $k\notin[0,n]$. An example of these channels is shown in Fig. 2 and analyzed in Section 2.

The throughput, Φ, is mainly used to evaluate the performance of a communication system. It is computed as:

$$\Phi=\frac{k}{n\overline{R}}, \tag{3}$$

where k/n represents the information rate of the code $\mathcal{C}$ and $\overline{R}$ defines the Average Number of Transmissions (ANT) required for receiving a sent word correctly.

If the system uses All Error Detecting (AED) codes, then the communication is error-free, and so Φ is the same as the theoretical information rate of the proposed ARQ communication schemes. In [4], the throughput for the Plain and DC T_1-H-ARQ using t-AEC/AAED codes based on the Hamming error model for Z-channel is analyzed. Here, we study the behavior of the VE-model through evaluating the overall throughput by computing the Average Number of Transmission (ANT) for both P-ARQ and DC T_1-H-ARQ protocols using (t-AEC/AAED) codes, for $t\geq 0$, over the m-ary Varshamov Z-channel defined by Eqs. (1) and (2).

Sections 3 and 4 analyze the P-ARQ and DC T_1-H-ARQ schemes respectively while Section 5 gives some concluding remarks.

2. An Asymmetric Varshamov (AV) channel model

In the VE-model defined on m-ary Z-channel, the error magnitude is taken into account and so Eq. (1) holds. Also, the channel is asymmetric and so for all $a_1,a_2\in\mathbb{Z}_m$, $c_{a_1,a_2}=0$ if, and only if, $a_1<a_2$. A Varshamov Z-channel example is the one represented in Fig. 2 whose channel transition probabilities are defined as:

$$Pr(a_2|a_1)=\binom{a_1}{a_2}\epsilon^{a_1-a_2}(1-\epsilon)^{a_2},\quad \text{for all } a_1,a_2\in\mathbb{Z}_m \text{ and } a_1\geq a_2. \tag{4}$$

This channel (which we may call as the Asymmetric Binomial Channel (ABC)) can be taken as the representative of all the Varshamov Z-channels satisfying Eq. (2) because for $\epsilon\simeq 0$,

$$\binom{x}{y}\epsilon^{x-y}(1-\epsilon)^{y}\simeq\binom{x}{y}\epsilon^{x-y},\quad \text{for all } x,y\in\mathbb{Z}_m \text{ and } x\geq y. \tag{5}$$

This channel could very well model simple physical communication systems in which the source is composed of $(m-1)$ independent and equal bi-stable devices and if the symbol $x\in\mathbb{Z}_m$ is transmitted, then x out of the $(m-1)$ devices are set to on for sending a signal (the remaining $(m-1)-x$ devices are set to off for not sending any signal); the receiver has the capability of only measuring how many bi-stable source devices have been set to on; and, each device signal cannot be created from nothing.

Under this ABC model, the analysis of both P-ARQ and DC-ARQ schemes is simplified considerably. In fact, if the communication channel can be modeled as the channel in Fig. 2 (or, any other equivalent channel such that Relation (5) holds) then we can regard any transmitted word $X=x_1x_2\ldots x_n\in\mathbb{Z}_m^n$ as an n basket array of n items. In position i, for all $i\in[1,n]$, there will be x_i marbles. So, the transmission process is equivalent to sending the x_i marbles in the sender's ith basket to

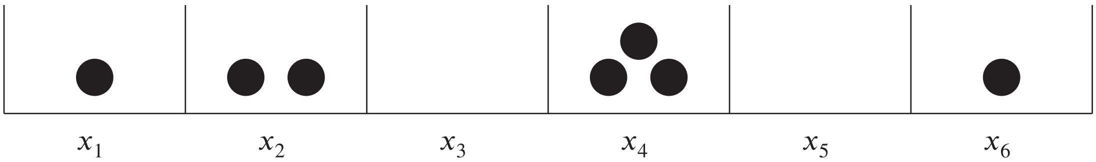


Fig. 3. The word $X = x_1x_2x_3x_4x_5x_6 = 120301$ in the binomial asymmetric channel model.

the receiver's ith basket where the probability that one marble may not reach the destination is ϵ and it is independent from the transmission of other marbles. For example, $X = 120301$ can be regarded as shown in Fig. 3. Now, note that the transmitted word $X = x_1x_2\dots x_n \in \mathbb{Z}_m^n$ is received error-free if, and only if, all the $w_{L_1}(X)$ marbles of X reach the destination, where $w_{L_1}(X)$ is the L_1 weight of X defined as:

$$w_{L_1}(X) \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_i.$$

In general, there has been exactly t Varshamov (or L_1) errors during the transmission of X if, and only if, $w_{L_1}(X) - t$ marbles (or t marbles) reach (or, do not reach, respectively) the destination, for any integer $t \in [0, w_{L_1}(X)]$. Recall that a marble reaches the destination with probability $1 - \epsilon$ and does not reach the destination with probability ϵ.

3. Plain ARQ (P-ARQ) protocol analysis using *t*-AEC/AAED codes over AV-channels

Suppose a system uses P-ARQ protocol using t-AEC/AAED code, $\mathcal{C}$, over the L_1 distance. Assume that X and Y are the transmitted and received words respectively. Furthermore, let $P_{ac}(X) \in [0, 1]$ be the probability of Y is accepted when X is transmitted. Similarly, let $P_{rej}(X) \in [0, 1]$ be the probability of Y is rejected when X is transmitted. When Y is accepted, the receiver sends the signal ACK to the transmitter, requesting it to send the next word. Similarly, when Y is rejected, the receiver sends the signal NAK to the receiver to resend the same word, X. Note that $P_{ac}(X) + P_{rej}(X) = 1$.

Let $NT^{(t)}(X) : \Omega \to \mathbb{N} - \{0\}$ be a random variable defined as the number of transmissions of X required by the transmitter to be accepted by the receiver. Here, $NT^{(t)}(X)$ is geometrically distributed and so,

$$\mathbb{E}\left[NT^{(t)}(X)\right] = \sum_{k=1}^{+\infty} k[P_{rej}(X)]^{k-1}P_{ac}(X) = P_{ac}(X)\sum_{k=1}^{+\infty} k[P_{rej}(X)]^{k-1} = \frac{P_{ac}(X)}{[1 - P_{rej}(X)]^2} = \frac{1}{P_{ac}(X)}; \tag{6}$$

this is because, $\sum_{j=1}^{+\infty} jx^{j-1} = 1/(1-x)^2$. To find $\mathbb{E}\left[NT^{(t)}(X)\right]$, $P_{ac}(X)$ should be computed first. In the P-ARQ protocol using t-AEC/AAED code, Y is accepted at the receiver, only when the number of Varshamov (or L_1) errors in Y is less than or equal to t. So,

$$P_{ac}(X) = \sum_{i=0}^{t} Pr(Y \text{ contains exactly } i\ L_1 \text{ errors}|X). \tag{7}$$

Since the communication channel is the ABC defined in Section 2, it follows that:

$$Pr(Y \text{ contains exactly } i\ L_1 \text{ errors}|X) =$$

$$Pr(\text{exactly } i \text{ marbles of } X \text{ have not reached the receiver} \mid X) = \binom{w_{L_1}(X)}{i}\epsilon^i(1-\epsilon)^{w_{L_1}(X)-i},$$

and so, from (7),

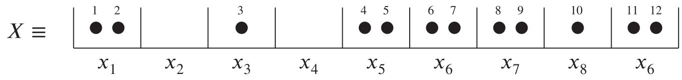
$$P_{ac}(X) = \sum_{i=0}^{t}\binom{w_{L_1}(X)}{i}\epsilon^i(1-\epsilon)^{w_{L_1}(X)-i}. \tag{8}$$

From Eqs. (6) and(8) it can be shown that in the case of P-ARQ protocol using $t - AEC/AAED$ code, the required number of transmission of X depends only on X through its L_1 weight and is equal to

$$NT^{(t)}(X) = 1\bigg/\left[\sum_{i=0}^{t}\binom{w_{L_1}(X)}{i}\epsilon^i(1-\epsilon)^{w_{L_1}(X)-i}\right]. \tag{9}$$

4. Diversity combining ARQ (DC-ARQ) protocol analysis using *t*-AEC/AAED codes over AV-channels

Let $0 < 1 < 2 < \dots < m-1$ be the total order in $\mathbb{Z}_m = \{0, 1, 2, \dots, m-1\}$ so that the max operation can be defined in $\mathbb{Z}_m$ as $\max(b, y) = y$ if $b < y$ and $\max(b, y) = b$ if $b \geq y$. Also, given $B, Y \in \mathbb{Z}_m^n$, $B \cup Y \in \mathbb{Z}_m^n$ indicates the word obtained from



r	Y_r	B_r	$d_{L_1}(X, B_r)$
0	–	$\underline{0}0\underline{0}0\underline{00000}$	12
1	$20\underline{0}0\underline{1}2\underline{0}1\underline{1}$	$20\underline{0}0\underline{1}2\underline{0}1\underline{1}$	5
2	$\underline{1}0\underline{0}02\underline{00}12$	$20\underline{0}022\underline{0}12$	3
3	$2010\underline{1}2\underline{0}12$	$201022\underline{0}12$	2
4	$\underline{1}0\underline{0}0\underline{1}2212$	201022212	0

Fig. 4. A sequence of transmissions example for the word $X = 201022212$.

B and Y by applying the max operation given above over their corresponding digits. For example, when $m = 5$, $B = 113301$ and $Y = 101334$, then $B \cup Y = 113334$.

Assume that $X \in \mathcal{C} \subseteq \mathbb{Z}_m^n$ is the transmitted word. Let Y_r represent the received word after $r - 1$ times of NAKs.

In DC-T_1-H ARQ protocol, the word B_r is either accepted or not accepted purely based on

$$B_r = \bigcup_{k=1}^{r} Y_k, \quad r \in \mathbb{N} \text{ and } r \geq 1.$$

In particular, when a t-AEC/AAED code is used, B_r is accepted when the number of L_1 errors in B_r, with respect to X is less than or equal to t. For example, Fig. 4 shows a sequence of Y_r and B_r when transmitting the word $X = 201022212 \in \mathbb{Z}_3^9$. In this example, if the system uses AAED then the received would accepts B_4, i. e. $B_4 = 201022212 = X$, the same transmitted word. However, if 2-AEC/AAED is used then the received word, $B_3 = 201022\underline{0}12$, is accepted and corrected as $201022212 = X$. Now, in the marble representation of any word $X \in \mathbb{Z}_m^n$ given in Section 2, let $NT_i(X) \in \mathbb{Z}_n - \{0\}$ be the number of transmissions required for receiving the ith marble of X, for all $i = 1, 2, \ldots, w_{L_1}(X)$ (= the L_1 weight of X). Thus, the total number of transmissions required to receive X without errors (i. e., the number of L_1 errors in $B_r \leq t$ with respect to X) using a t-AED/AAED code is the random variable:

$$NT^{(t)} = NT^{(t)}(X) \stackrel{\text{def}}{=} (t+1)\text{th largest element in the set } \{NT_1, NT_2, \ldots, NT_{w_{L_1}(X)}\};$$

where NT_i is the number of transmissions required so that the ith marble reaches the destination.

For the example presented in Fig. 4, assume that the $w = w_{L_1}(X) = 12$ marbles of X are numbered as in the top part of the figure. In the transmission

$$(NT_1, NT_2; NT_3; NT_4, NT_5; NT_6, NT_7; NT_8, NT_9; NT_{10}; NT_{11}, NT_{12}) = (1, 1; 3; 1, 2; 1, 1; 4, 4; 1; 1, 2).$$

Thus, $NT^{(0)} = 4$ retransmissions are required using AAED codes, $NT^{(1)} = 4$ retransmissions are required using 1-AEC/AAED codes, $NT^{(2)} = 3$ retransmissions are required using 2-AEC/AAED codes, $NT^{(3)} = 2$ retransmissions are required using 3-AEC/AAED codes, $NT^{(4)} = 2$ retransmissions are required using 4-AEC/AAED codes, and $NT^{(5)} = 1$ retransmission is required using 5-AEC/AAED codes. Hence, finding the ANT for the sent word, X, to be received correctly means finding the average of $NT^{(t)}$ defined above. Note that the ABC model defined in Section II implies that for the sent word, X, all NT_i's are independent and equally distributed with common cumulative distribution function (cdf) given by

$$F(r) = F_{NT_i}(r) = Pr(NT_i \leq r | X) = \sum_{\rho=1}^{r} Pr(NT_i = \rho | X),$$

where (recall that each of the $w = w_{L_1}(X)$ marbles of X reaches the destination with probability $1 - \epsilon$ and does not reach the destination with probability ϵ)

$$Pr(NT_i = \rho | X \text{ is being sent}) = Pr(NT_i = \rho | \text{the } i\text{-th marble of } X \text{ is being sent}) = \epsilon^{\rho - 1}(1 - \epsilon).$$

So, the NT_i's are all geometrically distributed with parameter ϵ and

$$F(r) = \sum_{k=1}^{r} \epsilon^{k-1}(1 - \epsilon) = 1 - \epsilon^r. \tag{10}$$

Now, when applying ordered statistics [25], the cdf of $NT^{(t)}$ becomes

$$F_{NT^{(t)}}(r) = Pr(NT^{(t)} \leq r | X) = \sum_{\lambda=0}^{t} \binom{w}{\lambda} [F(r)]^{w-\lambda} [1 - F(r)]^{\lambda},$$

where $w = w_L(X)$. Thus, from (10),

$$\begin{aligned} &Pr\left(NT^{(t)} = r|X \text{ is being sent}\right) \\ &\quad = F_{NT^{(t)}}(r) - F_{NT^{(t)}}(r-1) \\ &\quad = Pr\left(NT^{(t)} \le r|X\right) - P\left(NT^{(t)} \le r-1|X\right) = \sum_{\lambda=0}^{t}\binom{w}{\lambda}\left[(1-\epsilon^{r})^{w-\lambda}\epsilon^{\lambda r} - \left(1-\epsilon^{r-1}\right)^{w-\lambda}\epsilon^{\lambda(r-1)}\right] \\ &\quad = \sum_{k=1}^{w}\binom{w}{0}\binom{w}{k}(-1)^{k+1}\left[\epsilon^{k(r-1)} - \epsilon^{kr}\right] + \sum_{\lambda=1}^{t}\sum_{k=0}^{w-\lambda}\binom{w}{\lambda}\binom{w-\lambda}{k}(-1)^{k+1}\left[\epsilon^{(\lambda+k)(r-1)} - \epsilon^{(\lambda+k)r}\right]. \end{aligned}$$

The above relations imply

$$\mathbb{E}\left[NT^{(t)}(X)\right] = \sum_{r=1}^{+\infty} Pr\left(NT^{(t)} = r|X\right)r = S_1 + S_2, \tag{11}$$

where,

$$S_1 = \sum_{k=1}^{w}(-1)^{k+1}\binom{w}{k}S(\epsilon, k), \tag{12}$$

$$S_2 = \sum_{\lambda=1}^{t}\sum_{k=0}^{w-\lambda}(-1)^{k+1}\binom{w}{\lambda}\binom{w-\lambda}{k}S(\epsilon, \lambda+k) \tag{13}$$

and, for all $k = 1, 2, \ldots,$

$$S(\epsilon, k) = \sum_{r=1}^{+\infty}\left[\epsilon^{k(r-1)} - \epsilon^{kr}\right]r = \sum_{r=0}^{+\infty}\left(\epsilon^{k}\right)^{r}(r+1) - \sum_{r=1}^{+\infty}\left(\epsilon^{k}\right)^{r}r = \sum_{r=0}^{+\infty}\left(\epsilon^{k}\right)^{r} = 1 + \frac{\epsilon^{k}}{1-\epsilon^{k}}.$$

Now, if $\epsilon \simeq 0$ then

$$S(\epsilon, k) = 1 + \frac{\epsilon^{k}}{1-\epsilon^{k}} \simeq 1 + \epsilon^{k}; \tag{14}$$

and so, a good approximating simple formula can be found for $\mathbb{E}\left[NT^{(t)}(X)\right]$. Eqs. (12) and (14) imply that

$$\begin{aligned} S_1 &\simeq \sum_{k=1}^{w}(-1)^{k+1}\binom{w}{k}\left(1+\epsilon^{k}\right) = -\left[\sum_{k=1}^{w}\binom{w}{k}(-1)^{k} + \sum_{k=1}^{w}\binom{w}{k}(-\epsilon)^{k}\right] \\ &= -\left[\sum_{k=0}^{w}\binom{w}{k}(-1)^{k} - 1 + \sum_{k=0}^{w}\binom{w}{k}(-\epsilon)^{k} - 1\right] = 2 - \sum_{k=0}^{w}\binom{w}{k}(-\epsilon)^{k} = 2 - (1-\epsilon)^{w}. \end{aligned} \tag{15}$$

On the other hand, from (13) and (14), it similarly follows

$$\begin{aligned} S_2 &\simeq \sum_{\lambda=1}^{t}\sum_{k=0}^{w-\lambda}(-1)^{k+1}\binom{w}{\lambda}\binom{w-\lambda}{k}\left(1+\epsilon^{\lambda+k}\right) = -\sum_{\lambda=1}^{t}\sum_{k=0}^{w-\lambda}(-1)^{k}\binom{w}{\lambda}\binom{w-\lambda}{k}\epsilon^{\lambda+k} \\ &= -\sum_{\lambda=1}^{t}\binom{w}{\lambda}\epsilon^{\lambda}\sum_{k=0}^{w-\lambda}\binom{w-\lambda}{k}(-\epsilon)^{k} = -\sum_{\lambda=1}^{t}\binom{w}{\lambda}\epsilon^{\lambda}(1-\epsilon)^{w-\lambda}. \end{aligned} \tag{16}$$

So, from (11), (15) and (16), the average number of transmissions required for the word X in a DC t-AEC/AAED system over m-ary Varshamov ABC given in Fig. 2 has the following simple approximating expression which is valid for $\epsilon \simeq 0$:

$$\mathbb{E}\left[NT^{(t)}(X)\right] \simeq 2 - \sum_{\lambda=0}^{t}\binom{w_{L_1}(X)}{\lambda}\epsilon^{\lambda}(1-\epsilon)^{w_{L_1}(X)-\lambda} = 1 + Pr(d_{L_1}(X, Y) > t|X \text{ is sent and } Y \text{ is received}).$$

In general, from

$$S(\epsilon, k) = \sum_{r=0}^{+\infty}\left(\epsilon^{k}\right)^{r},$$

the exact formula

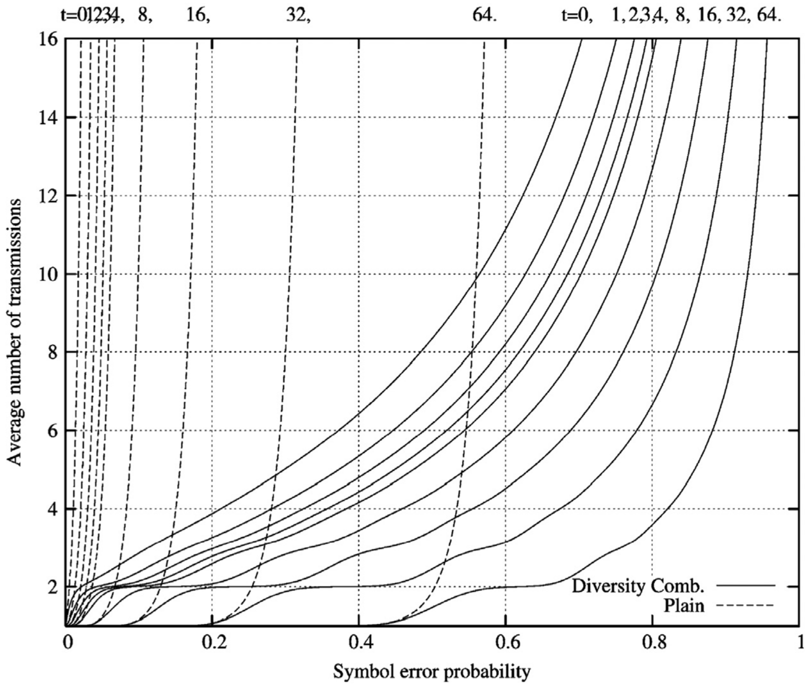


Fig. 5. Average Number of transmissions (ANT) for a constant weight $w_{L_1} = 128$ with P-ARQ and DC-ARQ schemes using t-AEC/AAED code, where $t = 0, 1, 2, 3, 4, 8, 16, 32$ and 64, over an m-ary Asymmetric Varshamov channel.

$$\mathbb{E}\left[NT^{(t)}(X)\right] = 1 + \sum_{r=1}^{+\infty}\left[1 - \sum_{\lambda=0}^{t}\binom{w_{L_1}(X)}{\lambda}(\epsilon^r)^\lambda(1-\epsilon^r)^{w_{L_1}(X)-\lambda}\right]. \tag{17}$$

can be derived.

Now, if all code words in $\mathcal{C}$ have equal chance of being transmitted, then the ANT is:

$$\mathbb{E}\left[NT^{(t)}(\mathcal{C})\right] = \frac{1}{|\mathcal{C}|}\sum_{X\in\mathcal{C}}\mathbb{E}\left[NT^{(t)}(X)\right].$$

From relations (9) and (17), for both P-ARQ and DC-ARQ schemes, $\mathbb{E}\left[NT^{(t)}(X)\right]$ is computed based on X through its L_1 weight; that is,

$$\mathbb{E}\left[NT^{(t)}(X)\right] = \overline{NT}^{(t)}(w_{L_1}(X)).$$

And so,

$$\overline{NT}^{(t)}(\mathcal{C}) \stackrel{\text{def}}{=} \mathbb{E}\left[NT^{(t)}(\mathcal{C})\right] = \frac{1}{|\mathcal{C}|}\sum_{w\in\mathbb{N}} A_w \overline{NT}^{(t)}(w),$$

where

$$A_w = |\{X\in\mathcal{C} :\ w_{L_1}(X) = w\}|, \quad w\in\mathbb{N},$$

represents the L_1 weight distribution of $\mathcal{C}$. So, for P-ARQ over the m-ary AV channel given in (2), the ANT of $\mathcal{C}$ is (see (9)),

$$\overline{NT}^{(t)}_{P-ARQ}(\mathcal{C}) = \frac{1}{|\mathcal{C}|}\sum_{w=0}^{n}\frac{A_w}{\sum_{j=0}^{t}\binom{w}{j}\epsilon^j(1-\epsilon)^{w-j}} = 1 + \frac{1}{|\mathcal{C}|}\sum_{w=0}^{n}A_w\sum_{r=1}^{+\infty}\left[1 - \sum_{j=0}^{t}\binom{w}{j}\epsilon^j(1-\epsilon)^{w-j}\right]^r. \tag{18}$$

If $\mathcal{C}$ is a constant L_1 weight w code, then

$$\overline{NT}^{(t)}_{P-ARQ}(\mathcal{C}) = \frac{1}{\sum_{j=0}^{t}\binom{w}{j}\epsilon^j(1-\epsilon)^{w-j}} = 1 + \sum_{r=1}^{+\infty}\left[1 - \sum_{j=0}^{t}\binom{w}{j}\epsilon^j(1-\epsilon)^{w-j}\right]^r. \tag{19}$$

For the DC-ARQ (see (17) and [4]),

$$\overline{NT}^{(t)}_{DC-ARQ}(\mathcal{C}) = 1 + \frac{1}{|\mathcal{C}|}\sum_{w=0}^{n} A_w \sum_{r=1}^{+\infty}\left[1 - \sum_{\lambda=0}^{t}\binom{w}{\lambda}(\epsilon^r)^{\lambda}(1-\epsilon^r)^{w-\lambda}\right]. \tag{20}$$

If $\mathcal{C}$ is a constant L_1 weight w code then

$$\overline{R}^{(t)}_{DC-ARQ}(\mathcal{C}) = 1 + \sum_{r=1}^{+\infty}\left[1 - \sum_{\lambda=0}^{t}\binom{w}{\lambda}(\epsilon^r)^{\lambda}(1-\epsilon^r)^{w-\lambda}\right]. \tag{21}$$

To summarize, the above analytical expressions (18)–(21) of the ANT for VE model are the same expressions given in (30), (31), (32) and (33) of [4] respectively obtained for the Hamming error model with the Hamming weight replaced by the L_1 weight. In [4], bounds, simple approximating formulae and plots are given to analyze the functions in (18)–(21). In particular, Fig. 5 shows that the plain ARQ scheme is inferior with respect to the diversity combining scheme, especially when ϵ is large and t is small. However, when ϵ is small or t is large, their performance is essentially similar.

5. Concluding remarks

This paper has presented a Divesting Combining Type-I Hybrid ARQ (DC-ARQ) system using t-Asymmetric Error Correcting/All Asymmetric Error Detecting (t-AEC/AAED) codes for the m-ary, $m \geq 2$, Asymmetric Varshamov (AV) channel model. We have also analyzed the Average Number of Transmissions (ANT) of a transmitted word X. We have demonstrated that the performance, in terms of the average number of transmissions, of the diversity combining ARQ protocol system is superior to that of the plain ARQ protocol system.

Conflict of interest

None.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.compeleceng.2019.06.015.

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