LE MATEMATICHE
Vol. LXXV (2020) – Issue I, pp. 57–66
doi: 10.4418/2020.75.1.3

# BREAKING THROUGH BORDERS WITH $\sigma$–HARMONIC MAPPINGS

GIOVANNI ALESSANDRINI - VINCENZO NESI

We consider mappings $U = (u^1, u^2)$, whose components solve an arbitrary elliptic equation in divergence form in dimension two, and whose respective Dirichlet data $\varphi^1, \varphi^2$ constitute the parametrization of a simple closed curve $\gamma$. We prove that, if the interior of the curve $\gamma$ is not convex, then we can find a parametrization $\Phi = (\varphi^1, \varphi^2)$ such that the mapping $U$ is not invertible.

*Dedicato a chi sconfina frontiere geografiche o ideologiche,*
*a chi travalica stereotipi e va oltre i pregiudizi.*

## 1. Introduction

Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ denote the unit disk. We denote by $\sigma = \sigma(x)$, $x \in B$, a possibly non–symmetric matrix having measurable entries and satisfying the ellipticity conditions

$$\begin{aligned} \sigma(x)\xi \cdot \xi &\geq K^{-1}|\xi|^2, \text{ for every } \xi \in \mathbb{R}^2 \ , x \in B \, , \\ \sigma^{-1}(x)\xi \cdot \xi &\geq K^{-1}|\xi|^2, \text{ for every } \xi \in \mathbb{R}^2 \ , x \in B \, , \end{aligned} \tag{1.1}$$

Received on June 28, 2019

*AMS 2010 Subject Classification:* 30C62, 35J55

*Keywords:* Elliptic equations, Beltrami operators, quasiconformal mappings.

G. A. has been supported by PRIN 2017 “Direct and inverse problems for partial differential equations: theoretical aspects and applications”.

V. N. has been supported by Sapienza - Ateneo 2018 Project “Stationary and Evolutive Problems in Mathematical Physics and Materials Science”.

for a given constant $K \geq 1$.

Given a homeomorphism $\Phi = (\varphi^1, \varphi^2)$ from the unit circle $\partial B$ onto a simple closed curve $\gamma \subset \mathbb{R}^2$, we denote by $D$ the bounded domain such that $\partial D = \gamma$.

Consider the mapping $U = (u^1, u^2) \in W^{1,2}_{loc}(B;\mathbb{R}^2) \cap C(\overline{B};\mathbb{R}^2)$ whose components are the solutions to the following Dirichlet problems

$$\begin{cases} \operatorname{div}(\sigma \nabla u^i) = 0, & \text{in} \quad B, \\ u^i = \varphi^i, & \text{on} \quad \partial B\ , i = 1,2\ . \end{cases} \tag{1.2}$$

We call such a $U$ a $\sigma$*-harmonic mapping*.

In the last two decades, it has been investigated, by the present authors and others, under which conditions can one assure that $U$ is an invertible mapping between $B$ and $D$.

The classical starting point for this issue is the celebrated Radò–Kneser–Choquet Theorem [10, 11, 13, 16] which asserts that assuming $\sigma = I$, the identity matrix, (that is: $u^1, u^2$ are harmonic) if $D$ is convex then $U$ is a homeomorphism. Generalizations to equations with variable coefficients have been obtained in [2, 7] and to certain nonlinear systems in [6, 8, 14]. Counterexamples [3, 10] show that if $D$ is not convex then the invertibility of $U$ may fail. In fact Choquet [10] proved that, whenever $D$ is not convex, there exists a homeomorphism $\Phi : \partial B \to \gamma$ such that the corresponding *harmonic* ($\sigma = I$) mapping $U$ is not invertible. The proof is crucially based on the classical mean value property of harmonic functions. Also the counterexample in [3] is limited to the purely harmonic case.

In [3, 5] the present authors investigated which additional conditions are needed for invertibility in the case of a possibly non–convex target $D$. Let us recall the main result in that direction.

**Theorem 1.1.** *Let $\Phi$ and $U$ be as above stated. Assume that the entries of $\sigma$ satisfy $\sigma_{ij} \in C^\alpha(\overline{B})$ for some $\alpha \in (0,1)$ and for every $i,j = 1,2$. Assume also that $U \in C^1(\overline{B};\mathbb{R}^2)$. The mapping $U$ is a diffeomorphism of $\overline{B}$ onto $\overline{D}$* if and only *if*

$$\det DU > 0 \quad \textit{everywhere on} \quad \partial B. \tag{1.3}$$

The object of the present note is to extend the construction by Choquet to $\sigma$-harmonic mappings with arbitrary coefficient matrix $\sigma$. The main result will be as follows.

**Theorem 1.2.** *Given a homeomorphism $\Psi : \partial B \to \gamma \subset \mathbb{R}^2$, let $D$ be the bounded domain such that $\partial D = \gamma$. Assume that $D$ is not convex. For every $\sigma = \sigma(x)$, satisfying (1.1), there exists a $C^\infty$ diffeomeomorphism $\Xi : \partial B \to \partial B$ such that, posing $\Phi = \Psi \circ \Xi$, the $\sigma$-harmonic mapping $U$ solving (1.2) is not invertible.*

Note that the parametrization $\Phi$ of the curve $\gamma$ is as much smooth as the original one $\Psi$. In particular, if $\Psi$ is $C^{1,\alpha}$ so is $\Phi$. Hence under the hypothesis of Hölder continuity of $\sigma$, it turns out that $U$ is $C^{1,\alpha}$ up to the boundary. As a consequence, we obtain that the hypothesis (1.3) in Theorem 1.1 is indeed non–trivial.

Let us illustrate what should be the features of a candidate counterexample: first we recall that Kneser [13] noticed that, in the purely harmonic case, if it is a–priori known that $U(B) \subset D$, then indeed $U$ is invertible, whether or not $D$ is convex. The observation by Kneser, is merely of topological nature, see also Duren [11, p. 31], and hence it actually extends to the $\sigma$-harmonic case, for any $\sigma$. That is, in order to violate invertibility in general, we must provide a mapping $U$ whose image exceeds $D$.

Viceversa, again by elementary topological arguments, if $U$ is one–to–one on all of $\overline{B}$, then it is an open mapping, hence a homeomorphism. Therefore it maps $\partial B$ onto $\gamma$ and $B$ onto $D$. In other terms, if $U$ maps some point of $B$ outside of $\overline{D}$, then it cannot be one–to–one.

In conclusion, in order to construct an example of a non–invertible $\sigma$–harmonic mapping $U$, whose boundary data $\Phi : \partial B \to \gamma$ is invertible, it is necessary and sufficient that $U$ *trespasses* the boundary $\gamma$, or in other words, that $U$ maps some interior point of $B$ outside of $\overline{D}$. This will be indeed the crux of our argument below.

## 2. $\sigma$–harmonic measure

Given $\sigma$ as in (1.1), and $\varphi \in C(\partial B)$, consider the scalar Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0, & \text{in} \quad B, \\ u = \varphi, & \text{on} \quad \partial B, \end{cases} \tag{2.1}$$

the, by now, classical theory of divergence structure elliptic equation tells us that there exists a unique weak solution $u \in W^{1,2}_{loc}(B) \cap C(\overline{B})$, see for instance [12, Theorem 8.30]. In particular the functional

$$C(\partial B) \ni \varphi \to u(0) \in \mathbb{R}$$

is bounded and linear. Hence there exists a Radon measure $\omega_\sigma$ on $\partial B$ such that

$$u(0) = \int_{\partial B} \varphi \mathrm{d}\omega_\sigma \,.$$

We call $\omega_\sigma$ the $\sigma$*–harmonic measure*. Note that, being $u \equiv 1$ the solution to (2.1) when $\varphi \equiv 1$, we trivially have $\omega_\sigma(\partial B) = 1$.

From examples due to Modica and Mortola and to Caffarelli, Fabes and Kenig [9, 15], it is known that the the $\sigma$–harmonic measure may not be absolutely continuous with the arclength measure. Still, some kind of continuity holds. For every $P \in \partial B$ and for every $r > 0$ let us denote

$$\Delta_r(P) = \partial B \cap B_r(P) \ .$$

We prove the following.

**Lemma 2.1.** *For every $P \in \partial B$ we have*

$$\lim_{r \to 0+} \omega_\sigma(\Delta_r(P)) = 0 \ . \tag{2.2}$$

*Proof.* Let $h_r$ be the Perron solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla h_r) = 0, & \text{in} \quad B, \\ h_r = \chi_{\Delta_r(P)}, & \text{on} \quad \partial B, \end{cases} \tag{2.3}$$

our aim is to prove that

$$\lim_{r \to 0+} h_r(0) = 0 \ .$$

We start considering the selfadjoint case, that is when $\sigma = \sigma^T$. We extend $\sigma = I$ outside of $B$.

Let $D_r$ be the annulus $B_2(P) \setminus \overline{B_r(P)}$, and let $c_r$ be the solution of the following Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla c_r) = 0, & \text{in} \quad D_r, \\ c_r = 0, & \text{on} \quad \partial B_2(P), \\ c_r = 1, & \text{on} \quad \partial B_r(P). \end{cases} \tag{2.4}$$

By the maximum principle, we have

$$0 \le h_r \le c_r \ , \text{on } B \setminus \overline{B_r(P)} \ .$$

Because of selfadjointness, we have

$$\int_{D_r} \sigma \nabla c_r \cdot \nabla c_r =$$

$$= \min \left\{ \int_{D_r} \sigma \nabla v \cdot \nabla v \,\middle|\, v \in W^{1,2}(D_r), v = 0 \text{ on } \partial B_2(P) \ , v = 1 \text{ on } \partial B_r(P) \right\} \ .$$

Choosing

$$v(x) = \frac{\log \frac{2}{|x-P|}}{\log \frac{2}{r}} \ ,$$

we compute

$$\int_{D_r} \sigma \nabla c_r \cdot \nabla c_r \leq K \int_{D_r} |\nabla v|^2 =$$

$$= 2\pi K \frac{1}{\log \frac{2}{r}} \to 0 \text{ as } r \to 0 \ .$$

Next we invoke a more or less standard form of Poincaré inequality, the emphasis being on the uniformity of the inequality with respect to the small radius $r$. A proof is outlined in Section 4 below.

**Lemma 2.2.** *For every $w \in W^{1,2}(D_r)$, having zero trace on $\partial B_2(P)$, we have*

$$\int_{D_r} w^2 \leq 16 \int_{D_r} |\nabla w|^2 \ .$$

Consequently we obtain $\|c_r\|_{W^{1,2}(D_r)} \to 0$ as $r \to 0$, and by an interior boundedness estimate [12, Theorem 8.17], $c_r(0) \to 0$, and the thesis follows.

Now we remove the symmetry assumption on $\sigma$.

It is well–known that there exists $k_r \in W^{1,2}(B)$, called the *stream function* of $h_r$ such that

$$\nabla k_r = J \sigma \nabla h_r \ , \tag{2.5}$$

where the matrix $J$ denotes the counterclockwise $90°$ rotation

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \tag{2.6}$$

see, for instance, [1]. Denoting

$$f = h_r + i k_r \ , \tag{2.7}$$

it is well–known that $f$ solves the Beltrami type equation

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \quad \text{in } B \ , \tag{2.8}$$

where, the so called complex dilatations $\mu, \nu$ are given by

$$\mu = \frac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{1 + \mathrm{Tr}\,\sigma + \det \sigma} \quad , \quad \nu = \frac{1 - \det \sigma + i(\sigma_{12} - \sigma_{21})}{1 + \mathrm{Tr}\,\sigma + \det \sigma} \ , \tag{2.9}$$

and satisfy the following ellipticity condition

$$|\mu| + |\nu| \leq k < 1 \ , \tag{2.10}$$

where the constant $k$ only depends on $K$, see [4, Proposition 1.8] and the notation $\mathrm{Tr}A$ is used for the trace of a square matrix $A$. We can also write

$$f_{\bar{z}} = \widetilde{\mu} f_z \text{ in } B \ ,$$

where $\widetilde{\mu}$ is defined almost everywhere by

$$\widetilde{\mu} = \mu + \frac{\overline{f_z}}{f_z}\nu \ ,$$

and consequently we obtain

$$\operatorname{div}(\widetilde{\sigma}\nabla h_r) = 0, \text{ in } B$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma} = \begin{bmatrix} \frac{|1-\widetilde{\mu}|^2}{1-|\widetilde{\mu}|^2} & -\frac{2\Im m(\widetilde{\mu})}{1-|\widetilde{\mu}|^2} \\ -\frac{2\Im m(\widetilde{\mu})}{1-|\widetilde{\mu}|^2} & \frac{|1+\widetilde{\mu}|^2}{1-|\widetilde{\mu}|^2} \end{bmatrix},$$

which satisfies uniform ellipticity conditions of the form (1.1) with a new constant $\widetilde{K}$ only dependent on $K$, see, for instance, [4], but in addition is symmetric. Hence we may proceed as before, just replacing $\sigma$ with $\widetilde{\sigma}$ in (2.3) and obtain again

$$\lim_{r\to 0+} h_r(0) = 0 \ . \qquad \square$$

The above Lemma can be seen as a continuity result for the cumulative distribution function associated to $\omega_\sigma$.

Given two points $P, Q \in \partial B$ we denote by $\widehat{PQ}$ the arc of the unit circle $\partial B$ which connects $P$ to $Q$, moving in the counterclockwise direction. The above Lemma, along with Harnack's inequality, implies the following straightforward consequence.

**Corollary 2.3.** *For every $P \in \partial B$, the function*

$$\partial B \ni Q \to \omega_\sigma(\widehat{PQ}) \in [0,1]$$

*is a strictly increasing, onto and continuous function, as $Q$ performs a full counterclockwise rotation on $\partial B$ starting from $P$ and ending on $P$ itself. Moreover, for every $P \in \partial B$, there exists exactly one point $Q \in \partial B$ such that*

$$\omega_\sigma(\widehat{PQ}) = \omega_\sigma(\widehat{QP}) = \frac{1}{2} \ .$$

## 3. Assembling a parametrization

Let us consider a given homeomorphism $\Psi : \partial B \to \gamma \subset \mathbb{R}^2$, let us fix two distinct points $a, b \in \gamma$. For any $\varepsilon > 0$ let $\alpha, \beta$ two disjoint simple open arcs in $\gamma$ such that

$$a \in \alpha \subset B_\varepsilon(a)\ ,\ b \in \beta \subset B_\varepsilon(b)\ .$$

Denote

$$A = \Psi^{-1}(a)\ ,\ B = \Psi^{-1}(b)\ ,$$

and

$$\widehat{A^-A^+} = \Psi^{-1}(\alpha)\ ,\ \widehat{B^-B^+} = \Psi^{-1}(\beta)\ .$$

Having fixed points $P, Q \in \partial B$ such that

$$\omega_\sigma(\widehat{PQ}) = \omega_\sigma(\widehat{QP}) = \frac{1}{2}$$

for any $r, 0 < r < 1$ we select a $C^\infty$ diffeomeomorphism $\Xi_r : \partial B \to \partial B$ such that

$$\Xi_r(\Delta_r(P)) = \widehat{A^+B^-}\ ,\ \Xi_r(\Delta_r(Q)) = \widehat{B^+A^-}\ .$$

In other words, setting $\widehat{P^-P^+} = \Delta_r(P)$, $\widehat{Q^-Q^+} = \Delta_r(Q)$, we need to construct a diffeomorphism $\Xi_r$ which maps the points $P^-, P^+, Q^-, Q^+$ to the points $A^+, B^-$, $B^+, A^-$ in their respective order. More generally, we can prove the following Lemma, whose proof is deferred to the next Section 4.

**Lemma 3.1.** *Let $N \geq 2$ and let $P_1, \ldots, P_N$ be distinct, cyclically ordered points on $\partial B$ and let $Q_1, \ldots, Q_N$ be another $N$–tuple of distinct, cyclically ordered points on $\partial B$. There exists a $C^\infty$ diffeomeomorphism $\Xi : \partial B \to \partial B$ such that $\Xi(P_n) = Q_n$ for every $n = 1, \ldots, N$.*

*Proof of Theorem 1.2.* We let $\Phi_r = \Psi \circ \Xi_r$ and consider $U = U_r$ as the solution to (1.2) when $\Phi = \Phi_r$. If $D$ is not convex, we may find two points $a, b \in \gamma$ such that the open segment with endpoints $a, b$ lies outside $\overline{D}$. In particular

$$\frac{1}{2}(a+b) \notin \overline{D}\ .$$

We have

$$U_r(0) = \int_{\partial B} \Phi_r \mathrm{d}\omega_\sigma$$

and we may split $\partial B$ into the four arcs $\widehat{P^-P^+}, \widehat{P^+Q^-}, \widehat{Q^-Q^+}, \widehat{Q^+P^-}$. Let $M > 0$ be such that $\gamma \subset B_M(0)$, then we evaluate

$$\left| \int_{\widehat{P^-P^+}} \Phi_r \mathrm{d}\omega_\sigma \right| \leq M \omega_\sigma(\Delta_r(P)) \to 0$$

as $r \to 0$ and, analogously,

$$\left| \int_{\widehat{Q^-Q^+}} \Phi_r \mathrm{d}\omega_\sigma \right| \leq M\omega_\sigma(\Delta_r(Q)) \to 0 \,.$$

Conversely, $\Phi_r(\widehat{P^+Q^-}) \subset \beta \subset B_\varepsilon(b)$ and $\Phi_r(\widehat{Q^+P^-}) \subset \alpha \subset B_\varepsilon(a)$, that is

$$|\Phi_r - b| < \varepsilon \text{ on } \widehat{P^+Q^-} \ , \ |\Phi_r - a| < \varepsilon \text{ on } \widehat{Q^+P^-} \ .$$

Note also that

$$\lim_{r\to 0+} \omega_\sigma(\widehat{P^+Q^-}) = \lim_{r\to 0+} \omega_\sigma(\widehat{Q^+P^-}) = \frac{1}{2} \,.$$

Hence we may find $r > 0$ small enough and a constant $C > 0$ such that

$$|U_r(0) - \frac{1}{2}(a+b)| \leq C\varepsilon$$

and, in conclusion, with $r, \varepsilon$ small enough, $U = U_r$ is such that

$$U(0) \notin \overline{D} \,. \qquad \square$$

## 4. Auxiliary proofs

*Proof of Lemma 2.2.* As is customary in this context, it suffices to consider $w \in C^1(\overline{D_r})$, $w(P + 2e^{i\vartheta}) = 0$ for all $\vartheta$. Hence, for every $\rho \in (r, 2)$ we have

$$w^2(P + \rho e^{i\vartheta}) = -\int_\rho^2 \frac{\partial}{\partial s} w^2(P + se^{i\vartheta})\mathrm{d}s \,,$$

hence

$$w^2(P + \rho e^{i\vartheta}) \leq 2\int_\rho^2 |w||\nabla w|(P + se^{i\vartheta})\mathrm{d}s \,.$$

Consequently

$$\int_{D_r} w^2 \leq 2\int_0^{2\pi} \mathrm{d}\vartheta \int_r^2 \rho \mathrm{d}\rho \int_\rho^2 |w||\nabla w|(P + se^{i\vartheta})\mathrm{d}s \,,$$

and, using the inequalities $0 < r \leq \rho \leq s$,

$$\int_{D_r} w^2 \leq 2\int_0^{2\pi} \mathrm{d}\vartheta \int_r^2 \mathrm{d}\rho \int_\rho^2 |w||\nabla w|(P + se^{i\vartheta})s\mathrm{d}s \ \leq$$
$$\leq 2\int_0^{2\pi} \mathrm{d}\vartheta \int_0^2 \mathrm{d}\rho \int_r^2 |w||\nabla w|(P + se^{i\vartheta})s\mathrm{d}s \ ,$$

that is

$$\int_{D_r} w^2 \leq 4\int_{D_r} |w||\nabla w|,$$

and by Schwarz inequality the thesis follows. $\square$

*Proof of Lemma 3.1.* Up to rotations, we may assume $P_n = e^{i\vartheta_n}, Q_n = e^{i\varphi_n}, n = 1,\dots,N$ where

$$0 = \vartheta_1 < \ldots < \vartheta_N < 2\pi \ , 0 = \varphi_1 < \ldots < \varphi_N < 2\pi \ .$$

We may construct a continuous, strictly increasing, piecewise linear function $f$ mapping the interval $[0,2\pi]$ onto itself, such that

$$f(\vartheta_n) = \varphi_n \text{ for every } n = 1,\dots,N \ ,$$

we may consider to extend $f$ to $\mathbb{R}$ in such a way that $f(\vartheta) - \vartheta$ is $2\pi$–periodic. We may also require that its corner points $\xi_1,\dots,\xi_J \in [0,2\pi]$ are distinct from the points

$$0 = \vartheta_1,\dots,\vartheta_N,\vartheta_{N+1} = 2\pi \ .$$

Let $\delta = \min\{|\vartheta_n - \xi_j||n = 1,\dots,N+1, j = 1,\dots,J\}$. Let $\chi_\varepsilon$ be a family of $C^\infty$, mollifying kernels, supported in $[-\varepsilon,\varepsilon]$, even symmetric with respect to 0. Fixing $\varepsilon < \delta$ and denoting

$$g = \chi_\varepsilon * f \ ,$$

we compute $g(\vartheta_n) = f(\vartheta_n)$ for all $n$, we obtain that $g$ is $C^\infty$ with positive derivative everywhere and we conclude that

$$\Xi(e^{i\vartheta}) = e^{ig(\vartheta)}$$

fulfils the thesis. □

## REFERENCES

[1] Alessandrini, G. & Magnanini, R.: Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions. SIAM J. Math. Anal., **25** 5, 1259–1268 (1994).

[2] Alessandrini, G. & Nesi, V.: Univalent $\sigma$-harmonic mappings. Arch. Ration. Mech. Anal., **158** 2, 155–171 (2001).

[3] Alessandrini, G. & Nesi, V.: Invertible harmonic mappings, beyond Kneser. Ann. Scuola Norm. Sup. Pisa, Cl. Sci. V, **8** 5, 451–468 (2009). Errata Corrige. Ann. Scuola Norm. Sup. Pisa, Cl. Sci. V, **17** 2, 815–818 (2017).

[4] Alessandrini, G. & Nesi, V.: Beltrami operators, non–symmetric elliptic equations and quantitative Jacobian bounds. Ann. Acad. Sci. Fenn. Math., **34** 1, 47–67 (2009).

[5] Alessandrini, G. & Nesi, V.: Globally diffeomorphic $\sigma$–harmonic mappings, arXiv:1906.00902, submitted.

[6] Alessandrini, G. & Sigalotti, M.: Geometric properties of solutions to the anisotropic p-Laplace equation in dimension two, Ann. Acad. Sci. Fenn. Math. **21**, 249–266 (2001).

[7] Bauman, P., Marini, A. & Nesi, V.: Univalent solutions of an elliptic system of partial differential equations arising in homogenization. Indiana Univ. Math. J., **50** 2, 747–757 (2001).

[8] Bauman, P. & Phillips, D.: Univalent minimizers of polyconvex functionals in two dimensions. Arch. Rational Mech. Anal., **126** 2, 161–181 (1994).

[9] Caffarelli, L., Fabes, E. & Kenig, C.: Completely Singular Elliptic-Harmonic Measures. Indiana University Mathematics Journal, 30(6), 917-924, (1981).

[10] Choquet, G.: Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, Bull. Sci. Math. (2) **69** 156–165, (1945).

[11] Duren, P.: Harmonic mappings in the plane. Cambridge Tracts in Mathematics, vol. 156, Cambridge University Press, Cambridge (2004).

[12] Gilbarg, D. and Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. Grundlehren der Mathematischen Wissenschaften, vol 224, 2nd edn, Springer, Berlin (1983).

[13] Kneser, H.: Lösung der Aufgabe 41. Jber. Deutsch. Math.-Verein., **35**, 123–124 (1926).

[14] Iwaniec, T., Koski, A. & Onninen, J.: Isotropic p-harmonic systems in 2D Jacobian estimates and univalent solutions. Rev. Mat. Iberoam. **32**, 57–77 (2016).

[15] Modica, L. & Mortola, S.: Construction of a singular elliptic-harmonic measure, Manuscripta Math. **33** 1, 81–98 (1980) .

[16] Radó, T.: Aufgabe 41, Jber. Deutsch. Math.-Verein. **35**, 49 (1926).

*GIOVANNI ALESSANDRINI*
*formerly Dipartimento di Matematica e Geoscienze*
*Università di Trieste*
*Via Valerio 12/b, 34100 Trieste, Italia*
*e-mail:* `55gioale@gmail.com`

*VINCENZO NESI*
*Dipartimento di Matematica "G. Castelnuovo"*
*Sapienza, Università di Roma*
*Piazzale A. Moro 2, 00185 Roma, Italia*
*e-mail:* `nesi@mat.uniroma1.it`