

RANDOM TIME-CHANGE WITH INVERSES OF MULTIVARIATE SUBORDINATORS: GOVERNING EQUATIONS AND FRACTIONAL DYNAMICS

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ABSTRACT. It is well-known that compositions of Markov processes with inverse subordinators are governed by integro-differential equations of generalized fractional type. This kind of processes are of wide interest in statistical physics as they are connected to anomalous diffusions. In this paper we consider a generalization; more precisely we mean componentwise compositions of \mathbb{R}^d -valued Markov processes with the components of an independent multivariate inverse subordinator. As a possible application, we present a model of anomalous diffusion in anisotropic medium, which is obtained as a weak limit of suitable continuous-time random walks.

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1. INTRODUCTION

A multivariate subordinator is a \mathbb{R}^d -valued Lévy process with non-decreasing (possibly dependent) marginal components. In [1], the authors studied the subordinated process $(X_1(H_1(t)), \dots, X_d(H_d(t)))$, where $(X_1(t), \dots, X_d(t))$ is a time-homogeneous \mathbb{R}^d -valued Markov process and $(H_1(t), \dots, H_d(t))$ is an independent multivariate subordinator. This generalized the well-known case of a common random time $H(t)$ for all the components (for further developments consult [39], [40], [41]). Time-changes of Markov processes with multivariate subordinators also appear in some references in the literature with applications to finance; see e.g. [24].

In the same spirit, this paper focuses on the time-changed process

$$(X_1(L_1(t)), \dots, X_d(L_d(t))), \quad t \geq 0 \quad (1.1)$$

where $(X_1(t), \dots, X_d(t))$ is a \mathbb{R}^d -valued Markov process, while, for each $j = 1, \dots, d$, $L_j(t)$ is the inverse (or right-continuous hitting time) of $H_j(t)$, i.e.

$$L_j(t) = \inf\{x > 0 : H_j(x) > t\},$$

which is assumed to be independent of $(X_1(t), \dots, X_d(t))$. Without loss of generality, we assume that $(X_1(0), \dots, X_d(0)) = (0, \dots, 0)$ almost surely.

The literature inspiring our study is vast. Indeed many works, such as [2], [19], [34], [32], [33], [35], [36], concern processes of the form

$$(X_1(L(t)), \dots, X_d(L(t))), \quad t \geq 0, \quad (1.2)$$

where each component of the original process is time-changed by the same random time $L(t) = \inf\{x > 0 : H(x) > t\}$, namely the inverse of the same univariate subordinator.

There exists a well established theory for the stochastic process in (1.2). An interesting fact is that the density $p(x, t)$ of (1.2), $x \in \mathbb{R}^d$, is governed by integro-differential equations of the form

$$\mathcal{D}_t p(x, t) = \mathcal{G}p(x, t), \quad x \neq 0, \quad (1.3)$$

where \mathcal{D}_t is the operator defined by

$$\mathcal{D}_t h(t) := \int_0^\infty (h(t) - h(t - \tau)) \nu(d\tau) \quad (1.4)$$

ν is the Lévy measure of the underlying subordinator $H(t)$ and \mathcal{G} is the dual to the generator of $(X_1(t), \dots, X_d(t))$. The operator \mathcal{D}_t is called generalized fractional derivative because, when $L(t)$ is an inverse stable subordinator of index $\alpha \in (0, 1)$, it reduces to the Caputo fractional derivative

$$\frac{d^\alpha}{dt^\alpha} f(t) = \int_0^\infty \frac{d}{d\tau} f(\tau) \frac{(t - \tau)^{-\alpha-1}}{\Gamma(1 - \alpha)} d\tau \quad (1.5)$$

(see e.g. [18] p. 92 for details). Moreover, if $(X_1(t), \dots, X_d(t))$ is a Lévy process, then processes defined in (1.2) can be seen as scaling limits of suitable continuous time random walks (hereafter CTRW). We recall that a CTRW is defined by a sequence of i.i.d. jumps $Y_i \in \mathbb{R}^d$ separated by i.i.d. inter-arrival times $J_i \in \mathbb{R}^+$.

In the special case where $(X_1(t), \dots, X_d(t))$ is a Brownian motion and $L(t)$ is the inverse of a stable subordinator, then the process (1.2) is a so-called subdiffusion, which has great interest in many areas of statistical physics (on this point, consult e.g. [26], [27] and [37]). See also [20] and [21], for the more general case where the external process is the fractional Brownian motion. Basically, a subdiffusion

models the case where the particle is subject to a trapping effect which delays the motion with respect to the simple Brownian process; clearly, in this model it is assumed that the trapping effect is the same in all directions (the external medium is isotropic) and thus the time-changed process is isotropic as well as the Brownian motion. It is natural, then, to ask what happens if the motion of the particle takes place in an anisotropic medium, where the trapping effect is different depending on the direction. This is the strongest physical argument that inspired our investigation of processes in (1.1).

To our knowledge, the study of the model (1.1) is a completely new problem. In this paper we restrict our analysis to the case $d = 2$. Our choice is motivated by technical problems; indeed, some of the results presented in this paper hold only for $d = 2$, and, moreover, some calculations are quite cumbersome even with this restriction. However, we are sure that the discussion of the 2-dimensional case, besides making the present exposure clearer, will be useful for possible future studies of the multidimensional counterpart.

Now we give a brief description of the main results and the outline of the paper. In Section 2 we recall the definition of bivariate subordinators and we find some auxiliary results. A crucial part of the paper is Section 3, where we study in depth some distributional properties and a governing equation of the biparameter process $(L_1(t_1), L_2(t_2))$; in this way we present an extension of the well-known theory of inverse subordinators to the 2-dimensional case. In Section 4 we focus on the biparameter process $(X_1(L_1(t_1)), X_2(L_2(t_2)))$ and we prove that its density $p(x_1, x_2, t_1, t_2)$ solves an equation of the following form

$$\mathcal{D}_{t_1, t_2} p(x_1, x_2, t_1, t_2) = \mathcal{G}p(x_1, x_2, t_1, t_2), \quad x_1, x_2 \neq 0, \quad (1.6)$$

where

$$\mathcal{D}_{t_1, t_2} h(t_1, t_2) := \int_0^\infty \int_0^\infty (h(t_1, t_2) - h(t_1 - \tau_1, t_2 - \tau_2)) \phi(d\tau_1, d\tau_2),$$

ϕ is the Lévy measure of the underlying bi-dimensional subordinator $(H_1(t_1), H_2(t_2))$ and \mathcal{G} is the dual to the generator of $(X_1(t), X_2(t))$. The operator \mathcal{D}_{t_1, t_2} in (1.6) is a bi-dimensional version of the generalized fractional derivative appearing in (1.3); it was already introduced in [30] in advection-dispersion equations governing multidimensional stable Lévy motions, though acting on the space variables.

Finally, in Section 5, we focus on the special case where $(X_1(t), X_2(t))$ is a Brownian motion $(B_1(t), B_2(t))$, while $(L_1(t_1), L_2(t_2))$ is the inverse of a bivariate stable subordinator. In particular, we illustrate how to construct a CTRW converging to $(B_1(L_1(t)), B_2(L_2(t)))$ under a suitable scaling limit.

Notation: For simplicity, throughout the paper we will often denote a process $\{X(t), t \geq 0\}$ by $X(t)$; moreover, we will often denote a bi-parameter process $\{(X(t_1), Y(t_2)), t_1 \geq 0, t_2 \geq 0\}$ by $(X(t_1), Y(t_2))$.

2. BIVARIATE SUBORDINATORS

This section is devoted to bivariate subordinators in the sense of [1]. In Subsection 2.1, we review some known facts and we connect them to the general theory of multivariate Lévy processes and their governing equation (see [34], Chapt. 6, and also [11] and [30] for the special case of multivariate stable processes). Then, in Subsection 2.2 we will present some original results.

2.1. Preliminaries. A Lévy process $\{(H_1(t), H_2(t)), t \geq 0\}$ is said to be a bivariate subordinator if its components are both a.s. non-decreasing; we restrict our attention to the case of pure jump subordinators, having no drift. The characteristic function of $(H_1(t), H_2(t))$ can be written (see [1]) as

$$\mathbb{E}e^{i(\alpha H_1(t) + \beta H_2(t))} = e^{t\sigma(\alpha, \beta)}, \quad \alpha, \beta \in \mathbb{R}, \quad (2.1)$$

where

$$\sigma(\alpha, \beta) = \int \int_{\mathbb{R}_+^2} (e^{i(\alpha x_1 + \beta x_2)} - 1) \phi(dx_1, dx_2) \quad (2.2)$$

and $\phi(dx_1, dx_2)$ denotes the Lévy measure; thus ϕ is a measure on $\mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ such that

$$\int \int_{\mathbb{R}_+^2} \min(\sqrt{x_1^2 + x_2^2}, 1) \phi(dx_1, dx_2) < \infty.$$

An equivalent formulation can be given in terms of the Laplace transforms, i.e.

$$\mathbb{E}e^{-\eta_1 H_1(t) - \eta_2 H_2(t)} = e^{-tS(\eta_1, \eta_2)} \quad \eta_1, \eta_2 \geq 0, \quad (2.3)$$

where

$$S(\eta_1, \eta_2) = \int \int_{\mathbb{R}_+^2} (1 - e^{-\eta_1 x_1 - \eta_2 x_2}) \phi(dx_1, dx_2) \quad (2.4)$$

is a bivariate Bernstein function (in the sense of [8], Chapt. 4). The marginal processes are univariate subordinators, hence

$$\mathbb{E}e^{-\eta_k H_k(t)} = e^{-tT_k(\eta_k)} \quad \eta_k \geq 0 \quad k = 1, 2, \quad (2.5)$$

where

$$T_k(\eta_k) = \int_0^\infty (1 - e^{-\eta_k x}) \nu_k(dx) \quad (2.6)$$

is a Bernstein function and ν_k is the Lévy measure associated to H_k . Throughout this paper we always assume that both H_1 and H_2 have infinite activity, namely

$$\nu_k[0, \infty) = \infty \quad k = 1, 2.$$

It is clear that, if H_1 and H_2 are independent, then the Lévy measure ϕ is supported on the coordinate axes, i.e. it has the form

$$\phi(dx_1, dx_2) = \nu_1(dx_1)\delta_0(dx_2) + \nu_2(dx_2)\delta_0(dx_1), \quad (2.7)$$

δ_0 denoting the Dirac delta measure; then

$$S(\eta_1, \eta_2) = T_1(\eta_1) + T_2(\eta_2). \quad (2.8)$$

Remark 2.1. *In view of some possible applications, we briefly recall some connections with the theory of copulas. It is well-known that a copula allows to separate the dependence structure of a random vector from its univariate marginal distributions. Moreover, since ϕ is a measure, it is possible to define a suitable notion of copula, that is the Lévy copula; see e.g. Section 3 in [15] for its definition and basic properties (in particular Theorem 3.6. in [15] can be considered as the analogue of the Sklar's Theorem for Lévy copulas). Lévy copulas characterize the dependence among components of vector valued Lévy processes and they are also used in some estimation problems (see e.g. [12] and [13]).*

In general, if $(H_1(t), H_2(t))$ has joint density $q_*(x_1, x_2, t)$, then (see [34], Chapt. 6) the following equation holds

$$\frac{\partial}{\partial t} q_*(x_1, x_2, t) = -\mathcal{D}_{x_1, x_2} q_*(x_1, x_2, t), \quad (2.9)$$

where

$$\mathcal{D}_{x_1, x_2} h(x_1, x_2) := \int \int_{\mathbb{R}_+^2} \left(h(x_1, x_2) - h(x_1 - y_1, x_2 - y_2) \right) \phi(dy_1, dy_2) \quad (2.10)$$

on a suitable class of functions h . The operator (2.10) can also be defined by means of its Laplace symbol:

$$\int_0^\infty \int_0^\infty e^{-\eta_1 x_1 - \eta_2 x_2} \mathcal{D}_{x_1, x_2} f(x_1, x_2) dx_1 dx_2 = S(\eta_1, \eta_2) \hat{f}(\eta_1, \eta_2) \quad (2.11)$$

where $\hat{f}(\eta_1, \eta_2) := \int_0^\infty \int_0^\infty e^{-\eta_1 x_1 - \eta_2 x_2} f(x_1, x_2) dx_1 dx_2$ and S has been defined in (2.4). Equation (2.9) comes up in the more general setting of Proposition 2.5 below.

The above facts extend the best known results holding for univariate subordinators; indeed, for each $i = 1, 2$, the marginal density $q_i(x, t)$ of $H_i(t)$ solves

$$\frac{\partial}{\partial t} q_i(x, t) = -\mathcal{D}_x^{(i)} q_i(x, t),$$

where the operator on the right-hand side is

$$\mathcal{D}_x^{(i)} h(x) = \int_0^\infty (h(x) - h(x - y)) \nu_i(dy), \quad (2.12)$$

whose Laplace symbol is defined by

$$\int_0^\infty e^{-\eta x} \mathcal{D}_x^{(i)} h(x) dx = T_i(\eta) \tilde{h}(\eta).$$

Now we present two examples of bivariate subordinators.

Example 2.2. Let $(H_1^\alpha(t), H_2^\alpha(t))$ be a bivariate stable subordinator of index $\alpha \in (0, 1)$, i.e. a bivariate stable process with non decreasing components. Its Lévy measure can be expressed in polar coordinates as

$$\phi_\alpha(dr, d\theta) = \frac{C\alpha}{r^{1+\alpha}} dr M(d\theta) \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad (2.13)$$

where $M(d\theta)$ is a measure on the arc of circle $\{(x, y) \in \mathbb{R}_+^2 : x^2 + y^2 = 1\}$. If $H_1^\alpha(t)$ and $H_2^\alpha(t)$ are independent, we can write

$$M(d\theta) = p\delta_0(d\theta) + (1 - p)\delta_{\frac{\pi}{2}}(d\theta)$$

for some $p \in (0, 1)$, where δ_x denotes the Dirac delta in x .

By simple calculations we see that the characteristic exponent (2.2) has the form

$$\begin{aligned} \sigma_\alpha(\gamma, \beta) &= \int \int_{\mathbb{R}_+^2} (e^{i(\gamma x_1 + \beta x_2)} - 1) \phi_\alpha(dx_1, dx_2) \\ &= \int_0^{\pi/2} \left(\int_0^\infty (e^{ir(\gamma \cos \theta + \beta \sin \theta)} - 1) \frac{C\alpha}{r^{\alpha+1}} dr \right) M(d\theta) \\ &= -C\Gamma(1 - \alpha) \int_0^{\pi/2} [-i(\gamma \cos \theta + \beta \sin \theta)]^\alpha M(d\theta) \end{aligned} \quad (2.14)$$

where, for the last equality, we have taken into account the well-known formula

$$\int_0^\infty (e^{i\xi x} - 1) \frac{\alpha dx}{x^{\alpha+1}} = -\Gamma(1-\alpha)(-i\xi)^\alpha.$$

Hence, the Laplace exponent has the form

$$S_\alpha(\eta_1, \eta_2) = C\Gamma(1-\alpha) \int_0^{\pi/2} (\eta_1 \cos \theta + \eta_2 \sin \theta)^\alpha M(d\theta). \quad (2.15)$$

Observe that $(\eta_1 \cos \theta + \eta_2 \sin \theta)$ is the Laplace symbol of $(\cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2})$, which is the directional derivative along the unit vector $(\cos \theta, \sin \theta)$. Then, by taking the Laplace inverse of (2.15), the operator in (2.10) here reduces to the following pseudo-differential operator

$$\mathcal{D}_{x_1, x_2}^\alpha f(x_1, x_2) = C\Gamma(1-\alpha) \int_0^{\pi/2} (\cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2})^\alpha M(d\theta). \quad (2.16)$$

Note that (2.16) represents the average under $M(d\theta)$ of the fractional power of the directional derivative along $(\cos \theta, \sin \theta)$. Thus equation (2.9) takes the form

$$\frac{\partial}{\partial t} q_*(x_1, x_2, t) = -\mathcal{D}_{x_1, x_2}^\alpha q_*(x_1, x_2, t)$$

which is a particular case of the equation governing multivariate stable processes (on this point consult [11] and [30]).

Example 2.3. Let $Y_1(t)$, $Y_2(t)$ and $Z(t)$ be independent subordinators, and let $T_1(\cdot)$, $T_2(\cdot)$ and $G(\cdot)$ be the respective Bernstein functions. For any $c_1, c_2 \in \mathbb{R}^+$, we consider the bivariate subordinator

$$\begin{cases} H_1(t) = Y_1(t) + c_1 Z(t) \\ H_2(t) = Y_2(t) + c_2 Z(t). \end{cases} \quad (2.17)$$

Such example is motivated by the model studied in [44] and [24]; here the authors considered multivariate subordinators such that each component is a sum of an idiosyncratic and a common term, which has some interest in finance modelling. It is easy to see that (2.17) is characterized by the bivariate Bernstein function

$$S(\eta_1, \eta_2) = T_1(\eta_1) + T_2(\eta_2) + G(c_1\eta_1 + c_2\eta_2).$$

Then the support of the Lévy measure is the union of the coordinate axes and the line with direction (c_1, c_2) passing through the origin; for the explicit expression of the Lévy measure consult Prop. 3.1 in [44].

It is interesting to note that, if $Y_1(t)$, $Y_2(t)$ and $Z(t)$ are stable subordinators with parameter $\alpha \in (0, 1)$, then the bivariate Bernstein function reads

$$S(\eta_1, \eta_2) = \eta_1^\alpha + \eta_2^\alpha + (c_1\eta_1 + c_2\eta_2)^\alpha;$$

thus the operator (2.10) reduces to the pseudo-differential operator

$$\mathcal{D}_{x_1, x_2}^\alpha = \frac{\partial^\alpha}{\partial x_1^\alpha} + \frac{\partial^\alpha}{\partial x_2^\alpha} + \left(c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} \right)^\alpha,$$

where the last summand is the fractional power of the directional derivative.

2.2. Some results. For technical reasons that will be clear in the next section, it is sometimes convenient to consider the related bi-parameter process

$$\{(H_1(t_1), H_2(t_2)), t_1 \geq 0, t_2 \geq 0\}$$

rather than the process $\{(H_1(t), H_2(t)), t \geq 0\}$. The following result characterizes its distribution.

Proposition 2.4. *The double Laplace transform of $(H_1(t_1), H_2(t_2))$ reads*

$$\begin{aligned} \mathbb{E}e^{-(\alpha H_1(t_1) + \beta H_2(t_2))} &= e^{-t_1 S(\alpha, \beta)} e^{-(t_2 - t_1) T_2(\beta)} \mathbf{1}_{\{t_2 \geq t_1\}} \\ &\quad + e^{-t_2 S(\alpha, \beta)} e^{-(t_1 - t_2) T_1(\alpha)} \mathbf{1}_{\{t_1 > t_2\}} \quad \alpha, \beta \geq 0, \end{aligned} \quad (2.18)$$

where S , T_1 and T_2 have been defined in (2.4) and (2.5), while $\mathbf{1}_A$ denotes the indicator function of the set A .

Proof. The desired equality can be checked as follows:

$$\begin{aligned} &\mathbb{E}e^{-(\alpha H_1(t_1) + \beta H_2(t_2))} \\ &= \mathbb{E}\left[e^{-(\alpha H_1(t_1) + \beta H_2(t_2))} e^{-\beta(H_2(t_2) - H_2(t_1))}\right] \mathbf{1}_{\{t_2 \geq t_1\}} \\ &\quad + \mathbb{E}\left[e^{-(\alpha H_1(t_2) + \beta H_2(t_2))} e^{-\alpha(H_1(t_1) - H_1(t_2))}\right] \mathbf{1}_{\{t_1 > t_2\}} \\ &= \mathbb{E}\left[e^{-(\alpha H_1(t_1) + \beta H_2(t_1))}\right] \mathbb{E}\left[e^{-\beta(H_2(t_2) - H_2(t_1))}\right] \mathbf{1}_{\{t_2 \geq t_1\}} \\ &\quad + \mathbb{E}\left[e^{-(\alpha H_1(t_2) + \beta H_2(t_2))}\right] \mathbb{E}\left[e^{-\alpha(H_1(t_1) - H_1(t_2))}\right] \mathbf{1}_{\{t_1 > t_2\}} \\ &= e^{-t_1 S(\alpha, \beta)} e^{-(t_2 - t_1) T_2(\beta)} \mathbf{1}_{\{t_2 \geq t_1\}} + e^{-t_2 S(\alpha, \beta)} e^{-(t_1 - t_2) T_1(\alpha)} \mathbf{1}_{\{t_1 > t_2\}}, \end{aligned}$$

where we repeatedly used independence of the increments. \square

From now on we assume that the random vector $(H_1(t), H_2(t))$ has joint and marginal densities, respectively defined by

$$\begin{aligned} P(H_1(t) \in dx_1, H_2(t) \in dx_2) &:= q_*(x_1, x_2, t) dx_1 dx_2, \quad t > 0 \\ P(H_i(t) \in dx_i) &:= q_i(x_i, t) dx_i \quad i = 1, 2 \quad t > 0. \end{aligned}$$

Moreover

$$P(H_1(0) \in dx_1, H_2(0) \in dx_2) := \delta_0(dx_1) \delta_0(dx_2). \quad (2.19)$$

Hence we introduce the joint density of $(H_1(t_1), H_2(t_2))$ by

$$P(H_1(t_1) \in dx_1, H_2(t_2) \in dx_2) := q(x_1, x_2, t_1, t_2) dx_1 dx_2 \quad \text{for all } t_1, t_2 > 0,$$

where

$$\begin{aligned} q(x_1, x_2, t_1, t_2) &= \int_0^\infty q_*(x_1, z, t_1) q_2(x_2 - z, t_2 - t_1) dz \mathbf{1}_{\{t_1 \leq t_2\}} \\ &\quad + \int_0^\infty q_*(z, x_2, t_2) q_1(x_1 - z, t_1 - t_2) dz \mathbf{1}_{\{t_1 > t_2\}}, \end{aligned}$$

which satisfies

$$q(x_1, x_2, t, t) = q_*(x_1, x_2, t).$$

Moreover we have that

$$\lim_{t_1 \rightarrow 0} q(x_1, x_2, t_1, t_2) = \delta_0(x_1) q_2(x_2, t_2) \quad (2.20)$$

$$\lim_{t_2 \rightarrow 0} q(x_1, x_2, t_1, t_2) = \delta_0(x_2) q_1(x_1, t_1) \quad (2.21)$$

in the sense of distributions.

The following proposition provides a governing equation for the bi-parameter process $(H_1(t_1), H_2(t_2))$, which involves the operator \mathcal{D}_{x_1, x_2} defined in (2.10). This extends (2.9) to the case where $t_1 \neq t_2$.

Proposition 2.5. *If $t_1 \neq t_2$, then joint density $q(x_1, x_2, t_1, t_2)$ is the fundamental solution of the equation*

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) q(x_1, x_2, t_1, t_2) = -\mathcal{D}_{x_1, x_2} q(x_1, x_2, t_1, t_2) \quad t_1 \neq t_2 \quad t_1, t_2 > 0, \quad (2.22)$$

under the initial conditions (2.20) and (2.21). In the case $t_1 = t_2 = t$, we have

$$\frac{\partial}{\partial t} q_*(x_1, x_2, t) = -\mathcal{D}_{x_1, x_2} q_*(x_1, x_2, t) \quad t > 0, \quad (2.23)$$

under the initial condition (2.19).

Proof. The Laplace transform given by Proposition 2.4 is differentiable for $t_1 \neq t_2$, and, by applying the operator $\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)$ to both sides, we have

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \mathbb{E} e^{-(\alpha H_1(t_1) + \beta H_2(t_2))} = -S(\alpha, \beta) \mathbb{E} e^{-(\alpha H_1(t_1) + \beta H_2(t_2))}, \quad t_1 \neq t_2. \quad (2.24)$$

Then, taking into account (2.11), by the Laplace inversion of both sides we get (2.22). Finally, for $t_1 = t_2 = t$, by (2.3) we have

$$\frac{\partial}{\partial t} \mathbb{E} e^{-\eta_1 H_1(t) - \eta_2 H_2(t)} = -S(\eta_1, \eta_2) e^{-tS(\eta_1, \eta_2)}$$

and the inverse Laplace transform of both sides gives (2.23). \square

We now present a further result that will be useful later. For $t_1, t_2 > 0$, we define the tail of the Lévy measure ϕ as

$$\bar{\phi}(t_1, t_2) = \phi((t_1, \infty) \times (t_2, \infty)) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} \phi(dx_1, dx_2). \quad (2.25)$$

Note that, in the case of independence, formula (2.7) yields $\bar{\phi}(t_1, t_2) = 0$ for each t_1 and t_2 , because ϕ is supported on the coordinate axes.

Proposition 2.6. *The double Laplace transform of (2.25) reads*

$$\int_0^{\infty} \int_0^{\infty} e^{-\eta_1 t_1 - \eta_2 t_2} \bar{\phi}(t_1, t_2) dt_1 dt_2 = \frac{T_1(\eta_1) + T_2(\eta_2) - S(\eta_1, \eta_2)}{\eta_1 \eta_2}, \quad (2.26)$$

where T_1, T_2 and S have been defined in (2.6) and (2.4).

Proof. It is sufficient to start from expression (2.4) and apply consecutively the integration by parts in both variables. The calculations exploit that

$$\nu_1(dx_1) = \int_0^{\infty} \phi(dx_1, dx_2) \quad \text{and} \quad \nu_2(dx_2) = \int_0^{\infty} \phi(dx_1, dx_2).$$

The result holds by simple algebraic manipulations. \square

3. INVERSES OF BIVARIATE SUBORDINATORS

Let $\{(H_1(x), H_2(x)), x \geq 0\}$ be a bivariate subordinator. We here define its inverse $\{(L_1(t), L_2(t)), t \geq 0\}$ as the process with marginal components

$$L_i(t) = \inf\{x : H_i(x) > t\}, \quad i = 1, 2.$$

We also consider the related bi-parameter process $\{(L_1(t_1), L_2(t_2)), t_1 \geq 0, t_2 \geq 0\}$, such that

$$L_i(t_i) = \inf\{x : H_i(x) > t_i\}, \quad i = 1, 2;$$

then

$$P(L_1(t_1) > x_1, L_2(t_2) > x_2) = P(H_1(x_1) \leq t_1, H_2(x_2) \leq t_2). \quad (3.1)$$

It is well-known from the theory of univariate subordinators and their inverses (see, for example, [19]) that the marginal densities $l_1(x_1, t_1)$ and $l_2(x_2, t_2)$ solve the generalized fractional Cauchy problems

$$\begin{cases} \frac{\partial}{\partial x_i} l_i(x_i, t_i) = -\mathcal{D}_{t_i} l_i(x_i, t_i) \\ l_i(0, t_i) = -\mathcal{D}_{t_i} \theta(t_i) = \bar{\nu}(t_i) \end{cases} \quad i = 1, 2, \quad (3.2)$$

where $\theta(\cdot)$ denotes the Heaviside function, \mathcal{D}_t is the operator defined in (1.4), while

$$\bar{\nu}_k(t_k) = \nu_k[t_k, \infty) \quad k = 1, 2$$

is the tail of the Lévy measure. Moreover, it is well-known that

$$\int_0^\infty e^{-\eta_i t_i} l_i(x_i, t_i) dt_i = \frac{T_i(\eta_i)}{\eta_i} e^{-T_i(\eta_i) x_i} \quad i = 1, 2$$

and the space-time Laplace transforms read

$$\int_0^\infty \int_0^\infty e^{-\eta_i t_i} e^{-\xi_i x_i} l_i(x_i, t_i) dx_i dt_i = \frac{T_i(\eta_i)}{\eta_i(\xi_i + T_i(\eta_i))} \quad i = 1, 2. \quad (3.3)$$

The aim of the next subsections is to find the bi-dimensional counterparts of such results.

3.1. Distributional properties. We here assume that $(H_1(x_1), H_2(x_2))$ is an absolutely continuous random vector with density $q(t_1, t_2, x_1, x_2)$.

In Proposition 3.1, we show that the distribution of $(L_1(t_1), L_2(t_2))$ has two components. The first one is absolutely continuous with respect to the bi-dimensional Lebesgue measure, namely

$$P(L_1(t_1) \in dx_1, L_2(t_2) \in dx_2) = l(x_1, x_2, t_1, t_2) dx_1 dx_2 \quad x_1 \neq x_2; \quad (3.4)$$

the second one has support on the bisector line $x_1 = x_2$ with one-dimensional Lebesgue density, i.e.

$$P(L_1(t_1) \in dx, L_2(t_2) \in dx) = l_*(x, t_1, t_2) dx. \quad (3.5)$$

The existence of the second component is due to the intuitive fact that the event $\{L_1(t_1) = L_2(t_2)\}$ has positive probability; in fact, since H_1 and H_2 have simultaneous jumps, the first time in which H_1 goes beyond the level t_1 may coincide with the first time in which H_2 goes beyond the level t_2 .

In view of what follows, we need to introduce the Laplace transforms of l and l_* with respect to the time-variables, i.e.

$$\tilde{l}(x_1, x_2, \eta_1, \eta_2) := \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} l(x_1, x_2, t_1, t_2) dt_1 dt_2,$$

$$\tilde{l}_*(x, \eta_1, \eta_2) := \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} l_*(x, t_1, t_2) dt_1 dt_2.$$

Proposition 3.1. *Assume that, for each $x_1, x_2 > 0$, $(H_1(x_1), H_2(x_2))$ has a density denoted by $q(t_1, t_2, x_1, x_2)$. Then, for any $t_1, t_2 > 0$, the distribution of $(L_1(t_1), L_2(t_2))$ has an absolutely continuous component with density*

$$l(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \int_0^{t_1} \int_0^{t_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2, \quad x_1 \neq x_2, \quad (3.6)$$

and its Laplace transform is

$$\begin{aligned} \tilde{l}(x_1, x_2, \eta_1, \eta_2) & \\ &= \frac{1}{\eta_1 \eta_2} T_2(\eta_2) (S(\eta_1, \eta_2) - T_2(\eta_2)) e^{-x_1 (S(\eta_1, \eta_2) - T_2(\eta_2))} e^{-x_2 T_2(\eta_2)} \mathbf{1}_{\{x_1 < x_2\}} \\ &+ \frac{1}{\eta_1 \eta_2} T_1(\eta_1) (S(\eta_1, \eta_2) - T_1(\eta_1)) e^{-x_2 (S(\eta_1, \eta_2) - T_1(\eta_1))} e^{-x_1 T_1(\eta_1)} \mathbf{1}_{\{x_1 > x_2\}}, \end{aligned} \quad (3.7)$$

where $\eta_1, \eta_2 \geq 0$ and S, T_1, T_2 have been defined in (2.4) and (2.6). Moreover, the distribution of $(L_1(t_1), L_2(t_2))$ also has a singular component supported on the line $x_1 = x_2$, having density

$$l_*(x, t_1, t_2) = \frac{\partial^2}{\partial x^2} \int_0^{t_1} \int_0^{t_2} q_*(\tau_1, \tau_2, x) d\tau_1 d\tau_2, \quad (3.8)$$

whose Laplace transform reads

$$\tilde{l}_*(x, \eta_1, \eta_2) = \frac{T_1(\eta_1) + T_2(\eta_2) - S(\eta_1, \eta_2)}{\eta_1 \eta_2} e^{-x S(\eta_1, \eta_2)}. \quad (3.9)$$

Proof. Since $(H_1(x_1), H_2(x_2))$ has a density, it is possible to differentiate both sides of (3.1) and we get (3.6). For $x_1 < x_2$, by using Proposition 2.4 and equation (3.6), we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} l(x_1, x_2, t_1, t_2) dt_1 dt_2 \\ &= \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \frac{\partial^2}{\partial x_1 \partial x_2} \int_0^{t_1} \int_0^{t_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2 dt_1 dt_2 \\ &= \frac{\partial^2}{\partial x_1 \partial x_2} \int_0^\infty \int_0^\infty q(\tau_1, \tau_2, x_1, x_2) \int_{\tau_1}^\infty \int_{\tau_2}^\infty e^{-\eta_1 t_1 - \eta_2 t_2} dt_1 dt_2 d\tau_1 d\tau_2 \\ &= \frac{1}{\eta_1 \eta_2} \frac{\partial^2}{\partial x_1 \partial x_2} \int_0^\infty \int_0^\infty e^{-\eta_1 \tau_1 - \eta_2 \tau_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2 \\ &= \frac{1}{\eta_1 \eta_2} \frac{\partial^2}{\partial x_1 \partial x_2} e^{-x_1 S(\eta_1, \eta_2) - (x_2 - x_1) T_2(\eta_2)} \\ &= \frac{1}{\eta_1 \eta_2} T_2(\eta_2) (S(\eta_1, \eta_2) - T_2(\eta_2)) e^{-x_1 (S(\eta_1, \eta_2) - T_2(\eta_2))} e^{-x_2 T_2(\eta_2)}. \end{aligned}$$

The result for the case $x_1 > x_2$ can be proved similarly. We also remark that

$$\int_0^\infty \int_0^\infty l(x_1, x_2, t_1, t_2) dx_1 dx_2 \neq 1,$$

as

$$\int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \left(\int_0^\infty \int_0^\infty l(x_1, x_2, t_1, t_2) dx_1 dx_2 \right) dt_1 dt_2 \neq \frac{1}{\eta_1 \eta_2}.$$

This proves that $P(L_1(t_1) = L_2(t_2)) > 0$. By putting $x_1 = x_2 = x$ in (3.1), and taking the derivatives with respect to x of both sides, equation (3.8) is obtained. To compute (3.9) we can use the following representation

$$l_*(x, t_1, t_2) = \int_0^{t_1} \int_0^{t_2} \bar{\phi}(t_1 - s_1, t_2 - s_2) q_*(s_1, s_2, x) ds_1 ds_2$$

and apply the convolution theorem for the Laplace transforms, by considering expressions (2.26) and (2.3). \square

Remark 3.2. *Of course, the following normalizing condition holds:*

$$\int_0^\infty \int_0^\infty l(x_1, x_2, t_1, t_2) dx_1 dx_2 + \int_0^\infty l_*(x, t_1, t_2) dx = 1. \quad (3.10)$$

In fact we can check (3.10) by using the Laplace transforms in Proposition 3.1, and we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \left(\int_0^\infty \int_0^\infty l(x_1, x_2, t_1, t_2) dx_1 dx_2 + \int_0^\infty l_*(x, t_1, t_2) dx \right) dt_1 dt_2 \\ &= \frac{1}{\eta_1 \eta_2}. \end{aligned}$$

Remark 3.3. *Proposition 3.1 gives the time-Laplace transform of the distribution, presenting three different expressions related to the regions $x_1 < x_2$, $x_1 > x_2$ and $x_1 = x_2$. These three components are jointly taken into account in the following space Laplace transform:*

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} e^{-\xi_1 x_1 - \xi_2 x_2} P(L_1(t_1) \in dx_1, L_2(t_2) \in dx_2) dt_1 dt_2 \\ &= \int_0^\infty \int_0^\infty e^{-\xi_1 x_1 - \xi_2 x_2} \tilde{l}(x_1, x_2, \eta_1, \eta_2) dx_1 dx_2 + \int_0^\infty e^{-(\xi_1 + \xi_2)x} \tilde{l}_*(x, \eta_1, \eta_2) dx \\ &= \frac{T_1(\eta_1) T_2(\eta_2)}{\eta_1 \eta_2 [\xi_1 + T_1(\eta_1)] [\xi_2 + T_2(\eta_2)]} + \frac{\xi_1 \xi_2 [T_1(\eta_1) + T_2(\eta_2) - S(\eta_1, \eta_1)]}{\eta_1 \eta_2 [\xi_1 + T_1(\eta_1)] [\xi_2 + T_2(\eta_2)] [\xi_1 + \xi_2 + S(\eta_1, \eta_1)]} \end{aligned} \quad (3.11)$$

for $\xi_1, \xi_2, \eta_1, \eta_2 > 0$. We remark that, in the case of independence, by using (2.8) we have that (3.11) reduces to the product of the univariate transforms given in (3.3).

In the following proposition we prove that the survival function of $(L_1(t_1), L_2(t_2))$ decays at least exponentially fast. Therefore all the mixed moments of integer order exist. We then compute the Laplace transform for the covariance between $L_1(t_1)$ and $L_2(t_2)$.

Proposition 3.4. *For any $\eta_1, \eta_2 \geq 0$ the following inequality holds*

$$\begin{aligned} P(L_1(t_1) \geq x_1, L_2(t_2) \geq x_2) &\leq e^{-x_1 S(\eta_1, \eta_2)} e^{-(x_2 - x_1) T_2(\eta_2)} e^{\eta_1 t_1} e^{\eta_2 t_2} \mathbf{1}_{\{x_2 \geq x_1\}} \\ &\quad + e^{-x_2 S(\eta_1, \eta_2)} e^{-(x_1 - x_2) T_1(\eta_1)} e^{\eta_1 t_1} e^{\eta_2 t_2} \mathbf{1}_{\{x_1 > x_2\}}. \end{aligned} \quad (3.12)$$

Therefore, all the mixed moments of integer order exist and, moreover,

$$\int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \text{cov}(L_1(t_1), L_2(t_2)) dt_1 dt_2 = \frac{T_1(\eta_1) + T_2(\eta_2) - S(\eta_1, \eta_2)}{\eta_1 \eta_2 T_1(\eta_1) T_2(\eta_2) S(\eta_1, \eta_2)}. \quad (3.13)$$

Proof. The proof of (3.12) is based on the so-called bivariate Markov inequality (see [38], Theorem 1). For $\eta_1, \eta_2 \geq 0$,

$$\begin{aligned} P(L_1(t_1) \geq x_1, L_2(t_2) \geq x_2) &= P(H_1(x_1) < t_1, H_2(x_2) < t_2) \\ &= P(e^{-\eta_1 H_1(x_1)} > e^{-\eta_1 t_1}, e^{-\eta_2 H_2(x_2)} > e^{-\eta_2 t_2}) \\ &\leq \frac{\mathbb{E}[e^{-\eta_1 H_1(x_1) - \eta_2 H_2(x_2)}]}{e^{-\eta_1 t_1} e^{-\eta_2 t_2}} \\ &= e^{-x_1 S(\eta_1, \eta_2)} e^{-(x_2 - x_1) T_2(\eta_2)} e^{\eta_1 t_1} e^{\eta_2 t_2} \mathbf{1}_{\{x_2 \geq x_1\}} \\ &\quad + e^{-x_2 S(\eta_1, \eta_2)} e^{-(x_1 - x_2) T_1(\eta_1)} e^{\eta_1 t_1} e^{\eta_2 t_2} \mathbf{1}_{\{x_1 > x_2\}} \end{aligned}$$

where the last equality holds by Proposition 2.4. The Laplace transform of $\mathbb{E}(L_1(t_1)L_2(t_2))$ can be computed as

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \mathbb{E}(L_1(t_1)L_2(t_2)) dt_1 dt_2 \\ &= \int_0^\infty \int_0^\infty x_1 x_2 \tilde{l}(x_1, x_2, \eta_1, \eta_2) dx_1 dx_2 + \int_0^\infty x^2 \tilde{l}_*(x, \eta_1, \eta_2) dx \end{aligned}$$

and then, exploiting Prop. 3.1, after some calculations we get

$$\int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \mathbb{E}(L_1(t_1)L_2(t_2)) dt_1 dt_2 = \frac{T_1(\eta_1) + T_2(\eta_2)}{\eta_1 \eta_2 T_1(\eta_1) T_2(\eta_2) S(\eta_1, \eta_2)}. \quad (3.14)$$

Finally, by taking into account that univariate inverse subordinators are such that

$$\int_0^\infty e^{-\eta_i t_i} \mathbb{E}L_i(t_i) dt_i = \frac{1}{\eta_i T_i(\eta_i)} \quad i = 1, 2,$$

we immediately obtain (3.13). \square

Remark 3.5. It is known that, if H is a univariate subordinator and L is its inverse, then $\mathbb{E}L(t)$ can be related to the potential density of H . We recall that, in the frame of potential theory (see e.g. [9]), the potential density of a subordinator H is the function $t \rightarrow u(t)$ such that $u(t)dt$ represents the mean sojourn time spent by H in the interval $[t, t + dt)$, namely

$$u(t)dt = \int_0^\infty P(H(x) \in dt) dx;$$

indeed the following relation holds

$$u(t) = \frac{d}{dt} \mathbb{E}L(t).$$

It is interesting to remark that a similar result holds for a bivariate subordinator (H_1, H_2) with inverse (L_1, L_2) . Indeed, let $u(t_1, t_2) dt_1 dt_2$ be the mean sojourn time spent by (H_1, H_2) in the set $[t_1, t_1 + dt_1) \times [t_2, t_2 + dt_2)$, i.e.

$$u(t_1, t_2) dt_1 dt_2 = \int \int_{\mathbb{R}_+^2} P(H_1(x_1) \in dt_1, H_2(x_2) \in dt_2) dx_1 dx_2;$$

then the following relation holds

$$\begin{aligned} \frac{\partial^2}{\partial t_1 \partial t_2} \mathbb{E}(L_1(t_1)L_2(t_2)) &= \frac{\partial^2}{\partial t_1 \partial t_2} \int \int_{\mathbb{R}_+^2} P(L_1(t_1) > x_1, L_2(t_2) > x_2) dx_1 dx_2 \\ &= \frac{\partial^2}{\partial t_1 \partial t_2} \int \int_{\mathbb{R}_+^2} P(H_1(x_1) \leq t_1, H_2(x_2) \leq t_2) dx_1 dx_2 \end{aligned}$$

$$= u(t_1, t_2).$$

3.2. Governing equation. The following theorem provides boundary value problems for both the densities $l(x_1, x_2, t_1, t_2)$ and $l_*(x, t_1, t_2)$ defined in (3.4) and (3.5). In this way we generalize the boundary value problems in (3.2). The arising equations are PDEs exhibiting ordinary derivatives in the space variables and the integro-differential operator \mathcal{D}_{t_1, t_2} in the time variables. We note that, for well-posedness reasons, we set two distinct boundary value problems for l , which are respectively defined on the open sets $x_2 > x_1 > 0$ and $x_1 > x_2 > 0$. We will denote again the Heaviside function by $\theta(\cdot)$.

Theorem 3.6. *The density $l(x_1, x_2, t_1, t_2)$ solves the equation*

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) l(x_1, x_2, t_1, t_2) = -\mathcal{D}_{t_1, t_2} l(x_1, x_2, t_1, t_2) \quad (3.15)$$

on the open set $x_2 > x_1 > 0$, under the boundary condition

$$l(0, x_2, t_1, t_2) = (\mathcal{D}_{t_1, t_2} - \mathcal{D}_{t_2}^{(2)}) \mathcal{D}_{t_2}^{(2)} \theta(t_1) P(L_2(t_2) \geq x_2), \quad x_2 > 0, \quad (3.16)$$

where $\mathcal{D}_x^{(i)}$ has been defined in (2.12). Moreover, on the open set $x_1 > x_2 > 0$, $l(x_1, x_2, t_1, t_2)$ solves the same equation (3.15) under the boundary condition

$$l(x_1, 0, t_1, t_2) = (\mathcal{D}_{t_1, t_2} - \mathcal{D}_{t_1}^{(1)}) \mathcal{D}_{t_1}^{(1)} \theta(t_2) P(L_1(t_1) \geq x_1) \quad x_1 > 0. \quad (3.17)$$

Furthermore, the density $l_*(x, t_1, t_2)$ solves the equation

$$\frac{\partial}{\partial x} l_*(x, t_1, t_2) = -\mathcal{D}_{t_1, t_2} l_*(x, t_1, t_2) \quad x > 0 \quad (3.18)$$

under the boundary condition

$$l_*(0, t_1, t_2) = \bar{\phi}(t_1, t_2), \quad (3.19)$$

where $\bar{\phi}$ has been defined in (2.25).

Proof. We start with the case $x_2 > x_1 > 0$; the case $x_1 > x_2 > 0$ can be treated similarly. We apply $\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ to both members of (3.6); by using Proposition 2.5 we have

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) l(x_1, x_2, t_1, t_2) = -\frac{\partial^2}{\partial x_1 \partial x_2} \int_0^{t_1} \int_0^{t_2} \mathcal{D}_{\tau_1, \tau_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2. \quad (3.20)$$

By using the definition of $\mathcal{D}_{\tau_1, \tau_2}$, Fubini's theorem gives

$$\int_0^{t_1} \int_0^{t_2} \mathcal{D}_{\tau_1, \tau_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2 = \mathcal{D}_{t_1, t_2} \int_0^{t_1} \int_0^{t_2} q(\tau_1, \tau_2, x_1, x_2) d\tau_1 d\tau_2 \quad (3.21)$$

which leads to equation (3.15). In order to check the boundary condition (3.16), we put $x_1 = 0$ in (3.7), which yields

$$\tilde{l}(0, x_2, \eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2} T_2(\eta_2) (S(\eta_1, \eta_2) - T_2(\eta_2)) e^{-x_2 T_2(\eta_2)}$$

and the inversion of the Laplace transform gives (3.16).

We finally conclude with the results for the density $l_*(x, t_1, t_2)$. In order to obtain equation (3.18) we have to apply $\frac{\partial}{\partial x}$ to both sides of (3.8) and use (2.9) together

with the fact that \mathcal{D}_{t_1, t_2} commutes with the integral. Finally, by putting $x = 0$ in (3.9) we get

$$\tilde{l}_*(0, \eta_1, \eta_2) = \frac{T_1(\eta_1) + T_2(\eta_2) - S(\eta_1, \eta_2)}{\eta_1 \eta_2};$$

by taking into account (2.26), the inverse Laplace transform gives the boundary condition (3.19). \square

Remark 3.7. *The boundary condition (3.19) means that, heuristically speaking, H_1 and H_2 perform initial jumps respectively greater than t_1 and t_2 if and only if the first hitting times of the levels t_1 and t_2 are both equal to 0.*

Remark 3.8. *We here discuss the well-posedness of the boundary value problems in Thm. 3.6. For the region $x_2 > x_1 > 0$, by applying the time-Laplace transform to (3.15), one has*

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \tilde{l}(x_1, x_2, \eta_1, \eta_2) = -S(\eta_1, \eta_2) \tilde{l}(x_1, x_2, \eta_1, \eta_2)$$

which is a damped wave equation in the variables x_1 and x_2 , under the boundary condition

$$\tilde{l}(0, x_2, \eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2} T_2(\eta_2) (S(\eta_1, \eta_2) - T_2(\eta_2) e^{-x_2 T_2(\eta_2)}).$$

Such a problem can be solved by the method of characteristics which gives the desired solution (3.7). Indeed the characteristic lines are $x_2 = x_1 + k$, with $k > 0$ which carry the boundary condition to the region $x_2 > x_1 > 0$, giving the solution

$$\tilde{l}(x_1, x_2, \eta_1, \eta_2) = \tilde{l}(0, x_2 - x_1, \eta_1, \eta_2) e^{-S(\eta_1, \eta_2) x_1}$$

which coincides with (3.7). In the same way, the characteristic lines $x_2 = x_1 + k$, with $k < 0$, carry the boundary condition $\tilde{l}(x_1, 0, \eta_1, \eta_2)$ to the region $x_1 > x_2 > 0$ and lead to the solution.

Finally, by passing again to Laplace transform in the time variables, it is straightforward to check the well-posedness of the boundary value problem for $l_*(x, t_1, t_2)$, as $\tilde{l}_*(x, \eta_1, \eta_2)$ is governed by an ordinary differential equation in the variable x .

4. TIME-CHANGE OF BIVARIATE MARKOV PROCESSES

Let $(X_1(t), X_2(t))$ be a \mathbb{R}^2 -valued Markov process such that $X_1(t)$ and $X_2(t)$ are independent. Without loss of generality, we assume that $(X_1(0), X_2(0)) = (0, 0)$ almost surely. Let $p_i(x_i, t)$, $i = 1, 2$, be the marginal density of $X_i(t)$, and let $p_*(x_1, x_2, t) = p_1(x_1, t)p_2(x_2, t)$ denote the joint density. The following forward equations hold:

$$\frac{\partial}{\partial t} p_i(x_i, t) = \mathcal{G}_i p_i(x_i, t) \quad i = 1, 2 \quad (4.1)$$

$$\frac{\partial}{\partial t} p_*(x_1, x_2, t) = (\mathcal{G}_1 + \mathcal{G}_2) p_*(x_1, x_2, t), \quad (4.2)$$

where the operators \mathcal{G}_1 and \mathcal{G}_2 are the duals to the generators of X_1 and X_2 . It is straightforward to verify that the bi-parameter process $(X_1(t_1), X_2(t_2))$ has a density $p(x_1, x_2, t_1, t_2) = p_1(x_1, t_1)p_2(x_2, t_2)$ satisfying

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) p(x_1, x_2, t_1, t_2) = (\mathcal{G}_1 + \mathcal{G}_2) p(x_1, x_2, t_1, t_2). \quad (4.3)$$

We are now interested in the time-changed process

$$\{(X_1(L_1(t_1)), X_2(L_2(t_2))), t_1 \geq 0, t_2 \geq 0\}, \quad (4.4)$$

where (L_1, L_2) is the inverse of a bivariate subordinator, which is independent on (X_1, X_2) . By a simple conditioning argument, the density of the random vector $(X_1(L_1(t_1)), X_2(L_2(t_2)))$ has the form

$$\begin{aligned} h(x_1, x_2, t_1, t_2) &= \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) dudv \\ &\quad + \int_0^\infty p_*(x_1, x_2, u) l_*(u, t_1, t_2) du, \end{aligned} \quad (4.5)$$

where the functions l and l_* have been defined in (3.4) and (3.5). The following theorem provides a governing equation for (4.5), which turns out to be a generalization of eq. (4.3) as the operator $\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}$ on the left hand side is replaced by \mathcal{D}_{t_1, t_2} defined in (2.10).

Theorem 4.1. *Let $(X_1(t), X_2(t))$ be a \mathbb{R}^2 -valued Markov process, with independent marginal components, such that $(X_1(0), X_2(0)) = (0, 0)$ almost surely. Moreover, let (4.1) and (4.2) hold. Then, for any $t_1, t_2 > 0$, the density (4.5) of the time-changed process $(X_1(L_1(t_1)), X_2(L_2(t_2)))$ satisfies the following equation*

$$\mathcal{D}_{t_1, t_2} h(x_1, x_2, t_1, t_2) = (\mathcal{G}_1 + \mathcal{G}_2) h(x_1, x_2, t_1, t_2) \quad x_1 \neq 0, x_2 \neq 0. \quad (4.6)$$

Proof. By applying \mathcal{D}_{t_1, t_2} to both sides of (4.5) and using Thm. 3.6 we have

$$\begin{aligned} &\mathcal{D}_{t_1, t_2} h(x_1, x_2, t_1, t_2) \\ &= - \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) \frac{\partial}{\partial u} l(u, v, t_1, t_2) dudv \\ &\quad - \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) \frac{\partial}{\partial v} l(u, v, t_1, t_2) dudv \\ &\quad - \int_0^\infty p_*(x_1, x_2, u) \frac{\partial}{\partial u} l_*(u, t_1, t_2) du. \end{aligned} \quad (4.7)$$

Now, we integrate by parts and use (4.1) and (4.2), together with the fact that $(X_1(0), X_2(0)) = (0, 0)$ almost surely; thus we get

$$\begin{aligned} &\mathcal{D}_{t_1, t_2} h(x_1, x_2, t_1, t_2) \\ &= \mathcal{G}_1 \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) dudv + \delta(x_1) \int_0^\infty p_2(x_2, v) l(0, v, t_1, t_2) dv + \\ &\quad + \mathcal{G}_2 \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) dudv + \delta(x_2) \int_0^\infty p_1(x_1, u) l(u, 0, t_1, t_2) du \\ &\quad + (\mathcal{G}_1 + \mathcal{G}_2) \int_0^\infty p_*(x_1, x_2, u) l_*(u, t_1, t_2) du + \delta(x_1) \delta(x_2) \bar{\phi}(t_1, t_2) \end{aligned}$$

where $\bar{\phi}(t_1, t_2)$ has been defined in (2.25). In the region $x_1 \neq 0, x_2 \neq 0$ we have

$$\begin{aligned} &\mathcal{D}_{t_1, t_2} h(x_1, x_2, t_1, t_2) \\ &= (\mathcal{G}_1 + \mathcal{G}_2) \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) dudv \\ &\quad + (\mathcal{G}_1 + \mathcal{G}_2) \int_0^\infty p_*(x_1, x_2, u) l_*(u, t_1, t_2) du \end{aligned}$$

$$= (\mathcal{G}_1 + \mathcal{G}_2)h(x_1, x_2, t_1, t_2),$$

which concludes the proof. \square

4.1. Time-change of continuous-time Markov chains. We specialize the framework of this section by considering the case where $X_1(t)$ and $X_2(t)$ are independent continuous-time Markov chains such that $(X_1(0), X_2(0)) = (0, 0)$ almost surely. In particular, they are defined as

$$X_l(t) = X_l^{(n)} \quad T_l^{(n)} \leq t < T_l^{(n+1)} \quad l = 1, 2, \quad (4.8)$$

where $X_1^{(n)}$ and $X_2^{(n)}$, $n \in \mathbb{N}_0$, denote the embedded Markov chains, with values on the discrete spaces $\mathcal{S}_1 \subset \mathbb{R}$ and $\mathcal{S}_2 \subset \mathbb{R}$ respectively, while all the inter-arrival times $T_l^{(n+1)} - T_l^{(n)}$ have exponential distribution with mean $1/\xi_l$. The transition matrices of the embedded chains are

$$P(X_1^{(n+1)} = j | X_1^{(n)} = i) := A_{ij}, \quad i, j \in \mathcal{S}_1 \quad (4.9)$$

and

$$P(X_2^{(n+1)} = j | X_2^{(n)} = i) := B_{ij}, \quad i, j \in \mathcal{S}_2. \quad (4.10)$$

Let $(L_1(t), L_2(t))$ be again the inverse of a bivariate subordinator. We are interested in the time-changed process $(X_1(L_1(t)), X_2(L_2(t)))$, with values in $\mathcal{S}_1 \times \mathcal{S}_2$, which is defined by

$$X_l(L_l(t)) = X_l^{(n)} \quad T_l^{(n)} \leq L_l(t) < T_l^{(n+1)}, \quad l = 1, 2,$$

or, equivalently, by

$$X_l(L_l(t)) = X_l^{(n)} \quad H_l(T_l^{(n)}) \leq t < H_l(T_l^{(n+1)}), \quad l = 1, 2.$$

We remark that, by Theorem 4.1, the density $p(x_1, x_2, t_1, t_2)$ of the random vector $(X_1(L_1(t_1)), X_2(L_2(t_2)))$ solves the following equation for $x_1, x_2 \neq 0$:

$$\begin{aligned} \mathcal{D}_{t_1 t_2} p(x_1, x_2, t_1, t_2) &= \xi_1 \sum_{k \in \mathcal{S}_1} (A_{k, x_1} - \delta_{k, x_1}) p(k, x_2, t_1, t_2) \\ &\quad + \xi_2 \sum_{k \in \mathcal{S}_2} (B_{k, x_2} - \delta_{k, x_2}) p(x_1, k, t_1, t_2), \end{aligned} \quad (4.11)$$

where the matrices A_{ij} and B_{ij} have been defined in (4.9) and (4.10).

Clearly, for each $l = 1, 2$, we have that $X_l(L_l(t))$ is a stepped semi-Markov process with inter-arrival times $\Delta_{l,n} = H_l(T_l^{(n+1)}) - H_l(T_l^{(n)})$. This class of processes has been widely studied in the literature concerning the classical time-change by inverse subordinators (see, in particular [36] and [42] for the subordination of continuous time Markov chains). Since H_l has stationary increments, the random variables $\Delta_{l,n}$, are i.i.d. copies of $H_l(W_l)$, where W_l is exponentially distributed with mean $1/\xi_l$.

Remark 4.2. *It is important to note that the time-change by (L_1, L_2) introduces dependence between the marginal components. Indeed, on the one hand, the Markov processes $X_1(t)$ and $X_2(t)$ are respectively characterized by the inter-arrival times following the laws of the random variables W_1 and W_2 cited above; on the other hand, the processes $X_1(L_1(t))$ and $X_2(L_2(t))$ exhibit new inter-arrival times respectively distributed as $H_1(W_1)$ and $H_2(W_2)$ (which are dependent because of the dependence between H_1 and H_2).*

The survival functions of $H_i(W_i)$, $i = 1, 2$ satisfy the following relation

$$P(H_i(W_i) > t) = \mathbb{E}e^{-\xi_i L_i(t)}. \quad (4.12)$$

For example, in the special case where $L_i(t)$ is the inverse of a stable subordinator with index $\alpha \in (0, 1)$, the following explicit expression holds

$$P(H_i(W_i) > t) = \mathcal{E}_\alpha(-\xi_i t^\alpha), \quad (4.13)$$

where $\mathcal{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}$ is the one-parameter Mittag-Leffler function.

It is interesting to note that the couple of dependent inter-arrival times $(H_1(W_1), H_2(W_2))$ has joint distribution such that

$$P(H_1(W_1) > t_1, H_2(W_2) > t_2) = \mathbb{E}e^{-\xi_1 L_1(t_1) - \xi_2 L_2(t_2)},$$

which clearly generalizes (4.12). The proof is straightforward:

$$\begin{aligned} & P(H_1(W_1) > t_1, H_2(W_2) > t_2) \\ &= \xi_1 \xi_2 \int_0^\infty \int_0^\infty P(H_1(w_1) > t_1, H_2(w_2) > t_2) e^{-\xi_1 w_1} e^{-\xi_2 w_2} dw_1 dw_2 \\ &= \xi_1 \xi_2 \int_0^\infty \int_0^\infty P(L_1(t_1) < w_1, L_2(t_2) < w_2) e^{-\xi_1 w_1} e^{-\xi_2 w_2} dw_1 dw_2 \\ &= \int_0^\infty \int_0^\infty P(L_1(t_1) \in dw_1, L_2(t_2) \in dw_2) e^{-\xi_1 w_1} e^{-\xi_2 w_2} \\ &= \mathbb{E}e^{-\xi_1 L_1(t_1) - \xi_2 L_2(t_2)}. \end{aligned}$$

4.1.1. *Bivariate fractional Poisson processes.* Let us consider the special case where $X_1(t)$ and $X_2(t)$ are Poisson processes $N_1(t)$ and $N_2(t)$ with intensities ξ_1 and ξ_2 , while $(L_1^\alpha(t), L_2^\alpha(t))$ is the inverse of a bivariate stable subordinator with index $\alpha \in (0, 1)$ (defined in Subsection 2.2).

Then, for each $i = 1, 2$, $N_i(L_i(t))$ is a fractional Poisson process, i.e. a counting renewal process with Mittag-Leffler inter-arrival times of type (4.13) (see e.g. [5], [6], [29], [22], [31] and also [7], [23], [28] for recent extensions). The discrete density $p_i(k, t) = P(N_i(L_i(t)) = k)$ is known to solve

$$\frac{d^\alpha}{dt^\alpha} p_i(k, t) = -\xi_i p_i(k, t) + \xi_i p_i(k-1, t), \quad p_i(k, 0) = \delta_0(k) \quad k \in \mathbb{N}_0, \quad (4.14)$$

where the fractional derivative has been defined in (1.5).

Now, the process $(N_1(L_1^\alpha(t)), N_2(L_2^\alpha(t)))$ is a bivariate renewal counting process in the sense of [14] and turns out to be an interesting bivariate generalization of the fractional Poisson process (other multivariate extensions had been recently proposed in [3] and [4]). In this case, equation (4.11) has the form

$$\begin{aligned} \mathcal{D}_{t_1 t_2}^\alpha p(k_1, k_2, t_1, t_2) &= -\xi_1 p(k_1, k_2, t_1, t_2) + \xi_1 p(k_1 - 1, k_2, t_1, t_2) \\ &\quad - \xi_2 p(k_1, k_2, t_1, t_2) + \xi_2 p(k_1, k_2 - 1, t_1, t_2), \quad k_1, k_2 \in \mathbb{N}_0, \end{aligned} \quad (4.15)$$

under the initial condition $p(k_1, k_2, 0, 0) = \delta_0(k_1)\delta_0(k_2)$. Eq (4.15) is a generalization of (4.14).

5. CTRW LIMITS AND ANOMALOUS DIFFUSION IN ANISOTROPIC MEDIA

Let $(B_1(t), B_2(t))$ be a standard bi-dimensional Brownian motion and let $L^\alpha(t)$ be the inverse of an independent univariate stable subordinator of index $\alpha \in (0, 1)$. The time-changed process

$$\{(B_1(L^\alpha(t)), B_2(L^\alpha(t))), t \geq 0\} \quad (5.1)$$

has great importance in statistical physics, since it models the so-called anomalous diffusion (see, for example, [25], [26], [27], [37] and also [10] for recent developments). The term "anomalous" refers to the fact that the moving particle is subject to a sort of trapping effect which, in some sense, delays the time with respect to what happens for $(B_1(t), B_2(t))$. Indeed, the mean square displacement grows as t^α , i.e. slower with respect to the Brownian motion. Such a process is known to have a density $p(x_1, x_2, t)$ solving the equation

$$\frac{\partial^\alpha}{\partial t^\alpha} p(x_1, x_2, t) = \frac{1}{2} \Delta p(x_1, x_2, t), \quad (5.2)$$

where the operator on the left-hand side is the Caputo derivative (see (1.5)).

It is important to note that the above model assumes that the trapping effect is the same along both the coordinate directions, i.e. the time-changed process is isotropic as well as $(B_1(t), B_2(t))$. Indeed, the time Laplace transform of the characteristic function of (5.1) has the form

$$\int_0^\infty e^{-\eta t} \mathbb{E}[e^{i(\xi_1 B_1(L^\alpha(t)) + \xi_2 B_2(L^\alpha(t)))}] dt = \frac{\eta^{\alpha-1}}{\eta^\alpha + (\xi_1^2 + \xi_2^2)/2}, \quad \xi_1, \xi_2 \in \mathbb{R}, \eta \geq 0$$

which is rotationally invariant in the variables ξ_1, ξ_2 .

Here we present a model of motion in an anisotropic medium, such that the trapping effect depends on the coordinate direction. Let $(B_1(t), B_2(t))$ be a standard bi-dimensional Brownian motion and let $(L_1^\alpha(t), L_2^\alpha(t))$ be the inverse of a bivariate stable subordinator with Lévy measure ϕ_α defined in (2.13); we construct the time-changed process

$$\{(B_1(L_1^\alpha(t)), B_2(L_2^\alpha(t))), t \geq 0\}. \quad (5.3)$$

We remark that the trajectories of the models (5.1) and (5.3) have a different behaviour. In fact, in the isotropic case (5.1) the test particle stays at rest simultaneously in both directions at the intervals in which the random time $L^\alpha(t)$ has a plateau, while in (5.3) the plateaux of L_1^α and L_2^α are not in general synchronized. For this reason we call this model anisotropic sub-diffusion. In conclusion, the motion for the second model (5.3) is given by the alternation of four different phases:

- i) The particle is at rest in the x direction but moves in the y direction (when only L_1^α has a plateau).
- ii) The particle is at rest in the y direction but moves in the x direction (when only L_2^α has a plateau).
- iii) The particle stays at rest in both direction (when both L_1^α and L_2^α have a plateau).
- iv) The particle moves in both directions (otherwise).

Remark 5.1. *Our model of anisotropic sub-diffusion is somewhat related to recent models of inhomogenous subdiffusion, which is governed by a time fractional*

equation of space-dependent order

$$\frac{\partial^{\alpha(x)}}{\partial t^{\alpha(x)}} p(x, t) = \frac{1}{2} \Delta p(x, t) \quad x \in \mathbb{R}^d.$$

Such equation has been recently studied in [16], [17], [42] and [43]. In this model the intensity of the trapping effect depends on the space position rather than on the direction. In particular, in [42] and [43] the inhomogeneous subdiffusion is constructed by means of a particular time-change such that the inverse subordinator and the Brownian motion exhibit a suitable stochastic dependence.

According to Theorem 4.1, the random vector $(B_1(L_1(t_1)), B_2(L_2(t_2)))$ has a density $p(x_1, x_2, t_1, t_2)$ which solves the equation

$$\mathcal{D}_{t_1, t_2}^\alpha p(x_1, x_2, t_1, t_2) = \frac{1}{2} \Delta p(x_1, x_2, t_1, t_2) \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\};$$

this is a generalization of (5.2), where the operator on the left-hand side is defined in (2.16). The time-Laplace transform of the characteristic function can be computed by using (3.11) and a simple conditioning argument:

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \mathbb{E}[e^{i(\xi_1 B_1(L_1(t_1)) + \xi_2 B_2(L_2(t_2)))}] dt_1 dt_2 \\ &= \int_0^\infty \int_0^\infty e^{-\eta_1 t_1 - \eta_2 t_2} \mathbb{E}[e^{-\frac{1}{2} \xi_1^2 L_1(t_1) - \frac{1}{2} \xi_2^2 L_2(t_2)}] dt_1 dt_2 \\ &= \frac{\eta_1^\alpha \eta_2^\alpha}{\eta_1 \eta_2 [\xi_1^2/2 + \eta_1^\alpha] [\xi_2^2/2 + \eta_2^\alpha]} + \\ & \quad + \frac{\xi_1^2 \xi_2^2 [\eta_1^\alpha + \eta_2^\alpha - S_\alpha(\eta_1, \eta_2)]}{4\eta_1 \eta_2 [\xi_1^2/2 + \eta_1^\alpha] [\xi_2^2/2 + \eta_2^\alpha] [\xi_1^2/2 + \xi_2^2/2 + S_\alpha(\eta_1, \eta_2)]}, \end{aligned}$$

where S_α has been defined in (2.15).

5.1. Functional limit results. As said in the introduction, a well established theory (see e.g. [2], [34], [32], [33], [35], [47]) shows that (5.1) arises as CTRW scaling limit. For the reader's convenience we first report the main aspects of such a theory (in the particular case of bi-dimensional CTRW); then we prove that, by constructing another suitable class of CTRWs, also anisotropic subdiffusion (5.3) arises as a particular scaling limit.

Let $(Y_i^1, Y_i^2), i = 1 \dots, n$, be i.i.d. random vectors in \mathbb{R}^2 representing the jumps of a particle and let $S_n = \sum_{k=1}^n (Y_k^1, Y_k^2)$ be the discrete-time random walk giving the location of the particle after n jumps, where $S_0 = (0, 0)$. Moreover, let $J_i \in \mathbb{R}^+$ be i.i.d. random variables representing the inter-arrival times between consecutive jumps and let $T_n = \sum_{k=1}^n J_k$ be the time of the n -th jump.

Let $N(t) = \max\{n \geq 0 : T_n \leq t\}$ be a renewal counting process giving the number of jumps up to time $t \in \mathbb{R}^+$. A CTRW is defined as the time-changed process

$$S_{N(t)} = \sum_{k=1}^{N(t)} (Y_k^1, Y_k^2), \quad t \in \mathbb{R}^+, \quad (5.4)$$

namely a process with jumps (Y_k^1, Y_k^2) separated by inter-arrival times J_k .

We now recall some functional limit results concerning the case of infinite mean inter-arrival times. Assume that the random variables J_i belong to the domain of attraction of a positively skewed stable law with index $\alpha \in (0, 1)$. Then

$$c^{-\frac{1}{\alpha}} T_{[ct]} \rightarrow H^\alpha(t), \quad \text{as } c \rightarrow \infty$$

in the J_1 topology, where $H^\alpha(t)$ is a α -stable subordinator and $[x]$ denotes the biggest integer less than (or equal to) x . Moreover

$$c^{-\alpha} N(ct) \rightarrow L^\alpha(t), \quad \text{as } c \rightarrow \infty$$

in the M_1 topology, where $L^\alpha(t) = \inf\{x : H^\alpha(x) > t\}$ is an inverse α stable subordinator. Moreover, if the (Y_i^1, Y_i^2) belong to the domain of attraction of a standard normal law, then $c^{-\frac{1}{2}} S_{[ct]}$ converges to a standard Brownian motion $(B_1(t), B_2(t))$ in J_1 topology. Then, by combining the above results,

$$c^{-\frac{\alpha}{2}} S_{N([ct])} \rightarrow (B_1(L^\alpha(t)), B_2(L^\alpha(t))) \quad \text{as } c \rightarrow \infty$$

in the M_1 topology, where the limit process is the subdiffusion defined by (5.1).

In the following subsection, we show how it is possible to construct a CTRW converging to the new process (5.3).

5.1.1. *Limits of CTRWs driven by bivariate renewal processes.* Let

$$(J_1^1, J_1^2), \dots, (J_n^1, J_n^2)$$

be i.i.d. random vectors in \mathbb{R}_+^2 (such that, for each i , J_i^1 and J_i^2 are possibly dependent) and let (T_n^1, T_n^2) be the simple random walk on \mathbb{R}_+^2 defined by

$$(T_n^1, T_n^2) = \left(\sum_{k=1}^n J_k^1, \sum_{k=1}^n J_k^2 \right) \quad n \in \mathbb{N}_0. \quad (5.5)$$

Let $(N^1(t), N^2(t))$ be a bivariate renewal counting process in the sense of [14], namely, for each $i = 1, 2$,

$$N^i(t) = \max\{n \in \mathbb{N}_0 : T_n^i \leq t\} \quad t \in \mathbb{R}^+. \quad (5.6)$$

$N^1(t)$ and $N^2(t)$ are possibly dependent. Consider now another random walk (S_n^1, S_n^2) , defined by

$$(S_n^1, S_n^2) = \left(\sum_{k=1}^n Y_k^1, \sum_{k=1}^n Y_k^2 \right) \quad n \in \mathbb{N}_0, \quad (5.7)$$

where $(Y_k^1, Y_k^2), k = 1, \dots, n$, are i.i.d random vectors in \mathbb{R}^2 .

We finally consider the CTRW defined by the bivariate time-change

$$(S_{N^1(t)}^1, S_{N^2(t)}^2) = \left(\sum_{k=1}^{N^1(t)} Y_k^1, \sum_{k=1}^{N^2(t)} Y_k^2 \right) \quad t \in \mathbb{R}^+. \quad (5.8)$$

We note that the marginal components of (5.8) do not have simultaneous jumps (unlike the ones of (5.4)). Moreover, if considered separately, both $S_{N^1(t)}^1$ and $S_{N^2(t)}^2$ are CTRWs of the type already treated in the literature and then we can refer to the classical theory to perform their scaling limits.

The novelty of the following theorem is to present functional limit results in the case of bivariate time-change. We will denote by $\xrightarrow{J_1}$ (resp. $\xrightarrow{M_1}$) the convergence in J_1 (resp. M_1) topology on the space $D^2([0, \infty))$.

Theorem 5.2. *Let (J_k^1, J_k^2) belong to the domain of attraction of a bivariate stable law with index $\alpha \in (0, 1)$ and support on \mathbb{R}_2^+ . Then*

$$c^{-\frac{1}{\alpha}}(T_{[ct]}^1, T_{[ct]}^2) \xrightarrow{J} (H_1^\alpha(t), H_2^\alpha(t)), \quad \text{as } c \rightarrow \infty \quad (5.9)$$

and

$$(c^{-\alpha}N_1(ct), c^{-\alpha}N_2(ct)) \xrightarrow{M_1} (L_1^\alpha(t), L_2^\alpha(t)), \quad \text{as } c \rightarrow \infty, \quad (5.10)$$

where $(H_1^\alpha(t), H_2^\alpha(t))$ is a bivariate stable subordinator (see Subsection 2.2) and $(L_1^\alpha(t), L_2^\alpha(t))$ is its inverse (in the sense explained in section 3). Moreover, if the random vectors (Y_i^1, Y_i^2) in (5.7) belong to the domain of attraction of a standard normal law, we have

$$(c^{-\frac{\alpha}{2}}S_{N^1([ct])}^1, c^{-\frac{\alpha}{2}}S_{N^2([ct])}^2) \xrightarrow{M_1} (B_1(L_1^\alpha(t)), B_2(L_2^\alpha(t))) \quad \text{as } c \rightarrow \infty \quad (5.11)$$

where $(B_1(t), B_2(t))$ denotes a standard Brownian motion.

Proof. Since (J_k^1, J_k^2) belong to the domain of attraction of a bivariate α -stable law, then, by Thm 6.21 at page 168 of [34], we have

$$c^{-\frac{1}{\alpha}}(T_{[ct]}^1, T_{[ct]}^2) \xrightarrow{d} (H_1^\alpha(t), H_2^\alpha(t)) \quad \text{for all } t \geq 0, \quad \text{as } c \rightarrow \infty, \quad (5.12)$$

where \xrightarrow{d} denotes convergence in distribution. Then, by using [46], Thm. 2.7, condition (5.12) implies (5.9).

The bivariate renewal process $(N^1(t), N^2(t))$ is a continuous functional in M_1 topology; then the continuous mapping argument provided by Thm. 1.6.13 in [45] (page 56) gives (5.10).

We finally observe that the set of discontinuities of the limit processes $(B_1(t), B_2(t))$ and $(H_1^\alpha(t), H_2^\alpha(t))$ are obviously disjoint as the Brownian motion has continuous sample paths; hence, another continuous mapping argument (i.e. a vector composition of processes in the sense of [45], Section 2.7) leads to (5.11). \square

Remark 5.3. *Theorem 5.2 is restricted to the case where the inter-arrival times belong to the domain of attraction of a stable law, while the jumps belong to the domain of attraction of a normal distribution. We trust that a more general result could be obtained. Indeed, by using triangular array convergence, one could remove such assumptions and study the case where a rescaled CTRW converges to a Lévy process time-changed by the inverse of a general bivariate subordinator. However, in this case, there are some difficulties in treating simultaneous jumps of the components of the limit processes. So, a further analysis is needed on these points.*

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