

Non-Standard Volterra Integral Equations: A Mean-Value Theorem Numerical Approach

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Abstract

In this paper, we present a novel and flexible numerical method to solve non-standard Volterra integral equations of the second kind. Starting from the mean-value theorem for integrals we give theoretical results that allow associating to each Volterra integral equation a system of non-linear equations that is solved by mean of a numerical method. The algorithm produces very accurate numerical solutions and it is very fast. To test the fitness of our method, we applied it to some examples.

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1 Introduction

Volterra and Fredholm integral equations of the first and the second kinds have practical applications in many fields, including engineering, biology, medicine and finance. In many cases find an analytical solution is unfeasible. Researchers, in the years, proposed several numerical algorithms to avoid the problem. Many mathematical papers have considered equations of the first kind; however, they are usually transformed to equations of the second kind

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using regularization method (see [13]). Aziz et al. [2] proposed new algorithms to solve non-linear Fredholm and Volterra integral equations of the second kind using Haar wavelets. Further, Doucet et al. [7] considered a standard Von Neumann expansion of the solution of a Fredholm integral equation of the second kind approximated using Markov chain Monte Carlo methods. In addition, in a mean-value theorem frameworks, algorithms were developed in order to solve Fredholm and Volterra integral equations [1] and [6]). Moreover, Aziz et al. [3] proposed a method based on Haar wavelet for the numerical solution of two-dimensional non-linear integral equations. Siraj-ul-Islam et al. [14] suggested a novel technique based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders.

Other methods involve the Runge-Kutta method, successive approximations method, Laplace transform method and Adomian decomposition method (see [5] and [16]).

In this paper, we solved non-standard Volterra integral equations of the second kind by applying the mean-value theorem for integrals. More precisely, we offer a theoretical result that allows to associate a system of non-linear equations to the following equation:

$$\phi(x) = f(x, \phi(x)) + \int_a^x k(x, t)\psi(x, t, \phi(x), \phi(t))dt, \quad (1)$$

where $x, t \in I = [a, b]$, $f(x, y)$ is a known function (continuous in both arguments), $k(x, t)$ is the kernel function continuous on $I \times I$ and ψ a continuous function.

The associate non-linear system is solved by means of a numerical method. To test the fitness of our method, we applied it to examples with known solutions. Particular advantages of our method include its simplicity, its flexibility and its ease of implementation, thus making the method applicable to Volterra integral equations with an unknown closed-form solution.

The remainder of the paper is organized as follows. In Section 2, after recalling some theoretical results, we apply them to present our numerical method. In Section 3, we present some numerical results. Finally, Section 4 concludes the paper.

2 The mean-value theorem approach

Let $I = [a, b]$ and let us consider a non-standard Volterra integral equation of the form (1).

Let n be a positive integer. Let us consider the following partition of the interval I into n intervals of equal length $\Delta = \frac{(b-a)}{n}$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b. \quad (2)$$

By means of additive properties for integrals, we can rewrite, for each x_i , $i = 0, 1, 2, \dots, n$, the equation (1) in the following way:

$$\phi(x_i) = f(x_i, \phi(x_i)) + \sum_{m=1}^i \int_{x_{m-1}}^{x_m} k(x_i, s) \psi(x_i, s, \phi(x_i), \phi(s)) ds. \quad (3)$$

Let us suppose the kernel function $k(x, t)$ is continuous in $I \times I$ and does not change its sign in both arguments, the function ϕ and ψ are continuous.

Then, by virtue of (3) and the Mean-Value Theorem for integrals, the following equality holds

$$\phi(x_i) = f(x_i) + \sum_{m=1}^i k(x_i, \xi_m) \psi(x_i, \xi_m, \phi(x_i), \phi(\xi_m)) \Delta, \quad (4)$$

with $\Delta = \int_{x_{m-1}}^{x_m} ds$. Clearly, these numbers ξ_m depend on x_{m-1} , x_m and on the unknown function ϕ , with $x_{m-1} \leq \xi_m \leq x_m$. Considering the problem treated in this section, it is very difficult to know, from the theory, the exact value for each of them. As we show in Proposition 2.1 below, it is not restrictive to assume $\xi_m(x_m) = \tilde{\xi}_m$, where $\tilde{\xi}_m$ are constants such that $x_{m-1} \leq \tilde{\xi}_m \leq x_m$.

For the sake of simplicity, we define the following operators for each fixed x_i :

$$(K\phi)(x_i) = \Delta \sum_{m=1}^i \psi(x_i, \xi_m, \phi(x_i), \phi(\xi_m)) k(x_i, \xi_m),$$

$$(\widetilde{K\phi})(x_i) = \Delta \sum_{m=1}^i \psi(x_i, \tilde{\xi}_m, \phi(x_i), \phi(\tilde{\xi}_m)) k(x_i, \tilde{\xi}_m).$$

The following result holds.

Theorem 2.1. *Let the kernel function $k(x, t)$ be continuous in $I \times I$ and satisfy the following Lipschitz condition:*

$$\|k(x, y_1) - k(x, y_2)\|_\infty \leq L \|y_1 - y_2\|_\infty. \quad (5)$$

where L is a positive constant. Let ψ and ϕ continuous and let M a constant such that $\|\psi\|_\infty \leq M$. Then, it follows

$$\|(\widetilde{K\phi})(x_i) - (K\phi)(x_i)\|_\infty \rightarrow 0 \quad \text{when} \quad n \rightarrow \infty. \quad (6)$$

Proof.

$$\begin{aligned} \|(K\phi)(x_i) - (\widetilde{K\phi})(x_i)\|_\infty &\leq \Delta \left\| \sum_{m=1}^i \psi(x_i, \xi_m, \phi(x_i), \phi(\xi_m)) k(x_i, \xi) \right. \\ &\quad \left. - \psi(x_i, \tilde{\xi}_m, \phi(x_i), \phi(\tilde{\xi}_m)) k(x_i, \tilde{\xi}_m) \right\|_\infty \leq \end{aligned}$$

$$\begin{aligned}
&\leq \Delta M \sum_{m=1}^i \|k(x_i, \xi_m) - k(x_i, \tilde{\xi}_m)\|_{\infty} \leq \\
&\leq \Delta M L \sum_{m=1}^i \|\xi_m - \tilde{\xi}_m\|_{\infty} \leq \\
&\leq \Delta M L \sum_{m=1}^i \Delta \leq \frac{T^2}{n^2} \cdot M \cdot L \cdot i.
\end{aligned}$$

As $n \rightarrow \infty$, it follows $|(K\phi)(x_i) - \widetilde{K\phi}(x_i)| \rightarrow 0$. We thus conclude the proof. \blacksquare

We provide the following algorithm in order to find the numerical solution.

Step 2.1. Let n be a positive integer. Let us consider the following partition Γ of the interval $[a, b]$ into n intervals of equal length $\Delta = \frac{(b-a)}{n}$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Under the hypothesis of the continuity of the functions k , ψ and ϕ , we can write

$$\phi(x_i) = f(x_i, \phi(x_i)) + \Delta \sum_{m=1}^i k(x_i, \xi_m) \psi(x_i, \xi_m, \phi(x_i), \phi(\xi_m)).$$

Let us suppose that Proposition 2.1 is verified.

Step 2.2. We randomly choose $\tilde{\xi}_m \in (x_{m-1}, x_m)$, for $m = 1, 2, \dots, n$, and insert the n -dimensional random vector $\{\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n\}$ into the following system.

$$\left\{ \begin{array}{l}
\phi(x_0) = f(x_0, \phi(x_0)) \\
\phi(\tilde{\xi}_1) = f(\tilde{\xi}_1, \phi(\tilde{\xi}_1)) + \Delta \psi(\tilde{\xi}_1, \tilde{\xi}_1, \phi(\tilde{\xi}_1), \phi(\tilde{\xi}_1)) k(\tilde{\xi}_1, \tilde{\xi}_1) \\
\phi(\tilde{\xi}_2) = f(\tilde{\xi}_2, \phi(\tilde{\xi}_2)) + \Delta \left[\psi(\tilde{\xi}_2, \tilde{\xi}_1, \phi(\tilde{\xi}_2), \phi(\tilde{\xi}_1)) k(\tilde{\xi}_2, \tilde{\xi}_1) + \psi(\tilde{\xi}_2, \tilde{\xi}_2, \phi(\tilde{\xi}_2), \phi(\tilde{\xi}_2)) k(\tilde{\xi}_2, \tilde{\xi}_2) \right] \\
\vdots \\
\phi(\tilde{\xi}_n) = f(\tilde{\xi}_n, \phi(\tilde{\xi}_n)) + \Delta \sum_{m=1}^n \psi(\tilde{\xi}_n, \tilde{\xi}_m, \phi(\tilde{\xi}_n), \phi(\tilde{\xi}_m)) k(\tilde{\xi}_n, \tilde{\xi}_m).
\end{array} \right.$$

The above non-linear system is solved by means of a numerical method, which gives the multivariate $(n+1)$ -dimensional vector $\{\phi(x_0), \phi(\tilde{\xi}_1), \dots, \phi(\tilde{\xi}_n)\}$.

Step 2.3. We choose a positive integer q , and repeating Step 2.2 q times we obtain the following $q \times (n + 1)$ -matrix:

$$\begin{bmatrix} \phi_1(x_0) & \phi_1(\tilde{\xi}_1) & \cdots & \phi_1(\tilde{\xi}_n) \\ \phi_2(x_0) & \phi_2(\tilde{\xi}_1) & \cdots & \phi_2(\tilde{\xi}_n) \\ \vdots & \dots & \ddots & \vdots \\ \phi_q(x_0) & \phi_q(\tilde{\xi}_1) & \cdots & \phi_q(\tilde{\xi}_n) \end{bmatrix}. \tag{7}$$

$\phi(x_0)$ is exactly known. For each $i = 1, \dots, n$ and for $j = 1, \dots, q$, by virtue of the matrix (7), we evaluate numerically $\phi(x_i)$, by solving the following equation

$$\phi(x_i) = f(x_i, \phi(x_i)) + \Delta \sum_{m=1}^i k(x_i, \tilde{\xi}_m) \psi(x_i, \tilde{\xi}_m, \phi(x_i), \phi_j(\tilde{\xi}_m)).$$

We obtain the following $q \times (n + 1)$ -matrix in which each row represents a possible approximation of the solution:

$$\begin{bmatrix} \tilde{\phi}_1(x_0) & \tilde{\phi}_1(x_1) & \cdots & \tilde{\phi}_1(x_n) \\ \tilde{\phi}_2(x_0) & \tilde{\phi}_2(x_1) & \cdots & \tilde{\phi}_2(x_n) \\ \vdots & \dots & \ddots & \vdots \\ \tilde{\phi}_q(x_0) & \tilde{\phi}_q(x_1) & \cdots & \tilde{\phi}_q(x_n) \end{bmatrix}. \tag{8}$$

It results that $\phi(x_0) = \tilde{\phi}_1(x_0) = \tilde{\phi}_2(x_0) = \cdots = \tilde{\phi}_q(x_0)$. The final approximated solution, for each x_i , for $i = 1, 2, \dots, n$, is obtained, starting from the second, by computing the mean value of each column of matrix (7):

$$\phi(x_i)_{approx} = \frac{\sum_{j=1}^q \tilde{\phi}_j(x_i)}{q} \tag{9}$$

for $i = 1, 2, \dots, n$, in agree with the weak law of large numbers.

3 Numerical results

In this section, we present two examples of our implemented method. Calculations were made by means of the software Matlab and run on a MacBook Pro with processor 2.6 GHz Intel Core i7 with 16-GB RAM. In our examples we chose $n = 50$ and $q = 100$.

Example 3.1. Let us consider the approximation of the early boundary exercise arising in the American option pricing problem. In the Black and Scholes framework, let us assume that the asset price $\{S(t), t \geq 0\}$ follows the log-normal diffusion process of the form

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{10}$$

where W_t is the standard Wiener process, r is the constant interest rate and σ is the constant volatility. Let us consider the interval $I = [0, T]$.

We denote with $\Phi(\cdot)$ the standard cumulative normal distribution function.

Let us denote with $\mathcal{B}(t)$ the early exercise boundary of an American put option.

Kim [11] showed that $\mathcal{B}(t)$ satisfies the following integral equation:

$$\begin{aligned} \mathcal{B}(t) = K - Ke^{-rt} &\Phi\left(-\frac{\log\left(\frac{\mathcal{B}(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{t}}\right) + \\ &+ \mathcal{B}(t)\Phi\left(-\frac{\log\left(\frac{\mathcal{B}(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{t}}\right) + \\ &- Kr \int_0^t e^{-r(t-s)} \Phi\left(-\frac{\log\left(\frac{\mathcal{B}(t)}{\mathcal{B}(s)}\right) + \left(r - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right) ds. \end{aligned} \quad (11)$$

The equation (11) represents a non-standard Volterra integral equation of the second kind.

Because Φ is a continuous and bounded function with $M = 1$, $\mathcal{B}(t)$ is continuous (see [11]) and the kernel function $k(t, s) = e^{-r(t-s)}$ is continuous in $I \times I$ and Lipschitzian respect to s for any fixed t with $M = re^{rT}$, it easy to check that the integral in equation (11) satisfies the Proposition 2.1. We are able to apply the algorithm in Section 2. By applying this method, when $s = t_i$, the indeterminate form, when we consider the number $\tilde{\xi}_i$, arises

$$\Phi\left(-\frac{\log\left(\frac{\mathcal{B}(\tilde{\xi}_i)}{\mathcal{B}(\tilde{\xi}_i)}\right) + \left(r - \frac{\sigma^2}{2}\right)(\tilde{\xi}_i - \tilde{\xi}_i)}{\sigma\sqrt{\tilde{\xi}_i - \tilde{\xi}_i}}\right) = \Phi\left(\frac{0}{0}\right). \quad (12)$$

In this case, it is easy to prove that

$$\lim_{s \rightarrow t^-} \Phi\left(-\frac{\log\left(\frac{\mathcal{B}(t)}{\mathcal{B}(s)}\right) + \left(r - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right) = \frac{1}{2}. \quad (13)$$

For our purpose, we can define the continuous function

$$\Phi\left(\frac{\log\left(\frac{u(t)}{u(s)}\right) + \left(r - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right) = \begin{cases} \Phi\left(\frac{\log\left(\frac{u(t)}{u(s)}\right) + \left(r - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right) & \iff 0 \leq s < t \\ \frac{1}{2} & \iff t = s. \end{cases}$$

Table 1: The value of $\mathcal{B}(t)$ for some values of t .

t	PSOR	BRQ	PRK	EKK	Zhu	SSCK	MVT
0.00001	99.70	99.68	99.71	99.69	99.51	99.69	99.70
0.00005	99.40	99.43	99.37	99.37	99.03	99.36	99.38
0.0001	99.20	99.11	99.18	99.14	98.72	99.11	98.18
0.0005	98.31	98.29	98.30	98.28	97.57	98.27	98.31
0.001	97.73	97.63	97.70	97.10	96.83	97.66	97.73
0.01	94.18	94.43	94.21	94.33	92.73	94.07	94.20
0.04	90.30	91.02	90.25	91.12	88.66	91.31	90.27
0.1	86.94	85.25	86.88	86.29	85.25	86.76	86.89

In Table 1 we report some numerical approximations of the early boundary exercise close to expiry. Some approximations here reported are taken from [12] and were considered $K = 100$, $\sigma = 0.3$ and $r = 0.1$.

In the first column, with $n = 1000$ time steps, PSOR indicates the projected successive over-relaxation method of Elliot and Ockendon [8], EKK refers to the analytical approximation for times close to expiration due to Evans et al. [9]. BRQ and PRK refer to respectively to the barycentric quadrature and the Pouzet-Runge Kutta method considered by Nedaiasl and Bastani in [12]. SSCH refers to an improved analytical approximation for the free boundary near the expiry due to Stamicar et al. [15]. Zhu refers to the method of Zhu [17] that suggested an analytical expression to the value of American put options and their optimal exercise boundary. Our algorithm, with $n = 50$ and $q = 100$, by virtue of the presence of the function Φ , took about 45 seconds.

MVT indicates the value of early exercise boundary obtained by means of the method described in this work. As shown in the table, our method produces results that are better or comparable with the others.

Example 3.2. Let us consider the following non-standard Volterra integral equation:

$$\phi(x) = \sqrt{2+x} - \frac{1}{2}(\log(5+3x) - \log(5-2x)) + \int_0^x \frac{5+x+t}{10+4x+2t} \frac{1}{1+\phi(x)^2+\phi(t)^2} dt, \tag{14}$$

where $x \in [0, 1]$. The exact solution is $\sqrt{2+x}$. The equation (14) is taken from [10] and solved numerically by Nedaiasl et al. [12] by means of Runge-Kutta and barycentric rational quadrature types. It is easy to see that Proposition 2.1 is verified. In fact the kernel function $k(x, t) = \frac{5+x+t}{10+4x+2t}$ verifies the Lipschitz condition with $L = \frac{1}{50}$ and the function $\psi(y, z) = \frac{1}{1+y^2+z^2}$ is bounded with $M = 1$. Nedaiasl et al. considering $n = 80$ time steps obtained an error of magnitude 10^{-7} (see Figure 2 in [12]). Applying our algorithm with the same number of time steps we obtained an error of magnitude 10^{-7} . So the

two methods are comparable. Our algorithm, with $n = 50$ and $q = 100$ took about 5 seconds.

n	$\ e\ _\infty$ (q=100)	$\ e\ _\infty$ (q=1000)
5	2.5291e-04	2.4080e-04
10	1.2654e-04	1.2204e-04
20	5.7578e-05	5.9815e-05
30	3.9196e-05	3.9910e-05
40	2.8482e-05	2.9407e-05
50	2.3445e-05	2.3752e-05

Table 2: Convergence analysis of equation 14.

To test the convergence of our algorithm, in Table 2 we report the error $\|e\|_\infty$ between the exact solution and the approximate ones obtained by virtue of the present method. We considered the cases $q = 100, 1000$.

The error size of order $1e - 5$ obtained when increasing the value of n with $q=1000$ assesses the goodness and the precision of the proposed algorithm. Furthermore, it is worth evidencing that the same error size order may be obtained by using $q=100$. Hence, the algorithm performs well already with $q=100$, thus allowing to reduce the computational execution time and, at the same time, maintain the same precision.

Through these tables, it is easy to see that the algorithm presented is very accurate.

4 Conclusion

This work deals with a computational method to obtain numerical solutions for a non-standard Volterra integral equations of the second kind. The approach proposed is able to numerically solve integral equations with unknown closed-form solutions in a way that is simple, flexible and easy to implement. The numerical computation given in order to compare the method with others mentioned in this paper highlights the effectiveness of the method proposed.

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