SAPIENZA
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PhD Thesis

Nonlinear elliptic equations with singularities and elliptic systems

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Contents

Introduction

This thesis is devoted to the study of nonlinear elliptic equations with homogeneous Dirichlet boundary conditions, with lower order terms that may be singular where the solution is zero and with irregular data, as well as of elliptic systems.
As regards singular elliptic equations our starting point is a class of problems whose simplest model is given by

$$
\begin{cases} -\Delta u = \dfrac{\mu}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, μ is a nonnegative datum and $\gamma > 0$. The equation in (1) is *singular*, that is the request of the solution to be zero on the boundary of the domain implies a blow-up of the right hand side.

The pioneering studies concerning problem (1) are contained in [**35**], [**57**] and [**70**]. In these works the authors consider the case of a smooth datum μ, proving the existence of a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (1). This solution does not belong to $C^2(\overline{\Omega})$ and, in [**57**], it is proved that $u \in W_0^{1,2}(\Omega)$ if and only if $\gamma < 3$ and that, if $\gamma > 1$, the solution does not belong nor to $C^1(\overline{\Omega})$. For further results on the Hölder continuity properties of the solution to (1) see [**52**].

As concerns data μ merely in $L^1(\Omega)$, we mainly refer to [**18**], where the authors prove the existence of a distributional solution to the problem working by approximation, desingularizing the right hand side of the equation. This solution belongs to $W_0^{1,2}(\Omega)$ if $\gamma = 1$ and it is only in $W_{loc}^{1,2}(\Omega)$ if $\gamma > 1$; finally, if $\gamma < 1$, it belongs to an homogeneous Sobolev space larger than $W_0^{1,2}(\Omega)$. In the case of measure data, we refer to [**40**], where the existence of a distributional solution is proved in the more general case of a quasilinear elliptic operator with quadratic coercivity and of a singular lower order term not necessarily non-increasing.

As one can expect, uniqueness of solutions to (1) is a challenging issue. If a solution to (1) belongs to $W_0^{1,2}(\Omega)$, uniqueness holds (see [**10**]). In [**71**], one can find a necessary and sufficient condition in order to have $W_0^{1,2}(\Omega)$ solutions to (1) if $\gamma > 1$ and $\mu \in L^1(\Omega)$ positive. If μ is a nonnegative function in $L^{\frac{2N}{N+2}}(\Omega)$ and the singular term is non-increasing, the solution to (1), defined through a transposition argument, is proved to be unique even if it belongs only to $W^{1,2}_{loc}(\Omega)$ (see [**49, 50**]). If $\gamma > 1$ and the datum is a diffuse measure, in [**65**] the authors prove a uniqueness result. Finally, if Ω has a sufficiently regular boundary, uniqueness of solutions belonging only to $W^{1,1}_{loc}(\Omega)$ is proved by means of a suitable Kato's type argument when μ is a general measure and H is a general non-increasing nonlinearity (see [**64**]).

In this thesis, more precisely in Chapter 3, we will study the following problem with a nonlinear principal operator

$$\begin{cases} -\Delta_p u = H(u)\mu & \text{in}\,\Omega, \\ u > 0 & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{2}$$

where, for $1 < p < N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator, μ is a nonnegative bounded Radon measure on Ω and $H(s)$ is a nonnegative, continuous and finite function outside the origin, which, roughly speaking, behaves as $s^{-\gamma}$ ($\gamma \geq 0$) near zero.

In presence of a nonlinear principal operator the literature is more limited. We refer to [**38**] for the existence of a distributional solution when $H(s) = s^{-\gamma}$ and $\mu \in L^1(\Omega)$ while, in case of a general singular nonlinearity H and $\mu \in L^{(p^*)'}(\Omega)$, we mention [**42**]. Furthermore, in [**27**], the uniqueness of solutions which belong to $W^{1,p}_{loc}(\Omega)$ is proved if $\mu \in L^1(\Omega)$. This uniqueness result holds true in full generality in case of a star-shaped domain, while some more regularity on f is needed if $\gamma > 1$, $1 < p \leq N$ and the domain is more general. Besides uniqueness of solutions belonging to $W^{1,p}_0(\Omega)$, which can be proved as in the case of a linear operator, many of the techniques used to prove uniqueness in the linear case $p = 2$ can not be extended to the general case $p > 1$.

We stress that uniqueness for solutions to (2) is an hard issue even if $H \equiv 1$. Indeed, in general, having a distributional solution is not sufficient to deduce uniqueness which holds in the framework of the so-called *renormalized solutions* (see Definition 3.1 below, given in the case of a general H). The notion of renormalized solution formally selects a particular solution among the distributional ones. We also highlight that the existence of a renormalized solution for a continuous and finite function H is given in [**62**] when $p = 2$; this solution is also unique if H is non-increasing and μ is diffuse with respect to the harmonic capacity. We refer the interested reader to [**37**] for a complete account on the renormalized framework for problems whose model is given by (2) with $H \equiv 1$ and the positivity requirement on u is removed (μ is not necessarily nonnegative).

Without the aim to be complete, we refer to various works treating different aspects of problems as in (1) and in (2). The literature concerning the case of linear operators is [**2, 3, 5, 21, 23, 24, 28, 33, 34, 44, 47**]. For more general operators we refer to [**38, 40, 51, 55, 63**]. Finally, also symmetry of solutions is considered in [**25, 26, 72**].

Here we show the existence of a distributional solution u to (2) despite a nonlinear operator, a measure as datum and a general lower order term.
The most interesting fact is that u turns out also to be a renormalized solution to the singular problem if $\gamma \le 1$. This is strictly related to the fact that, in this case, the truncations of u at any level k, $T_k(u)$, belong to the space of finite energy, differently to the case $\gamma > 1$, where $T_k(u)$ is, in general, only in $W^{1,p}_{loc}(\Omega)$.
As already stressed, the existence of a renormalized solution is linked to the uniqueness of the solution to (2). Indeed, in case of a diffuse measure datum and of a non-increasing H, without requiring any additional assumption on Ω and on μ, we are able to prove that the renormalized solution is unique even in presence of a principal operator which can be way more general than the p-Laplacian.
It is worth noting that, at the best of our knowledge, our result is new even in case of a continuous and finite nonlinearity H (i.e., if $\gamma = 0$), so that we are also providing an extension of the results of [**62**] to the case $p \neq 2$.
These results are contained in [**39**].

Let us observe that, if we consider in (1) $\mu = f$ a bounded nonnegative function, we can perform the change of variable

$$v = \frac{u^{\gamma+1}}{\gamma+1},$$

formally transforming (1) into the quasilinear singular equation with singular and gradient quadric lower order term

$$\begin{cases} -\Delta v + \dfrac{\gamma}{\gamma+1}\dfrac{|\nabla v|^2}{v} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Equation (3) is a particular case of the quasilinear singular equation

$$\begin{cases} -\Delta v + B\dfrac{|\nabla v|^2}{v^\rho} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $B > 0$ and $\rho > 0$.
One usually says that the quadratic growth in ∇v of (4) is *natural* as this growth is invariant under the simple change of variable $w = F(v)$, where F is a smooth function. Also in this case the equation (4) is *singular* since the lower order term is singular where the solution is zero.

Problem (4) has been recently studied in [**8, 61**] and [**4**], where existence of positive solutions has been proved. More precisely, in [**8, 61**] existence of solutions is proved for every $B > 0$ if $\rho < 1$, and for $B < 1$ if $\rho = 1$, under the assumption that f is *nonnegative* in Ω, while in [**4**] existence is proved for every $B > 0$ and for every $\rho < 2$ under the assumption that f is *strictly positive* in Ω. Moreover existence of positive solutions in the same framework of [**4**], under a weaker assumption on f, that is f *strictly positive* in a

neighborhood of $\partial\Omega$, it is proved in [**29**]. In other words, the case $B = 1$ and $\rho = 1$ is a borderline case, requiring stronger assumptions on the datum in order to prove existence of positive solutions. One wonders whether this stronger assumption is really necessary, or if it is only technical.
Since the case $B = 1$ and $\rho = 1$ can be seen as the limit case as γ tends to infinity of equation (3), and since this (model) equation is connected to equation (1), one can try to study problem (4), in the borderline case $B = 1$ and $\rho = 1$, through the asymptotic behavior, as γ tends to infinity, of the solutions of (1) under the assumption that f is either nonnegative or strictly positive.

In Chapter 4 we show that if f is *strictly positive* in Ω, then, by our approximation, letting γ tend to infinity we prove that there is no limit equation to (1) and we find a positive solution to

$$\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{v} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5}$$

recovering the existence result contained in [**4**].
If we assume f only *nonnegative* we prove by a one-dimensional example the nonexistence of positive solutions of (5) obtained by approximation. To our knowledge there are no results on nonexistence of positive solutions in literature, this implies that the existence results contained in [**61**], [**4**] and [**29**] are sharp.
These results are contained in [**46**].

Finally, in Chapter 5, we focus on the following class of elliptic systems

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + A\varphi^{\theta+1}|u|^{r-2}u = f, & u \in W_0^{1,p}(\Omega), \\ -\mathrm{div}(|\nabla \varphi|^{p-2}\nabla \varphi) = |u|^r\varphi^\theta, & \varphi \in W_0^{1,p}(\Omega), \end{cases} \tag{6}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with $N \geq 2$, $1 < p < N$, $A > 0$, $r > 1$ and $0 \leq \theta < p - 1$.

We have been motivated by the work of Benci and Fortunato [**6**]. In that work the authors, investigating the eigenvalue problem for the Schrödinger operator coupled with the electromagnetic field, studied the existence for the following system of Schrödinger-Maxwell equations in $\mathbb{R}^3$

$$\begin{cases} -\frac{1}{2}\Delta u + \varphi u = \omega u, \\ -\Delta\varphi = 4\pi u^2. \end{cases} \tag{7}$$

The existence of a solution of (7) is proved by using a variational approach: the equations of the system are the Euler-Lagrange equations of a suitable functional that is neither bounded from below nor from above but has a critical point of saddle type.

Starting from this work, first Boccardo in [**9**] then Boccardo and Orsina in [**20**] studied the related Dirichlet problem with a source term f

$$\begin{cases} -\Delta u + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,2}(\Omega), \\ -\Delta\varphi = |u|^r, & \varphi \in W_0^{1,2}(\Omega), \end{cases} \tag{8}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with $N > 2$, $A > 0$ and $r > 1$.

In [**9**] the existence of a weak solution (u, φ) in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ is proved if f in $L^m(\Omega)$, with $m \geq \dfrac{2N}{N+2} = (2^*)'$, where 2^* is the Sobolev exponent, using once again that (u, φ) is a critical point of a suitable functional. The author proves that if $(2^*)' \leq m < \dfrac{2Nr}{N+2+4r}$, with $r > 2^*-1$, the second equation of (8) admits finite energy solutions even if the datum $|u|^r$ does not belong to the dual space $L^{\frac{2N}{N+2}}(\Omega)$.
In [**20**] the authors improve this result by proving a regularizing effect also on the solution u of the first equation of (8). Existence of a solution (u, φ) in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ is proved if $r > 2^*$ and f belongs to $L^m(\Omega)$, with $m \geq r'$. Then, in the case $r' \leq m < (2^*)'$, the authors find a finite energy solution u of the first equation of (8) with data f possibly not belonging to the dual space.

If $\theta = 0$ in (6), we show how the regularizing effect proved in [**20**] can be improved, proving the existence of a weak solution u in $W_0^{1,p}(\Omega)$ of the first equation of (6) with f belonging to $L^m(\Omega)$, with $(r+1)' \leq m < (p^*)'$. Conversely, in the case $p = 2$ and $0 < \theta < 1$ the second equation of the system (6) is sublinear. This fact does not allow us to use the same method as the previous case and we are not able to prove the regularizing effect on u. However, we prove a regularizing effect on φ generalizing the results proved in [**9**] (in which we recall that $p = 2$ and $\theta = 0$).

Without the aim to be complete, we refer to various developments of the paper [**6**] in which the equations are defined in $\mathbb{R}^3$ and the right hand side of the first equation of (7) is replaced with a nonlinear function $g(x, u)$ with polynomial growth in u (see e.g. [**1**], [**30**], [**32**], [**36**], [**53**], [**56**], [**67**]).
As concerns semilinear elliptic systems we refer to [**43**], where the author proves existence, multiplicity and symmetry of solutions. In the case of elliptic systems with singular lower order terms see [**19**], [**41**].
These results are contained in [**45**].

Below a short plan of the thesis, chapter by chapter.

In Chapter 1 we introduce the notations and some well known results concerning the functional spaces used in the sequel.

In Chapter 2 we give an overview on the existence, uniqueness and regularity results for the elliptic equations used in this thesis.
In Section 2.1 and Section 2.2 we present well known results for linear and nonlinear elliptic equations. In Section 2.3 and 2.4 we focus on known results for singular elliptic problems.

In Chapter 3 we show the existence and the uniqueness of a renormalized solution to (2).
In Section 3.1 we provide the assumptions, the notions of solutions we are adopting and the statements of the existence and uniqueness theorems. In Section 3.2 we prove the existence theorem when H is bounded. In Section 3.3 we provide the approximation scheme and the main tools in preparation of the proof of the theorems when H can blow up at the origin. In Section 3.4 we apply all tools of the previous section to deduce the existence and uniqueness theorems in their full generality. Finally, in Section 3.5, we provide some further results concerning the regularity of a solution to (2) when $H(s)$ can degenerate (i.e., becomes zero) at some point $s > 0$.

In Chapter 4 we study the existence and nonexistence of positive solutions to (5).
In Section 4.1 we state the results that will be proved in the chapter. In Section 4.2 we prove a priori estimates that allow us to pass to the limit in (1) and (3) as γ tends to infinity. In Section 4.3 we pass to the limit in (1). In Section 4.4 we pass to the limit in (3), in the case f strictly positive, obtaining the existence of positive solutions of (5). In Section 4.5 we show, if f is only nonnegative, the one-dimensional example of nonexistence of positive solutions to (5). To conclude, in Section 4.6 we present some open problems.

In Chapter 5 we study the regularizing effect on the existence of solutions to (6).
Section 5.1 is devoted to introducing the problem. In Section 5.2 we deal with a regular datum for the first equation in (6). We define the following functional
$$J(z,\eta) = \frac{1}{p}\int_\Omega |\nabla z|^p - \frac{A(\theta+1)}{pr}\int_\Omega |\nabla \eta|^p + \frac{A}{r}\int_\Omega (\eta^+)^{\theta+1}|z|^r - \int_\Omega fz,$$
and we prove existence of a saddle point (u,φ) of J in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ which is a weak solution of (6).
In Section 5.3 we provide the approximation scheme that gives us estimates in the case $\theta = 0$ and, by these estimates, we prove that there exists a solution in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ of the system (6) with f possibly not belonging to the dual space. We give also a summability result on the solution u of the first equation.
Section 5.4 is devoted to the case $0 < \theta < p-1$. Once again by an approximation scheme we prove estimates that allow us to pass to the limit in the approximate equations and to prove the existence of a weak solution of (6), with the datum f in the dual space.

CHAPTER 1

Preliminary tools and basic results

We begin by giving some notations and recalling the properties of the topological spaces that we will use throughout the thesis.

1.1. Notations

Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 1$. We denote by $\partial\Omega$ its boundary, by $|A|$ the Lebesgue measure of A, where A is a Lebesgue measurable subset of $\mathbb{R}^N$, and by $\mathbb{R}^+$ the set $\mathbb{R} \cap \{x \in \mathbb{R} \text{ s.t. } x > 0\}$. Moreover we define $\text{diam}(\Omega) = \sup\{|x-y| : x, y \in \Omega\}$. By $C_b(\mathbb{R})$ we mean the space of continuous and bounded functions on $\mathbb{R}$, by $C_c(\Omega)$ the space of continuous functions with compact support in Ω and by $C_0(\Omega)$ the space of continuous functions in Ω that are zero on $\partial\Omega$. Analogously, if $k \geq 1$, $C_c^k(\Omega)$ (resp. $C_0^k(\Omega)$) is the space of C^k functions with compact support in Ω (resp. C^k functions that are zero on $\partial\Omega$).
If no otherwise specified, we will denote by C several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance C may depend on Ω, N) but they will never depend on the indexes of the sequences we will introduce. Moreover, in order to take into account the order of the limits, we will denote by $\epsilon(n, r, \nu)$ any quantity such that

$$\limsup_{\nu \to 0} \limsup_{r \to \infty} \limsup_{n \to \infty} \epsilon(n, r, \nu) = 0.$$

For a fixed $k > 0$, we introduce the truncation functions T_k and G_k

$$T_k(s) = \max(-k, \min(s, k)),$$

$$G_k(s) = (|s| - k)^+ \operatorname{sign}(s),$$

and we also define the functions $\pi_k : \mathbb{R} \to \mathbb{R}$ and $\theta_k : \mathbb{R} \to \mathbb{R}$

$$\pi_k(s) = \frac{T_k(s - T_k(s))}{k}, \tag{1.1}$$

$$\theta_k(s) = 1 - |\pi_k(s)|. \tag{1.2}$$

From now onwards, when employing functions denoted by π_k or θ_k, we will mean the previous functions.
We also mention the definition of the Gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \, dt \,, \tag{1.3}$$

where z is a complex number with positive real part.
We define $\phi_\lambda : \mathbb{R} \to \mathbb{R}$, with $\lambda > 0$, the following function

$$\phi_\lambda(s) \;=\; s\, e^{\lambda s^2}\,, \tag{1.4}$$

in what follows we will use that for every $a, b > 0$ we have, if $\lambda > \dfrac{b^2}{4a^2}$, that

$$a\,\phi_\lambda'(s) - b\,|\phi_\lambda(s)| \;\geq\; \frac{a}{2}\,.$$

1.2. Functional spaces and Radon measures

For $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is the space of the almost everywhere equivalence classes of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \; dx\right)^{\frac{1}{p}} < \infty.$$

$L^\infty(\Omega)$ consists of the almost everywhere equivalence classes of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |u| = \inf\big\{M \geq 0 : |u(x)| \leq M \text{ a.e. in}\,\Omega\big\} < \infty.$$

A function $u \in L^p(\Omega)$ has a weak partial derivative in the direction x_i if there exists a function $v_i \in L^p(\Omega)$ such that

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i} = -\int_\Omega v_i\varphi, \quad \forall\varphi \in C_c^\infty(\Omega).$$

The function v_i is denoted by $\dfrac{\partial u}{\partial x_i}$. If u has a weak derivative in every direction, then we denote the weak gradient as the vector

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N}\right)\,.$$

By the weak gradient of a L^p-function ($p \geq 1$) we can define the Sobolev space $W^{1,p}(\Omega)$ as follows

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\}\,.$$

$W^{1,p}(\Omega)$ equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{(L^p(\Omega))^N}$$

is a Banach space, reflexive for every $1 < p < +\infty$, and separable for every $1 \leq p < +\infty$. $W^{1,2}(\Omega)$ is a separable Hilbert space.
We define, for $1 \leq p < +\infty$, $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$. $W_0^{1,\infty}(\Omega)$ is the space of the functions belonging to $W^{1,\infty}(\Omega) \cap C_0(\Omega)$.
All these spaces have a local counterpart. For instance, let $p \geq 1$, then u belongs to $L^p_{loc}(\Omega)$ (or to $W^{1,p}_{loc}(\Omega)$) if u belongs to $L^p(\omega)$ (or to $W^{1,p}(\omega)$) for all $\omega \subset\subset \Omega$. In the same way we say that a sequence u_n converges to u in $L^p_{loc}(\Omega)$ (or in $W^{1,p}_{loc}(\Omega)$) if u_n converges

to u in $L^p(\omega)$ (or in $W^{1,p}(\omega)$) for any $\omega \subset\subset \Omega$.
For $p \leq 1 < +\infty$, the dual space of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$, where $p' = \dfrac{p}{p-1}$ is the Hölder conjugate exponent of p (if $p = 1$ then $1' = \infty$), while the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$.

To be complete, for $0 < p < \infty$, we introduce the Marcinkiewicz space $M^p(\Omega)$ as follows

$$M^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable s.t. } |\{x \in \Omega : |u(x)| \geq k\}| \leq \frac{c}{k^p}, \forall k > 0, \text{ for some } c > 0 \right\}.$$

$M^p(\Omega)$ equipped with the norm

$$\|u\|_{M^p(\Omega)} = \inf \left\{ c > 0 \, : \, |\{x \in \Omega \, : \, |u(x)| \geq k\}| \leq \frac{c}{k^p}, \forall k > 0 \right\}^{\frac{1}{p}}$$

is a Banach space. Moreover the following continuous embeddings hold

$$L^1(\Omega) \hookrightarrow M^1(\Omega),$$

and, if $p > 1$,

$$L^p(\Omega) \hookrightarrow M^p(\Omega) \hookrightarrow L^{p-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, p-1].$$

Let $f \in L^1_{loc}(\Omega)$ then $x \in \Omega$ is a Lebesgue point of f if there exists $\widehat{f}(x) \in \mathbb{R}$ such that

$$\lim_{\rho \to 0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \left| f - \widehat{f}(x) \right| = 0.$$

By the Lebesgue differentiation Theorem, almost every point $x \in \Omega$ is a Lebesgue point of f and $f(x) = \widehat{f}(x)$. We denote as $\mathcal{L}_f$ the set of Lebesgue points of a function $f \in L^1_{loc}(\Omega)$.

We introduce the space of Radon measures, that is the space of the real valued, additive and regular set functions defined on the σ-algebra of Borel subsets of Ω (the smallest σ-algebra that contains all of the open sets). If μ is a Radon measure, by the Hahn decomposition theorem, we know that there exists a unique nonnegative pair (μ^+, μ^-) of Radon measures such that $\mu = \mu^+ - \mu^-$. The measure μ^+, μ^- are called respectively positive and negative part of the measure μ. We define $|\mu|(\Omega) = \mu^+(\Omega) + \mu^-(\Omega)$ as the total variation of the measure μ. We say that μ is a bounded Radon measure if $|\mu|(\Omega)$ is bounded. We define as $\mathcal{M}(\Omega)$ the space of the bounded Radon measure equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega).$$

$\mathcal{M}(\Omega)$ is a Banach space which is, by Riesz representation theorem, the dual space of $C_c(\Omega)$ with the topology of the uniform convergence.
We say that μ is concentrated on a Borel set E, that is $\mu\lfloor E$, if $\mu(B) = \mu(E \cap B)$, for every Borel set B in Ω. We mean that μ is absolutely continuous with respect to $\lambda \in \mathcal{M}(\Omega)$ if $\lambda(E) = 0$ implies $\mu(E) = 0$, for every Borel set E. We denote this property with $\mu \ll \lambda$. Conversely we say that μ is orthogonal to λ, that is $\mu \perp \lambda$, if there exists a set E such that $\mu(E) = 0$ and $\lambda = \lambda\lfloor E$.
We give the following decomposition theorem.

Theorem 1.1. *Let $\mu\,,\lambda$ belong to $\mathcal{M}(\Omega)$. Then there exists a unique pair (μ_0,μ_1) in $\mathcal{M}(\Omega)\times\mathcal{M}(\Omega)$ such that*

$$\mu=\mu_0+\mu_1\,,\qquad \textit{where } \mu_0\ll\lambda \textit{ and } \mu_1\perp\lambda\,.$$

We recall that a sequence of measures μ_n converges to μ in the narrow topology of $\mathcal{M}(\Omega)$ if

$$\lim_{n\to\infty}\int_\Omega \varphi d\mu_n=\int_\Omega \varphi d\mu\,,\quad \forall\varphi\in C_b(\Omega).$$

It is possible to prove that μ_n narrow converges to μ if and only if μ_n *-weakly converges to μ in $\mathcal{M}(\Omega)$ and $\mu_n(\Omega)$ converges to $\mu(\Omega)$.

Finally we define the standard p-capacity of a Borel set $E\subset\Omega$ as

$$\mathrm{cap}(E,\Omega)=\inf\left\{\int_\Omega|\nabla u|^p\ \text{ with } u\in W_0^{1,p}(\Omega)\,:\,u\geq 1 \text{ a.e. in a neighborhood of E}\right\}.$$

A function u is said to be cap_p-quasi continuous if for every $\varepsilon>0$ there exists an open set $E\subset\Omega$ such that $\mathrm{cap}(E)<\varepsilon$ and $u|_{\Omega\setminus E}$ is continuous in $\Omega\setminus E$.
Moreover for every $u\in W^{1,p}(\Omega)$ there exists a cap_p-quasi continuous representative $\tilde u$ yielding $u=\tilde u$ almost everywhere in Ω and if $\widehat u$ is another cap_p-quasi continuous representative of u, then $\widehat u=\tilde u$ cap_p-almost everywhere in Ω. We will always refer to the cap_p-quasi continuous representative when dealing with functions in $W^{1,p}(\Omega)$.
We note that the set function cap_p is not a measure on Ω since it lacks, in general, of the property of additivity on disjoint sets, but it is an outer measure and the definitions given for the measures also apply in this case. Thus $\mu\in\mathcal{M}(\Omega)$ is said to be diffuse (or absolutely continuous) with respect to the p-capacity if for every Borel set $B\subset\Omega$ such that $\mathrm{cap}_p(B)=0$ it results $\mu(B)=0$. Moreover μ is said to be concentrated on a set E of zero p-capacity if $\mu(B)=\mu(E\cap B)$ for every $B\subset\Omega$, with $\mathrm{cap}_p(E)=0$.
We have again a decomposition theorem contained in [**48**].

Theorem 1.2. *If $\mu\in\mathcal{M}(\Omega)$, then it can be uniquely decomposed as*

$$\mu=\mu_d+\mu_c,$$

where μ_d is diffuse with respect to the p-capacity and μ_c is concentrated on a set of zero p-capacity. Moreover, if $\mu\geq 0$, then $\mu_d,\mu_c\geq 0$.

Furthermore, in [**13**], is proved the following decomposition result

$$\mu\in\mathcal{M}(\Omega) \text{ is diffuse if and only if } \mu=f-\mathrm{div}(F)\ \text{ with } f\in L^1(\Omega), F\in L^{p'}(\Omega)^N.$$

The latter decomposition is not unique since $L^1(\Omega)\cap W^{-1,p'}(\Omega)\neq\{0\}$.

1.3. Useful basic results

In this section we give some basic results that we will often use in the proofs of our theorems.

We begin with well known inequalities and convergence theorems for functions belonging to Lebesgue spaces.

LEMMA 1.3 (*Generalized Young's inequality*). *Let $p > 1$ and $\varepsilon > 0$. Then*

$$|ab| \leq \frac{1}{\varepsilon^p}\frac{|a|^p}{p} + \varepsilon^{p'}\frac{|b|^{p'}}{p'}, \quad \forall\, a, b \in \mathbb{R}\,.$$

LEMMA 1.4 (*Hölder's inequality*). *Let $p \geq 1$ and let f be a function in $L^p(\Omega)$ and g a function in $L^{p'}(\Omega)$. Then fg belongs to $L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\,\|g\|_{L^{p'}(\Omega)}\,.$$

LEMMA 1.5 (*Fatou's lemma*). *Let $f_n : \Omega \to \mathbb{R}$ be a sequence of measurable and nonnegative functions such that f_n converges to f almost everywhere in Ω. Then*

$$\int_\Omega f \leq \liminf_{n\to\infty} \int_\Omega f_n\,.$$

THEOREM 1.6 (*Beppo Levi theorem*). *Let $f_n : \Omega \to \mathbb{R}$ be an increasing sequence of measurable and nonnegative functions such that f_n converges to f almost everywhere in Ω. Then*

$$\lim_{n\to\infty} \int_\Omega f_n = \int_\Omega f\,.$$

THEOREM 1.7 (*Vitali's theorem*). *Let $1 \leq p < \infty$ and let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that f_n converges to f almost everywhere in Ω. If*

$$\lim_{|E|\to 0} \sup_n \int_E |f_n| = 0\,,$$

then f belongs to $L^p(\Omega)$ and f_n strongly converges to f in $L^p(\Omega)$.

THEOREM 1.8 (*Lebesgue theorem*). *Let $1 \leq p < \infty$ and let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that f_n converges to f almost everywhere in Ω. If there exists a function g belonging to $L^p(\Omega)$ such that $|f_n| \leq g$ for every n in $\mathbb{N}$, then f belongs to $L^p(\Omega)$ and f_n strongly converges to f in $L^p(\Omega)$.*

As consequence of the previous theorems we have the following proposition.

PROPOSITION 1.9. *Let $p > 1$ and let $\{f_n\}$ be a bounded sequence in $L^p(\Omega)$ such that f_n converges to f almost everywhere in Ω. Then f_n strongly converges to f in $L^q(\Omega)$, for every $1 \leq q < p$.*

We recall also the following very well known consequence of the Egorov Theorem.

LEMMA 1.10. *Let f_n be a sequence converging to f weakly in $L^1(\Omega)$ and let g_n be a sequence converging to g almost everywhere in Ω and *-weakly in $L^\infty(\Omega)$. Then*

$$\lim_{n\to+\infty} \int_\Omega f_n g_n = \int_\Omega fg.$$

Now we give some well known results concerning Sobolev spaces.

Theorem 1.11 (Poincaré inequality). *Let $p \geq 1$. Then there exists a positive constant $\mathcal{C} = \mathcal{C}(N, p, \Omega)$ such that*

$$\|u\|_{L^p(\Omega)} \leq \mathcal{C}\|\nabla u\|_{L^p(\Omega)^N}, \quad \forall\, u \in W_0^{1,p}(\Omega).$$

In particular $\|\nabla u\|_{L^p(\Omega)^N}$ is an equivalent norm on $W_0^{1,p}(\Omega)$.
We state two famous embedding theorems.

Theorem 1.12 (Sobolev embedding theorem). *Let $p \geq 1$ and $\partial\Omega$ be of class C^1. Then there are the following continuous embeddings*

$$\begin{aligned}
&\text{if} \quad p < N \quad W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \ \text{where } p^* = \frac{pN}{N-p}, \\
&\text{if} \quad p = N \quad W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega), \ \text{for every } 1 \leq q < \infty, \\
&\text{if} \quad p > N \quad W^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega), \ \text{where } \gamma = 1 - \frac{N}{p},
\end{aligned}$$

with $C^{0,\gamma}(\Omega)$ denoting the space of the Hölder continuous functions of exponent γ.

Theorem 1.13 (Rellich-Kondrachov's theorem). *Let $p \geq 1$ and $\partial\Omega$ be of class C^1. Then there are the following compact embeddings*

$$\begin{aligned}
&\text{if} \quad p < N \quad W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \text{for every } 1 \leq q < p^*, \\
&\text{if} \quad p = N \quad W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \text{for every } 1 \leq q < \infty, \\
&\text{if} \quad p > N \quad W^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega).
\end{aligned}$$

These theorems are still valid if we replace $W^{1,p}(\Omega)$ with $W_0^{1,p}(\Omega)$ and we do not require any regularity assumption on the boundary of Ω.
Finally we give a fundamental result proved by Guido Stampacchia that we will use continuously in the thesis.

Theorem 1.14. *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. If u belongs to $W_0^{1,p}(\Omega)$, then $G(u)$ belongs to $W_0^{1,p}(\Omega)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in Ω.*

Now we recall results on the space of Radon measures and concerning the p-capacity.

Theorem 1.15 (Lebesgue theorem for general measure). *Let $1 \leq p < \infty$, let μ be a measure in $\mathcal{M}(\Omega)$ and let $\{f_n\} \subset L^p(\Omega, \mu)$ be a sequence such that f_n converges to f μ-almost everywhere in Ω. If there exists g in $L^p(\Omega, \mu)$ such that $|f_n| \leq g$ for every n in $\mathbb{N}$ and μ-almost everywhere in Ω, then f belongs to $L^p(\Omega, \mu)$ and f_n strongly converges to f in $L^p(\Omega, \mu)$.*

We collect some results contained in [**7**] and [**37**].

Lemma 1.16. *Let λ be a nonnegative bounded Radon measure concentrated on a set E such that $\mathrm{cap}_p(E) = 0$. Then, for every $\nu > 0$, there exists a compact subset $K_\nu \subset E$ and a function $\Psi_\nu \in C_c^\infty(\Omega)$ such that the following hold*

$$\lambda(E \setminus K_\nu) < \nu, \ \ 0 \leq \Psi_\nu \leq 1 \ in\ \Omega, \ \ \Psi_\nu \equiv 1 \ in\ K_\nu, \ \lim_{\nu \to 0} \|\Psi_\nu\|_{W_0^{1,p}(\Omega)} = 0.$$

In the entire thesis we will denote by Ψ_ν a function with the properties of the previous lemma.

LEMMA 1.17. *Let $u : \Omega \to \mathbb{R}$ be a measurable function almost everywhere finite on Ω such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\le k\}} \quad \textit{for every } k > 0,$$

and we define the gradient of u as $\nabla u = v$. Moreover, if

$$\int_\Omega |\nabla T_k(u)|^p \le C(k+1) \quad \forall k > 0,$$

then u is cap_p-almost everywhere finite, i.e. $\mathrm{cap}_p\{x \in \Omega : |u(x)| = +\infty\} = 0$, and there exists a cap_p-quasi continuous representative $\tilde{u}$ of u, namely a function $\tilde{u}$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u}$ is cap_p-quasi continuous.

In what follows, when dealing with a function u that satisfies the assumptions of the previous Lemma, we will always consider its cap_p-quasi continuous representative.

LEMMA 1.18. *Let μ_d be a nonnegative diffuse measure with respect to the p-capacity and let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a nonnegative function. Then, up to the choice of its cap_p-quasi continuous representative, u belongs to $L^\infty(\Omega, \mu_d)$ and*

$$\int_\Omega u\, d\mu_d \le \|u\|_{L^\infty(\Omega)}\mu_d(\Omega).$$

We conclude this chapter with two results on Banach spaces that we will use to prove the existence of solutions for our equations and with two technical lemmas that we will use in Chapter 4.

THEOREM 1.19 (Schauder fixed point theorem). *Let X be a Banach space, $F : X \to X$ be a continuous map such that $\overline{F(C)}$ is compact for every $C \subset X$ bounded and K be a convex, closed and bounded subset of X that is invariant for F. Then F has at least a fixed point in K.*

THEOREM 1.20 (Generalized Weierstrass's theorem). *Let X be a reflexive Banach space and $K \subset X$ be a weakly closed subset. If $J : K \to \mathbb{R}$ is a coercive and weakly lower semicontinuous functional, then*

$$\exists\, u \in K \textit{ s. t. } J(u) = \min_{v\in K} J(v)\,.$$

LEMMA 1.21. Let $m(j,r) : [0,+\infty) \times [0,R_0) \to \mathbb{R}$ be a function such that $m(\cdot,r)$ is nonincreasing and $m(j,\cdot)$ is nondecreasing. Moreover, suppose that there exist $k_0 \ge 0$, $C,\nu,\delta > 0$ and $\mu > 1$ satisfying

$$m(j,r) \le C\frac{m(k,R)^\mu}{(j-k)^\nu(R-r)^\delta} \qquad \forall\ j > k \ge k_0,\ 0 \le r < R < R_0.$$

Then, for every $0 < \sigma < 1$*, there exists* $d > 0$ *such that*

$$m(k_0 + d, (1-\sigma)R_0) = 0,$$

where $d^\nu = \dfrac{2^{(\nu+\delta)\frac{\mu}{\mu-1}} C m(k_0, R_0)^{\mu-1}}{\sigma^\delta R_0^\delta}$.

Proof. See [**69**], Lemma 5.1. □

LEMMA 1.22. *Let* $g : [0, +\infty) \to [0, +\infty)$ *be a continuous and increasing function, with* $g(0) = 0$*, such that*

$$t \in (0, +\infty) \mapsto \frac{g(t)}{t} \text{ is increasing and } \int^{+\infty} \frac{1}{\sqrt{tg(t)}} < +\infty.$$

Then, for any $C > 0$ *and* $\delta \geq 0$*, there exists a function* $\varphi : [0,1] \to [0,1]$ *depending on* g, C, δ *with* $\varphi \in C^1([0,1])$, $\sqrt{\varphi} \in C^1([0,1])$, $\varphi(0) = \varphi'(0) = 0$, $\varphi(1) = 1$, $\varphi(\sigma) > 0$ *for every* $\sigma > 0$ *and satisfying*

$$t^{\delta+1} \frac{\varphi'(\sigma)^2}{\varphi(\sigma)} \leq \frac{1}{C} t^\delta g(t) \varphi(\sigma) + 1, \qquad \forall\ 0 \leq \sigma \leq 1,\ t \geq 0.$$

Proof. See [**58**], Lemma 1.1. □

CHAPTER 2

Results on elliptic PDEs

In this chapter, without the aim to be complete, we give some well known results on existence, uniqueness and regularity of solutions of elliptic PDEs that are the starting point of our studies.

2.1. Linear and nonlinear equations with irregular data

2.1.1. Linear equations. We begin with linear elliptic equations.
Let $M(x)$ be a matrix which satisfies, for some positive constants $0 < \alpha \leq \beta$, a.e. in $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$ the following assumptions:

$$M(x)\,\xi \cdot \xi \geq \alpha|\xi|^2 \qquad \text{and} \qquad |M(x)| \leq \beta\,.$$

Let us consider the linear problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

We give the definitions of weak and distributional solution to (2.1) that we will recover also in the nonlinear case.

Definition 2.1. *Let f be a function in $L^{\frac{2N}{N+2}}(\Omega)$. A function u in $W_0^{1,2}(\Omega)$ is a weak solution of* (2.1) *if*

$$\int_\Omega M(x)\nabla u \cdot \nabla \varphi = \int_\Omega f\varphi\,, \quad \forall \varphi \in W_0^{1,2}(\Omega)\,.$$

Definition 2.2. *Let $f = \mu$ be a measure in $\mathcal{M}(\Omega)$. A function u in $W_0^{1,1}(\Omega)$ is a distributional solution of* (2.1) *if*

$$\int_\Omega M(x)\nabla u \cdot \nabla \varphi = \int_\Omega \varphi\, d\mu\,, \quad \forall \varphi \in C_c^1(\Omega)\,.$$

Furthermore we have another definition of solution of (2.1) due to Guido Stampacchia.

Definition 2.3. *Let $f = \mu$ be a measure in $\mathcal{M}(\Omega)$. A function u in $L^1(\Omega)$ is a duality solution of* (2.1) *if*

$$\int_\Omega u\, g = \int_\Omega v\, d\mu\,,$$

for every g in $L^\infty(\Omega)$, where v is the weak solution of

$$\begin{cases} -\operatorname{div}(M^*(x)\nabla v) = g & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $M^*(x)$ the adjoint matrix of $M(x)$.

Remark 2.4. *It is possible to prove that if u is a weak solution then is a duality solution, and if u is a duality solution then is a distributional solution.*

We have the following existence, uniqueness and regularity results for solutions of (2.1) proved once again by Guido Stampacchia.

Theorem 2.5. *Let f be a function in $L^m(\Omega)$. Then the following hold:*

i) *if $m > \frac{N}{2}$, then there exists a unique weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;*
ii) *if $\frac{2N}{N+2} \le m < \frac{N}{2}$, then there exists a unique weak solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$;*
iii) *if $1 < m < \frac{2N}{N+2}$, then there exists a unique duality solution $u \in W_0^{1,m^*}(\Omega)$,*
iv) *if $m = 1$ or $f = \mu \in \mathcal{M}(\Omega)$, then there exists a unique duality solution $u \in W_0^{1,q}(\Omega)$ for $q < \frac{N}{N-1}$.*

Proof. See [**69**]. □

Remark 2.6. *If $\partial\Omega$ is of class C^1 and $f \in L^m(\Omega)$ with $m > \frac{N}{2}$ then the solution u to* (2.1) *belongs to $C(\overline{\Omega})$ and it is such that*

$$\|u\|_{C(\overline{\Omega})} \le C(m)\|f\|_{L^m(\Omega)}\,.$$

We underline that if f belongs to $L^m(\Omega)$, with $1 \le m < \frac{2N}{N+2}$, or f belongs to $\mathcal{M}(\Omega)$, the uniqueness result obtained in Theorem 2.5 holds only for duality solutions. Indeed a distributional solution of (2.1), in general, may not be unique (see [**68**]). Unfortunately the idea of duality solutions is strongly related to the linearity of the problem, so that we lose this notion and the consequent uniqueness result in the nonlinear case.

2.1.2. Nonlinear equations. Now we study the nonlinear elliptic problem with Dirichlet boundary conditions.
Fix $p > 1$. Let $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying Leray-Lions structure conditions, that is, for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, there exist $\alpha, \beta > 0$ such that

$$a(x,\xi)\cdot\xi \ge \alpha|\xi|^p\,, \qquad |a(x,\xi)| \le \beta|\xi|^{p-1}\,,$$

and

$$(a(x,\xi) - a(x,\eta))\cdot(\xi-\eta) > 0, \quad \xi \ne \eta.$$

The assumptions on the function a imply that $A(u) = -\operatorname{div}(a(\cdot,\nabla u))$ is a differential operator continuous, coercive and monotone acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. The model case is the p-laplacian operator $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which corresponds to the

choice $a(x,\xi) = |\xi|^{p-2}\xi$ (for $p = 2$ is the classical laplacian operator).
Let f belong to a suitable Lebesgue space. We consider the nonlinear problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = f & \text{in } \Omega\,,\\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.2}$$

Our notions of solution are a generalization of those given in the linear case.

DEFINITION 2.7. *Let f be a function in $L^{(p^*)'}(\Omega)$. A function u in $W_0^{1,p}(\Omega)$ is a weak solution of* (2.2) *if*

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi = \int_\Omega f\,\varphi\,,\quad \forall\varphi\in W_0^{1,p}(\Omega).$$

DEFINITION 2.8. *Let $f = \mu$ be a measure in $\mathcal{M}(\Omega)$. A measurable function $u:\Omega\to\mathbb{R}$ such that $|\nabla u|^{p-1}\in L^1_{loc}(\Omega)$ is a distributional solution of* (2.2) *if $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$ and*

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi = \int_\Omega \varphi\,d\mu\,,\quad \forall\varphi\in C^1_c(\Omega).$$

We observe that the condition $T_k(u)\in W_0^{1,p}(\Omega)$ gives meaning to the boundary condition of (2.2).
We give an existence and uniqueness result for weak solutions duo to Leray and Lions.

THEOREM 2.9 (Leray-Lions theorem). *Under the above assumptions on $a(x,\xi)$, the differential operator $A(u): W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega)$ is surjective. Hence, for all f in $W^{-1,p'}(\Omega)$, there exists a function $u\in W_0^{1,p}(\Omega)$ such that $A(u)=f$, that is there exists a weak solution of* (2.2).

PROOF. See [**59**]. □

REMARK 2.10. *It is easy to prove that if f belongs to $W^{-1,p'}(\Omega)$, then there exists a unique weak solution. Moreover, if $p>N$ then, by Sobolev embedding theorem, $\mathcal{M}(\Omega)\hookrightarrow W^{-1,p'}(\Omega)$ and so, if $f=\mu$ belongs to $\mathcal{M}(\Omega)$, there exists a unique weak solution of* (2.2).

We focus on the case $p<N$.
Now we give an existence result of distributional solutions of (2.2).

THEOREM 2.11. *Let $f=\mu$ be in $\mathcal{M}(\Omega)$. Then there exists a distributional solution of* (2.2). *Moreover u belongs to $M^{\frac{N(p-1)}{N-p}}(\Omega)$ and $|\nabla u|$ belongs to $M^{\frac{N(p-1)}{N-1}}(\Omega)$. In particular, if $2-\dfrac{1}{N}<p<N$, then u belongs to $W_0^q(\Omega)$, for every $1\le q<\dfrac{N(p-1)}{N-1}$.*

PROOF. See [**7**], Theorem 6.1. □

We have the following regularity results for these solutions.

THEOREM 2.12. *Let $f\in L^m(\Omega)$ with $m\ge 1$. Then*

i) *if $m>\frac{N}{p}$, then the weak solution u, given by Theorem 2.9, belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$;*

ii) *if* $(p^*)' \le m < \frac{N}{p}$ *and* $1 < p < N$, *then the weak solution* u, *given by Theorem 2.9, belongs to* $W_0^{1,p}(\Omega) \cap L^s(\Omega)$, *with* $s = \frac{Nm(p-1)}{N-mp}$;

iii) *if* $1 < m < (p^*)'$ *and* $2 - \frac{1}{N} < p < N$, *then the distributional solution* u, *given by Theorem 2.11, belongs to* $W_0^{1,(p-1)m^*}(\Omega)$.

Proof. For (i) see [**14**], Theorem 1; for (ii) see [**14**], Theorem 5; for (iii) see [**12**], Theorem 3. □

If f does not belongs to $W^{-1,p'}(\Omega)$, Theorem 2.11 guarantees the existence but not the uniqueness of solutions of (2.2). We need a new notion of solution to recover the uniqueness. In [**37**] the authors presented the renormalized solution for (2.2) in four equivalents definitions. This notion allows to prove uniqueness at least for diffuse measure data. Here we give only the definition of renormalized solution that we will use in Chapter 3.

Definition 2.13. *Let* $f = \mu \in \mathcal{M}(\Omega)$, μ_d *be its absolutely continuous part with respect to the p-capacity and* $\mu_c = \mu_c^+ + \mu_c^-$ *be its singular part. A function* u *such that* $T_k(u) \in W_0^{1,p}(\Omega)$ *for every* $k > 0$, *is a renormalized solution to problem* (2.2) *if* $|\nabla u|^{p-1} \in L^q(\Omega)$ *for every* $1 \le q < \dfrac{N}{N-1}$ *and if the following conditions hold:*
(i) for every $\varphi \in C_b(\Omega)$ *it results*

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le u\le 2n\}} a(x,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi\, d\mu_c^+$$
$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n\le u\le -n\}} a(x,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi\, d\mu_c^- \,;$$

(ii) for every $S \in W^{1,\infty}(\mathbb{R})$ *with compact support in* $\mathbb{R}$ *and for every* $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *such that* $S(u)\varphi \in W_0^{1,p}(\Omega)$ *it results*

$$\int_\Omega a(x,\nabla u)\cdot\nabla u\, S'(u)\varphi + \int_\Omega a(x,\nabla u)\cdot\nabla\varphi\, S(u) = \int_\Omega S(u)\varphi\, d\mu_d\,.$$

Theorem 2.14. *Let* $f = \mu$ *be in* $\mathcal{M}(\Omega)$. *Then there exists a renormalized solution* u *of* (2.2). *Moreover, if* μ *is diffuse with respect to the p-capacity, then* u *is unique.*

Remark 2.15. *It is possible to prove that if* u *is a weak solution then* u *is a renormalized solution, and that, if* u *is a renormalized solution, then* u *is a distributional solution.*

Finally we conclude this subsection with two lemmas that are fundamental to prove the previous existence results of solutions of (2.2).

Lemma 2.16. *Assume that* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *is such that*

$$u_n \to u \text{ weakly in} W_0^{1,p}(\Omega)\,,$$
$$u_n \to u \text{ a.e. in } \Omega\,,$$
$$\lim_{n\to\infty} \int_\Omega \big(a(x,\nabla u_n) - a(x,\nabla u)\big)\cdot\nabla(u_n - u) = 0\,.$$

Then

$$u_n \to u \text{ strongly in } W_0^{1,p}(\Omega)\,.$$

PROOF. See [**16**], Lemma 5. □

LEMMA 2.17. *Let* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *be a sequence of solutions to the following problem*

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_n)) = g_n & \text{in } \Omega\,,\\ u_n = 0 & \text{on } \partial\Omega\,,\end{cases}$$

where $\{g_n\}$ *is bounded in* $L^1_{loc}(\Omega)$. *If we assume that* $\{u_n\}$ *is bounded in* $W^{1,p}_{loc}(\Omega)$ *and that* u_n *converges almost everywhere to a function* u *belonging to* $W^{1,p}_{loc}(\Omega)$, *then* ∇u_n *strongly converges to* ∇u *in* $L^q_{loc}(\Omega)^N$, *for every* $1 \le q < p$.

PROOF. See [**15**], Theorem 2.1. □

2.2. Nonlinear equations with "sublinear" right hand side

In this section we give existence, uniqueness and regularity results for a particular class of nonlinear elliptic equations. These results will be used in Chapter 5.

Let $\rho : \Omega \to \mathbb{R}$ be a measurable function, and suppose that there exists a positive constant ρ_0 and an open subset Ω' in Ω, with $\Omega' \subset\subset \Omega$, such that

$$\rho(x) \ge \rho_0 > 0 \quad \text{almost everywhere in } \Omega'\,. \tag{2.3}$$

Let $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a continuous, increasing function such that $g(0) = 0$, and

$$\exists \delta > 0\,, 0 \le \theta < p-1 \text{ s.t. } g(t) \le \delta t^\theta\,, \forall t \ge 0\,, \quad \text{and} \quad \lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} > \frac{\lambda_1}{\rho_0}\,, \tag{2.4}$$

where λ_1 is the first eigenvalue of $-\Delta_p$ on Ω'. We are focused on the following problem

$$\begin{cases} -\Delta_p u = \rho\, g(u) & \text{in } \Omega\,,\\ u \ge 0 & \text{in } \Omega\,,\\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.5}$$

The model case of this type of equations is exactly $g(t) = t^\theta$. If $p = 2$ the problem (2.5) is called *sublinear*.
Our notion of weak solution for (2.5) is the following:

DEFINITION 2.18. *Let* ρ *be a function in* $L^{\left(\frac{p^*}{\theta+1}\right)'}(\Omega)$. *A nonnegative function* u *in* $W_0^{1,p}(\Omega)$ *is a weak solution of* (2.5) *if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int_\Omega \rho g(u)\varphi\,, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,.$$

We give the existence and regularity result of a weak solution due to Lucio Boccardo and Luigi Orsina.

Theorem 2.19. *Let g be a function satisfying* (2.4) *and let ρ be a function satisfying* (2.3). *Let ρ be in $L^s(\Omega)$, with $s \geq \left(\frac{p^*}{\theta+1}\right)'$. Then there exists a weak solution u of* (2.5). *Moreover, if $\left(\frac{p^*}{\theta+1}\right)' \leq s < \frac{N}{p}$, then u belongs to $L^q(\Omega)$ with $q = \frac{Ns(p-1-\theta)}{N-sp}$. If $s > \frac{N}{p}$, then u belongs to $L^\infty(\Omega)$.*

Proof. See [**17**], Theorem 5.5. □

Remark 2.20. *If ρ is nonnegative and belongs to $L^\infty(\Omega)$, then, by the strong maximum principle, we have that $u > 0$ in Ω.*

Once again we ask when a weak solution is unique.
We assume that $\rho \not\equiv 0$ belongs to $L^\infty(\Omega)$ and we define $f(x,t) = \rho(x)g(t)$. Then we have that $f(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$ is a function such that

$$\begin{cases} t \mapsto f(x,t) \text{ is a continuous function on } [0,\infty) \text{ for almost every } x \in \Omega\,, \\ t \mapsto \dfrac{f(x,t)}{t^{p-1}} \text{ is decreasing on } (0,\infty) \text{ for almost every } x \in \Omega\,, \\ x \mapsto f(x,t) \text{ belongs to } L^\infty(\Omega) \text{ for each } s \geq 0\,. \end{cases} \tag{2.6}$$

We have the following uniqueness result by Brezis and Oswald.

Theorem 2.21. *Let $p = 2$ and let f be a function satisfying* (2.6). *Then there exists at most one weak solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of*

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Proof. See [**22**], Theorem 1. □

For our purposes we need a similar result in the general case $p > 1$ and for ρ nonnegative, hence for f nonnegative. We state the following proposition:

Proposition 2.22. *Let f be a nonnegative function satisfying* (2.6). *Then there exists at most one weak solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of*

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.7}$$

Proof. We suppose that there exist u, v in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ weak solutions of (2.7). Fix $\varepsilon > 0$. If we define

$$\varphi = \frac{(u+\varepsilon)^p - (v+\varepsilon)^p}{(u+\varepsilon)^{p-1}}\,,$$

and

$$\psi = \frac{(u+\varepsilon)^p - (v+\varepsilon)^p}{(v+\varepsilon)^{p-1}}\,,$$

then φ, ψ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\nabla\varphi = \nabla u \,-\, p\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p-1}\nabla v \,+\, (p-1)\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p}\nabla u\,,$$

and

$$\nabla\psi = -\nabla v \,+\, p\left(\frac{u+\varepsilon}{v+\varepsilon}\right)^{p-1}\nabla u \,-\, (p-1)\left(\frac{u+\varepsilon}{v+\varepsilon}\right)^{p}\nabla v\,.$$

We choose φ as test function for (2.7) with solution u and ψ as test function for (2.7) with solution v, then, subtracting the equation for v to that for u, we obtain

$$\begin{aligned}
&\int_\Omega \left\{|\nabla u|^p - \left|\frac{u+\varepsilon}{v+\varepsilon}\nabla v\right|^p - p\left(\frac{u+\varepsilon}{v+\varepsilon}\right)^{p-1}|\nabla v|^{p-2}\nabla v\cdot\left(\nabla u - \frac{u+\varepsilon}{v+\varepsilon}\nabla v\right)\right\}\\
&+\int_\Omega \left\{|\nabla v|^p - \left|\frac{v+\varepsilon}{u+\varepsilon}\nabla u\right|^p - p\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p-1}|\nabla u|^{p-2}\nabla u\cdot\left(\nabla v - \frac{v+\varepsilon}{u+\varepsilon}\nabla u\right)\right\}\\
&=\int_\Omega \left\{\left[\frac{f(x,u)}{(u+\varepsilon)^{p-1}} - \frac{f(x,v)}{(v+\varepsilon)^{p-1}}\right][(u+\varepsilon)^p - (v+\varepsilon)^p]\right\}\,.
\end{aligned}$$

As a consequence of the strict convexity of the function $w \mapsto |w|^p$ acting between $\mathbb{R}^N$ and $\mathbb{R}$, the left hand side of the previous equality is strictly positive, hence we deduce that

$$\int_\Omega \left\{\left[\frac{f(x,u)}{(u+\varepsilon)^{p-1}} - \frac{f(x,v)}{(v+\varepsilon)^{p-1}}\right][(u+\varepsilon)^p - (v+\varepsilon)^p]\right\} > 0\,.$$

We define for almost every x in Ω

$$h_\varepsilon(x) = \left[\frac{f(x,u)}{(u+\varepsilon)^{p-1}} - \frac{f(x,v)}{(v+\varepsilon)^{p-1}}\right][(u+\varepsilon)^p - (v+\varepsilon)^p]\,,$$

and

$$h(x) = \left(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}}\right)(u^p - v^p)\,,$$

so that, recalling that u, v are positive in Ω, we have that h_ε converges to h almost everywhere in Ω. The assumptions on f imply that $h \le 0$. Therefore we can decompose

$$0 \le \int_\Omega h_\varepsilon = \int_\Omega h_\varepsilon\,\chi_{\{h_\varepsilon>0\}} - \int_\Omega (-h_\varepsilon)\,\chi_{\{h_\varepsilon\le 0\}}. \tag{2.8}$$

Since $h \le 0$, letting ε tend to 0^+, we obtain that $h_\varepsilon\,\chi_{\{h_\varepsilon>0\}}$ converges to 0 and $-h_\varepsilon\,\chi_{\{h_\varepsilon\le 0\}}$ converges to $-h$ almost everywhere in Ω. Moreover, using that f, u, v are nonnegative and belong to $L^\infty(\Omega)$, we have $h_\varepsilon\,\chi_{\{h_\varepsilon>0\}} \le C$, so, by Lebesgue theorem,

$$\lim_{\varepsilon\to 0^+}\int_\Omega h_\varepsilon\,\chi_{\{h_\varepsilon>0\}} \,=\, 0\,.$$

As regards the second term on the right hand side of (2.8), we can apply Fatou's lemma to obtain

$$-\limsup_{\varepsilon\to 0^+}\int_\Omega h_\varepsilon\,\chi_{\{h_\varepsilon\le 0\}} \;=\; \liminf_{\varepsilon\to 0^+}\int_\Omega (-h_\varepsilon)\,\chi_{\{h_\varepsilon\le 0\}} \;\ge\; \int_\Omega -h\,,$$

hence

$$0\le \limsup_{\varepsilon\to 0^+}\int_\Omega h_\varepsilon \;=\; \limsup_{\varepsilon\to 0^+}\int_\Omega h_\varepsilon\,\chi_{\{h_\varepsilon\le 0\}} \;\le\; \int_\Omega h\,.$$

Recalling that $h\le 0$, we deduce that

$$\left(\frac{f(x,u)}{u^{p-1}}-\frac{f(x,v)}{v^{p-1}}\right)(u^p-v^p) \;=\; 0\,,$$

almost everywhere in Ω. If we assume that $u\ne v$, by (2.6) we deduce that

$$\left(\frac{f(x,u)}{u^{p-1}}-\frac{f(x,v)}{v^{p-1}}\right)(u^p-v^p) \;<\; 0$$

so that there is a contradiction and, hence, $u=v$ almost everywhere in Ω. □

As a consequence of Theorem 2.19 and Proposition 2.22 we have the following theorem:

THEOREM 2.23. *Let ρ be a nonnegative function in $L^\infty(\Omega)$. Then there exists a unique weak solution of* (2.5).

2.3. Semilinear equations with singular lower order terms

In this section we give known results on semilinear elliptic equations with lower order terms that are singular where the solution is zero. The starting point of the weak existence's theory of solutions for this type of problems is the paper [**18**] due to Lucio Boccardo and Luigi Orsina. We underline that this work has greatly influenced this thesis.

Let $M(x)$ be a matrix which satisfies, for some positive constants $0<\alpha\le\beta$, a.e. in $x\in\Omega$ and $\forall\xi\in\mathbb{R}^N$ the following assumptions:

$$M(x)\,\xi\cdot\xi\ge\alpha|\xi|^2 \qquad\text{and}\qquad |M(x)|\le\beta\,. \tag{2.9}$$

Let $\gamma>0$ be a real number. We consider the following semilinear elliptic problem with a singular nonlinearity

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)=\dfrac{f}{u^\gamma} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

As always we give the definition of solution for (2.10).

DEFINITION 2.24. *Let f be a function in $L^1(\Omega)$. A function u in $W^{1,1}_{loc}(\Omega)$ such that*

$$\begin{cases} u\in W^{1,1}_0(\Omega) & \textit{if } \gamma<1,\\ u^{\frac{\gamma+1}{2}}\in W^{1,2}_0(\Omega) & \textit{if } \gamma\ge 1, \end{cases}$$

is a distributional solution of (2.10) *if the following conditions are satisfied:*

$$\forall \omega \subset\subset \Omega \ \exists\, c_{\omega,\gamma} \,:\, u \,\geq\, c_{\omega,\gamma} \,>\, 0 \ \ in\ \omega\,,$$

and

$$\int_\Omega M(x)\nabla u \cdot \nabla \varphi \,=\, \int_\Omega \frac{f\,\varphi}{u^\gamma}\,, \quad \forall\, \varphi \in C^1_c(\Omega)\,.$$

We underline that, if $\gamma > 1$, the condition $u^{\frac{\gamma+1}{2}} \in W^{1,2}_0(\Omega)$ gives meaning to the boundary condition of (2.10).
In [**18**], existence results for distributional solutions of (2.10) have been proved. To be more precise, we have the following theorem in the case $\gamma > 1$, which is relevant to our purposes in Chapter 4.

THEOREM 2.25. *Let $\gamma > 1$, and let f be in $L^\infty(\Omega)$, with $f \geq 0$ in Ω, f not identically zero. Then there exists a distributional solution u of* (2.10)*, with u in $W^{1,2}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$. Moreover we can extend the class of test functions in the sense that*

$$\int_\Omega M(x)\nabla u \cdot \nabla \varphi = \int_\Omega \frac{f\,\varphi}{u^\gamma}, \qquad \forall \varphi \in W^{1,2}_0(\Omega) \text{ with compact support.} \tag{2.11}$$

Sketch of the proof of Theorem 2.25. Let m in $\mathbb{N}$ and consider the approximated problems

$$\begin{cases} -{\rm div}(M(x)\nabla u_m) = \dfrac{f}{(u_m + \frac{1}{m})^\gamma} & \text{in } \Omega, \\ u_m > 0 & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

The existence of a solution u_m can be easily proved by means of the Schauder fixed point theorem. Since the sequence $g_m(s) = \dfrac{1}{(s + \frac{1}{m})^\gamma}$ is increasing in m, standard elliptic estimates imply that the sequence $\{u_m\}$ is increasing, so that $u_m \geq u_1$, and there exists the pointwise limit u of u_m. Since (by the maximum principle) for every $\omega \subset\subset \Omega$ there exists $c_{\omega,\gamma} > 0$ such that $u_1 \geq c_{\omega,\gamma}$ in ω, it then follows that u_m (and so u) has the same property.
Choosing u_m^γ as test function in (2.12) we obtain, using (2.9), that

$$\frac{4\alpha\gamma}{(\gamma+1)^2} \int_\Omega |\nabla u_m^{\frac{\gamma+1}{2}}|^2 \leq \gamma \int_\Omega M(x)\nabla u_m \cdot \nabla u_m\, u_m^{\gamma-1} = \int_\Omega \frac{f\, u_m^\gamma}{(u_m + \frac{1}{m})^\gamma} \leq \int_\Omega f\,.$$

Therefore, $\{u_m^{\frac{\gamma+1}{2}}\}$ is bounded in $W^{1,2}_0(\Omega)$. Choosing $u_m\,\varphi^2$ as test function in (2.12), with φ in $C^1_0(\Omega)$, we obtain, using again (2.9),

$$\alpha \int_\Omega |\nabla u_m|^2\, \varphi^2 + 2\int_\Omega M(x)\nabla u_m \nabla \varphi\, u_m\, \varphi \leq \int_\Omega \frac{f\, u_m\, \varphi^2}{(u_m + \frac{1}{m})^\gamma}\,.$$

Hence, if $\omega = \{\varphi \neq 0\}$, recalling that $u_m \geq c_{\omega,\gamma} > 0$ in ω, we have, by Young's inequality,

$$\alpha \int_\Omega |\nabla u_m|^2 \varphi^2 \leq \frac{\alpha}{2} \int_\Omega |\nabla u_m|^2 \varphi^2 + C \int_\Omega |\nabla \varphi|^2 u_m^2 + \frac{\|f\varphi^2\|_{L^\infty(\Omega)}}{c_{\omega,\gamma}^\gamma} \int_\Omega u_m \,.$$

Since u_m is bounded in $L^2(\Omega)$ (recall that $u_m^{\frac{\gamma+1}{2}}$ is bounded in $W_0^{1,2}(\Omega)$, so that $u_m^{\gamma+1}$ is bounded in $L^1(\Omega)$ by Poincaré inequality, and that $\gamma > 1$), we thus have

$$\int_\Omega |\nabla u_m|^2 \varphi^2 \leq C \,,$$

so that the sequence $\{u_m\}$ is bounded in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Let now $k > 1$, choose $G_k(u_m)$ as test function in (2.12). We obtain, using (2.9),

$$\alpha \int_\Omega |\nabla G_k(u_m)|^2 \leq \int_\Omega \frac{f\, G_k(u_m)}{(u_m + \frac{1}{m})^\gamma} \leq \frac{1}{k^\gamma} \int_\Omega f\, G_k(u_m) \,,$$

so that

$$\alpha \int_\Omega |\nabla G_k(u_m)|^2 \leq \int_\Omega f\, G_k(u_m) \,, \qquad \forall k \geq 1 \,.$$

Starting from this inequality, and reasoning as in [**69**], we can prove that u_m is bounded in $L^\infty(\Omega)$, so that u belongs to $L^\infty(\Omega)$ as well.
Once we have the *a priori* estimates on u_m, we can pass to the limit in the approximate equation with test functions φ in $W_0^{1,2}(\Omega)$ with compact support; indeed

$$\lim_{m \to +\infty} \int_\Omega M(x)\, \nabla u_m \cdot \nabla \varphi = \int_\Omega M(x)\, \nabla u \cdot \nabla \varphi \,,$$

since u_m is weakly convergent to u in $W^{1,2}_{\mathrm{loc}}(\Omega)$, and

$$\lim_{n \to +\infty} \int_\Omega \frac{f\, \varphi}{(u_m + \frac{1}{m})^\gamma} = \int_\Omega \frac{f\,\varphi}{u^\gamma} \,,$$

by the Lebesgue theorem, since $u_m \geq c_{\{\varphi\neq 0\},\gamma} > 0$ on the support of φ. ☐

To be complete we give the existence result also in the case $\gamma \leq 1$. The proof is very similar to the previous one.

THEOREM 2.26. *Let f be a nonnegative function in $L^m(\Omega)$ with $m \geq 1$. Then there exists a distributional solution u of* (2.10) *such that*

i) *if $\gamma = 1$ then $u \in W_0^{1,2}(\Omega)$;*
ii) *if $\gamma < 1$ and $m \geq \left(\frac{2^*}{1-\gamma}\right)'$ then $u \in W_0^{1,2}(\Omega)$, otherwise if $1 \leq m < \left(\frac{2^*}{1-\gamma}\right)'$ then $u \in W_0^{1,\frac{Nm(\gamma+1)}{N-m(1-\gamma)}}(\Omega)$.*

PROOF. For i) see [**18**],Theorem 3.2; for ii) see [**18**], Theorem 5.2 and Theorem 5.6. ☐

As regards the uniqueness results we refer to [**10**] where the authors prove the following theorem:

THEOREM 2.27. *Let u in $W_0^{1,2}(\Omega)$ be a distributional solution of (2.10). Then u is the unique weak solution in the sense that*

$$\frac{f\,\varphi}{u^\gamma} \in L^1(\Omega), \quad \forall\,\varphi \in W_0^{1,2}(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi \;=\; \int_\Omega \frac{f\,\varphi}{u^\gamma}\,, \quad \forall\,\varphi\in W_0^{1,2}(\Omega)\,.$$

PROOF. See [**10**], Theorem 2.2 and Theorem 2.4. □

We observe that, if we consider (2.5) with $p = 2$, $\rho = f$ nonnegative and $g(t) = t^\theta$, then the problem (2.5) becomes

$$\begin{cases} -\Delta u = f\,u^\theta & \text{in } \Omega\,,\\ \quad u>0 & \text{in } \Omega\,,\\ \quad u=0 & \text{on } \partial\Omega\,,\end{cases} \tag{2.13}$$

where $0 \le \theta < 1$. Hence we can unify the existence and uniqueness theorems of weak solutions for the sublinear problem (Theorem 2.19 and Theorem 2.23) and for the singular problem with $\gamma \le 1$ (Theorem 2.26 and Theorem 2.27) under a unique theorem.

THEOREM 2.28. *Let $-1 \le \theta < 1$. Let f be a nonnegative function (not identically zero) in $L^m(\Omega)$, with $m \ge \left(\dfrac{2^*}{1+\theta}\right)'$ (if $\theta = -1$ we define $(\infty)' = 1$). Then there exists a weak solution u of (2.13). Moreover, if $-1 \le \theta \le 0$ or if $0 < \theta < 1$ and f belongs to $L^\infty(\Omega)$, then u is unique.*

REMARK 2.29. *If we suppose formally that u is a classical solution of (2.10), by the change of variable $v = \dfrac{u^{\gamma+1}}{\gamma+1}$, we have*

$$\nabla v \;=\; u^\gamma \nabla u\,,$$

and, using that u is a solution of (2.10), we deduce

$$\operatorname{div}(M(x)\nabla v) \;=\; \gamma u^{\gamma-1} M(x)\nabla u\cdot\nabla u \;+\; u^\gamma \operatorname{div}(M(x)\nabla u) \;=\; \gamma u^{\gamma-1}M(x)\nabla u\cdot\nabla u \;-\; f\,.$$

Observing that

$$\gamma u^{\gamma-1}M(x)\nabla u\cdot\nabla u \;=\; \gamma\frac{M(x)u^\gamma\nabla u\cdot u^\gamma\nabla u}{u^{\gamma+1}} \;=\; \frac{\gamma}{\gamma+1}\frac{M(x)\nabla v\cdot\nabla v}{v}\,,$$

we conclude that

$$-\operatorname{div}(M(x)\nabla v) \;+\; \frac{\gamma}{\gamma+1}\frac{M(x)\nabla v\cdot\nabla v}{v} \;=\; f\,.$$

Thus, formally, v is a solution of

$$\begin{cases} -\operatorname{div}(M(x)\nabla v) + b\dfrac{M(x)\nabla v\cdot\nabla v}{v} = f & \text{in } \Omega,\\ \qquad v = 0 & \text{on } \partial\Omega,\end{cases}$$

where $b = \dfrac{\gamma}{\gamma+1}$ *belongs to* $(0,1)$.

Remark 2.29 leads us to study in the next section this type of elliptic equations.

2.4. Quasilinear equations with singular and gradient quadratic lower order terms

Here we are focus on the existence of solutions for quasilinear elliptic problems with singular lower order terms that have natural growth with respect to the gradient.

Let $M(x,s) = (m_{ij}(x,s))$, for $i,j = 1,...,N$, be a matrix whose coefficients $m_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e., $m_{ij}(\cdot, s)$ is measurable on Ω for every $s \in \mathbb{R}$, and $m_{ij}(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$) such that there exist constants $0 < \alpha \le \beta$ satisfying

$$M(x,s)\,\xi\cdot\xi \ge \alpha|\xi|^2 \quad \text{and} \quad |M(x,s)| \le \beta\,, \quad \text{for a.e. } x \in \Omega\,, \forall\,(s,\xi) \in \mathbb{R}\times\mathbb{R}^N\,. \tag{2.14}$$

Let $b > 0$ and $\rho > 0$ be real numbers. We consider the following quasilinear elliptic problem with singular and gradient quadratic lower order term

$$\begin{cases} -\mathrm{div}(M(x,u)\nabla u) + b\,\dfrac{|\nabla u|^2}{u^\rho} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

We give the definition of weak solution for (2.15).

Definition 2.30. *A function* u *in* $W_0^{1,2}(\Omega)$ *is a weak solution of* (2.15) *if the following conditions are satisfied*

i) $u > 0$ *almost everywhere in* Ω,

ii) $\dfrac{|\nabla u|^2}{u^\rho}$ *belongs to* $L^1(\Omega)$,

iii) *it holds*

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\varphi + b\int_\Omega \frac{|\nabla u|^2}{u^\rho}\,\varphi = \int_\Omega f\,\varphi\,, \quad \forall\,\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Even if we are interested only in weak solutions of (2.15), to be complete we give the definition of distributional solution.

Definition 2.31. *A function* u *in* $W_0^{1,1}(\Omega)$ *is a distributional solution of* (2.15) *if the following conditions are satisfied*

i) $u > 0$ *almost everywhere in* Ω,

ii) $\dfrac{|\nabla u|^2}{u^\rho}$ *belongs to* $L^1(\Omega)$,

iii) *it holds*

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\varphi + b\int_\Omega \frac{|\nabla u|^2}{u^\rho}\,\varphi = \int_\Omega f\,\varphi\,, \quad \forall\,\varphi \in C_c^1(\Omega)\,.$$

During recent years the existence and nonexistence of a weak solution for (2.15) has been widely studied. We can summarize these results in the following three theorems.

THEOREM 2.32. *Let $0 < \rho < 1$, $b > 0$ and let f be a nonnegative function in $L^m(\Omega)$, with $m \geq \left(\frac{2^*}{\rho}\right)'$. Then there exists u weak solution of* (2.15)*.*

PROOF. See [**8**], Theorem 3.1. □

THEOREM 2.33. *Let $\rho = 1$, $0 < b < \alpha$ (where α is given by* (2.14)*) and let f be a nonnegative function in $L^m(\Omega)$, with $m \geq \frac{2N}{N+2}$. Then there exists u weak solution of* (2.15)*.*

PROOF. See [**61**], Theorem 1.1 or, if $b \leq \frac{\alpha}{2}$, see [**8**], Theorem 4.1. □

THEOREM 2.34. *Let f be a nonnegative function in $L^m(\Omega)$, with $m > \frac{N}{2}$. Suppose that for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $f \geq c_\omega$ in ω. Then there exists a weak solution of* (2.15) *if and only if $0 < \rho < 2$. Moreover, let λ_1 be the first eigenvalue of the Laplacian in the N-dimensional unit ball, assume $f \in L^\infty(\Omega)$, $M(x,s) \equiv I$, $b = 1$, and either*

$$\rho > 2 \quad or \quad \rho = 2 \text{ and } \|f\|_{L^\infty(\Omega)} < \frac{\lambda_1}{\operatorname{diam}(\Omega)^2}\,.$$

Then the sequence $\{u_n\}$ of solutions of

$$\begin{cases} -\Delta u_n + \dfrac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\rho} = f & in\ \Omega, \\ \qquad u = 0 & on\ \partial\Omega, \end{cases}$$

tends to 0 in $W_0^{1,2}(\Omega)$, and the sequence $\left\{ \dfrac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\rho} \right\}$ converges to f in the $*$-weak topology of measures.

PROOF. See [**4**], Theorem 1.5. □

So, in the case f only *nonnegative* and $\rho = 1$, if $b = \alpha$ we lack theorems on existence or nonexistence for weak solutions of (2.15). Chapter 4 is devoted to fill this big hole.

CHAPTER 3

Existence and uniqueness for nonlinear elliptic equations with possibly singular right hand side and measure data

In this chapter we are concerned with the existence of a distributional solution and of a renormalized solution for a singular elliptic problem modelled by

$$\begin{cases} -\Delta_p u = H(u)\mu & \text{in}\, \Omega, \\ u > 0 & \text{in}\, \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, for $1 < p < N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator, μ is a nonnegative bounded Radon measure on Ω and $H(s)$ is a nonnegative, continuous and finite function outside the origin, which, roughly speaking, behaves as $s^{-\gamma}$ ($\gamma \geq 0$) near zero.

The idea to deal with this type of singular problems is first to approximate these problems with nonsingular ones, truncating the singular lower order term, and to find a priori estimates on the sequence of approximate solutions; then passing to the limit of the approximations to obtain at least a distributional solution.
There are two main difficulties with this method.
The first is that we have a nonlinear left hand side so that the weak convergence of the approximating solutions is not sufficient to pass to the limit in the distributional formulation.
The second is that we look for positive solutions in the domain to give sense to the right hand side, so that we need a property of uniform local positivity of the approximations which also holds to the limit.

We overcome these problems and we show the existence of a distributional solution and then, if $\gamma \leq 1$, we prove the existence of a renormalized solution. As noted in the Introduction the existence of a renormalized solution implies that this solution obtained by approximation is unique.

3.1. Main assumptions and results

We will consider the following problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = H(u)\mu & \text{in}\, \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions structure conditions for $1<p<N$, namely

$$a(x,\xi)\cdot\xi\geq\alpha|\xi|^p,\quad \alpha>0, \tag{3.2}$$

$$|a(x,\xi)|\leq\beta|\xi|^{p-1},\quad \beta>0, \tag{3.3}$$

$$(a(x,\xi)-a(x,\xi'))\cdot(\xi-\xi')>0, \tag{3.4}$$

for every $\xi\neq\xi'$ in $\mathbb{R}^N$ and for almost every x in Ω.
Moreover μ is a nonnegative bounded Radon measure on Ω uniquely decomposed as the sum $\mu_d+\mu_c$, where μ_d is a diffuse measure with respect to the p-capacity and μ_c is a measure concentrated on a set of zero p-capacity. We underline that (see Remark 3.7 below) we will always assume

$$\mu_d\not\equiv 0. \tag{3.5}$$

Finally, if not otherwise specified, $H:(0,+\infty)\to(0,+\infty)$ is a continuous function, possibly blowing up at the origin, such that the following properties hold true

$$\exists \lim_{s\to\infty}H(s):=H(\infty)<\infty \tag{3.6}$$

$$\exists\, C,s_0>0,\gamma\geq 0 \text{ s.t. } H(s)\leq\frac{C}{s^\gamma} \text{ if } s<s_0. \tag{3.7}$$

We emphasize that, since we are allowing γ to be zero, we are taking into account also the case of a bounded H. Moreover the assumption on the strict positivity of H is a technical one needed to handle the case in which the singular part of the measure is not identically zero, as widely explained in Section 3.5.

First of all it is worth to clarify what we mean by *solution* to problem (3.1). We provide two different notions of solution.

Definition 3.1. *Let a satisfy* (3.2), (3.3), (3.4), *let μ be a nonnegative bounded Radon measure and let H satisfy* (3.6) *and* (3.7). *A positive function u, which is almost everywhere finite on Ω, is a* renormalized solution *to problem* (3.1) *if $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$ and if the following hold*

$$H(u)S(u)\varphi\in L^1(\Omega,\mu_d) \textit{ and}$$

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi S(u)+\int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)\varphi=\int_\Omega H(u)S(u)\varphi d\mu_d \tag{3.8}$$

$$\forall S\in W^{1,\infty}(\mathbb{R}) \textit{ with compact support and } \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}}a(x,\nabla u)\cdot\nabla u\varphi=H(\infty)\int_\Omega\varphi d\mu_c\quad\forall\varphi\in C_b(\Omega). \tag{3.9}$$

Definition 3.2. *Let a satisfy* (3.2), (3.3), (3.4), *let μ be a nonnegative bounded Radon measure and let H satisfy* (3.6) *and* (3.7). *A positive and measurable function u such that*

$|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ *is a* distributional solution *to problem* (3.1) *if* $H(u) \in L^1_{loc}(\Omega, \mu_d)$*, and the following hold*

$$T_k^{\frac{\tau-1+p}{p}}(u) \in W_0^{1,p}(\Omega) \quad \forall k > 0, \quad \textit{where} \quad \tau = \max(1, \gamma), \tag{3.10}$$

and

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi = \int_\Omega H(u)\varphi d\mu_d + H(\infty) \int_\Omega \varphi d\mu_c \quad \forall \varphi \in C_c^1(\Omega). \tag{3.11}$$

The notion of renormalized solution is way more general than the distributional one. Indeed, if $\gamma \leq 1$, it results that the former implies the latter one.

LEMMA 3.3. *Let* $\gamma \leq 1$ *and let* u *be a renormalized solution to* (3.1). *Then* u *is also a distributional solution to* (3.1).

PROOF. It follows from the definition of renormalized solution that (3.10) holds. Taking as test functions in (3.8) $S = \theta_t$, where θ_t is defined in (1.2), and $\varphi = T_k(u)$, with $s_0 < k < t$, we obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_k(u)\theta_t(u) \leq \frac{k}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u + \int_\Omega H(u)T_k(u)\theta_t(u)d\mu_d.$$

Using (3.2) and (3.7), we find

$$\begin{aligned}\alpha \int_\Omega |\nabla T_k(u)|^p &\leq \frac{k}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u + \int_{\{u<s_0\}} H(u)T_k(u)\theta_t(u)d\mu_d \\ &+ \int_{\{u\geq s_0\}} H(u)T_k(u)\theta_t(u)d\mu_d \leq \frac{k}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u \\ &+ C s_0^{1-\gamma} \|\mu_d\|_{\mathcal{M}(\Omega)} + k\|H\|_{L^\infty([s_0,+\infty))} \|\mu_d\|_{\mathcal{M}(\Omega)},\end{aligned}$$

so that, passing to the limit as $t \to \infty$, we find that there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla T_k(u)|^p \leq C(k+1), \quad \forall k > 0. \tag{3.12}$$

By (3.12), using Lemma 1.17 we deduce that u is cap_p-almost everywhere finite and cap_p-quasi continuous and, using Lemma 4.2 of [**7**], we deduce moreover that $|\nabla u|^{p-1} \in L^1(\Omega)$. Now taking $\varphi \in C_c^1(\Omega)$ and $S = \theta_t$ in (3.8) we obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi \theta_t(u) = \frac{1}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u \varphi + \int_\Omega H(u)\varphi\theta_t(u)d\mu_d. \tag{3.13}$$

By (3.8) it results $H(u)\theta_1(u)\varphi \in L^1(\Omega, \mu_d)$, and so, using Lemma 1.18, we find

$$\begin{aligned}\int_\Omega |H(u)\varphi| d\mu_d &= \int_{\{u<1\}} H(u)|\varphi|d\mu_d + \int_{\{u\geq 1\}} H(u)|\varphi|d\mu_d \\ &\leq \int_\Omega H(u)\theta_1(u)|\varphi|d\mu_d + \|H\|_{L^\infty([1,+\infty))} \|\varphi\|_{L^\infty(\Omega)} \|\mu_d\|_{\mathcal{M}(\Omega)} \leq C,\end{aligned}$$

that implies $H(u) \in L^1_{loc}(\Omega, \mu_d)$. Letting t go to infinity in (3.13) we obtain, applying Lebesgue's Theorem for general measures and (3.9), that (3.11) holds. Hence u is a distributional solution to (3.1). □

We will prove the following results.

THEOREM 3.4. *Let a satisfy* (3.2), (3.3), (3.4), *and let μ be a nonnegative bounded Radon measure which satisfies* (3.5). *If H satisfies* (3.6) *and* (3.7) *with $\gamma \le 1$, there exists a renormalized solution u to problem* (3.1). *Moreover,*

i) *if $p > 2 - \frac{1}{N}$ then $u \in W_0^{1,q}(\Omega)$ $\forall\, q < \frac{N(p-1)}{N-1}$;*
ii) *if $1 < p \le 2 - \frac{1}{N}$ then $u^{p-1} \in L^q(\Omega)$ $\forall\, q < \frac{N}{N-p}$ and $|\nabla u|^{p-1} \in L^q(\Omega)$ $\forall\, q < \frac{N}{N-1}$.*

Finally, if H is non-increasing and $\mu_c \equiv 0$, u is unique.

THEOREM 3.5. *Let a satisfy* (3.2), (3.3), (3.4), *and let μ be a nonnegative bounded Radon measure which satisfies* (3.5). *If H satisfies* (3.6) *and* (3.7), *there exists a distributional solution u to problem* (3.1) *such that*

$$u^{p-1} \in L^q_{loc}(\Omega)\ \forall\, q < \frac{N}{N-p} \quad \text{and} \quad |\nabla u|^{p-1} \in L^q_{loc}(\Omega)\ \forall\, q < \frac{N}{N-1}.$$

REMARK 3.6. *From Theorems 3.4, 3.5 and Lemma 3.3, we deduce that, for any nonlinearity H satisfying* (3.6) *and* (3.7) *with $\gamma \le 1$, we are able to find a renormalized solution that is also a distributional one. Otherwise, if H blows up too fast at the origin (i.e. $\gamma > 1$ in* (3.7)*), the solution loses the weak trace in the classical Sobolev sense and we are only able to prove the existence of a distributional solution. We underline that the renormalized framework seems to be the natural one associated to this kind of problems, since it is well posed with respect to uniqueness, at least in case of a non-increasing nonlinearity H.*

REMARK 3.7. *As concerns the assumption* (3.5), *we underline that, if $H(0) < \infty$, we can prove the existence of a renormalized solution to* (3.1) *even if it results $\mu_d \equiv 0$, since we never use that $\mu_d \not\equiv 0$ in the proof of Theorem 3.4 (cf. Section 3.2). If instead $H(0) = \infty$, we do not treat the case $\mu_d \equiv 0$ to avoid nonexistence results (in the approximation sense) analogous to the ones of Section 3.3 of* [**40**]. *Furthermore, in case $\mu_d \equiv 0$, our notions of solution formally lead us to the following problem with linear lower order term*

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = H(\infty)\mu_c & \text{in}\, \Omega, \\ u = 0 & \text{on}\ \partial\Omega, \end{cases}$$

which could be analyzed using classical tools.

3.2. Proof of existence in case of a finite H

We start proving the existence of a renormalized solution in case of a finite nonlinearity H, namely assuming $\gamma = 0$ in (3.7).

We introduce the following scheme of approximation

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_n)) = H(u_n)\mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where $\mu_n = \mu_{n,d} + \mu_{n,c} = f_n - \operatorname{div}(F_n) + \mu_{n,c}$. Following [**13**] we suppose that:

$$\begin{aligned} &0 \le f_n \in L^\infty(\Omega), \quad f_n \to f \text{ weakly in } L^1(\Omega), \\ &F_n \in W_0^{1,\infty}(\Omega)^N, \quad F_n \to F \text{ in } L^{p'}(\Omega)^N, \\ &0 \le \mu_{n,c} \in L^\infty(\Omega), \quad \mu_{n,c} \to \mu_c \text{ in the narrow topology of } \mathcal{M}(\Omega). \end{aligned} \tag{3.15}$$

Moreover it results that $\|\mu_n\|_{L^1(\Omega)} \le C$.
Since H is a continuous function satisfying (3.6) and (3.7) with $\gamma = 0$ and a satisfies (3.2), (3.3) and (3.4) with $1 < p < N$, the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is guaranteed by [**59**]. Furthermore, since H and μ_n are nonnegative functions, we also have that u_n is nonnegative. Taking $S(u_n)\varphi$ as test function in the weak formulation of (3.14) where $S \in W^{1,\infty}(\mathbb{R})$ and has compact support and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi S(u_n) + \int_\Omega a(x,\nabla u_n)\cdot\nabla u_n S'(u_n)\varphi = \int_\Omega H(u_n)S(u_n)\varphi\mu_n. \tag{3.16}$$

Moreover, since $a(x,\nabla u_n)\cdot\nabla u_n \in L^1(\Omega)$, we deduce

$$\lim_{t\to\infty} \frac{1}{t}\int_{\{t<u_n<2t\}} a(x,\nabla u_n)\cdot\nabla u_n\varphi = 0 \quad \forall\varphi \in C_b(\Omega), \tag{3.17}$$

namely u_n is also a renormalized solution to (3.14). We need some a priori estimates on u_n.

LEMMA 3.8. *Let u_n be a solution to* (3.14)*. Then $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$. Moreover:*

i) *if $p > 2 - \frac{1}{N}$, u_n is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$;*
ii) *if $1 < p \le 2 - \frac{1}{N}$, u_n^{p-1} is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-p}$ and $|\nabla u_n|^{p-1}$ is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-1}$.*

Finally u_n converges almost everywhere in Ω to a function u, which is cap_p-almost everywhere finite and cap_p-quasi continuous.

PROOF. We take $T_k(u_n)$ in the weak formulation of (3.14) obtaining

$$\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n) = \int_\Omega H(u_n)T_k(u_n)\mu_n.$$

Then, using (3.2) and (3.15), we find

$$\alpha\int_\Omega |\nabla T_k(u_n)|^p \le k\|H\|_{L^\infty(\mathbb{R})}\|\mu_n\|_{L^1(\Omega)} \le Ck, \tag{3.18}$$

namely $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ with respect to n.
Then, if $p > 2 - \frac{1}{N}$, by the computations of Subsection II.4 in **[11]**, it follows that u_n is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$. So there exists a nonnegative function u belonging to $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$ such that u_n converges to u almost everywhere in Ω and weakly in $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$.
Otherwise, if $1 < p \leq 2 - \frac{1}{N}$, it results that $0 < \frac{N(p-1)}{N-1} \leq 1$ and we cannot proceed as before. Anyway, from (3.18), using Lemma 4.1 and Lemma 4.2 of **[7]** we deduce that u_n is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-p}}(\Omega)$ and that $|\nabla u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-1}}(\Omega)$. In particular u_n^{p-1} is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-p}$ and $|\nabla u_n|^{p-1}$ is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-1}$. Furthermore, by (3.18) we deduce that $T_k(u_n)$ is a Cauchy sequence in $L^p(\Omega)$ for all $k > 0$, so that, up to subsequences, it is a Cauchy sequence in measure for each $k > 0$. Then, using the Marcinkiewicz estimates on u_n, we find that u_n is a Cauchy sequence in measure. To prove this property we begin by observing that for all $k, \sigma > 0$ and for all $n, m \in \mathbb{N}$, it results that

$$\{|u_n - u_m| > \sigma\} \subseteq \{|u_n| \geq k\} \cup \{|u_m| \geq k\} \cup \{|T_k(u_n) - T_k(u_m)| > \sigma\}. \tag{3.19}$$

Now, if $\varepsilon > 0$ is fixed, the Marcinkiewicz estimates imply that there exists a $\overline{k} > 0$ such that

$$|\{|u_n| > k\}| < \frac{\varepsilon}{3}, \;\; |\{|u_m| > k\}| < \frac{\varepsilon}{3} \;\forall n, m \in \mathbb{N}, \;\forall k > \overline{k},$$

while, using that $T_k(u_n)$ is a Cauchy sequence in measure for each $k > 0$ fixed, we deduce that there exists $\eta_\varepsilon > 0$ such that

$$|\{|T_k(u_n) - T_k(u_m)| > \sigma\}| < \frac{\varepsilon}{3} \;\forall n, m > \eta_\varepsilon, \;\forall \sigma > 0.$$

Thus, if $k > \overline{k}$, from (3.19) we obtain that

$$|\{|u_n - u_m| > \sigma\}| < \varepsilon \quad \forall n, m \geq \eta_\varepsilon, \;\forall \sigma > 0,$$

and so that u_n is a Cauchy sequence in measure. Then, in case $1 < p \leq 2 - \frac{1}{N}$, there exists a nonnegative measurable function $u : \Omega \to \mathbb{R}$ to which u_n converges almost everywhere in Ω. Since u_n^{p-1} is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-p}$, thanks to the almost everywhere convergence and Vitali's Theorem, we find that $u^{p-1} \in L^q(\Omega)$ for every $q < \frac{N}{N-p}$. This implies that the limit function u is almost everywhere finite.
Hence, in all cases, it results

$$T_k(u_n) \to T_k(u) \text{ weakly in } W_0^{1,p}(\Omega) \text{ for every } k > 0 \text{ and a.e. in } \Omega. \tag{3.20}$$

Finally, thanks to (3.18), by weak lower semicontinuity we deduce

$$\int_\Omega |\nabla T_k(u)|^p \leq C(k+1) \quad \forall k > 0,$$

and so, by the previous and Lemma 1.17, we conclude that the function u is cap_p-almost everywhere finite and cap_p-quasi continuous. □

The previous lemma guarantees only the weak convergence of $T_k(u_n)$ towards $T_k(u)$ in $W_0^{1,p}(\Omega)$. In the next lemma we prove the strong convergence of truncations in $W_0^{1,p}(\Omega)$, which, in turn, will assure the almost everywhere convergence of ∇u_n to ∇u in Ω.

LEMMA 3.9. *Let u_n be a solution to* (3.14). *Then $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$.*

PROOF. We follow the lines of Step 2 of the proof of Theorem 2.10 in [**62**]. We want to show that

$$\lim_{n\to\infty}\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\big)\cdot\nabla(T_k(u_n)-T_k(u)) = 0 \tag{3.21}$$

in order to apply [**16**, Lemma 5] and to conclude the proof.
In (3.16) we take $\varphi = (T_k(u_n)-T_k(u))(1-\Psi_\nu)$ and $S=\theta_r$, where $r>k$ and Ψ_ν is as in Lemma 1.16, obtaining

$$\begin{aligned}
&\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))(1-\Psi_\nu)\\
&= -\int_{\{k<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))\theta_r(u_n)(1-\Psi_\nu) \quad \text{(a)}\\
&+\frac{1}{r}\int_{\{r<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla u_n(T_k(u_n)-T_k(u))(1-\Psi_\nu) \quad \text{(b)}\\
&+\int_\Omega H(u_n)\theta_r(u_n)(T_k(u_n)-T_k(u))(1-\Psi_\nu)\mu_n \quad \text{(c)}\\
&+\int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\nu(T_k(u_n)-T_k(u))\theta_r(u_n). \quad \text{(d)}
\end{aligned} \tag{3.22}$$

For (a), we note that the term $\{a(x,\nabla u_n)\theta_r(u_n)\}$ is bounded in $L^{p'}(\Omega)^N$ with respect to n. Moreover we have that $|\nabla T_k(u)|\chi_{\{u_n>k\}}$ converges to zero in $L^p(\Omega)$, which allows us to deduce that

$$\text{(a)} \le C\int_\Omega |a(x,\nabla u_n)\theta_r(u_n)||\nabla T_k(u)|\chi_{\{u_n>k\}} = \epsilon(n). \tag{3.23}$$

In the same way, we observe that $\{a(x,\nabla u_n)\cdot\nabla\Psi_\nu\theta_r(u_n)\}$ is bounded in $L^{p'}(\Omega)$ and that, by (3.20), $T_k(u_n)$ strongly converges to $T_k(u)$ in $L^p(\Omega)$, and so we arrive to

$$\text{(d)} \le \int_\Omega |a(x,\nabla u_n)\cdot\nabla\Psi_\nu\theta_r(u_n)||(T_k(u_n)-T_k(u))| = \epsilon(n). \tag{3.24}$$

Now we focus on (c), finding, by (3.15), that

$$\begin{aligned}
\text{(c)} &\le \|H\|_{L^\infty(\mathbb{R})}\int_\Omega |T_k(u_n)-T_k(u)|\mu_{n,d}\\
&+\int_\Omega H(u_n)\theta_r(u_n)(T_k(u_n)-T_k(u))(1-\Psi_\nu)\mu_{n,c}.
\end{aligned} \tag{3.25}$$

Since $T_k(u_n) - T_k(u)$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and converges to zero almost everywhere in Ω, by Lemma 1.10, the first term of the right hand side of (3.25) converges to zero as n goes to infinity. As regards the second term we have that

$$\int_\Omega H(u_n)\theta_r(u_n)(T_k(u_n) - T_k(u))(1-\Psi_\nu)\mu_{n,c} \le 2k\|H\|_{L^\infty(\mathbb{R})} \int_\Omega (1-\Psi_\nu)\mu_{n,c},$$

which, through the narrow convergence of $\mu_{n,c}$ to μ_c and Lemma 1.16, implies

$$\text{(c)} \le \epsilon(n,r,\nu). \tag{3.26}$$

Gathering (3.23), (3.24), (3.26) in (3.22) we deduce

$$\begin{aligned} &\int_\Omega a(x,\nabla T_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))(1-\Psi_\nu) \\ &\le \epsilon(n,r,\nu) + \frac{2k}{r}\int_{\{r<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla u_n(1-\Psi_\nu). \end{aligned} \tag{3.27}$$

Let us take $\varphi = \pi_r(u_n)(1-\Psi_\nu)$ and $S = \theta_t$ in (3.16), where $r, k, t \in \mathbb{N}$, $r > k$, and $\pi_r(s)$ is given by (1.1). It results

$$\begin{aligned} &\frac{1}{r}\int_{\{r<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla u_n \theta_t(u_n)(1-\Psi_\nu) \\ &= \frac{1}{t}\int_{\{t<u_n<2t\}} a(x,\nabla u_n)\cdot\nabla u_n \pi_r(u_n)(1-\Psi_\nu) \quad \text{(a}'\text{)} \\ &+ \int_\Omega H(u_n)\pi_r(u_n)\theta_t(u_n)(1-\Psi_\nu)\mu_n \quad \text{(b}'\text{)} \\ &+ \int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\nu \pi_r(u_n)\theta_t(u_n). \quad \text{(c}'\text{)} \end{aligned} \tag{3.28}$$

As regards (c$'$), thanks to Lebesgue Theorem, it results

$$\lim_{t\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\nu\pi_r(u_n)\theta_t(u_n) = \int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\nu\pi_r(u_n).$$

Recalling that $\operatorname{supp}(\pi_r(s)) = \{|s| \ge r\}$, that u is almost everywhere finite and $|\nabla u_n|^{p-1}$ is bounded in $L^q(\Omega)$ for each $q < \frac{N}{N-1}$, then it follows from the Hölder inequality with exponents q and q', where $q < \frac{N}{N-1}$ is fixed, that

$$\begin{aligned} \left|\int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\nu\pi_r(u_n)\right| &\le \|\nabla\Psi_\nu\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u_n|^{(p-1)q}\right)^{\frac{1}{q}} |\{u_n \ge r\}|^{\frac{1}{q'}} \\ &\le C\ |\{u_n \ge r\}|^{\frac{1}{q'}} = \epsilon(n,r), \end{aligned}$$

which implies

$$\text{(c}'\text{)} \le \epsilon(t,n,r). \tag{3.29}$$

As concerns (b′) we have

(3.30)
$$\int_\Omega H(u_n)\pi_r(u_n)\theta_t(u_n)(1-\Psi_\nu)(\mu_{n,d}+\mu_{n,c}) \le \|H\|_{L^\infty(\mathbb{R})}\int_\Omega \pi_r(u_n)(1-\Psi_\nu)(\mu_{n,d}+\mu_{n,c}).$$

Finally we consider (a′). Letting t go to infinity and recalling (3.17), we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u_n<2t\}} a(x,\nabla u_n)\cdot\nabla u_n \pi_r(u_n)(1-\Psi_\nu)\\ &\le \lim_{t\to\infty}\frac{1}{t}\int_{\{t<u_n<2t\}} a(x,\nabla u_n)\cdot\nabla u_n = 0.\end{aligned} \tag{3.31}$$

As t goes to infinity in (3.28) and, by (3.29), (3.30), (3.31), we obtain

$$\frac{1}{r}\int_{\{r<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla u_n(1-\Psi_\nu) \le \epsilon(n,r) + \|H\|_{L^\infty(\mathbb{R})}\int_\Omega \pi_r(u_n)(1-\Psi_\nu)(\mu_{n,d}+\mu_{n,c}).$$

Since $\pi_r(u_n)$ converges to its almost everywhere limit weakly* in $L^\infty(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$, we deduce, by Lemma 1.10, that

$$\lim_{n\to\infty}\int_\Omega \pi_r(u_n)(1-\Psi_\nu)\mu_{n,d} = \int_\Omega \pi_r(u)(1-\Psi_\nu)d\mu_d.$$

As u is cap_p almost everywhere finite, $\pi_r(u)$ converges to zero μ_d-almost everywhere as $r\to\infty$; then, using Lebesgue Theorem for general measure, we obtain that

$$\int_\Omega \pi_r(u)(1-\Psi_\nu)d\mu_d = \epsilon(r,\nu).$$

Moreover it follows from the narrow convergence of $\mu_{n,c}$ to μ_c and from Lemma 1.16 that

$$\lim_{n\to\infty}\int_\Omega \pi_r(u_n)(1-\Psi_\nu)\mu_{n,c} \le \lim_{n\to\infty}\int_\Omega (1-\Psi_\nu)\mu_{n,c} = \int_\Omega (1-\Psi_\nu)d\mu_c \le C\nu.$$

Thus we obtain

$$\frac{1}{r}\int_{\{r<u_n<2r\}} a(x,\nabla u_n)\cdot\nabla u_n(1-\Psi_\nu) \le \epsilon(n,r,\nu), \tag{3.32}$$

and then, going back to (3.27), we conclude that

$$\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))(1-\Psi_\nu) \le \epsilon(n,r,\nu).$$

Now we reason as follows

$$
\begin{aligned}
&\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\big)\cdot\nabla(T_k(u_n)-T_k(u))\\
&= \int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\big)\cdot\nabla(T_k(u_n)-T_k(u))\Psi_\nu\\
&+ \int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))(1-\Psi_\nu)\\
&- \int_\Omega a(x,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))(1-\Psi_\nu)\\
&\le C\int_\Omega \Big(|\nabla T_k(u_n)|^p + |\nabla T_k(u)|^p\Big)\Psi_\nu + \epsilon(n,r,\nu).
\end{aligned}
\tag{3.33}
$$

Now choosing as test function $(k-u_n)^+\Psi_\nu$ in the weak formulation (3.14) we have

$$
\begin{aligned}
&-\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\Psi_\nu + \int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla\Psi_\nu(k-u_n)^+\\
&= \int_\Omega H(u_n)(k-u_n)^+\Psi_\nu\mu_{n,d} + \int_\Omega H(u_n)(k-u_n)^+\Psi_\nu\mu_{n,c},
\end{aligned}
$$

which implies, using $\mu_{n,d}\ge 0$ and (3.2),

$$
\alpha\int_\Omega |\nabla T_k(u_n)|^p\Psi_\nu + \int_\Omega H(u_n)(k-u_n)^+\Psi_\nu\mu_{n,c} \le \int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla\Psi_\nu(k-u_n)^+.
\tag{3.34}
$$

Moreover, since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, it follows by an application of the Hölder inequality and by Lemma 1.16 that

$$
\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla\Psi_\nu(k-u_n)^+ \le k\|T_k(u_n)\|_{W_0^{1,p}(\Omega)}\|\Psi_\nu\|_{W_0^{1,p}(\Omega)} \le \epsilon(n,\nu).
\tag{3.35}
$$

By (3.34) and (3.35) we obtain

$$
\int_\Omega |\nabla T_k(u_n)|^p\Psi_\nu = \epsilon(n,\nu)
\tag{3.36}
$$

and

$$
\int_\Omega H(u_n)(k-u_n)^+\Psi_\nu\mu_{n,c} = \epsilon(n,\nu).
\tag{3.37}
$$

Finally, by (3.33) and (3.36), we have

$$
\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\big)\cdot\nabla(T_k(u_n)-T_k(u)) \le \epsilon(n,r,\nu),
$$

which is (3.21) as desired. In conclusion it holds

$$
T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega) \text{ for every fixed } k>0,
$$

yielding also that ∇u_n converges almost everywhere in Ω to ∇u. □

REMARK 3.10. *It follows from Lemma 3.8 and Lemma 3.9 that, if $p > 2-\frac{1}{N}$, u_n converges to u strongly in $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$. Otherwise, if $1 < p \leq 2-\frac{1}{N}$, u_n^{p-1} converges to u^{p-1} strongly in $L^q(\Omega)$ for every $q < \frac{N}{N-p}$ and $|\nabla u_n|^{p-1}$ converges to $|\nabla u|^{p-1}$ strongly in $L^q(\Omega)$ for every $q < \frac{N}{N-1}$. In all cases we have*

$$a(x, \nabla u_n) \to a(x, \nabla u) \text{ strongly in } L^q(\Omega)^N \text{ for every } q < \frac{N}{N-1}. \tag{3.38}$$

Now we are ready to prove Theorem 3.4 in case $\gamma = 0$, namely when $H(0) < \infty$.

PROOF OF THEOREM 3.4 IN CASE $\gamma = 0$. In order to prove the existence part of the theorem we only need to show that u, almost everywhere limit of the solutions u_n to (3.14), is a renormalized solution to (3.1). Indeed we already know, by Lemma 3.8, that $T_k(u) \in W_0^{1,p}(\Omega)$. If $S \in W^{1,\infty}(\mathbb{R})$ with $\text{supp}(S) \subset [-M, M]$ and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, taking $S(u_n)\varphi$ as test function in the weak formulation of (3.14) we obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla\varphi S(u_n) + \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n S'(u_n)\varphi = \int_\Omega H(u_n)S(u_n)\varphi\mu_n. \tag{3.39}$$

It follows from Lemma 3.9 that we have

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n S'(u_n)\varphi &= \lim_{n\to\infty} \int_\Omega a(x, \nabla T_M(u_n)) \cdot \nabla T_M(u_n) S'(T_M(u_n))\varphi \\
&= \int_\Omega a(x, \nabla T_M(u)) \cdot \nabla T_M(u) S'(T_M(u))\varphi \\
&= \int_\Omega a(x, \nabla u) \cdot \nabla u S'(u)\varphi,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla\varphi S(u_n) &= \lim_{n\to\infty} \int_\Omega a(x, \nabla T_M(u_n)) \cdot \nabla\varphi S(T_M(u_n)) \\
&= \int_\Omega a(x, \nabla T_M(u)) \cdot \nabla\varphi S(T_M(u)) \\
&= \int_\Omega a(x, \nabla u) \cdot \nabla\varphi S(u).
\end{aligned}$$

Hence, in order to deduce (3.8), we need to pass to the limit the right hand side of (3.39). We split it as follows

$$\int_\Omega H(u_n)S(u_n)\varphi\mu_n = \int_\Omega H(u_n)S(u_n)\varphi\mu_{n,d} + \int_\Omega H(u_n)S(u_n)\varphi\mu_{n,c}, \tag{3.40}$$

treating the two terms in the right hand side of the previous separately.
Let $H_j(s)$ be a sequence of functions in $C^1(\mathbb{R}^+)$ such that

$$H_j' \in L^\infty(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \|H_j - H\|_{L^\infty(\mathbb{R}^+)} \leq \frac{1}{j}.$$

Since u is cap_p-quasi continuous, H, H_j and S are continuous and finite functions on $\mathbb{R}$, then $H_j(u)S(u)\varphi$ and $H(u)S(u)\varphi$ are μ_d-measurable. Then we have

$$\begin{aligned}
&\left|\int_\Omega H(u_n)S(u_n)\varphi\mu_{n,d} - \int_\Omega H(u)S(u)\varphi d\mu_d\right| \le \left|\int_\Omega (H(u_n) - H_j(u_n))S(u_n)\varphi\mu_{n,d}\right| \\
&+ \left|\int_\Omega (H_j(u) - H(u))S(u)\varphi d\mu_d\right| + \left|\int_\Omega H_j(u_n)S(u_n)\varphi\mu_{n,d} - H_j(u)S(u)\varphi d\mu_d\right| \\
&\le \frac{C}{j} + \left|\int_\Omega H_j(u_n)S(u_n)\varphi\mu_{n,d} - H_j(u)S(u)\varphi d\mu_d\right|.
\end{aligned} \tag{3.41}$$

Now, thanks to the assumptions on the functions H_j, S and φ and to (3.18), it is easy to verify that $H_j(u_n)S(u_n)\varphi$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with respect to $n \in \mathbb{N}$ and its almost everywhere limit is given by $H_j(u)S(u)\varphi$. Then, by Lemma 1.10 and (3.15), we get

$$\lim_{n\to\infty} \int_\Omega H_j(u_n)S(u_n)\varphi\mu_{n,d} = \int_\Omega H_j(u)S(u)\varphi d\mu_d.$$

Now, using the Lebesgue Theorem for general measures and the assumptions on the sequence H_j, we are able to pass to the limit also with respect to j, concluding that

$$\lim_{j\to\infty}\lim_{n\to\infty} \int_\Omega H_j(u_n)S(u_n)\varphi\mu_{n,d} = \int_\Omega H(u)S(u)\varphi d\mu_d$$

and that $H(u)S(u)\varphi \in L^1(\Omega, \mu_d)$. As regards the second term in the right hand side of (3.40), we first observe that, since S has compact support, there exist $k > 0$ and $c_k > 0$ such that $S(s) \le c_k(k-s)^+$ for every $s \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\int_\Omega H(u_n)S(u_n)\varphi\mu_{n,c} = \int_\Omega H(u_n)S(u_n)\varphi\Psi_\nu\mu_{n,c} + \int_\Omega H(u_n)S(u_n)\varphi(1-\Psi_\nu)\mu_{n,c} \\
&\le \|\varphi\|_{L^\infty(\Omega)} c_k \int_\Omega H(u_n)(k-u_n)^+\Psi_\nu\mu_{n,c} + \|H\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\Omega)}\|S\|_{L^\infty(\mathbb{R})}\int_\Omega (1-\Psi_\nu)\mu_{n,c}.
\end{aligned}$$

So, by Lemma 1.16 and (3.37), letting first n go to infinity and then ν go to zero, we obtain

$$\lim_{n\to\infty} \int_\Omega H(u_n)S(u_n)\varphi\mu_{n,c} = 0,$$

which proves (3.8), as desired.
Now we want to prove that (3.9) holds true.
First we need to prove that u is a distributional solution of (3.1). If $\varphi \in C_c^1(\Omega)$, we have

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi = \int_\Omega H(u_n)\varphi\mu_{n,d} + \int_\Omega H(u_n)\varphi\mu_{n,c}. \tag{3.42}$$

For the left hand side of the previous, by (3.38) we deduce

$$\lim_{n\to\infty} \int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi.$$

Concerning the first term on the right hand side of (3.42), we reason as in (3.41) yielding

$$\begin{aligned}\left|\int_\Omega H(u_n)\varphi\mu_{n,d} - \int_\Omega H(u)\varphi d\mu_d\right| \leq & \left|\int_\Omega (H(u_n) - H_j(u_n))\varphi\mu_{n,d}\right| \\ & + \left|\int_\Omega (H_j(u) - H(u))\varphi d\mu_d\right| \\ & + \left|\int_\Omega H_j(u_n)\varphi\mu_{n,d} - H_j(u)\varphi d\mu_d\right| \\ \leq & \frac{C}{j} + \left|\int_\Omega H_j(u_n)\varphi\mu_{n,d} - H_j(u)\varphi d\mu_d\right|.\end{aligned}$$

To prove that the last term in the previous formula goes to zero with respect to n, it is sufficient to show that $H_j(u_n)\varphi$ is bounded with respect to n, with j fixed, in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Clearly $H_j(u_n)\varphi$ is bounded, with respect to n, in $L^\infty(\Omega)$. To show the boundedness in $W_0^{1,p}(\Omega)$ of $H_j(u_n)\varphi$, we take $\theta_k(u_n)\int_0^{T_{2k}(u_n)}\left|H_j'(s)\right| ds$ as test function in the weak formulation of (3.14). Then we find

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\cdot\nabla T_{2k}(u_n)\left|H_j'(T_{2k}(u_n))\right|\theta_k(u_n) = & \int_\Omega H(u_n)\left(\theta_k(u_n)\int_0^{T_{2k}(u_n)}\left|H_j'(s)\right| ds\right)\mu_n \\ & + \frac{1}{k}\int_{\{k<u_n<2k\}} a(x,\nabla u_n)\cdot\nabla u_n\left(\int_0^{T_{2k}(u_n)}\left|H_j'(s)\right| ds\right) \\ & \leq \|H\|_{L^\infty(\mathbb{R})}\|H_j\|_{L^\infty(\mathbb{R})}\|\mu_n\|_{L^1(\Omega)} + \epsilon(k) \\ & \leq C + \epsilon(k),\end{aligned}$$

since $H_j' \in L^1(\mathbb{R}^+)$ and (3.17) holds. Then, by (3.2), we deduce

$$\int_\Omega |\nabla T_{2k}(u_n)|^p\left|H_j'(T_{2k}(u_n))\right|\theta_k(u_n) \leq C + \epsilon(k)$$

namely

$$\int_\Omega |\nabla u_n|^p\left|H_j'(u_n)\right|\theta_k(u_n) \leq C + \epsilon(k).$$

Letting $k \to \infty$ in the previous and using Fatou Lemma, we find

$$\frac{1}{\|H_j'\|_{L^\infty(\mathbb{R})}^{p-1}}\int_\Omega |\nabla H_j(u_n)|^p \leq \int_\Omega |\nabla u_n|^p\left|H_j'(u_n)\right| \leq C,$$

which implies that $H_j(u_n)\varphi$ is bounded in $W_0^{1,p}(\Omega)$ with respect to n.
Now we go back to the second term on the right hand side of (3.42). By (3.15), recalling

that $\varphi \in C_c^1(\Omega)$, it results

$$
\begin{aligned}
(3.43)\qquad \left|\int_\Omega H(u_n)\varphi\mu_{n,c} - \int_\Omega H(\infty)\varphi d\mu_c\right| \le & \left|\int_\Omega H(u_n)\varphi\mu_{n,c} - \int_\Omega H(\infty)\varphi\mu_{n,c}\right| \\
& + \left|\int_\Omega H(\infty)\varphi\mu_{n,c} - \int_\Omega H(\infty)\varphi d\mu_c\right| \\
\le & \|\varphi\|_{L^\infty(\Omega)} \int_\Omega |H(u_n) - H(\infty)|\mu_{n,c} + \epsilon(n).
\end{aligned}
$$

By (3.6), for every $\eta > 0$ there exist $s_\eta > 0$ and $L_\eta > 0$ such that

$$
(3.44)\qquad |H(s) - H(\infty)| \le \eta, \qquad \forall s > s_\eta
$$

and, using that $H(s) > 0$ for $s \ge 0$, we have

$$
(3.45)\qquad |H(s) - H(\infty)| \le H(s)L_\eta(2s_\eta - s)^+, \qquad \forall s \in [0, s_\eta].
$$

It follows from (3.44), (3.45), (3.15) and applying (3.37) with $k = 2s_\eta$ that

$$
\begin{aligned}
\int_\Omega |H(u_n) - H(\infty)|\mu_{n,c} = & \int_\Omega |H(u_n) - H(\infty)|\Psi_\nu\mu_{n,c} + \int_\Omega |H(u_n) - H(\infty)|(1 - \Psi_\nu)\mu_{n,c} \\
\le & \eta \int_{\{u_n > s_\eta\}} \Psi_\nu\mu_{n,c} + L_\eta \int_{\{u_n \le s_\eta\}} H(u_n)(2s_\eta - u_n)^+\Psi_\nu\mu_{n,c} \\
& + 2\|H\|_{L^\infty(\mathbb{R})} \int_\Omega (1 - \Psi_\nu)\mu_{n,c} \\
\le & \epsilon(n, \nu, \eta).
\end{aligned}
$$

Hence, by (3.43), we have

$$
\left|\int_\Omega H(u_n)\varphi\mu_{n,c} - \int_\Omega H(\infty)\varphi d\mu_c\right| \le \epsilon(n, \nu, \eta),
$$

which implies that

$$
(3.46)\qquad \lim_{n\to\infty} \int_\Omega H(u_n)\varphi\mu_{n,c} = H(\infty) \int_\Omega \varphi d\mu_c,
$$

then (3.11) is proved.
Now taking $S = \theta_t$ and $\varphi \in C_c^1(\Omega)$ in (3.8) we obtain

$$
\frac{1}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u\varphi = -\int_\Omega H(u)\theta_t(u)\varphi d\mu_d + \int_\Omega a(x, \nabla u) \cdot \nabla\varphi\theta_t(u).
$$

Now, using that θ_t belongs to $C_b(\mathbb{R})$ and that u is cap_p-almost everywhere defined, by Lebesgue's Theorem for general measures we obtain

$$
\lim_{t\to\infty} \frac{1}{t} \int_{\{t<u<2t\}} a(x, \nabla u) \cdot \nabla u\varphi = -\int_\Omega H(u)\varphi d\mu_d + \int_\Omega a(x, \nabla u) \cdot \nabla\varphi,
$$

which implies, by (3.11), that

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi = H(\infty)\int_\Omega \varphi d\mu_c \qquad \forall\varphi\in C_c^1(\Omega). \tag{3.47}$$

By the density of $C_c^1(\Omega)$ in $C_c(\Omega)$, (3.47) is true when $\varphi\in C_c(\Omega)$.
Now, if $\varphi\in C_b(\Omega)$, we have $\varphi\Psi_\nu\in C_c(\Omega)$ and then

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\Psi_\nu\varphi = H(\infty)\int_\Omega \varphi\Psi_\nu d\mu_c \qquad \forall\varphi\in C_b(\Omega). \tag{3.48}$$

Applying (3.32) with $r=t$, and letting n go to infinity, we find

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u(1-\Psi_\nu)\varphi = \epsilon(\nu) \qquad \forall\varphi\in C_b(\Omega). \tag{3.49}$$

Then, by (3.48) and (3.49), we deduce

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi = H(\infty)\int_\Omega \varphi\Psi_\nu d\mu_c+\epsilon(\nu) \qquad \forall\varphi\in C_b(\Omega).$$

Letting ν go to zero, by Lemma 1.16, we obtain (3.9).
Now we further ask that H is non-increasing and that $\mu_c\equiv 0$ and we prove the uniqueness of a renormalized solution to (3.1).
Let u and v be two renormalized solutions of (3.1). We can choose $S=\theta_t$ and $\varphi=\theta_t(v)\,T_k(u-v)$ in the equation of u, and $S=\theta_t$ and $\varphi=\theta_t(u)\,T_k(u-v)$ in the equation of v to obtain, subtracting the equations, that

$$\begin{aligned}
&\int_\Omega (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla T_k(u-v)\,\theta_t(u)\theta_t(v)\\
&=\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\,T_k(u-v)\theta_t(v)\\
&-\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla v)\cdot\nabla v\,T_k(u-v)\theta_t(u)\\
&+\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla u)\cdot\nabla v\,T_k(u-v)\theta_t(u)\\
&-\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla v)\cdot\nabla u\,T_k(u-v)\theta_t(v)\\
&+\int_\Omega (H(u)-H(v))\,T_k(u-v)\,\theta_t(u)\theta_t(v)\,d\mu_d\,.
\end{aligned} \tag{3.50}$$

It follows from the definition of renormalized solution and by the assumption $\mu_c\equiv 0$ that

$$\lim_{t\to\infty}\left|\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\,T_k(u-v)\theta_t(v)\right| \le \lim_{t\to\infty}\frac{k}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\;=\;0\,,$$

and, in the same way, that

$$\lim_{t\to\infty}\left|\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla v)\cdot\nabla v\, T_k(u-v)\theta_t(u)\right| \le \lim_{t\to\infty}\frac{k}{t}\int_{\{t<v<2t\}} a(x,\nabla v)\cdot\nabla v \;=\; 0\,.$$

Now we focus on the third term on the right hand side of (3.50). Using Hölder inequality, the definition of θ_t, (3.2) and (3.3), we obtain

$$\begin{aligned}
&\left|\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla u)\cdot\nabla v\, T_k(u-v)\theta_t(u)\right| \\
&\le k\int_{\{t<v<2t\}\cap\{u<2t\}} |a(x,\nabla u)\cdot\nabla v| \\
&\le k\left(\frac{1}{t}\int_{\{u<2t\}} |a(x,\nabla u)|^{p'}\right)^{\frac{1}{p'}}\left(\frac{1}{t}\int_{\{t<v<2t\}} |\nabla v|^p\right)^{\frac{1}{p}} \\
&\le C\,k\left(\frac{1}{t}\int_{\{u<2t\}} |\nabla u|^p\right)^{\frac{1}{p'}}\left(\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla v)\cdot\nabla v\right)^{\frac{1}{p}}\,.
\end{aligned}$$

As a consequence of the definition of renormalized solution with $\mu_c\equiv 0$ we have that $\left\{\frac{1}{t}\,|\nabla T_{2t}(u)|^p\right\}$ is bounded in $L^1(\Omega)$ with respect to t and that $\left\{\frac{1}{t}\,a(x,\nabla v)\cdot\nabla v\right\}$ strongly converges to 0 in $L^1(\Omega)$. Thus we have

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<v<2t\}} a(x,\nabla u)\cdot\nabla v\, T_k(u-v)\theta_t(u) \;=\; 0\,,$$

and, interchanging the roles of u and v, that

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla v)\cdot\nabla u\, T_k(u-v)\theta_t(v) \;=\; 0\,.$$

Moreover, by the assumption that H is nonincreasing, we obtain that

$$(H(u)-H(v))\,T_k(u-v)\,\theta_t(u)\theta_t(v)\;\le\;0\,,$$

cap_p-almost everywhere. So that we deduce from (3.50) that

$$\limsup_{t\to\infty}\int_\Omega (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla T_k(u-v)\,\theta_t(u)\theta_t(v)\;\le\;0\,.$$

Applying Fatou's lemma we have

$$\int_\Omega (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla T_k(u-v)\;\le\;0\,,$$

for every $k>0$. So that, by (3.4) and letting k tend to infinity we obtain $\nabla u=\nabla v$ and then $u=v$ almost everywhere in Ω.
This concludes the proof of Theorem 3.4 if $\gamma=0$. □

3.3. The approximation scheme if H is singular

In this section we collect some properties of the solutions to the scheme of approximation which will be the basis to prove Theorems 3.4, 3.5 in case $\gamma > 0$, namely when the function H can blow up at the origin.
We will find a solution to the problem passing to the limit in the following approximation

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_{n,m})) = H_n(u_{n,m})(\mu_d + \mu_m) & \text{in } \Omega, \\ u_{n,m} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.51}$$

where $H_n = T_n(H)$ and μ_m is, once again, a sequence of nonnegative functions in $L^\infty(\Omega)$, bounded in $L^1(\Omega)$, that converges to μ_c in the narrow topology of measures. We recall that H satisfies (3.6) and (3.7) with $\gamma > 0$ and that a is a Carathéodory function such that (3.2), (3.3) and (3.4) with $1 < p < N$ hold true.
The existence of a nonnegative renormalized solution $u_{n,m}$ to problem (3.51) is guaranteed by the result proven in Section 3.2. Moreover it follows from Lemma 3.3 that $u_{n,m}$ is also a distributional solution to (3.51).

For the sake of simplicity, since until the passage to the limit it will be not necessary to distinguish between n and m, we will consider the following approximation in place of (3.51)

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_n)) = H_n(u_n)(\mu_d + \mu_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.52}$$

The first step is proving the local uniform positivity for u_n, which will assure that the possibly singular right hand side is locally integrable with respect to μ_d.

LEMMA 3.11. *Let u_n be a solution to (3.52). Then*

$$\forall\, \omega \subset\subset \Omega \quad \exists\, c_\omega > 0 : u_n \ge c_\omega \quad cap_p\text{-a.e. in } \omega, \quad \forall n \ge n_0, \tag{3.53}$$

for some $n_0 > 0$.

PROOF. The proof is similar to the one of Lemma 3.4 in [**40**] given for $p = 2$. For this reason we just sketch it. For some $n_0 \in \mathbb{N}$, it is possible to construct a non-increasing function $h \in C_b(\mathbb{R})$ such that $h(s) \le H_n(s)$ for every $n \ge n_0$ and for all $s \ge 0$.
Then we can consider the following problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla v)) = h(v)\mu_d & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.54}$$

for which the existence of a nonnegative renormalized solution $v \not\equiv 0$ follows once again from Section 3.2. It can be proven that there exists $\overline{r} > 0$ such that $\mu_d\lfloor_{\{v<\overline{r}\}} \not\equiv 0$ and that $h(v)\mu_d$ is a diffuse measure respect to p-capacity. Then, from Definition 2.29 and Remark 2.32 in [**37**], we deduce that $T_{\overline{r}}(v) \in W_0^{1,p}(\Omega)$ solves the following

$$-\operatorname{div}(a(x,\nabla T_{\overline{r}}(v))) = h(v)\mu_d\lfloor_{\{v<\overline{r}\}} + \lambda_{\overline{r}} \ge 0 \quad \text{in } \Omega,$$

where $\lambda_{\overline{r}}$ is a nonnegative diffuse measure concentrated on the set $\{v = \overline{r}\}$. Hence we can apply the strong maximum principle (see, for instance, Theorem 1.2 of [**73**]), obtaining

$$\forall\, \omega \subset\subset \Omega \quad \exists\, C_{\omega,\overline{r}} > 0 : v \geq T_{\overline{r}}(v) \geq c_\omega := C_{\omega,\overline{r}} > 0 \ \text{ a.e. in } \ \omega\,.$$

Now we consider the renormalized formulations of (3.52) and of (3.54), taking $S = \theta_k$ in both equations and $\varphi = \theta_k(v)T_r(v-u_n)^+$ in (3.52), $\varphi = \theta_k(u_n)T_r(v-u_n)^+$ in (3.54), where $r > 0$ is fixed.
We have
(3.55)

$$\begin{aligned}
&\int_\Omega \big(a(x,\nabla v) - a(x,\nabla u_n)\big)\cdot \nabla T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)\\
&= \frac{1}{k}\int_{\{k<u_n<2k\}} a(x,\nabla v)\cdot\nabla u_n T_r(v-u_n)^+\theta_k(v) - \frac{1}{k}\int_{\{k<v<2k\}} a(x,\nabla u_n)\cdot\nabla v T_r(v-u_n)^+\theta_k(u_n)\\
&+ \frac{1}{k}\int_{\{k<v<2k\}} a(x,\nabla v)\cdot\nabla v T_r(v-u_n)^+\theta_k(u_n) - \frac{1}{k}\int_{\{k<u_n<2k\}} a(x,\nabla u_n)\cdot\nabla u_n T_r(v-u_n)^+\theta_k(v)\\
&+ \int_\Omega \big(h(v) - H_n(u_n)\big)T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)d\mu_d - \int_\Omega H_n(u_n)T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)\mu_n.
\end{aligned}$$

Since the concentrated part of the datum is zero both in (3.52) and in (3.54), from the definition of renormalized solution we obtain that the third and the fourth term of the right hand side of (3.55) go to zero as k goes to infinity. With the same argument, after an application of the Hölder inequality, we deduce that the first and the second term of the right hand side of the previous go to zero as k goes to infinity. Since the last term of (3.55) is nonpositive and h is non-increasing, we deduce that

$$\begin{aligned}
&\int_\Omega \big(a(x,\nabla v) - a(x,\nabla u_n)\big)\cdot \nabla T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)\\
&\leq \int_{\{v\geq u_n\}} \big(h(v) - H_n(u_n)\big)T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)d\mu_d\\
&\leq \int_{\{v\geq u_n\}} \big(h(u_n) - H_n(u_n)\big)T_r(v-u_n)^+\theta_k(v)\theta_k(u_n)d\mu_d.
\end{aligned}$$

Since $h \leq H_n$ for every $n \geq n_0$, h and H_n are continuous and u_n is cap_p-almost everywhere defined, we have $\big(h(u_n) - H_n(u_n)\big) \leq 0$ cap_p-almost everywhere in Ω if $n \geq n_0$. Moreover, applying in the previous the Fatou Lemma first in k and then in r, we deduce

$$\int_\Omega \big(a(x,\nabla v) - a(x,\nabla u_n)\big)\cdot\nabla(v-u_n)\chi_{\{v\geq u_n\}} \leq 0,$$

which, by (3.4), implies

$$\chi_{\{v\geq u_n\}} \equiv 0 \ \text{ if } n \geq n_0.$$

Hence we have proved that (3.53) holds almost everywhere in Ω.
Now, if $\omega \subset\subset \Omega$ and $k_\omega > c_\omega$, then

$$T_{k_\omega}(u_n) \geq c_\omega \text{ a.e. in } \omega. \tag{3.56}$$

Using the definition of the set of Lebesgue points of a function f applied with the choice $f = T_{k_\omega}(u_n)|_\omega$ and Lebesgue differentiation Theorem, we deduce that

$$T_{k_\omega}(u_n) \geq c_\omega \text{ in } \mathcal{L}_{T_{k_\omega}(u_n)|_\omega}.$$

Since $T_{k_\omega}(u_n) \in W^{1,p}(\omega)$, using Proposition 8.6 of **[66]** we obtain that $\text{cap}_p(\omega \backslash \mathcal{L}_{T_{k_\omega}(u_n)|_\omega}) = 0$. In particular (3.56) holds cap_p-almost everywhere on ω and, since $u_n \geq T_{k_\omega}(u_n)$, we conclude that (3.53) holds cap_p-almost everywhere in ω. □

Now we are interested in providing some a priori estimates up to the boundary in order to give a weak sense to the Dirichlet datum.

LEMMA 3.12. *Let u_n be a solution to (3.52). Then $T_k^{\frac{\tau-1+p}{p}}(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$ where $\tau = \max(1, \gamma)$.*

PROOF. We take as test functions in the renormalized formulation of (3.52) $S = \theta_r$ and $\varphi = T_k^\tau(u_n)$ where $r > k$. We let $r \to \infty$ and use that the concentrated part of the datum in (3.52) is zero. Then we obtain the following

(3.57)
$$\begin{aligned}\int_\Omega |\nabla T_k^{\frac{\tau-1+p}{p}}(u_n)|^p &\leq C s_0^{\tau-\gamma} \int_{\{u_n < s_0\}} (d\mu_d + \mu_n) + C k^\tau \|H\|_{L^\infty([s_0,+\infty))} \int_{\{u_n \geq s_0\}} (d\mu_d + \mu_n) \\ &\leq C(k^\tau + 1),\end{aligned}$$

as desired. □

REMARK 3.13. *Let us underline that, in case $\gamma > 1$, $T_k(u_n)$ is bounded in $W^{1,p}_{loc}(\Omega)$ with respect to $n \in \mathbb{N}$ for n large enough and for every fixed $k > 0$. Indeed, it follows from Lemma 3.11 and Lemma 3.12 that for every $\omega \subset\subset \Omega$ it results*

$$\begin{aligned}&\left(\frac{\gamma+p-1}{p}\right)^p c_\omega^{\gamma-1} \int_\omega |\nabla T_k(u_n)|^p = \left(\frac{\gamma+p-1}{p}\right)^p \int_\Omega T_k(u_n)^{\gamma-1} |\nabla T_k(u_n)|^p \\ &= \int_\Omega |\nabla T_k(u_n)^{\frac{\gamma+p-1}{p}}|^p \leq C(1+k^\gamma).\end{aligned}$$

We prove local a priori estimates for u_n.

LEMMA 3.14. *Let u_n be a solution to (3.52). Then:*

i) *if $p > 2 - \frac{1}{N}$, u_n is bounded in $W^{1,q}_{loc}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$;*
ii) *if $1 < p \leq 2 - \frac{1}{N}$, u_n^{p-1} is bounded in $L^q_{loc}(\Omega)$ for every $q < \frac{N}{N-p}$ and $|\nabla u_n|^{p-1}$ is bounded in $L^q_{loc}(\Omega)$ for every $q < \frac{N}{N-1}$.*

Moreover there exists an almost everywhere finite function u such that u_n converges almost everywhere to u in Ω, u is locally cap_p-almost everywhere finite, locally cap_p-quasi continuous and such that

$$\forall\, \omega \subset\subset \Omega \quad \exists\, c_\omega > 0 : u \geq c_\omega \ cap_p\text{-a.e. in } \omega, \tag{3.58}$$

$$H(u) \in L^\infty(\omega; \mu_d) \quad \forall \omega \subset\subset \Omega.$$

Proof. By Lemma 3.12 and Remark 3.13, we have that $T_k(u_n)$ is bounded in $W^{1,p}_{loc}(\Omega)$ with respect to $n \in \mathbb{N}$ for each $k > 0$ fixed and for all $\gamma > 0$. Then, localizing the proof Lemma 3.8, we deduce immediately that i) and ii) hold true and that there exists an almost everywhere finite function u such that u_n converges almost everywhere to u in Ω. Moreover, using (3.57), once again Remark 3.13 and localizing Lemma 1.17, we obtain that u is locally cap_p-almost everywhere finite and locally cap_p-quasi continuous. Now, letting $n \to \infty$ in (3.53), we deduce that

$$\forall\, \omega \subset\subset \Omega \quad \exists\, c_\omega > 0 : u \geq c_\omega \quad a.e. \text{ in } \omega, \tag{3.59}$$

and, since $T_k(u) \in W^{1,p}_{loc}(\Omega)$, we can proceed as at the end of the proof of Lemma 3.11 to conclude that (3.59) holds also cap_p-almost everywhere in ω, that is (3.58). Using (3.58) and the fact that $H(s)$ is finite if $s > 0$, we deduce $H(u) \in L^\infty(\omega; \mu_d)$ for every $\omega \subset\subset \Omega$. □

Remark 3.15. *Recalling Lemma 3.12, in the case $\gamma \leq 1$ we can improve the previous Lemma obtaining that i) and ii) hold true globally in Ω and that u is cap_p-almost everywhere finite and cap_p-quasi continuous.*

The next Lemma is a strong convergence result for the truncations, this time (compare with Lemma 3.9, see also [**40**] for $p = 2$) in the local space $W^{1,p}_{loc}(\Omega)$.

Lemma 3.16. *Let u_n be a solution to* (3.52). *Then $T_k(u_n)$ converges to $T_k(u)$ in $W^{1,p}_{loc}(\Omega)$ for every $k > 0$.*

Proof. The proof is similar to the one of Lemma 3.9. It suffices to take $\varphi = (T_k(u_n) - T_k(u))(1 - \Psi_\nu)\psi$ and $S = \theta_r$ $(r > k)$ in the renormalized formulation of (3.52) where $\psi \in C^1_c(\Omega)$ such that for $\omega \subset\subset \Omega$ we have

$$\begin{cases} 0 \leq \psi \leq 1 \text{ on } \Omega, \\ \psi \equiv 1 \text{ on } \omega \subset\subset \Omega. \end{cases}$$

Hence, through the local estimates and proceeding in an analogous way as to prove the strong convergence of truncations in Lemma 3.9, we obtain

$$\lim_{n\to\infty} \int_\Omega \big(a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))\big) \cdot \nabla (T_k(u_n) - T_k(u))\psi = 0,$$

so that, by [**16**, Lemma 5], we have that $T_k(u_n)$ converges to $T_k(u)$ strongly in $W^{1,p}_{loc}(\Omega)$ for every $k > 0$ and ∇u_n converges to ∇u almost everywhere in Ω. This concludes the proof. □

Remark 3.17. *Analogously to Remark 3.10, from Lemma 3.14 and Lemma 3.16 we deduce that if $p > 2 - \frac{1}{N}$ then u_n converges to u strongly in $W^{1,q}_{loc}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$. Otherwise if $1 < p \leq 2 - \frac{1}{N}$ then u_n^{p-1} converges to u^{p-1} strongly in $L^q_{loc}(\Omega)$ for every $q < \frac{N}{N-p}$ and $|\nabla u_n|^{p-1}$ converges to $|\nabla u|^{p-1}$ strongly in $L^q_{loc}(\Omega)$ for every $q < \frac{N}{N-1}$. In all cases we have*

$$a(x, \nabla u_n) \to a(x, \nabla u) \textit{ strongly in } L^q_{loc}(\Omega)^N \textit{ for every } q < \frac{N}{N-1}. \tag{3.60}$$

3.4. Proof of the existence and uniqueness results

In this section we first prove Theorem 3.5, and then Theorem 3.4 in full generality, namely for $\gamma > 0$.
Indeed, in order to prove Theorem 3.4, we need that the scheme of approximation actually takes to a distributional solution to (3.1), which is the content of Theorem 3.5.

Proof of Theorem 3.5. Let $u_{n,m}$ be a renormalized solution to (3.51). We need to prove that its almost everywhere limit u, whose existence is guaranteed by Lemma 3.14, is a distributional solution to (3.1).
It follows from Lemma 3.12 that (3.10) holds. Hence we just need to show (3.11), namely we have to pass to the limit first in m and then in n the following weak formulation

$$\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\varphi = \int_\Omega H_n(u_{n,m})\varphi d\mu_d + \int_\Omega H_n(u_{n,m})\varphi\mu_m, \quad \forall\varphi\in C^1_c(\Omega). \tag{3.61}$$

Thanks to (3.60), we are able to pass to the limit the first term on left hand side of the previous as $n,m\to\infty$. Now we pass to the right hand side of (3.61). For $n\in\mathbb{N}$ fixed and proceeding as to deduce (3.46), we find that

$$\lim_{m\to\infty}\int_\Omega H_n(u_{n,m})\varphi\mu_m = H_n(\infty)\int_\Omega \varphi d\mu_c,$$

and, since for $n\in\mathbb{N}$ large enough it results $H_n(\infty)=H(\infty)$, we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega H_n(u_{n,m})\varphi\mu_m = H(\infty)\int_\Omega \varphi d\mu_c.$$

For the first term on the right hand side of (3.61) we observe that, by Lemma 3.16, it yields that $T_k(u_{n,m})$ strongly converges to $T_k(u)$ in $W^{1,p}_{loc}(\Omega)$. This implies (see Lemma 3.5 of [**54**]) that $T_k(u_{n,m})$ converges to $T_k(u)$ cap_p-almost everywhere in ω for each $k>0$ fixed and for $\omega\subset\subset\Omega$. Being $u_{n,m}$ and u cap_p-almost everywhere finite functions, we deduce that $u_{n,m}$ converges cap_p-almost everywhere to u in ω for each $\omega\subset\subset\Omega$. Hence $H_n(u_{n,m})$ converges to $H(u)$ cap_p-almost everywhere in $\text{supp}(\varphi)$. Thus we are in position to apply the Lebesgue Theorem for general measures since

$$|H_n(u_{n,m})\varphi| \le \|H\|_{L^\infty([c_{\text{supp}(\varphi)},\infty))}\|\varphi\|_{L^\infty(\Omega)} \in L^1(\Omega,\mu_d),$$

where we have used that, by Lemma 3.11, $u_{n,m}\ge c_{\text{supp}(\varphi)}$ cap_p-almost everywhere on $\text{supp}(\varphi)$ for n and m large enough. Hence we have proved that it results

$$\lim_{n,m\to\infty}\int_\Omega H_n(u_{n,m})\varphi d\mu_d = \int_\Omega H(u)\varphi d\mu_d,$$

and then u is a distributional solution to (3.1). This concludes the proof. □

PROOF OF THEOREM 3.4 IN CASE $\gamma > 0$. Let $u_{n,m}$ be a renormalized solution to (3.51), then it follows from the proof of Theorem 3.5 that its almost everywhere limit u is a distributional solution to (3.1). We have that $u_{n,m}$ is such that

$$\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\varphi S(u_{n,m}) + \int_\Omega a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}S'(u_{n,m})\varphi = \int_\Omega H_n(u_{n,m})S(u_{n,m})\varphi d\mu_d + \int_\Omega H_n(u_{n,m})S(u_{n,m})\varphi\mu_m, \tag{3.62}$$

where $S\in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp}(S)\subset[-M,M]$ and $\varphi\in C^1_c(\Omega)$.
As regards the left hand side of (3.62), since, by Lemma 3.16, $T_M(u_{n,m})$ strongly converges to $T_M(u)$ in $W^{1,p}_{loc}(\Omega)$, by (3.3) and Vitali's Theorem, we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\left(\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\varphi S(u_{n,m}) + \int_\Omega a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}S'(u_{n,m})\varphi\right) = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi S(u) + \int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)\varphi.$$

For the first term on the right hand side of (3.62) we observe that, using once again Lemma 3.11, it results

$$H_n(u_{n,m})S(u_{n,m})\varphi \le \|H\|_{L^\infty([c_{\operatorname{supp}(\varphi)},\infty))}\|\varphi\|_{L^\infty(\Omega)}\|S\|_{L^\infty(\mathbb{R})}\in L^1(\Omega,\mu_d).$$

Then, thanks to the cap_p-almost everywhere convergence of $u_{n,m}$ to u, we can apply the Lebesgue Theorem for general measures, obtaining

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega H_n(u_{n,m})S(u_{n,m})\varphi d\mu_d = \int_\Omega H(u)S(u)\varphi d\mu_d.$$

For the second term on the right hand side of (3.62) we have, proceeding as in the proof of Theorem 3.4 in the case $\gamma = 0$, that there exist $k>0$ and $c_k>0$ such that $S(s)\le c_k(k-s)^+$ for every $s\in\mathbb{R}$ and

$$\int_\Omega H_n(u_{n,m})S(u_{n,m})\varphi\mu_m \le c_k\|\varphi\|_{L^\infty(\Omega)}\int_\Omega H_n(u_{n,m})(k-u_{n,m})^+\Psi_\nu\mu_m + \|H\|_{L^\infty([c_{\operatorname{supp}(\varphi)},\infty))}\|S\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\Omega)}\int_\Omega(1-\Psi_\nu)\mu_m. \tag{3.63}$$

Using $S(s) = (k-|s|)^+$ and $\varphi = \Psi_\nu$ in the renormalized formulation of (3.51) and dropping positive terms we obtain

$$\int_\Omega H_n(u_{n,m})(k-u_{n,m})^+\Psi_\nu\mu_m \le \int_\Omega a(x,\nabla T_k(u_{n,m}))\cdot\nabla\Psi_\nu(k-u_{n,m})^+ \le k\|T_k(u_{n,m})\|_{W^{1,p}(\operatorname{supp}(\Psi_\nu))}\|\Psi_\nu\|_{W^{1,p}_0(\Omega)}. \tag{3.64}$$

Then, from (3.63) and (3.64), we deduce, applying Lemma 1.16, Lemma 3.12, Remark 3.13 and letting $\nu\to 0$, that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega H_n(u_{n,m})S(u_{n,m})\varphi\mu_m = 0.$$

Hence we have proved

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi S(u)+\int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)\varphi=\int_\Omega H(u)S(u)\varphi d\mu_d, \tag{3.65}$$

for every $S\in W^{1,\infty}(\mathbb{R})$ with compact support and for every $\varphi\in C_c^1(\Omega)$, namely (3.8) for a smaller class of test functions $\varphi\in C_c^1(\Omega)$. Note that (3.65) holds true also if $\gamma>1$. Now we take $S=\theta_t$ in (3.65) and we obtain

$$\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi=-\int_\Omega H(u)\theta_t(u)\varphi d\mu_d+\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\theta_t(u).$$

We pass to the limit in t obtaining

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi=-\int_\Omega H(u)\varphi d\mu_d+\int_\Omega a(x,\nabla u)\cdot\nabla\varphi,$$

which implies, since u is a distributional solution to (3.1), that

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi=H(\infty)\int_\Omega\varphi d\mu_c \qquad \forall\varphi\in C_c^1(\Omega). \tag{3.66}$$

By the density of $C_c^1(\Omega)$ in $C_c(\Omega)$, (3.66) is true when $\varphi\in C_c(\Omega)$. Now, if $\varphi\in C_b(\Omega)$, we have $\varphi\Psi_\nu\in C_c(\Omega)$ and then

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\Psi_\nu\varphi=H(\infty)\int_\Omega\varphi\Psi_\nu d\mu_c \qquad \forall\varphi\in C_b(\Omega). \tag{3.67}$$

We want to prove that

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u(1-\Psi_\nu)\varphi=\epsilon(\nu) \qquad \forall\varphi\in C_b(\Omega). \tag{3.68}$$

Choosing in the renormalized formulation of (3.51) $\varphi=\pi_t(u_{n,m})(1-\Psi_\nu)$ and $S=\theta_r$, with $t>1$, we obtain

$$\begin{aligned}
&\frac{1}{t}\int_{\{t<u_{n,m}<2t\}} a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}\theta_r(u_{n,m})(1-\Psi_\nu)\\
=&\frac{1}{r}\int_{\{r<u_{n,m}<2r\}} a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}\pi_t(u_{n,m})(1-\Psi_\nu) \quad (a)\\
&+\int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})\theta_r(u_{n,m})(1-\Psi_\nu)d\mu_d \quad (b)\\
&+\int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})\theta_r(u_{n,m})(1-\Psi_\nu)\mu_m \quad (c)\\
&+\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\Psi_\nu\pi_t(u_{n,m})\theta_r(u_{n,m}). \quad (d)
\end{aligned} \tag{3.69}$$

As concerns (d), thanks to the Lebesgue Theorem, we deduce

$$\lim_{r\to\infty}\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\Psi_\nu\pi_t(u_{n,m})\theta_r(u_{n,m})=\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\Psi_\nu\pi_t(u_{n,m}).$$

Recalling that u is almost everywhere finite, that $|\nabla u_{n,m}|^{p-1}$ is bounded in $L^q(\omega)$ for each $q < \frac{N}{N-1}$ where $\omega :=$supp(Ψ_ν), using (3.3) and Hölder inequality with exponents q and q', with $1 < q < \frac{N}{N-1}$ fixed, we find

$$\left|\int_\Omega a(x,\nabla u_{n,m})\cdot\nabla\Psi_\nu\pi_t(u_{n,m})\right| \le \|\nabla\Psi_\nu\|_{L^\infty(\Omega)}\left(\int_\omega |\nabla u_{n,m}|^{(p-1)q}\right)^{\frac{1}{q}} |\{x\in\omega : u_{n,m}(x)\ge t\}|^{\frac{1}{q'}}$$
$$\le C\ |\{x\in\omega : u_{n,m}(x)\ge t\}|^{\frac{1}{q'}} = \epsilon(m,n,t).$$

Then

$$(d) \le \epsilon(m,n,t). \tag{3.70}$$

Concerning (b) and (c), once again by Lebesgue Theorem, we deduce that

$$\int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})\theta_r(u_{n,m})(1-\Psi_\nu)d\mu_d \le \|H\|_{L^\infty([1,+\infty))}\int_\Omega \pi_t(u_{n,m})(1-\Psi_\nu)d\mu_d = \epsilon(m,n,t), \tag{3.71}$$

and that

$$\lim_{r\to\infty}\int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})\theta_r(u_{n,m})(1-\Psi_\nu)\mu_m = \int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})(1-\Psi_\nu)\mu_m.$$

By the narrow convergence of μ_m and Lemma 1.16, we obtain

$$\int_\Omega H_n(u_{n,m})\pi_t(u_{n,m})(1-\Psi_\nu)\mu_m \le \|H\|_{L^\infty([1,+\infty))}\int_\Omega(1-\Psi_\nu)\mu_m = \epsilon(m,\nu) \tag{3.72}$$

Finally, by (3.17), we obtain

$$\begin{aligned}&\frac{1}{r}\int_{\{r<u_{n,m}<2r\}} a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}\pi_t(u_{n,m})(1-\Psi_\nu)\\ &\le \frac{1}{r}\int_{\{r<u_{n,m}<2r\}} a(x,\nabla u_{n,m})\cdot\nabla u_{n,m} = \epsilon(r).\end{aligned} \tag{3.73}$$

Letting r go to infinity in (3.69) and using (3.70), (3.71),(3.72) and (3.73), we get

$$\frac{1}{t}\int_{\{t<u_{n,m}<2t\}} a(x,\nabla u_{n,m})\cdot\nabla u_{n,m}(1-\Psi_\nu) = \epsilon(m,n,t,\nu).$$

Then, by Vitali's Theorem, letting m, n and t go to infinity we deduce (3.68). As a consequence of (3.67) and (3.68), letting ν go to zero, by Lemma 1.16 we have

$$\lim_{t\to\infty}\frac{1}{t}\int_{\{t<u<2t\}} a(x,\nabla u)\cdot\nabla u\varphi = H(\infty)\int_\Omega \varphi d\mu_c, \tag{3.74}$$

for all $\varphi \in C_b(\Omega)$. Hence (3.9) holds and, in order to deduce that u is a renormalized solution, we just need to show that (3.65) holds for a larger class of test functions, namely for $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

It follows from Remark 3.15 that $T_k(u) \in W_0^{1,p}(\Omega)$, for every $k > 0$. Now let $\phi_n \in C_c^1(\Omega)$

be a sequence of nonnegative functions that converges in $W_0^{1,p}(\Omega)$ to a nonnegative $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let ρ_η be a smooth mollifier. We take $\varphi = \rho_\eta * (v \wedge \phi_n) \in C_c^1(\Omega)$ in (3.65) where $v \wedge \phi_n := \inf(v, \phi_n)$, obtaining

$$\begin{aligned}&\int_\Omega a(x,\nabla u)\cdot\nabla(\rho_\eta * (v\wedge\phi_n))S(u) + \int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)\rho_\eta * (v\wedge\phi_n)\\ &= \int_\Omega H(u)S(u)(\rho_\eta * (v\wedge\phi_n))d\mu_d.\end{aligned} \tag{3.75}$$

We assume that $\text{supp}(S) \subset [-M, M]$ and we analyze the three terms in (3.75) separately. As concerns the first term on the left hand side of (3.75), using that

$$a(x,\nabla u)S(u) = a(x,\nabla T_M(u))S(T_M(u)) \in L^{p'}(\Omega)^N,$$

that $\rho_\eta * (v\wedge\phi_n)$ strongly converges to $v\wedge\phi_n$ in $W_0^{1,p}(\Omega)$ as $\eta \to 0$ and that $v\wedge\phi_n$ strongly converges to v in $W_0^{1,p}(\Omega)$ as $n \to \infty$, we deduce

$$\int_\Omega a(x,\nabla u)\cdot\nabla(\rho_\eta * (v\wedge\phi_n))S(u) = \int_\Omega a(x,\nabla u)\cdot\nabla v S(u) + \epsilon(\eta,n). \tag{3.76}$$

We consider now the second term on the left hand side of (3.75). Since

$$a(x,\nabla u)\cdot\nabla u S'(u) = a(x,\nabla T_M(u))\cdot\nabla T_M(u)S'(T_M(u)) \in L^1(\Omega)$$

and $\rho_\eta * (v\wedge\phi_n)$ converges to v weakly* in $L^\infty(\Omega)$ as $\eta \to 0$ and $n \to \infty$, we have that

$$\int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)\rho_\eta * (v\wedge\phi_n) = \int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)v + \epsilon(\eta,n). \tag{3.77}$$

Finally we consider the right hand side of (3.75). Since $\rho_\eta * (v\wedge\phi_n)$ converges to $v\wedge\phi_n$ cap_p-almost everywhere as $\eta \to 0$ and the following inequality holds true cap_p-almost everywhere

$$H(u)S(u)(\rho_\eta * (v\wedge\phi_n)) \le \|H\|_{L^\infty([c_{\text{supp}(\phi_n)},\infty))}\|S\|_{L^\infty(\mathbb{R})}\|v\wedge\phi_n\|_{L^\infty(\Omega)} \in L^1(\Omega,\mu_d)$$

by Lebesgue's Theorem for general measure we find

$$\int_\Omega H(u)S(u)(\rho_\eta * (v\wedge\phi_n))d\mu_d = \int_\Omega H(u)S(u)(v\wedge\phi_n)d\mu_d + \epsilon(\eta). \tag{3.78}$$

Hence, putting together (3.76), (3.77) and (3.78), we find

$$\int_\Omega a(x,\nabla u)\cdot\nabla v S(u) + \int_\Omega a(x,\nabla u)\cdot\nabla u S'(u)v = \int_\Omega H(u)S(u)(v\wedge\phi_n)d\mu_d + \epsilon(\eta,n) \tag{3.79}$$

Now, since we can write S as $S^+ - S^-$, where S^+ and S^- are the positive and the negative part of S, we can assume, without loss of generality, that $S \ge 0$. In particular, $H(u)S(u)(v\wedge\phi_n)$ is a sequence of nonnegative and μ_d-measurable functions (recall that ϕ_n has compact support for each $n \in \mathbb{N}$) that converges cap_p-almost everywhere to $H(u)S(u)v$. Hence we can apply Fatou's Lemma in (3.79) obtaining

$$\begin{aligned}\int_\Omega H(u)S(u)v d\mu_d &\le \liminf_{n\to\infty} \int_\Omega H(u)S(u)(v \wedge \phi_n) d\mu_d \\ &= \int_\Omega a(x,\nabla u)\cdot \nabla v S(u) + \int_\Omega a(x,\nabla u)\cdot \nabla u S'(u) v + \epsilon(\eta, n).\end{aligned}$$

The latter one implies that

$$H(u)S(u)v \in L^1(\Omega, \mu_d) \quad \forall v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \text{ s.t. } v \ge 0.$$

Then, since

$$H(u)S(u)(v\wedge \phi_n) \underset{n\to\infty}{\longrightarrow} H(u)S(u)v \quad \mu_d\text{-a.e.}$$

and

$$H(u)S(u)(v\wedge \phi_n) \le H(u)S(u)v \quad \mu_d\text{-a.e.}$$

by Lebesgue's Theorem we deduce that

$$\lim_{n\to\infty} \int_\Omega H(u)S(u)(v\wedge\phi_n) d\mu_d = \int_\Omega H(u)S(u)v d\mu_d.$$

In conclusion, passing to the limit first as $\eta \to 0$ and then as $n \to \infty$ in (3.79), we obtain

$$\int_\Omega a(x,\nabla u)\cdot \nabla v S(u) + \int_\Omega a(x,\nabla u)\nabla u S'(u) v = \int_\Omega H(u)S(u)v d\mu_d \tag{3.80}$$

for every $S \in W^{1,\infty}(\mathbb{R})$ with compact support and for every nonnegative $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Since it is possible to write each $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as the difference between its positive and its negative part (as done before for the test function S), we trivially deduce that (3.80) holds for all $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Hence, recalling also (3.74), we conclude that u is a renormalized solution to (3.1).
Once again, if H is non-increasing and $\mu_c \equiv 0$, it follows with the same proof given in case of $\gamma = 0$ that the renormalized solution is unique. This concludes the proof. ☐

3.5. Some remarks when H degenerates

It is worth to analyze more in depth what kind of phenomena could appear in case of a nonnegative function H, namely if we remove the request of strict positivity for H.
We recall that the problem is given by

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = H(u)\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.81}$$

Here we assume that μ is a nonnegative bounded Radon measure on Ω such that $\mu_c \equiv 0$ and that the function a satisfies (3.2), (3.3) and (3.4). Concerning the function H : $(0,+\infty) \to [0,+\infty)$, we will assume that is continuous, such that (3.6) and (3.7) hold and that it is zero for some $s > 0$.
We will prove that, under these assumptions on the lower order term, there exists a

solution to (3.81) that is bounded and that belongs, at least locally, to the energy space. This kind of remark has already been done in [**42**] for more regular data. We state the results and give just a brief idea of the proofs.

THEOREM 3.18. *Let us assume that* $\mu_c \equiv 0$ *and that* $0 \leq \gamma \leq 1$. *If* $s_1 > 0$ *is the smallest positive value such that* $H(s_1) = 0$, *then there exists a renormalized solution* u *to* (3.81) *with* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $\|u\|_{L^\infty(\Omega)} \leq s_1$.

THEOREM 3.19. *Let us assume that* $\mu_c \equiv 0$. *If* $s_1 > 0$ *is the smallest positive value such that* $H(s_1) = 0$, *then there exists a distributional solution* u *to* (3.81) *with* $u \in W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $\|u\|_{L^\infty(\Omega)} \leq s_1$.

Our first observation is that the assumption $H(s) > 0$ for all $s \geq 0$ is used in the proof of Theorems 3.4 and 3.5 only to show that the solution blows up on the support of μ_c (see (3.45)).
Hence, if $\mu_c \equiv 0$, the proofs of Theorems 3.4 and 3.5 remain valid even if H is just nonnegative and, in order to prove Theorems 3.18 and 3.19, we only need to show the improvement in the regularity of the solution.
Precisely, we will show that, under these assumptions on the lower order term, the schemes of approximation (3.14) and (3.52) (i.e. the approximations that led us to the existence results, respectively, in case $\gamma = 0$ and $\gamma > 0$), admit a sequence of solutions that is, respectively, bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ if $\gamma \leq 1$ and in $W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ if $\gamma > 1$.

We recall that the scheme of approximation (3.14), used in the case $\gamma = 0$, is given by

$$\begin{cases} -\operatorname{div}(a(x, \nabla u_n)) = H(u_n)\mu_n & \text{in}\, \Omega, \\ u_n = 0 & \text{on}\ \partial\Omega, \end{cases} \tag{3.82}$$

where H is bounded and $\mu_n = \mu_{n,d} \in L^\infty(\Omega)$ is bounded in $L^1(\Omega)$ and such that (3.15) holds.
We define on $[0, +\infty)$ the continuous function H^* as follows

$$H^*(s) = \begin{cases} H(s) & \text{if}\ s < s_1, \\ 0 & \text{if}\ s \geq s_1, \end{cases} \tag{3.83}$$

and we consider the following problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u_n^*)) = H^*(u_n^*)\mu_n & \text{in}\, \Omega, \\ u_n^* = 0 & \text{on}\ \partial\Omega. \end{cases} \tag{3.84}$$

The latter problem has a weak solution $u_n^* \in W_0^{1,p}(\Omega)$, that is also nonnegative. Now taking $G_{s_1}(u_n^*)$ as test function in (3.84), we immediately find

$$\int_\Omega |\nabla G_{s_1}(u_n^*)|^p = 0$$

which implies $u_n^* \le s_1$ almost everywhere in Ω. Hence, recalling (3.83), we conclude that u_n^* solves also (3.82). Moreover, having in mind the L^∞-estimate for u_n^* and taking u_n^* itself as test function in the weak formulation of (3.82), we deduce that u_n^* is bounded in $W_0^{1,p}(\Omega)$. This is sufficient to deduce Theorem 3.18 if $\gamma = 0$.

The scheme of approximation introduced to prove Theorems 3.4 and 3.5 in the case $\gamma > 0$ is instead given by

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_n)) = H_n(u_n)\mu_d & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.85}$$

where $H_n = T_n(H)$. In this case we consider the following problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u_n^*)) = H_n^*(u_n^*)\mu_d & \text{in } \Omega, \\ u_n^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.86}$$

with $H_n^*(s) = T_n(H^*(s))$ for each $n \in \mathbb{N}$. Applying Theorem 3.18 in the case $\gamma = 0$, we deduce that, if $n \in \mathbb{N}$ is fixed, there exists a renormalized solution $u_n^* \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.86).
To prove the positivity of the sequence u_n^*, proceeding as done to deduce (3.53), it is sufficient to construct on $[0,+\infty)$ a nonnegative function h that is not identically zero, non-increasing, continuous, bounded and such that

$$h(s) \le H_n^*(s) \text{ for all } s > 0 \text{ and for } n \text{ large enough.}$$

Since $H(s)$ is continuous for each $s > 0$ and s_1, with $s_1 > s_0 > 0$, is the smallest zero of H, there exists $s^* \in [0, s_0]$ such that

$$H(s^*) = \min_{[0,s_0]} H(s) > 0.$$

A good candidate for h is then the following function

$$h(s) = \begin{cases} H(s^*) & \text{if } 0 \le s < s^*, \\ \dfrac{H(s^*)}{(s_0 - s^*)}(s_0 - s) & \text{if } s^* \le s \le s_0, \\ 0 & \text{if } s > s_0. \end{cases}$$

From this point onwards, we can proceed as in Lemma 3.11 to prove that

$$\forall\, \omega \subset\subset \Omega \quad \exists\, c_\omega > 0 : u_n^* \ge c_\omega \ \text{ cap}_p\text{-a.e. in } \omega \text{ for } n \text{ large enough.}$$

Since, once again taking $G_{s_1}(u_n^*)$, it is possible to prove that $u_n^* \le s_1$ almost everywhere in Ω, the function u_n^* turns out to be a solution to (3.85).
Now we take as test function in the renormalized formulation of (3.85) the following ones

$$\begin{cases} S = \theta_r, \ \varphi = u_n^* & \text{if } \gamma \le 1, \\ S = \theta_r, \ \varphi = (u_n^*)^\gamma & \text{if } \gamma > 1, \end{cases}$$

where $r > 0$.
In case $\gamma \leq 1$, as $r \to \infty$ we find

$$\alpha \int_\Omega |\nabla u_n^*|^p \leq \int_\Omega a(x, \nabla u_n^*) \cdot \nabla u_n^* = \int_{\{u_n^* < s_0\}} H_n(u_n^*) u_n^* d\mu_d + \int_{\{u_n^* \geq s_0\}} H_n(u_n^*) u_n^* d\mu_d$$
$$\leq \left(C s_0^{1-\gamma} + \|H\|_{L^\infty([s_0, s_1))} s_1 \right) \|\mu_d\|_{\mathcal{M}(\Omega)},$$

namely that u_n^* is bounded in $W_0^{1,p}(\Omega)$.
If $\gamma > 1$, we find instead

$$\alpha \gamma c_\omega^{\gamma-1} \int_\omega |\nabla u_n^*|^p \leq \gamma \int_\Omega a(x, \nabla u_n^*) \cdot \nabla u_n^* (u_n^*)^{\gamma-1}$$
$$\leq \left(C + \|H\|_{L^\infty([s_0, s_1))} s_1^\gamma \right) \|\mu_d\|_{\mathcal{M}(\Omega)},$$

i.e. that u_n^* is bounded in $W_{loc}^{1,p}(\Omega)$. From now on, we can proceed as in the proof of Theorems 3.4 and 3.5 in order to obtain Theorem 3.18 for $\gamma > 0$ and Theorem 3.19.

CHAPTER 4

Existence and nonexistence for quasilinear elliptic equations with singular quadratic growth terms

In a recent paper [**18**], existence and regularity of the nonnegative solution of the following semilinear singular problem was studied:

$$
\begin{cases} -\operatorname{div}(M(x)\nabla u) = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N > 2$, M is a uniformly elliptic and bounded matrix, f is a nonnegative function belonging to some Lebesgue space, and $\gamma > 0$. In particular, existence of positive solutions for every $\gamma > 0$ (see Theorem 2.25) was proved. This problem with $M(x) \equiv I$, as shown in the Introduction, is strictly connected, setting $v = \dfrac{u^{\gamma+1}}{\gamma+1}$, with the following problem

$$
\begin{cases} -\Delta v + \dfrac{\gamma}{\gamma+1}\dfrac{|\nabla v|^2}{v} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}
$$

Formally, letting γ tend to infinity, the equation (4.2) becomes

$$
\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{v} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}
$$

In this chapter we show how, if we let γ tend to infinity in (4.1) and (4.2), the assumptions f strictly positive or f only nonnegative influenced the existence of a limit equation for the first problem and of positive solutions for (4.3).

More precisely, thanks to a priori estimates from below and from above for the distributional solution of (4.1), we prove the existence of a limit equation for (4.1) if f is zero in a neighborhood of the boundary of the domain, and the nonexistence of a limit equation if f is strictly positive.
Moreover, if f is strictly positive, we recover the existence of positive solutions for (4.3) given in Theorem 2.34. If f is zero in a neighborhood of the boundary we present a one-dimensional example in which the solution of (4.3) obtained as limit of our approximation is zero where f is zero, so that we have a nonexistence result of positive solutions for (4.3).

4.1. Main assumptions and statement of the results

We will study first the behaviour of the sequence $\{u_n\}$ of solutions of

$$(4.4)\qquad \begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \dfrac{f(x)}{u_n^n} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

as n tends to infinity. Here Ω is an open bounded subset of $\mathbb{R}^N$, $N > 2$, f a fixed non-negative $L^\infty(\Omega)$ function and $M(x)$ a matrix which satisfies, for some positive constants $0 < \alpha \le \beta$, a.e. in $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$ the following assumptions :

$$(4.5)\qquad M(x)\,\xi\cdot\xi \ge \alpha|\xi|^2 \qquad \text{and} \qquad |M(x)| \le \beta.$$

Then we fix $M(x) \equiv I$ and we study the sequence $\left\{v_n = \dfrac{u_n^{n+1}}{n+1}\right\}$ of solutions of

$$(4.6)\qquad \begin{cases} -\Delta v_n + \dfrac{n}{n+1}\dfrac{|\nabla v_n|^2}{v_n} = f(x) & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Our results are the following:

Theorem 4.1. *Let f be a nonnegative $L^\infty(\Omega)$ function. Suppose that there exists $\omega \subset\subset \Omega$ such that $f = 0$ in $\Omega \setminus \omega$, and such that for every $\omega' \subset\subset \omega$ there exists $c_{\omega'} > 0$ such that $f \ge c_{\omega'}$ in ω'. Let $\{u_n\}$ be a sequence of solutions, given by Theorem 2.25, of*

$$(4.7)\qquad \begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \dfrac{f(x)}{u_n^n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\{u_n\}$ is bounded in $L^\infty(\Omega)$, so that it converges, up to subsequences, to a bounded function u which is identically equal to 1 in ω. Furthermore, the sequence of right hand sides $\{f(x)/u_n^n\}$ is bounded in $L^1(\Omega)$, and if μ is the $*$-weak limit in the sense of measures of the right hand sides $f(x)/u_n^n$, μ is concentrated on $\partial\omega$, and u in $W_0^{1,2}(\Omega)$ is the solution of

$$(4.8)\qquad \begin{cases} -\mathrm{div}(M(x)\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 4.2. *Let f be a nonnegative $L^\infty(\Omega)$ function. Suppose that for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $f \ge c_\omega$ in ω. Let $\{\omega_n\}$ be an increasing sequence of compactly contained subsets of Ω such that their union is Ω, and let u_n be the solution of*

$$(4.9)\qquad \begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \dfrac{f(x)\,\chi_{\omega_n}}{u_n^n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\{u_n\}$ is bounded in $L^\infty(\Omega)$, so that it converges, up to subsequences, to a bounded function u, which is identically equal to 1 in Ω. Furthermore, the sequence of right hand sides $\{f(x)\chi_{\omega_n}/u_n^n\}$ is unbounded in $L^1(\Omega)$, and there is no limit equation for u.

Starting from these results and considering the sequence $\{v_n\}$ of solutions of (4.6) we prove the following existence theorem for (4.10) in the case f *strictly positive*.

THEOREM 4.3. *Let f be a nonnegative $L^\infty(\Omega)$ function. Suppose that for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $f \geq c_\omega$ in ω. Then $\{v_n\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, so that it converges, up to subsequences, to a bounded nonnegative function v. Moreover v is a weak solution of*

$$\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{v} = f & \text{in } \Omega, \\ \qquad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

On the other hand, if f is *nonnegative*, more precisely if f is zero in a neighborhood of $\partial\Omega$, we show, with a one-dimensional explicit example, nonexistence of positive solutions for (4.10) obtained by approximation.
We prove the following result:

THEOREM 4.4. *Let $\Omega = (-2, 2)$ and $\omega = (-1, 1)$. Let u_n in $W_0^{1,2}((-2, 2))$ be the weak solution, given by Theorem 2.25, of*

$$\begin{cases} -u_n''(t) = \dfrac{\chi_{(-1,1)}}{u_n^n} & \text{in } (-2, 2)\,, \\ u_n(\pm 2) \, = \, 0\,. \end{cases} \tag{4.11}$$

Let $v_n = \dfrac{u_n^{n+1}}{n+1}$ be a weak solution of

$$\begin{cases} -v_n'' + \dfrac{n}{n+1}\dfrac{|v_n'|^2}{v_n} \, = \, \chi_{(-1,1)} & \text{in } (-2, 2), \\ v_n(\pm 2) \, = \, 0\,, \end{cases} \tag{4.12}$$

then v_n weakly converges to a function v in $W_0^{1,2}((-2, 2))$ and v, belonging to $C_0^\infty((-1, 1))$, is a classical solution of

$$\begin{cases} -v'' + \dfrac{|v'|^2}{v} \, = \, 1 & \text{in } (-1, 1), \\ v(\pm 1) \, = \, 0\,. \end{cases} \tag{4.13}$$

Moreover $v(t) = \dfrac{2}{\pi^2}\cos^2\left(\dfrac{\pi}{2}t\right)$ in $(-1, 1)$ and $v(t) \equiv 0$ in $[-2, -1] \cup [1, 2]$.

As a direct consequence of Theorem 4.4 we have that the assumption f *strictly positive* is necessary (and not only technical) to have positive solutions on the whole Ω. Hence the results contained in **[4]** and **[29]** are sharp.

4.2. Estimates from above and from below

Since the formulation of distributional solution for (4.1) given in Section 2.3 is not suitable for our purposes, we are going to better specify the class of test functions which are admissible for the problem (4.1) to obtain estimates from above for u. We start with the following theorem:

THEOREM 4.5. *The solution u of (4.1) given by Theorem 2.25 is such that:*

i) *$u^{\gamma+1}$ belongs to $W_0^{1,2}(\Omega)$;*

ii) *u is such that*

$$\int_\Omega M(x)\,\nabla\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\cdot\nabla v \le \int_\Omega f\,v\,, \qquad \forall v\in W_0^{1,2}(\Omega)\,,\ v\ge 0\,; \tag{4.14}$$

iii) *u is such that*

$$\|u\|_{L^\infty(\Omega)} \le [C\,(\gamma+1)\,\|f\|_{L^\infty(\Omega)}\,]^{\frac{1}{\gamma+1}}\,, \tag{4.15}$$

for some constant $C>0$, independent on γ.

Proof. We begin by observing that, using the boundedness in $L^\infty(\Omega)$ of the sequence u_m of solutions of (2.12), and the boundedness of $u_m^{\frac{\gamma+1}{2}}$ in $W_0^{1,2}(\Omega)$, the sequence u_m^p is bounded in $W_0^{1,2}(\Omega)$ for every $p\ge\frac{\gamma+1}{2}$. In particular, $\{u_m^{\gamma+1}\}$ is bounded in $W_0^{1,2}(\Omega)$. This yields that $u^{\gamma+1}$ belongs to $W_0^{1,2}(\Omega)$ as well; i.e., i) is proved.
We now fix a positive φ in $C_0^1(\Omega)$ and take $u_m^\gamma\,\varphi$ as test function in (2.12). We obtain

$$\gamma\int_\Omega M(x)\nabla u_m\cdot\nabla u_m\,u_m^{\gamma-1}\varphi+\int_\Omega M(x)\nabla u_m\cdot\nabla\varphi\,u_m^\gamma\le\int_\Omega f\,\varphi.$$

Dropping the first term (which is positive), we obtain

$$\int_\Omega M(x)\nabla\Big(\frac{u_m^{\gamma+1}}{\gamma+1}\Big)\cdot\nabla\varphi\le\int_\Omega f\,\varphi\,.$$

Letting m tend to infinity, and using the boundedness of $u_m^{\gamma+1}$ in $W_0^{1,2}(\Omega)$, we obtain

$$\int_\Omega M(x)\nabla\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\cdot\nabla\varphi\le\int_\Omega f\,\varphi\,, \qquad \forall\varphi\in C_0^1(\Omega)\,,\ \varphi\ge 0\,.$$

Since $u^{\gamma+1}$ belongs to $W_0^{1,2}(\Omega)$, we obtain by density

$$\int_\Omega M(x)\nabla\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\cdot\nabla v\le\int_\Omega f\,v\,, \qquad \forall v\in W_0^{1,2}(\Omega)\,,\ v\ge 0\,,$$

which is (4.14). We now choose

$$v=G_k\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\,,$$

as test function in (4.14) (recall that $u \geq 0$, so that $v \geq 0$ as well). We obtain, setting $A_\gamma(k) = \{u^{\gamma+1} \geq (\gamma+1)\,k\} = \{v \geq 0\}$,

$$\int_{A_\gamma(k)} M(x)\nabla\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\cdot\nabla G_k\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big) \leq \int_{A_\gamma(k)} f\,G_k\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big).$$

Recalling (4.5) we therefore have

$$\alpha\int_{A_\gamma(k)} \Big|\nabla G_k\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big)\Big|^2 \leq \int_{A_\gamma(k)} f\,G_k\Big(\frac{u^{\gamma+1}}{\gamma+1}\Big).$$

From this inequality, reasoning once again as in [**69**], we obtain that there exists $C > 0$ such that

$$\Big\|\frac{u^{\gamma+1}}{\gamma+1}\Big\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^\infty(\Omega)}\ ,$$

which then yields (4.15). □

REMARK 4.6. *We observe that if we also assume that $\omega = \{f > 0\}$ is compactly contained in Ω in Theorem 2.25, then u belongs to $W_0^{1,2}(\Omega)$ and $\frac{f}{u^\gamma}$ belongs to $L^1(\Omega)$. As a matter of fact, taking u_m as test function in* (2.12), *we have*

$$\alpha\int_\Omega |\nabla u_m|^2 \leq \int_\Omega \frac{f\,u_m}{(u_m + \frac{1}{m})^\gamma} \leq \frac{\|f\|_{L^\infty(\Omega)}}{c_{\omega,\gamma}^{\gamma-1}},$$

so that u belongs to $W_0^{1,2}(\Omega)$. Moreover, using the Lebesgue theorem and that $u_m \geq c_{\omega,\gamma}$, we deduce that $\frac{f}{u_m^\gamma}$ strongly converges to $\frac{f}{u^\gamma}$ in $L^1(\Omega)$. As a consequence we can extend the class of test functions for (2.11) *to $W_0^{1,2}(\Omega)$.*

REMARK 4.7. *Under the assumptions of the Remark 4.6, thanks to the results contained in* [**10**], *it follows that u is the unique weak solution of* (4.1).

From now on, $\gamma = n$, and we will denote by u_n be the solution of (4.7); therefore, by the results of Theorem 4.5, we have that u_n^{n+1} belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and that

$$\|u_n\|_{L^\infty(\Omega)} \leq (C(n+1)\|f\|_{L^\infty(\Omega)})^{\frac{1}{n+1}},$$

which in particular implies that

$$\limsup_{n\to+\infty} \|u_n\|_{L^\infty(\Omega)} \leq 1. \tag{4.16}$$

We now turn to the estimates from below on the sequence $\{u_n\}$.

THEOREM 4.8. *Let u_n be the solution of* (4.7), *and let $\omega \subset\subset \Omega$ be such that for every $\omega' \subset\subset \omega$ there exists $c_{\omega'} > 0$ satisfying $f \geq c_{\omega'}$ in ω'. Then there exists $M_{\omega'} > 0$ such that*

$$u_n \geq (n+1)^{\frac{1}{n+1}} e^{-\frac{M_{\omega'}}{n+1}} \qquad \text{in } \omega'. \tag{4.17}$$

Proof. Let $\omega'' \subset\subset \omega' \subset\subset \omega$, by the assumptions we have that

$$m_{\omega'} = \inf_{x\in\omega'} f(x) > 0\,. \tag{4.18}$$

Let η in $C_0^1(\Omega)$ be such that

$$\eta(x) = \begin{cases} 1 & \text{in } \omega'', \\ 0 & \text{in } \Omega\setminus\overline{\omega'}. \end{cases}$$

We consider the function $\varphi \in C^1([0,1])$ given by Lemma 1.22, in correspondence of $g(t) = \mathrm{e}^t - 1$, $\delta = 1$ and of an arbitrary constant $C > 0$. Define

$$\xi(x) = \sqrt{\varphi(\eta(x))} \in C_0^1(\Omega)\,,$$

$$z_n = -\log\left(\frac{u_n^{n+1}}{n+1}\right),$$

and, for $k > 0$,

$$v_n = \frac{G_k(z_n^+)}{u_n}\,.$$

Note that $v_n \geq 0$ is well defined, since where $z_n^+ > k$ one has $u_n \neq 0$. We have

$$\nabla\xi = \frac{\varphi'(\eta)}{2\sqrt{\varphi(\eta)}}\,\nabla\eta\,. \tag{4.19}$$

Since

$$\nabla z_n = -\frac{(n+1)\nabla u_n}{u_n}$$

we obtain

$$\nabla v_n = -\frac{\nabla u_n}{u_n^2}\,G_k(z_n^+) + \frac{1}{u_n}\,\nabla z_n\,\chi_{A_n(k)} = -\frac{\nabla u_n}{u_n^2}\,G_k(z_n^+) - \frac{(n+1)\nabla u_n}{u_n^2}\chi_{A_n(k)}\,,$$

where $A_n(k) = \{z_n^+ \geq k\} = \{G_k(z_n^+) \neq 0\}$. Therefore, since u_n belongs to $W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ and it is locally positive, z_n and v_n belong to $W^{1,2}_{\mathrm{loc}}(\Omega)$. So that the positive function $v_n\,\xi^2$ belongs to $W_0^{1,2}(\Omega)$, has compact support and can be chosen as test function in (2.11), with $\gamma = n$, to obtain

$$-\int_{A_n(k)} M(x)\nabla u_n\cdot\nabla u_n\,\frac{G_k(z_n^+)\,\xi^2}{u_n^2} - \int_{A_n(k)} M(x)\nabla u_n\cdot\nabla u_n\,\frac{(n+1)\,\xi^2}{u_n^2}$$
$$+2\int_{A_n(k)} M(x)\nabla u_n\cdot\nabla\xi\,\frac{G_k(z_n^+)\,\xi}{u_n} = \int_{A_n(k)} \frac{f\,G_k(z_n^+)\,\xi^2}{u_n^{n+1}}\,.$$

Since

$$\frac{n+1}{u_n^{n+1}} = \mathrm{e}^{z_n}\,,$$

the previous identity can be rewritten as

$$
\begin{aligned}
&-\frac{1}{n+1}\int_{A_n(k)} M(x)\nabla z_n\cdot\nabla z_n\, G_k(z_n^+)\,\xi^2-\int_{A_n(k)} M(x)\nabla z_n\cdot\nabla z_n\,\xi^2\\
&\quad -2\int_{A_n(k)} M(x)\nabla z_n\cdot\nabla\xi\, G_k(z_n^+)\,\xi=\int_{A_n(k)} f\,\mathrm{e}^{z_n^+}G_k(z_n^+)\,\xi^2\,.
\end{aligned}
$$

Since the first term is negative, we have, using (4.5) and (4.18), as well as the fact that $G_k(s^+)\le s^+$, that

$$
\alpha\int_{A_n(k)}|\nabla z_n|^2\,\xi^2+m_{\omega'}\int_{A_n(k)}\mathrm{e}^{G_k(z_n^+)}G_k(z_n^+)\,\xi^2\le 2\beta\int_{A_n(k)}|\nabla z_n||\nabla\xi|G_k(z_n^+)\,\xi\,.
$$

Using Young's inequality in the right hand side, we have

$$
2\beta\int_{A_n(k)}|\nabla z_n||\nabla\xi|G_k(z_n^+)\,\xi\le\frac{\alpha}{2}\int_{A_n(k)}|\nabla z_n|^2\,\xi^2+\frac{2\beta^2}{\alpha}\int_{A_n(k)}|\nabla\xi|^2\,G_k(z_n^+)^2\,,
$$

so that we have

$$
\frac{\alpha}{2}\int_{A_n(k)}|\nabla G_k(z_n^+)|^2\,\xi^2+m_{\omega'}\int_{A_n(k)}\mathrm{e}^{G_k(z_n^+)}G_k(z_n^+)\,\xi^2\le\frac{2\beta^2}{\alpha}\int_{A_n(k)}|\nabla\xi|^2\,G_k(z_n^+)^2\,.
$$

Observing that

$$
\frac{\alpha}{4}|\nabla(G_k(z_n^+)\xi)|^2\le\frac{\alpha}{2}|\nabla G_k(z_n^+)|^2\,\xi^2+\frac{\alpha}{2}|\nabla\xi|^2\,G_k(z_n^+)^2\,,
$$

we obtain

$$
\frac{\alpha}{4}\int_{A_n(k)}|\nabla(G_k(z_n^+)\xi)|^2+m_{\omega'}\int_{A_n(k)}\mathrm{e}^{G_k(z_n^+)}G_k(z_n^+)\,\xi^2\le\frac{4\beta^2+\alpha^2}{2\alpha}\int_{A_n(k)}|\nabla\xi|^2\,G_k(z_n^+)^2\,.
$$

Using that $\xi=\sqrt{\varphi(\eta)}$ and (4.19), we deduce

$$
\begin{aligned}
&\frac{\alpha}{4}\int_{A_n(k)}|\nabla(G_k(z_n^+)\xi)|^2+m_{\omega'}\int_{A_n(k)}\mathrm{e}^{G_k(z_n^+)}G_k(z_n^+)\,\varphi(\eta)\\
&\le\frac{4\beta^2+\alpha^2}{8\alpha}\|\nabla\eta\|^2_{L^\infty(\Omega)}\int_{A_n(k)}G_k(z_n^+)^2\,\frac{\varphi'(\eta)^2}{\varphi(\eta)}\,.
\end{aligned}
$$

Applying Lemma 1.22, with $t=G_k(z_n^+)$, and choosing the constant C as

$$
C=\frac{4\beta^2+\alpha^2}{4\alpha m_{\omega'}}\,\|\nabla\eta\|^2_{L^\infty(\Omega)}\ ,
$$

we have

$$
\begin{aligned}
&\frac{4\beta^2+\alpha^2}{8\alpha}\|\nabla\eta\|^2_{L^\infty(\Omega)}\int_{A_n(k)}G_k(z_n^+)^2\,\frac{\varphi'(\eta)^2}{\varphi(\eta)}\\
&\le\frac{m_{\omega'}}{2}\int_{A_n(k)}G_k(z_n^+)\,(\mathrm{e}^{G_k(z_n^+)}-1)\,\varphi(\eta)+\frac{4\beta^2+\alpha^2}{8\alpha}\|\nabla\eta\|^2_{L^\infty(\Omega)}|A_n(k)\cap\omega'|\,,
\end{aligned}
$$

where $|A_n(k)\cap\omega'|$ is the Lebesgue measure of $A_n(k)\cap\omega'$. Hence, we obtain

$$\frac{\alpha}{4}\int_{A_n(k)}|\nabla(G_k(z_n^+)\xi)|^2+\frac{m_{\omega'}}{2}\int_{A_n(k)}\mathrm{e}^{G_k(z_n^+)}G_k(z_n^+)\,\varphi(\eta)$$
$$+\frac{m_{\omega'}}{2}\int_{A_n(k)}G_k(z_n^+)\,\varphi(\eta)\le\frac{4\beta^2+\alpha^2}{8\alpha}\|\nabla\eta\|^2_{L^\infty(\Omega)}|A_n(k)\,\cap\,\omega'|\,.$$

Dropping the positive terms in the left hand side, we have

$$\int_{A_n(k)}|\nabla(G_k(z_n^+)\xi)|^2\le\frac{4\beta^2+\alpha^2}{2\alpha^2}\|\nabla\eta\|^2_{L^\infty(\Omega)}|A_n(k)\,\cap\,\omega'|\,.$$

Moreover, denoting with $\mathcal{S}$ the constant given by the Sobolev embedding theorem and recalling that $\xi\equiv 1$ in ω'', we deduce, for $j>k>0$, that

$$(j-k)^2\,|A_n(j)\,\cap\,\omega''|^{\frac{2}{2^*}}\le\left(\int_{A_n(j)\,\cap\,\omega''}|G_k(z_n^+)|^{2^*}\right)^{\frac{2}{2^*}}$$
$$\le\left(\int_{A_n(k)\,\cap\,\omega'}|G_k(z_n^+)\xi|^{2^*}\right)^{\frac{2}{2^*}}\le\mathcal{S}^2\,\frac{4\beta^2+\alpha^2}{2\alpha^2}\|\nabla\eta\|^2_{L^\infty(\Omega)}|A_n(k)\,\cap\,\omega'|\,.$$

Defining $c_0^{\frac{2}{2^*}}=\mathcal{S}^2\dfrac{4\beta^2+\alpha^2}{2\alpha^2}$, we have, for all $\omega''\subset\subset\omega'\subset\subset\omega$, that

$$|A_n(j)\,\cap\,\omega''|\le c_0\frac{\|\nabla\eta\|^{2^*}_{L^\infty(\Omega)}|A_n(k)\,\cap\,\omega'|^{\frac{2^*}{2}}}{(j-k)^{2^*}}\,. \tag{4.20}$$

Now we consider $R_0=\mathrm{dist}(\omega'',\omega)$. Define

$$\omega_r=\{x\in\Omega\,:\,\mathrm{dist}(x,\omega'')<r\}$$

and

$$m(k,r)=|A_n(k)\,\cap\,\omega_r|\,,$$

for every $0<r<R_0$ and $k>0$. Choosing $0\le r<R<R_0$ and η such that $\|\nabla\eta\|_{L^\infty(\Omega)}\le\dfrac{c_1}{R-r}$ and taking $\omega''=\omega_r$ and $\omega'=\omega_R$ in (4.20), we deduce

$$m(j,r)\le c_2\,\frac{m(k,R)^{\frac{2^*}{2}}}{(j-k)^{2^*}(R-r)^{2^*}}\,,$$

where $c_2=c_0\,c_1^{2^*}$. From this inequality it follows, applying Lemma 1.21, that there exists $M_{\omega'}>0$ (independent on n) such that

$$\|z_n^+\|_{L^\infty(\omega')}\le M_{\omega'}\,.$$

Recalling the definition of z_n in terms of u_n, we therefore have

$$u_n=(n+1)^{\frac{1}{n+1}}\mathrm{e}^{-\frac{z_n}{n+1}}\ge(n+1)^{\frac{1}{n+1}}\mathrm{e}^{-\frac{M_{\omega'}}{n+1}}\quad\text{in }\omega',$$

which is (4.17). $\square$

We conclude this section with the following remark:

REMARK 4.9. *As a consequence of the estimates* (4.16) *and* (4.17), *we thus have*

$$\lim_{n\to+\infty} u_n = 1 \quad \textit{uniformly in } \omega'.$$

Repeating this argument for every ω' contained in ω, we have that u_n converges to 1 on ω.

4.3. Proofs of Theorems 4.1 and 4.2

We start with the proof of Theorem 4.1, in which we recall that $\omega = \{f > 0\}$ is compactly contained in Ω.

Proof of Theorem 4.1. We have already proved that

$$\|u_n\|_{L^\infty(\Omega)} \le (C(n+1)\|f\|_{L^\infty(\Omega)})^{\frac{1}{n+1}}, \tag{4.21}$$

so that u_n is bounded in $L^\infty(\Omega)$. This implies that there exists u in $L^\infty(\Omega)$ such that u_n *-weakly converges to u in $L^\infty(\Omega)$ and, by Remark 4.9, $u \equiv 1$ in ω. We are now going to prove that the right hand side of (4.7) is bounded in $L^1(\Omega)$ uniformly in n. As a matter of fact, if u_n is the solution of (4.7), from Theorem 2.25 and Remark 4.6, it follows that $u_n \in W_0^{1,2}(\Omega)$, $u_n \ge c_{\omega,n} > 0$ in ω and $\dfrac{f}{u_n^n}$ belongs to $L^\infty(\Omega)$. Then we have, by the results in [**69**], that

$$u_n(x) = \int_\Omega G(x,y)\,\frac{f(y)}{u_n^n(y)}\,dy\,, \qquad \forall x \in \Omega\,,$$

where $G(x,\cdot)$ is the Green function of the linear differential operator defined by the adjoint matrix $M^*(x)$ of $M(x)$, i.e., the unique duality solution of

$$\begin{cases} -\mathrm{div}(M^*(x)\nabla G(x,\cdot)) = \delta_x & \text{in } \Omega, \\ G(x,\cdot) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_x is the Dirac delta concentrated at x in Ω. It is well-known (see for example [**60**]), that for every $\omega' \subset\subset \Omega$ there exists $K > 0$ such that

$$G(x,y) \ge \frac{K}{|x-y|^{N-2}}\,, \qquad \forall x,\ y \in \omega'\,. \tag{4.22}$$

Fix now $\overline{x}$ in $\Omega \setminus \overline{\omega}$, let $\omega'' \subset\subset \Omega$ be such that $\omega \subset \omega''$ and $\overline{x}$ belongs to ω'', and let K be such that (4.22) holds. We then have

$$\begin{aligned}(C(n+1)\|f\|_{L^\infty(\Omega)})^{\frac{1}{n+1}} \ge u_n(\overline{x}) &= \int_\Omega G(\overline{x},y)\,\frac{f(y)}{u_n^n(y)}\,dy \\ &\ge \int_\Omega \frac{K}{|\overline{x}-y|^{N-2}}\,\frac{f(y)}{u_n^n(y)}\,dy \\ &\ge \frac{K}{\mathrm{diam}(\Omega)^{\mathrm{N}-2}}\int_\omega \frac{f(y)}{u_n^n(y)}\,dy\,.\end{aligned}$$

Therefore, there exists $M > 0$ such that

$$\int_\omega \frac{f(x)}{u_n^n} = \int_\Omega \frac{f(x)}{u_n^n} \leq M\,, \tag{4.23}$$

i.e., the right hand side of (4.7) is bounded in $L^1(\Omega)$. Observe now that for every $\omega' \subset\subset \omega$ there exists $M_{\omega'}$ such that

$$u_n(x) \geq (n+1)^{\frac{1}{n+1}}\, \mathrm{e}^{-\frac{M_{\omega'}}{n+1}}, \quad \text{in } \omega'.$$

Therefore,

$$\int_{\omega'} \frac{f(x)}{u_n^n} \leq \frac{|\omega'| \mathrm{e}^{\frac{nM_{\omega'}}{n+1}} \|f\|_{L^\infty(\Omega)}}{(n+1)^{\frac{n}{n+1}}},$$

so that

$$\lim_{n\to+\infty} \int_{\omega'} \frac{f(x)}{u_n^n} = 0, \tag{4.24}$$

i.e., the right hand side converges to zero in $L^1_{\text{loc}}(\omega)$. Let now μ be the bounded Radon measure such that

$$\frac{f(x)}{u_n^n} \to \mu, \quad \text{in the } *\text{-weak topology of measures.}$$

Clearly, by the assumption on f, $\mu \llcorner (\Omega \setminus \overline{\omega}) = 0$, and, by (4.24), $\mu \llcorner \omega = 0$, so that $\mu = \mu \llcorner \partial\omega$. Moreover, by Remark 4.6, we can take u_n as test function in (4.7) and we obtain, using (4.5), (4.21) and (4.23), that

$$\int_\Omega |\nabla u_n|^2 \leq \int_\Omega \frac{f(x)\, u_n}{u_n^n} \leq \|u_n\|_{L^\infty(\Omega)} \int_\Omega \frac{f(x)}{u_n^n} \leq C\,,$$

then u_n weakly converges to u in $W_0^{1,2}(\Omega)$ as n tends to infinity. Recalling that, by Remark 4.6, u_n is the (unique) weak solution of (4.7), that is

$$\int_\Omega M(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega \frac{f\,\varphi}{u_n^n}, \qquad \forall \varphi \in W_0^{1,2}(\Omega)\,, \tag{4.25}$$

we obtain, letting n tend to infinity, that

$$\int_\Omega M(x)\nabla u \cdot \nabla\varphi = \int_\Omega \varphi\, d\mu\,, \qquad \forall \varphi \in C_0^1(\Omega)\,, \tag{4.26}$$

so that u is a distributional solution with finite energy of the limit problem (4.8). □

Remark 4.10. *We observe that u_n is also the unique duality solution of* (4.7), *i.e.*

$$\int_\Omega u_n\, g = \int_\Omega \frac{f}{u_n^n}\, v\,, \qquad \forall g \in L^\infty(\Omega)\,, \tag{4.27}$$

where $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is the unique weak solution of

$$\begin{cases} -\mathrm{div}(M^*(x)\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.28}$$

This implies, letting n tend to infinity in (4.27) *and using the standard results contained in* [**69**], *that u is the unique duality solution of* (4.8).

Now we prove Theorem 4.2. Here let us recall that for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $f \geq c_\omega$ in ω and that $\{\omega_n\}$ is an increasing sequence of compactly contained subsets of Ω such that their union is Ω.

Proof of Theorem 4.2. Let be u_n the solution of (4.9). It follows, from the fact that $f(x)\,\chi_{\omega_n}(x)$ has compact support in Ω and using Remark 4.6, that u_n belongs to $W_0^{1,2}(\Omega)$ and $\dfrac{f(x)\,\chi_{\omega_n}(x)}{u_n^n}$ belongs to $L^1(\Omega)$. Once again as a consequence of Theorem 4.5 we have that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Then there exists u in $L^\infty(\Omega)$ such that u_n *-weakly converges to u in $L^\infty(\Omega)$. Moreover, by Remark 4.9, we deduce that u_n uniformly converges to 1 in ω, for every $\omega \subset\subset \Omega$, hence $u \equiv 1$ in Ω. If we assume that the sequence $\left\{\dfrac{f(x)\,\chi_{\omega_n}(x)}{u_n^n}\right\}$ is bounded in $L^1(\Omega)$, then it *-weakly converges to μ in the topology of measure. Repeating the same argue contained in Remark 4.10 we obtain

$$\int_\Omega u\,g = \int_\Omega v\,d\mu\,, \qquad \forall g \in L^\infty(\Omega)\,,$$

where v in $W_0^{1,2}(\Omega)$ is the weak solution of (4.28). Then u in $L^\infty(\Omega)$ is the duality solution of (4.8), so that u belongs to $W_0^{1,1}(\Omega)$. Since $u \equiv 1$ in Ω, there is a contradiction. Hence, the right hand side of (4.9) is not bounded in $L^1(\Omega)$ and there cannot be any limit equation. □

4.4. One-dimensional solutions and Proof of Theorem 4.3

First we prove a result that makes the link between a distributional solution of (4.4) and a finite energy solution of (4.6) rigorous.

PROPOSITION 4.11. *Let f be a nonnegative function belonging to $L^\infty(\Omega)$. If u_n is a solution of* (4.4) *given by Theorem 2.25, then $v_n = \dfrac{u_n^{n+1}}{n+1}$ is a distributional solution of* (4.6) *with finite energy.*

PROOF. We already know, by Theorem 4.5, that u_n^{n+1} belongs to $W_0^{1,2}(\Omega)$, so that v_n belongs to $W_0^{1,2}(\Omega)$. With the same argument we have that u_n^n belongs to $W_0^{1,2}(\Omega)$. Let φ be a function in $C_c^1(\Omega)$, we have that $u_n^n\varphi$ is a function in $W_0^{1,2}(\Omega)$ with compact support ($\omega = \mathrm{supp}(\varphi)$). Then we can take $u_n^n\varphi$ as test function in (2.11) and we obtain that

$$\int_\Omega \nabla u_n \cdot \nabla\varphi\, u_n^n + n\int_\Omega \nabla u_n \cdot \nabla u_n\, u_n^{n-1}\varphi = \int_\Omega f\varphi \tag{4.29}$$

If we rewrite (4.29), using that $u_n \geq c_{\omega,n}$ in ω, we have

$$\int_\Omega \nabla\left(\frac{u_n^{n+1}}{n+1}\right)\cdot\nabla\varphi + n\int_\Omega |\nabla u_n|^2\,\frac{u_n^{2n}}{u_n^{n+1}}\varphi = \int_\Omega f\varphi.$$

Hence, by definition of v_n, we deduce that

$$\int_\Omega \nabla v_n \cdot \nabla \varphi + \frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\varphi = \int_\Omega f\varphi,$$

that is v_n is a distributional solution with finite energy of (4.6). □

REMARK 4.12. *We note that for every $\omega \subset\subset \Omega$ we know, by Theorem 2.25, that $u_n \geq c_{\omega,n}$ in ω. Then $v_n \geq \dfrac{c_{\omega,n}^{n+1}}{n+1}$ in ω. Using this property and that v_n has finite energy we can extend the class of test functions for (4.6) from $C_c^1(\Omega)$ to $W_0^{1,2}(\Omega)$ with compact support.*

Now we study (4.4) in the one-dimensional case to better understand what happens, if f is *strictly positive*, to u_n and to the related v_n by passing to the limit for n tending to infinity.

Fix n in $\mathbb{N}$. We consider (4.7) with $\Omega = (-R, R)$, $R > 0$, $M(x) \equiv I$ and $f \equiv 1$ in $(-R, R)$. So that we have

$$\begin{cases} -u_n'' = \dfrac{1}{u_n^n} & \text{in } (-R, R), \\ u_n(\pm R) = 0. \end{cases} \tag{4.30}$$

In order to study (4.30) we focus on the solutions y_n of the following Cauchy problem

$$\begin{cases} -y_n''(t) = \dfrac{1}{y_n^n(t)} & \text{for } t \geq 0, \\ y_n(0) = \alpha_n, \\ y_n'(0) = 0, \end{cases} \tag{4.31}$$

where α_n is a positive real number that we will choose later. Defining $w_n = \dfrac{y_n}{\alpha_n}$, we can rewrite (4.31) as

$$\begin{cases} -w_n''(t) = \dfrac{1}{\alpha_n^{n+1} w_n^n(t)} & \text{for } t \geq 0, \\ w_n(0) = 1, \\ w_n'(0) = 0. \end{cases} \tag{4.32}$$

Since $\dfrac{1}{\alpha_n^{n+1} s^n}$ is Lipschitz continuous near $s = 1$, then there exists a unique solution w_n locally near $t = 0$. It is easy, by a classical iteration argument, to extend the definition interval of w_n to $[0, T_n)$, where $T_n < +\infty$ is the first zero of w_n (i.e. $w_n(T_n) = 0$) when it occurs, otherwise $T_n = +\infty$. Hence w_n is concave ($w_n''(t) < 0$), decreasing ($w_n'(t) < 0$) and $0 < w_n(t) \leq 1$ for $t \in [0, T_n)$ and it belongs to $C^\infty((0, T_n))$.
Now multiplying the equation by $w_n'(t)$ we have

$$-\frac{[w_n'(t)^2]'}{2} = \frac{w_n'(t)}{\alpha_n^{n+1} w_n^n(t)},$$

hence, integrating on $[0, s]$, with $0 < s < T_n$, and recalling that $w_n'(0) = 0$, we have

$$w_n'(s)^2 = \frac{2}{(n-1)\alpha_n^{n+1}}(w_n^{1-n}(s) - 1).$$

Since $w_n'(s) < 0$ we deduce

$$w_n'(s) = -\sqrt{\frac{2}{(n-1)\alpha_n^{n+1}}}\,(w_n^{1-n}(s) - 1)^{\frac{1}{2}}, \tag{4.33}$$

therefore we can divide (4.33) by $(w_n^{1-n}(s) - 1)^{\frac{1}{2}}$ and integrate on $[0, t]$, with $0 \le t < T_n$, to obtain

$$\int_0^t \frac{w_n'(s)}{(w_n^{1-n}(s) - 1)^{\frac{1}{2}}}\, ds = -\sqrt{\frac{2}{(n-1)\alpha_n^{n+1}}}\ t. \tag{4.34}$$

Setting $r = w_n(s)$ in the first integral of (4.34) and recalling that $w_n(0) = 1$, we have

$$\int_{w_n(t)}^1 \frac{r^{\frac{n-1}{2}}}{(1 - r^{n-1})^{\frac{1}{2}}}\, dr = \sqrt{\frac{2}{(n-1)\alpha_n^{n+1}}}\ t.$$

Once again we can perform the change of variable $h = 1 - r^{n-1}$ to deduce

$$\int_0^{1-w_n^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}\,(1-h)^{\frac{n-3}{2(n-1)}}}\, dh = \sqrt{\frac{2(n-1)}{\alpha_n^{n+1}}}\, t. \tag{4.35}$$

Define $I_n(t) := \displaystyle\int_0^{1-w_n^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}(1-h)^{\frac{n-3}{2(n-1)}}}\, dh$ for $t \ge 0$, then $I_n(0) = 0$ and I_n is a continuous positive and increasing function in $[0, T_n)$, so that $I_n(t) \le I_n(T_n)$. It is a well known result that

$$I_n(T_n) = \int_0^1 \frac{1}{h^{\frac{1}{2}}(1-h)^{\frac{n-3}{2(n-1)}}}\, dh = \sqrt{\pi}\frac{\Gamma\left(\frac{1}{2} + \frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)}, \tag{4.36}$$

where $\Gamma(s)$ is defined in (1.3). Thus we can extend $I_n(t)$ in $[0, T_n]$ and it is uniformly bounded for every $n \in \mathbb{N}$ and $t \in [0, T_n]$. Moreover, from (4.36) and computing (4.35) for $t = T_n$, we have

$$T_n = \sqrt{\frac{\pi\,\alpha_n^{n+1}}{2(n-1)}}\,\frac{\Gamma\left(\frac{1}{2} + \frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)}. \tag{4.37}$$

We observe that T_n and α_n are such that if α_n tends to infinity also T_n tends to infinity. Recalling that we want a solution for (4.30) that is zero if $t = R$, imposing $T_n = R$ for every n in $\mathbb{N}$ we find that

$$\alpha_n = \left(\frac{2R^2\,(n-1)\,\Gamma^2\left(\frac{n}{n-1}\right)}{\pi\,\Gamma^2\left(\frac{1}{2} + \frac{1}{n-1}\right)}\right)^{\frac{1}{n+1}}. \tag{4.38}$$

Hence, with this value of α_n, $w_n(R) = 0$ for every n in $\mathbb{N}$ and w_n belongs to $C^2((0,R))$. Thanks to the initial condition $w_n'(0) = 0$, we can extend w_n to an even function $\tilde{w}_n$ on $[-R,R]$ in the following way

$$\tilde{w}_n(t) = \begin{cases} w_n(t) & \text{for } t \in [0,R] \\ w_n(-t) & \text{for } t \in [-R,0)\,. \end{cases}$$

So $\tilde{w}_n$ belongs to $C_0^2((-R,R))$ and is the classical solution of

$$\begin{cases} -\tilde{w}_n''(t) = \dfrac{1}{\alpha_n^{n+1}\tilde{w}_n^n(t)} & \text{for } t \geq 0, \\ \tilde{w}_n(\pm R) = 0\,. \end{cases} \tag{4.39}$$

Setting $u_n(t) = \alpha_n\,\tilde{w}_n(t)$ for t in $[-R,R]$ we have that u_n belongs to $C_0^2((-R,R))$ and is the classical solution of (4.30). This implies that $v_n(t) = \dfrac{u_n(t)^{n+1}}{n+1}$ is a classical solution (in $C_0^2((-R,R))$) of

$$\begin{cases} -v_n'' + \dfrac{n}{n+1}\dfrac{|v_n'|^2}{v_n} = 1 & \text{in } (-R,R), \\ v_n(\pm R) = 0, \end{cases} \tag{4.40}$$

that is (4.6) in the one-dimensional case. Multiplying the equation (4.40) by v_n and integrating by parts on $(-R,R)$ we obtain that $\{v_n\}$ is bounded in $W_0^{1,2}((-R,R))$. By definition of v_n, this implies that $\{\tilde{w}_n^{n+1}\}$ is bounded in $W_0^{1,2}((-R,R))$. Using the Rellich-Kondrachov's theorem we deduce that there exist a subsequence, still indexed by $\tilde{w}_n^{n+1}$, and a function $g : (-R,R) \to [0,1]$ in $C_0((-R,R))$ such that $\tilde{w}_n^{n+1}$ uniformly converges to g in $(-R,R)$. We want to make g explicit.
By definition of $\tilde{w}_n$ it follows that

$$\lim_{n\to\infty} w_n^{n-1}(t) = \lim_{n\to\infty} \left(w_n^{n+1}(t)\right)^{\frac{n-1}{n+1}} = g(t)\,,$$

uniformly in $(0,R)$. Combining (4.35) and (4.38) we obtain

$$\int_0^{1-w_n^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}\,(1-h)^{\frac{n-3}{2(n-1)}}}\,dh = \frac{\sqrt{\pi}}{R}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)}\,t. \tag{4.41}$$

Computing (4.41) as n tends to infinity we obtain the explicit expression of g. Indeed we have, by Lebesgue theorem and from well known result of integral calculus, that

$$2\arcsin(\sqrt{1-g(t)}) = \lim_{n\to\infty}\int_0^{1-w_n^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}\,(1-h)^{\frac{n-3}{2(n-1)}}}\,dh = \lim_{n\to\infty}\frac{\sqrt{\pi}}{R}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)}\,t = \frac{\pi}{R}\,t.$$

It follows that

$$g(t) = 1 - \sin^2\left(\frac{\pi}{2R}\,t\right) = \cos^2\left(\frac{\pi}{2R}\,t\right).$$

So that g is an even C^∞ function defined on $\mathbb{R}$, in particular on $[-R, R]$.
Fix now t in $(-R, R)$. We want to prove that $\tilde{w}_n(t)$ tends to 1 as n tends to infinity. We assume, by contradiction, that

$$\lim_{n\to\infty} \tilde{w}_n(t) = \beta < 1.$$

Defining $\varepsilon := \dfrac{1-\beta}{2}$, we deduce, for n large enough, that $\tilde{w}_n(t) \le 1-\varepsilon$. So that

$$\tilde{w}_n^{n+1}(t) \le (1-\varepsilon)^{n+1},$$

and, letting n tend to infinity, we obtain $\cos^2\left(\frac{\pi}{2R}\,t\right) = 0$. Since $t \neq \pm R$, we find a contradiction, then $\tilde{w}_n(t)$ tends to 1, as n tends to infinity, for every t in $(-R, R)$.
Now we return to problem (4.30) recalling that $u_n(t) = \alpha_n\,\tilde{w}_n(t)$. From (4.38) and using that $\tilde{w}_n(t)$ tends to 1, as n tends to infinity, for t in $(-R, R)$, it follows that

$$\lim_{n\to\infty} u_n(t) = 1, \qquad \forall t \in (-R, R).$$

This result is exactly the one-dimensional version of Remark 4.9. From (4.38), we deduce that

$$v_n(t) = \frac{2R^2\,(n-1)\,\Gamma^2\left(\frac{n}{n-1}\right)}{\pi\,(n+1)\,\Gamma^2\left(\frac{1}{2}+\frac{1}{n-1}\right)}\,\tilde{w}_n^{n+1}(t),$$

so that we have that there exists a limit function $v : [-R, R] \to \mathbb{R}$ such that

$$v(t) = \lim_{n\to\infty} v_n(t) = \frac{2R^2}{\pi^2}\,\cos^2\left(\frac{\pi}{2R}\,t\right).$$

After a little algebra we obtain that v is a classical solution of

$$\begin{cases} -v'' + \dfrac{|v'|^2}{v} = 1 & \text{in } (-R, R), \\ v_n(\pm R) = 0, \end{cases}$$

that is (4.10). Thus we have proved Theorem 4.3 in the one-dimensional case.
Finally we prove Theorem 4.3 in the N-dimensional case, here we recall that f is *strictly positive*.

Proof of Theorem 4.3. Let u_n be the solution of (4.4) given by Theorem 2.25. It follows from Proposition 4.11 that v_n are distributional solutions of (4.6).
By assumption for every $\omega \subset\subset \Omega$ there exists a positive constant c_ω such that $f \ge c_\omega$. This implies, by Theorem 4.8, that

$$u_n \ \ge\ (n+1)^{\frac{1}{n+1}}\,\mathrm{e}^{-\frac{M_\omega}{n+1}},$$

then

$$v_n \ \ge\ \mathrm{e}^{-M_\omega}, \qquad \forall\,\omega \subset\subset \Omega, \tag{4.42}$$

with M_ω a positive constant depending only on ω. So that v_n is locally uniformly positive. Moreover, by Theorem 4.5, we have that v_n belongs to $W_0^{1,2}(\Omega)$ and

$$\|v_n\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)} ,$$

where C is a positive constant.
Choosing a nonnegative φ belonging to $C_c^1(\Omega)$ as test function in (4.6) and dropping the nonnegative integral involving the quadratic gradient term, we deduce that

$$\int_\Omega \nabla v_n \cdot \nabla\varphi \le \int_\Omega f\,\varphi\,. \tag{4.43}$$

As a consequence of the density of $C_c^1(\Omega)$ in $W_0^{1,2}(\Omega)$ we can extend (4.43) for every nonnegative φ in $W_0^{1,2}(\Omega)$. Choosing v_n as test function and using Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\int_\Omega |\nabla v_n|^2 \le \int_\Omega f\,v_n \le \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\,\|v_n\|_{L^{2^*}(\Omega)} \le \mathcal{S}\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\,\|v_n\|_{W_0^{1,2}(\Omega)}\,,$$

where $\mathcal{S}$ is the Sobolev constant. Hence $\{v_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Thus, up to a subsequence, it follows that there exists v belonging to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that

$$\begin{aligned} &v_n \to v \text{ weakly in } W_0^{1,2}(\Omega) \text{ and weakly-* in } L^\infty(\Omega),\\ &v_n \to v \text{ strongly in } L^q(\Omega),\ \forall\, q<+\infty, \text{ and a.e. in } \Omega. \end{aligned} \tag{4.44}$$

In order to pass to the limit in (4.6) we first prove that v_n strongly converges to v in $W_{\mathrm{loc}}^{1,2}(\Omega)$, that is

$$\lim_{n\to+\infty}\int_\Omega |\nabla(v_n-v)|^2\varphi = 0, \qquad \forall\,\varphi\in C_c^1(\Omega) \text{ with } \varphi\ge 0\,. \tag{4.45}$$

We consider the function $\phi_\lambda(s)$ defined in (1.4) and, choosing $\phi_\lambda(v_n-v)\varphi$ as test function in (4.6), we obtain

$$\begin{aligned} &\int_\Omega \nabla v_n\cdot\nabla(v_n-v)\,\phi_\lambda'(v_n-v)\,\varphi + \int_\Omega \nabla v_n\cdot\nabla\varphi\,\phi_\lambda(v_n-v)\\ &+\frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\,\phi_\lambda(v_n-v)\varphi = \int_\Omega f\ \phi_\lambda(v_n-v)\,\varphi \end{aligned}$$

It follows from (4.44) and using Lebesgue theorem that

$$\lim_{n\to+\infty}\int_\Omega \nabla v_n\cdot\nabla\varphi\,\phi_\lambda(v_n-v) = 0 \quad \text{and} \quad \lim_{n\to+\infty}\int_\Omega f\ \phi_\lambda(v_n-v)\,\varphi = 0.$$

Thus

$$\int_\Omega \nabla v_n\cdot\nabla(v_n-v)\,\phi_\lambda'(v_n-v)\,\varphi + \frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\,\phi_\lambda(v_n-v)\varphi = \epsilon(n). \tag{4.46}$$

Moreover, setting $\omega_\varphi = \operatorname{supp}(\varphi)$ and using (4.42), we deduce that

$$\begin{aligned}\frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\,\phi_\lambda(v_n - v)\varphi &\geq -\frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\,|\phi_\lambda(v_n - v)|\varphi \\ &\geq -\,\mathrm{e}^{M_{\omega_\varphi}}\int_\Omega |\nabla v_n|^2\,|\phi_\lambda(v_n - v)|\,\varphi,\end{aligned}$$

so that

$$\int_\Omega \nabla v_n \cdot \nabla(v_n - v)\,\phi'_\lambda(v_n - v)\,\varphi \;-\; \mathrm{e}^{M_{\omega_\varphi}}\int_\Omega |\nabla v_n|^2\,|\phi_\lambda(v_n - v)|\,\varphi = \epsilon(n). \tag{4.47}$$

We can add to (4.47)

$$-\int_\Omega \nabla v \cdot \nabla(v_n - v)\,\phi'_\lambda(v_n - v)\,\varphi$$

to obtain, noting that this quantity by (4.44) tends to 0 letting n go to infinity, that

$$\int_\Omega |\nabla(v_n - v)|^2\,\phi'_\lambda(v_n - v)\,\varphi \;-\; \mathrm{e}^{M_{\omega_\varphi}}\int_\Omega |\nabla v_n|^2\,|\phi_\lambda(v_n - v)|\,\varphi = \epsilon(n). \tag{4.48}$$

Since, by Young's inequality and using once again (4.44), we have

$$\begin{aligned}&\int_\Omega |\nabla v_n|^2\,|\phi_\lambda(v_n - v)|\,\varphi \leq 2\int_\Omega |\nabla(v_n - v)|^2\,|\phi_\lambda(v_n - v)|\,\varphi \\ &+ 2\int_\Omega |\nabla v|^2\,|\phi_\lambda(v_n - v)|\,\varphi \;=\; 2\int_\Omega |\nabla(v_n - v)|^2\,|\phi_\lambda(v_n - v)|\,\varphi \;+\; \epsilon(n)\,,\end{aligned}$$

we deduce that

$$\int_\Omega |\nabla(v_n - v)|^2\;\left\{\phi'_\lambda(v_n - v) - 2\mathrm{e}^{M_{\omega_\varphi}}|\phi_\lambda(v_n - v)|\right\}\,\varphi \;=\; \epsilon(n).$$

Choosing $\lambda \geq \mathrm{e}^{2M_{\omega_\varphi}}$, we have that $\{\phi'_\lambda(v_n - v) - 2\mathrm{e}^{M_{\omega_\varphi}}|\phi_\lambda(v_n - v)|\} \geq \frac{1}{2}$, hence (4.45) holds and

$$v_n \to v \text{ strongly in } W^{1,2}_{\mathrm{loc}}(\Omega). \tag{4.49}$$

Now we pass to the limit in (4.6) with test functions φ belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with compact support. We have, by (4.45), that

$$\lim_{n\to+\infty}\int_\Omega \nabla v_n \cdot \nabla\phi = \int_\Omega \nabla v \cdot \nabla\varphi,$$

and, using (4.49), (4.42) with $\omega = \operatorname{supp}(\varphi)$ and Lebesgue theorem, we deduce

$$\lim_{n\to+\infty}\frac{n}{n+1}\int_\Omega \frac{|\nabla v_n|^2}{v_n}\,\varphi \;=\; \int_\Omega \frac{|\nabla v|^2}{v}\,\varphi,$$

so that

$$\int_\Omega \nabla v \cdot \nabla\varphi \;+\; \int_\Omega \frac{|\nabla v|^2}{v}\,\varphi \;=\; \int_\Omega f\,\varphi\,, \tag{4.50}$$

for all φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with compact support.
Let φ be a nonnegative function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Let $\{\varphi_m\}$ in $C_c^1(\Omega)$ be a sequence of nonnegative functions that converges to φ strongly in $W_0^{1,2}(\Omega)$. Taking $\varphi_m\wedge\varphi$, which belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with compact support, as test function in (4.50), we obtain

$$\int_\Omega \frac{|\nabla v|^2}{v}\,(\varphi_m\wedge\varphi) = \int_\Omega f\,(\varphi_m\wedge\varphi) - \int_\Omega \nabla v\cdot\nabla(\varphi_m\wedge\varphi)\,. \tag{4.51}$$

Since $\varphi_m\wedge\varphi$ strongly converges to φ in $W_0^{1,2}(\Omega)$ we have

$$\lim_{m\to+\infty}\int_\Omega\left\{f\,(\varphi_m\wedge\varphi) - \int_\Omega \nabla v\cdot\nabla(\varphi_m\wedge\varphi)\right\} = \int_\Omega f\,\varphi - \int_\Omega\nabla v\cdot\nabla\varphi\,. \tag{4.52}$$

Moreover $\dfrac{|\nabla v|^2}{v}\,(\varphi_m\wedge\varphi)$ is a nonnegative function that converges to $\dfrac{|\nabla v|^2}{v}\,\varphi$ almost everywhere in Ω. Applying Fatou's lemma on the left hand side of (4.51) and using (4.52) we deduce that

$$\int_\Omega\frac{|\nabla v|^2}{v}\,\varphi \le \liminf_{m\to+\infty}\int_\Omega\frac{|\nabla v|^2}{v}\,(\varphi_m\wedge\varphi) = \int_\Omega f\,\varphi - \int_\Omega\nabla v\cdot\nabla\varphi\,,$$

so that $\dfrac{|\nabla v|^2}{v}\,\varphi$ belongs to $L^1(\Omega)$. Since $\dfrac{|\nabla v|^2}{v}\,(\varphi_m\wedge\varphi)\le\dfrac{|\nabla v|^2}{v}\,\varphi$, by Lebesgue theorem, we have

$$\lim_{m\to+\infty}\int_\Omega\frac{|\nabla v|^2}{v}\,(\varphi_m\wedge\varphi) = \int_\Omega\frac{|\nabla v|^2}{v}\,\varphi\,. \tag{4.53}$$

As a consequence of (4.52) and (4.53) we obtain

$$\int_\Omega\nabla v\cdot\nabla\varphi + \int_\Omega\frac{|\nabla v|^2}{v}\,\varphi = \int_\Omega f\,\varphi\,,\quad \forall\,\varphi\ge 0 \text{ in } W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,. \tag{4.54}$$

Furthermore, taking $\dfrac{T_\varepsilon(v)}{\varepsilon}$ as test function in (4.54) and dropping a positive term, we deduce

$$\int_\Omega\frac{|\nabla v|^2}{v}\,\frac{T_\varepsilon(v)}{\varepsilon} \le \int_\Omega f\,\frac{T_\varepsilon(v)}{\varepsilon}\,. \tag{4.55}$$

Applying respectively Fatou's lemma on the left hand side and Lebesgue theorem on the right hand side of (4.55) we have

$$\int_\Omega\frac{|\nabla v|^2}{v} \le \liminf_{m\to+\infty}\int_\Omega\frac{|\nabla v|^2}{v}\,\frac{T_\varepsilon(v)}{\varepsilon} \le \int_\Omega f\,,$$

so that $\dfrac{|\nabla v|^2}{v}$ belongs to $L^1(\Omega)$. Since we can write each $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ as the difference between its positive and its negative part, we trivially deduce that (4.54) holds for all $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, so that v is a weak solution of (4.10). □

Remark 4.13. *We note that we can also consider test functions only belonging to* $W_0^{1,2}(\Omega)$ *in* (4.54). *Indeed let* φ *be in* $W_0^{1,2}(\Omega)$*, then* $T_k(\varphi^+)$ *is a positive function belonging to* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *that strongly converges to* φ^+ *in* $W_0^{1,2}(\Omega)$ *as* k *tends to infinity. Taking* $T_k(\varphi^+)$ *as test function in* (4.54) *and letting* k *tend to infinity, by Lebesgue theorem and Beppo Levi theorem, we deduce*

$$\int_\Omega \nabla v \cdot \nabla \varphi^+ + \int_\Omega \frac{|\nabla v|^2}{v} \varphi^+ = \int_\Omega f \varphi^+ . \tag{4.56}$$

In the same way we obtain

$$\int_\Omega \nabla v \cdot \nabla \varphi^- + \int_\Omega \frac{|\nabla v|^2}{v} \varphi^- = \int_\Omega f \varphi^- , \tag{4.57}$$

so that subtracting (4.57) *to* (4.56) *we have that* (4.54) *holds for every* φ *belonging to* $W_0^{1,2}(\Omega)$.

Remark 4.14. *To prove that* $\{v_n\}$ *is bounded in* $W_0^{1,2}(\Omega)$ *and* (4.44) *we only used that* f *is nonnegative and belongs to* $L^\infty(\Omega)$.

4.5. Nonexistence of positive solutions

Here we prove Theorem 4.4. As a consequence, if f is only a *nonnegative* function, it follows a nonexistence result for positive solutions obtained by approximation of (4.10).

Proof of Theorem 4.4. First we study the behaviour of u_n weak solution, given by Theorem 2.25, of (4.11), that is (4.7) in the case $N = 1$, $\Omega = (-2, 2)$, $M(x) \equiv I$ and $f = \chi_{(-1,1)}$. In order to study u_n we use the construction of one-dimensional solutions done in the previous section, in which we have proved that there exists a function w_n in $C^2((0, T_n))$ classical solution of

$$\begin{cases} -w_n''(t) = \dfrac{1}{\alpha_n^{n+1} w_n^n(t)} & \text{in } (0, T_n), \\ w_n(0) = 1, \\ w_n'(0) = 0, \end{cases} \tag{4.58}$$

where T_n is the first zero of w_n. We recall that $0 < w_n(t) < 1$, w_n is concave ($w_n''(t) < 0$) and decreasing ($w_n'(t) < 0$) for every t in $(0, T_n)$. Moreover we have obtained that

$$w_n'(t) = -\sqrt{\frac{2}{(n-1)\alpha_n^{n+1}}}\,(w_n^{1-n}(t) - 1)^{\frac{1}{2}}, \tag{4.59}$$

and, by integrating, that

$$S_n(1 - w_n^{n-1}(t)) := \int_0^{1-w_n^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}\,(1-h)^{\frac{n-3}{2(n-1)}}}\, dh = \sqrt{\frac{2(n-1)}{\alpha_n^{n+1}}}\, t\,, \tag{4.60}$$

for every t in $[0,T_n)$. So that $S_n : [0,1) \to [0, S_n(1))$ is a nonnegative, continuous and strictly increasing function. Recalling (4.36) we have that $S_n(1) = I_n(T_n)$, that is uniformly bounded, thus we can extend S_n in 1 to have $S_n : [0,1] \to [0, S_n(1)]$. Then there exists the inverse function $S_n^{-1} : [0, S_n(1)] \to [0,1]$. Furthermore we recall that

$$T_n = \sqrt{\frac{\pi\, \alpha_n^{n+1}}{2(n-1)} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)}}. \tag{4.61}$$

In order to have $1 < T_n < +\infty$ for every n we can choose $\alpha_n = (c_n(n-1))^{\frac{1}{n+1}}$, with c_n a positive constant such that

$$c_n > \frac{2\,\Gamma^2\left(\frac{n}{n-1}\right)}{\pi\,\Gamma^2\left(\frac{1}{2} + \frac{1}{n-1}\right)} =: \underline{c}_n\,, \qquad \forall\, n \text{ in } \mathbb{N}. \tag{4.62}$$

Now we consider the following Cauchly problem

$$\begin{cases} -y_n''(t) = \dfrac{\chi_{(0,1)}}{c_n(n-1)y_n^n(t)} & \text{for } t \geq 0\,, \\ y_n(0) = 1, \\ y_n'(0) = 0. \end{cases} \tag{4.63}$$

For every t in $(0,1)$ we have that (4.58) and (4.63) are the same problem, so that there exists $y_n(t) \equiv w_n(t)$ classical solution of (4.63) in $(0,1)$. Since $y_n''(t) = 0$ for every $t \geq 1$, we deduce that $y_n(t) = y_n(1) + y_n'(1)(t-1) = w_n(1) + w_n'(1)(t-1)$ in $[1,2)$. It follows from (4.59) and by the definition of α_n that

$$w_n'(1) = -\sqrt{\frac{2}{c_n(n-1)^2}}\,(w_n^{1-n}(1) - 1)^{\frac{1}{2}}\,. \tag{4.64}$$

Since we want that $y_n(2) = 0$ for every n in $\mathbb{N}$, we search for c_n such that $w_n'(1) = -w_n(1)$. With a little algebra it follows from (4.64) and (4.60) that is possible if and only if, for every fixed n, we have

$$w_n^{n+1}(1) = \frac{2}{c_n(n-1)^2}\,(1 - w_n^{n-1}(1)) \;=\; \frac{2}{c_n(n-1)^2}\, S_n^{-1}\left(\sqrt{\frac{2}{c_n}}\right). \tag{4.65}$$

By Lemma 4.16 below there exists a sequence $\{c_n\}$ such that (4.65) holds for every n, hence we have that y_n belonging to $C^1((0,2))$ is such that

$$y_n(t) \equiv w_n(t) \text{ in } [0,1]\,, \quad y_n(t) = w_n(1)(2-t) \text{ in } (1,2]\,, \quad y_n'(0) = y_n(2) = 0\,. \tag{4.66}$$

We want that $w_n(t) \leq y_n(t)$ in $[0,T_n]$. This is true if and only if $T_n \leq 2$. If, by contradiction, $T_n > 2$ we have $w_n(t) \equiv y_n(t)$ in $[0,1]$ and $-y_n''(t) < -w_n''(t)$ in $(1,2]$, so that, by $w_n'(1) = y_n'(1)$, we deduce $w_n(t) < y_n(t)$ in $(1,2]$. It follows from $y_n(2) = 0$ that $0 < w_n(2) < 0$, that is a contradiction. Then we obtain $T_n \leq 2$, $w_n(t) \leq y_n(t)$ in $[0,T_n]$

and, by (4.61), that

$$c_n \le \frac{8\,\Gamma^2\left(\frac{n}{n-1}\right)}{\pi\,\Gamma^2\left(\frac{1}{2}+\frac{1}{n-1}\right)} =: \overline{c}_n\,, \qquad \forall\, n \text{ in } \mathbb{N}\,. \tag{4.67}$$

Thus $\{c_n\}$ is bounded and, up to subsequences, there exists a positive real number c_∞ such that

$$\frac{2}{\pi^2} = \lim_{n\to+\infty} \underline{c}_n \le c_\infty := \lim_{n\to+\infty} c_n \le \lim_{n\to+\infty} \overline{c}_n = \frac{8}{\pi^2}\,,$$

and, respectively,

$$1 \le T_\infty := \lim_{n\to+\infty} T_n = \pi\sqrt{\frac{c_\infty}{2}} \le 2\,.$$

As shown in the previous section, it follows from (4.60) that

$$\lim_{n\to+\infty} w_n^{n+1}(t) = \cos^2\left(\frac{\pi}{2\,T_\infty}t\right) \quad \text{and} \quad \lim_{n\to+\infty} w_n(t) = 1, \qquad \text{for } t\in(0,T_\infty)\,. \tag{4.68}$$

Now we suppose that $T_\infty > 1$. Fix $\beta = \dfrac{T_\infty - 1}{2} > 0$, so that $1+\beta < T_\infty$. We know that for n large enough

$$w_n(1+\beta) \le y_n(1+\beta) = w_n(1)(1-\beta)\,.$$

By passing to the limit as n tends to infinity and using (4.68) we obtain $1 \le 1-\beta$, that is $\beta \le 0$. This is a contradiction, then $T_\infty = 1$ and, therefore, $c_\infty = \dfrac{2}{\pi^2}$.
Recalling that $y_n(t) \equiv w_n(t)$ in $(0,1)$ and using, once again, (4.68) we have

$$\lim_{n\to+\infty} y_n^{n+1}(t) = \cos^2\left(\frac{\pi}{2}t\right) \quad \text{and} \quad \lim_{n\to+\infty} y_n(t) = 1\,, \qquad \text{for } t\in(0,1)\,. \tag{4.69}$$

It follows from (4.65) and using that $y_n(1) = w_n(1)$ for every n that

$$\lim_{n\to+\infty} y_n^{n+1}(1) = 0 \quad \text{and} \quad \lim_{n\to+\infty} y_n(1) = 1\,, \tag{4.70}$$

hence, by (4.66), we obtain that $y_n^{n+1}(t) = w_n(1)^{n+1}(2-t)^{n+1}$ and that

$$\lim_{n\to+\infty} y_n^{n+1}(t) = 0 \quad \text{and} \quad \lim_{n\to+\infty} y_n(t) = (2-t)\,, \qquad \text{for } t\in(1,2]\,. \tag{4.71}$$

Therefore, by the initial condition $y_n'(0) = 0$, we can extend y_n to an even function defined in $(-2,2)$ as follows

$$\tilde{y}_n(t) = \begin{cases} (c_n(n-1))^{\frac{1}{n+1}} y_n(t) & \text{for } t\in[0,2],\\ (c_n(n-1))^{\frac{1}{n+1}} y_n(-t) & \text{for } t\in[-2,0),\end{cases}$$

so that $\tilde{y}$ belonging to $C_0^1((-2,2))$ is a weak solution of (4.11). By Remark 4.7 there is a unique weak solution of (4.11), hence $\tilde{y}_n(t) \equiv u_n(t)$ for every t in $(-2,2)$ and n in $\mathbb{N}$.
Moreover, by Proposition 4.11, setting $v_n(t) = \dfrac{u_n^{n+1}(t)}{n+1}$, we have that v_n in $C_0^1((-2,2))$ is a weak solution of (4.12) and, by Remark 4.14, that there exists a function v such that v_n

weakly converges to v in $W_0^1((-2,2))$ and almost everywhere in $(-2,2)$. As a consequence of (4.69), (4.70) and (4.71) we deduce that

$$v(t) = \begin{cases} \dfrac{2}{\pi^2}\cos^2\left(\dfrac{\pi}{2}t\right) & \text{for } t \in (-1,1), \\ 0 & \text{for } t \in [-2,-1] \cup [1,2], \end{cases}$$

so that v belongs to $C_0^1(-2,2) \cap C_0^\infty(-1,1)$. Furthermore, with a little algebra, it follows that v is a classical solution of (4.13). □

REMARK 4.15. *From the proof of Theorem 4.4 we deduce that u_n pointwise converges to u defined as follows*

$$u(t) = \begin{cases} (2-t) & \textit{for } t \in [1,2], \\ 1 & \textit{for } t \in (-1,1), \\ (2+t) & \textit{for } t \in [-2,-1]. \end{cases}$$

Moreover, by Theorem 4.1, u_n weakly converges to u in $W_0^{1,2}((-2,2))$. Hence we have that

$$u'(t) = \begin{cases} -1 & \textit{for } t \in (1,2), \\ 0 & \textit{for } t \in (-1,1), \\ 1 & \textit{for } t \in (-2,-1), \end{cases}$$

and u is a distributional solution of

$$\begin{cases} -u'' = -\delta_{-1} + \delta_1 & \textit{in } (-2,2), \\ \qquad u(\pm 2) = 0\,. \end{cases}$$

So that we have completely recovered the result of Theorem 4.1.

To be complete we show the technical lemma that we needed to prove the theorem.
Fix n in $\mathbb{N}$.

LEMMA 4.16. *Let c belong to $(c_0,+\infty)$, with $c_0 = \dfrac{2\,\Gamma^2\left(\frac{n}{n-1}\right)}{\pi\,\Gamma^2\left(\frac{1}{2}+\frac{1}{n-1}\right)}$. Let $w_c(t)$ be the classical solution of*

$$\begin{cases} -w_c''(t) = \dfrac{1}{c(n-1)\,w_c^n(t)} & \textit{for } t \geq 0, \\ w_c(0) = 1, \\ w_c'(0) = 0. \end{cases} \tag{4.72}$$

Let T_c be the first zero of w_c. Then there exists a unique $\tilde{c}$ in $(c_0,+\infty)$ such that $T_{\tilde{c}} > 1$ and

$$w_{\tilde{c}}^{n+1}(1) = \frac{2}{\tilde{c}(n-1)^2}\,S_{\tilde{c}}^{-1}\left(\sqrt{\frac{2}{\tilde{c}}}\right), \tag{4.73}$$

where $S_c : [0,1] \to [0, S_c(1)]$ *is defined as*

$$S_c(1 - w_c^{n-1}(t)) := \int_0^{1-w_c^{n-1}(t)} \frac{1}{h^{\frac{1}{2}}\,(1-h)^{\frac{n-3}{2(n-1)}}}\,dh\,,$$

for t *in* $[0, T_c]$.

PROOF. It follows from the proof of Theorem 4.4 that if $c > c_0$ then there exists $w_c(t)$ classical solution of (4.72) in $[0, T_c]$, with $T_c > 1$.
Now we define $F : (c_0, +\infty) \to \mathbb{R}$ as

$$F(c) \;=\; w_c^{n+1}(1) - \frac{2}{c(n-1)^2}\,S_c^{-1}\left(\sqrt{\frac{2}{c}}\right).$$

It is obvious that $w_c(t)$ is continuous on $(c_0, +\infty)$ for every t in $[0, T_c)$, so that F is continuous. Fix $c_0 < c_1 < c_2$. Recalling that

$$T_c \;=\; \sqrt{\frac{\pi\, c}{2}}\,\frac{\Gamma\left(\frac{1}{2}+\frac{1}{n-1}\right)}{\Gamma\left(\frac{n}{n-1}\right)},$$

we deduce $T_{c_1} < T_{c_2}$. Moreover we state that $w_{c_1}(t) < w_{c_2}(t)$ for every t in $(0, T_{c_1}]$. Indeed, since $-w''_{c_1}(t) > -w''_{c_2}(t)$ near $t = 0$ and using the initial conditions, we obtain that $w_{c_1}(t) < w_{c_2}(t)$ near $t = 0$. If, by contradiction, there exists s in $(0, T_{c_1})$ such that $w_{c_1}(s) = w_{c_2}(s)$ we have that $w'_{c_1}(s) \geq w'_{c_2}(s)$. We know, by (4.59), that

$$w'_{c_1}(s) = -\sqrt{\frac{2}{(n-1)^2 c_1}}\,(w_{c_1}^{1-n}(s) - 1)^{\frac{1}{2}} \;<\; -\sqrt{\frac{2}{(n-1)^2 c_2}}\,(w_{c_1}^{1-n}(s) - 1)^{\frac{1}{2}} = w'_{c_2}(s)\,,$$

that is a contradiction. Hence we have that $w_c(t)$ is monotone increasing in c. This implies that F also is monotone increasing in c. By letting c tend to the boundary of $(c_0, +\infty)$ and recalling that

$$\lim_{c\to c_0} w_c^{n+1}(1) \;= 0 \quad \text{and} \quad \lim_{c\to+\infty} w_c^{n+1}(1) \;=\; 1\,,$$

we deduce

$$\lim_{c\to c_0} F(c) \;=\; -\frac{2}{c_0(n-1)^2}\,S_{c_0}^{-1}\left(\sqrt{\frac{2}{c_0}}\right) \;<\; 0 \quad \text{and} \quad \lim_{c\to+\infty} F(c) \;=\; 1.$$

Applying Bolzano's theorem we obtain that there exists $\tilde{c}$ such that $F(\tilde{c}) = 0$, that is (4.73). Since F is monotone increasing, $\tilde{c}$ is unique. □

REMARK 4.17. *It follows from Theorem 4.4 that if* f *is nonnegative we cannot obtain by approximation a positive solution of* (4.10). *This implies that the existence results contained in* **[4]** *and* **[29]** *are sharp.*

4.6. Open problems

We are now studying the nonexistence of positive solutions of (4.10) in the N-dimensional case with f only *nonnegative*. More precisely we assume that f is a nonnegative $L^\infty(\Omega)$ function and that there exists $\omega \subset\subset \Omega$ such that $f = 0$ in $\Omega \setminus \omega$, and such that for every $\omega' \subset\subset \omega$ there exists $c_{\omega'} > 0$ such that $f \geq c_{\omega'}$ in ω'.
We observe that from Remark 4.15 it follows that u, given by Theorem 4.4, is a classical solution of

$$\begin{cases} -u'' = 0 & \text{in } (-2,-1) \cup (1,2)\,, \\ u(\pm 1) = 1\,, \\ u(\pm 2) = 0\,. \end{cases}$$

Our conjecture is that it is true also for $N > 1$. More precisely we think that the following result holds.

CONJECTURE 4.18. *Let u be the function given by Theorem 4.1, with $M(x) \equiv I$. Then u is a classical solution of*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ u = 1 & \text{on } \partial\omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The difficulty is to prove that $u = 1$ in $\partial\omega$. A possible way is to prove that $\{u_n\}$ is a sequence of functions uniformly continuous in n but until now we have not been able to prove it.
With a similar idea we think that Theorem 4.4 holds for $N > 1$. We state that:

CONJECTURE 4.19. *Let u_n be the solution of* (4.7) *given by Theorem 2.25, with $M(x) \equiv I$. Let $v_n = \dfrac{u_n^{n+1}}{n+1}$ be the sequence of solutions of* (4.6)*. Then $\{v_n\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, so that it converges, up to subsequences, to a bounded nonnegative function v. Moreover v is a weak solution of*

$$\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{v} = f & \text{in } \omega, \\ v = 0 & \text{on } \partial\omega, \end{cases}$$

and $v \equiv 0$ in $\Omega \setminus \omega$.

We think that, starting to the one-dimensional function v obtained by Theorem 4.4, we can prove Conjecture 4.19 in the radial case $\omega = B_1(0) \subset \Omega = B_2(0) \subset \mathbb{R}^N$, with $N > 1$.

CHAPTER 5

Existence and regularizing effect for p-Laplacian systems

5.1. Introduction and main assumptions

In this chapter we are concerned with the existence of solutions for the following nonlinear elliptic system

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + A\varphi^{\theta+1}|u|^{r-2}u = f, & u \in W_0^{1,p}(\Omega), \\ -\operatorname{div}(|\nabla \varphi|^{p-2}\nabla \varphi) = |u|^r\varphi^\theta, & \varphi \in W_0^{1,p}(\Omega), \end{cases} \tag{5.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with $N \geq 2$, $1 < p < N$, $A > 0$, $r > 1$ and $0 \leq \theta < p - 1$.
In the case $\theta = 0$ the system (5.1) becomes

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,p}(\Omega), \\ -\operatorname{div}(|\nabla \varphi|^{p-2}\nabla \varphi) = |u|^r, & \varphi \in W_0^{1,p}(\Omega). \end{cases} \tag{5.2}$$

For such value of θ, we show that there exists a regularizing effect for the existence of solutions with finite energy. Indeed we prove the existence of a weak solution u in $W_0^{1,p}(\Omega)$ of the first equation of (5.2) with f belonging to $L^m(\Omega)$, with $(r+1)' \leq m < (p^*)'$.
Conversely, in the case $p = 2$ and $0 < \theta < 1$ the second equation of the system (5.1) is sublinear. This fact does not allow us to use the same method as the previous case and we are not able to prove the regularizing effect on u. However, we prove a regularizing effect on the existence of finite energy solution for the second equation of (5.1), that is we find finite energy solutions even if the right hand side of the second equation does not belong to the dual space.

Once again we reason by approximation. We first prove the existence of finite energy solutions for (5.1) if f is regular using that for regular data the system (5.1) has variational nature. Then we prove the regularizing effect by passing to the limit in our approximation.

5.2. Regular data

Let us firstly prove the existence of a weak solution (u, φ) of (5.1) with data f in $L^m(\Omega)$, $m > \frac{N}{p}$. This solution is a saddle point of a functional defined on $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

Proposition 5.1. *Let f in $L^m(\Omega)$, with $m > \frac{N}{p}$, and let $A > 0$, $r > 1$ and $0 \leq \theta < p-1$. Then there exists a weak solution (u, φ) of (5.1). Moreover, u and φ are in $L^\infty(\Omega)$, $\varphi \geq 0$*

and (u,φ) is a saddle point of the functional defined on $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ as

(5.3)
$$J(z,\eta)=\begin{cases}\frac{1}{p}\int_\Omega|\nabla z|^p-\frac{A(\theta+1)}{pr}\int_\Omega|\nabla\eta|^p+\frac{A}{r}\int_\Omega(\eta^+)^{\theta+1}|z|^r-\int_\Omega fz & \text{if } \int_\Omega(\eta^+)^{\theta+1}|z|^r<+\infty,\\ +\infty & \text{otherwise.}\end{cases}$$

PROOF. Fix $\psi\in W_0^{1,p}(\Omega)$ and let I_1 be the functional defined on $W_0^{1,p}(\Omega)$ as $I_1(z):=J(z,\psi)$. We have, by Hölder's inequality and denoting by C_s the constant of the Sobolev embedding theorem, that

$$I_1(z)\ge\frac{1}{p}\|z\|^p_{W_0^{1,p}(\Omega)}-\frac{A(\theta+1)}{pr}\|\psi\|^p_{W_0^{1,p}(\Omega)}-C_s\|f\|_{L^{(p^*)'}(\Omega)}\|z\|_{W_0^{1,p}(\Omega)}.$$

This implies that I_1 is coercive. Now we prove that I_1 is weakly lower semicontinuous, which is that if $z_n\rightharpoonup z$ in $W_0^{1,p}(\Omega)$ then

$$I_1(z)\le\liminf_{n\to\infty}I_1(z_n). \tag{5.4}$$

Since $f\in L^m(\Omega)\subset L^{(p^*)'}(\Omega)$ we have that that

$$\lim_{n\to\infty}\int_\Omega fz_n=\int_\Omega fz.$$

As a consequence of Fatou's lemma, it also yields

$$\frac{A}{r}\int_\Omega(\psi^+)^{\theta+1}|z|^r\le\liminf_{n\to\infty}\frac{A}{r}\int_\Omega(\psi^+)^{\theta+1}|z_n|^r.$$

Then, by the weakly lower semicontinuity of the norm, we deduce (5.4). Hence there exists a minimum v of I_1 on $W_0^{1,p}(\Omega)$. Moreover, by the classical theory of elliptic equations, v is the unique weak solution of the Euler-Lagrange equation

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v)+A(\psi^+)^{\theta+1}|v|^{r-2}v=f,\quad v\in W_0^{1,p}(\Omega). \tag{5.5}$$

We have, thanks to the results in [**69**], that

$$\|v\|_{W_0^{1,p}(\Omega)}+\|v\|_{L^\infty(\Omega)}\le C_1\|f\|^{\frac{1}{p-1}}_{L^m(\Omega)}, \tag{5.6}$$

where C_1 is a positive constant not depending on f. We define $S:W_0^{1,p}(\Omega)\to W_0^{1,p}(\Omega)$ as the operator such that $v=S(\psi)$. Now we consider the functional defined on $W_0^{1,p}(\Omega)$ as $I_2(\eta):=J(v,\eta)$. As before, since $\theta<p-1$, we have that $-I_2$ is coercive and weakly lower semicontinuous. Then there exists a minimum ζ of $-I_2$, that is a maximum of I_2 on $W_0^{1,p}(\Omega)$. Let I_3 be a functional defined on $W_0^{1,p}(\Omega)$ as

$$I_3(\eta):=\frac{\theta+1}{p}\int_\Omega|\nabla\eta|^p-\int_\Omega(\eta^+)^{\theta+1}|v|^r.$$

Since ζ is a maximum of I_2, we have

$$\begin{aligned}\frac{A}{r}I_3(\zeta) &= -I_2(\zeta) + \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega fv \\ &\le -I_2(\eta) + \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega fv = \frac{A}{r}I_3(\eta), \quad \forall \eta \in W_0^{1,p}(\Omega),\end{aligned}$$

so that ζ is a minimum of I_3. We observe that $\zeta \ge 0$ and $\zeta \not\equiv 0$ in Ω. In fact we have

$$I_3(\zeta) = \frac{\theta+1}{p}\int_\Omega |\nabla\zeta|^p - \int_\Omega (\zeta^+)^{\theta+1}|v|^r \le \frac{\theta+1}{p}\int_\Omega |\nabla\zeta^+|^p - \int_\Omega (\zeta^+)^{\theta+1}|v|^r = I_3(\zeta^+),$$

then $\|\zeta\|_{W_0^{1,p}(\Omega)} \le \|\zeta^+\|_{W_0^{1,p}(\Omega)}$ and so ζ^- is zero almost everywhere in Ω. Now we show that $\zeta \not\equiv 0$. We consider λ_1 to be the first eigenvalue of $-\Delta_p$ while φ_1 in $W_0^{1,p}(\Omega)$ is the associated eigenfunction, that is

$$\begin{cases} -\text{div}(|\nabla\varphi_1|^{p-2}\nabla\varphi_1) = \lambda_1|\varphi_1|^{p-2}\varphi_1 & \text{in } \Omega, \\ \varphi_1 > 0 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $t > 0$; computing I_3 in $t\varphi_1$, we obtain

$$\begin{aligned}I_3(t\varphi_1) &= \frac{(\theta+1)t^p}{p}\int_\Omega |\nabla\varphi_1|^p - t^{\theta+1}\int_\Omega \varphi_1^{\theta+1}|v|^r \\ &= \frac{(\theta+1)\lambda_1 t^p}{p}\int_\Omega \varphi_1^p - t^{\theta+1}\int_\Omega \varphi_1^{\theta+1}|v|^r = c_1t^p - c_2t^{\theta+1},\end{aligned}$$

where $c_1 := \frac{(\theta+1)\lambda_1}{p}\int_\Omega \varphi_1^p \in (0,+\infty)$ and $c_2 := \int_\Omega \varphi_1^{\theta+1}|v|^r \in (0,+\infty]$. By taking t such that $c_1t^{p-\theta-1} - c_2 < 0$, that is $t < \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-\theta-1}}$, we have $I_3(t\varphi_1) < 0$. Then $I_3(\zeta) < 0 = I_3(0)$ and $\zeta \not\equiv 0$. Since ζ is a nonnegative minimum of I_3, by Proposition 2.22, it is the unique weak solution of the Euler-Lagrange equation

$$-\text{div}(|\nabla\zeta|^{p-2}\nabla\zeta) = |v|^r\zeta^\theta, \quad \zeta \in W_0^{1,p}(\Omega). \tag{5.7}$$

Following **[17]**, we have that

$$\|\zeta\|_{W_0^{1,p}(\Omega)} + \|\zeta\|_{L^\infty(\Omega)} \le C_2\|v\|_{L^\infty(\Omega)}^{\frac{r}{p-\theta-1}}, \tag{5.8}$$

and we deduce, using (5.6), that

$$\|\zeta\|_{W_0^{1,p}(\Omega)} + \|\zeta\|_{L^\infty(\Omega)} \le C\|f\|_{L^m(\Omega)}^{\frac{r}{(p-1)(p-\theta-1)}} =: R, \tag{5.9}$$

where C and C_2 are positive constants not depending on f and v. Now we define $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ as the operator such that $\zeta = T(v) = T(S(\psi))$. We want to prove that $T \circ S$ has a fixed point by Schauder's fixed point theorem. By (5.9) we have that $\overline{B_R(0)} \subset W_0^{1,p}(\Omega)$ is invariant for $T \circ S$. Let $\{\psi_n\} \subset W_0^{1,p}(\Omega)$ be a sequence weakly convergent to some ψ and let $v_n = S(\psi_n)$. As a consequence of (5.6), there exists a subsequence indexed by v_{n_k} such that

$$v_{n_k} \to v \text{ weakly in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega, \qquad (5.10)$$
$$v_{n_k} \to v \text{ weakly-* in } L^\infty(\Omega).$$

Moreover, we have

$$-\mathrm{div}(|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}) = f - A(\psi_{n_k}^+)^{\theta+1}|v_{n_k}|^{r-2}v_{n_k} =: g_{n_k},$$

and, using Hölder's inequality, the Poincaré inequality and (5.6), we obtain

$$\|g_{n_k}\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)} + A\|v_{n_k}\|_{L^\infty(\Omega)}^{r-1}\|\psi_{n_k}\|_{L^{\theta+1}(\Omega)}^{\theta+1} \le \|f\|_{L^1(\Omega)} + AC_1\|f\|_{L^m(\Omega)}^{\frac{r-1}{p-1}}\|\psi_n\|_{W_0^{1,p}(\Omega)}^{\theta+1} \le C.$$

Then, by Theorem 2.1 in [**15**], we obtain that ∇v_{n_k} converges to ∇v almost everywhere in Ω. Since

$$\||\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\|_{(L^{p'}(\Omega))^N} = \|v_{n_k}\|_{W_0^{1,p}(\Omega)}^{p-1} \le C_1\|f\|_{L^m(\Omega)},$$

we deduce that

$$|\nabla v_{n_k}|^{p-2}\nabla v_{n_k} \to |\nabla v|^{p-2}\nabla v \text{ weakly in } (L^{p'}(\Omega))^N. \qquad (5.11)$$

We recall that v_{n_k} satisfies

$$\int_\Omega |\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\cdot\nabla w + A\int_\Omega (\psi_{n_k}^+)^{\theta+1}|v_{n_k}|^{r-2}v_{n_k}w = \int_\Omega fw, \quad \forall w \in W_0^{1,p}(\Omega).$$

Letting k tend to infinity, by (5.10), (5.11) and Vitali's theorem, we have that

$$\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla w + A\int_\Omega (\psi^+)^{\theta+1}|v|^{r-2}vw = \int_\Omega fw, \quad \forall w \in W_0^{1,p}(\Omega),$$

so that v is the unique weak solution of (5.5) and it does not depend on the subsequence. Hence $v_n = S(\psi_n)$ converges to $v = S(\psi)$ weakly in $W_0^{1,p}(\Omega)$ and weakly-* in $L^\infty(\Omega)$. Then

$$|v_n|^r \to |v|^r \text{ strongly in } L^q(\Omega)\ \forall q < +\infty \text{ and } \||v_n|^r\zeta_n^\theta\|_{L^1(\Omega)} \le C. \qquad (5.12)$$

Using (5.9), (5.12) and proceeding in the same way, we obtain that

$$\zeta_n = T(v_n) \to \zeta = T(v) \text{ weakly in } W_0^{1,p}(\Omega), \text{ and weakly-* in } L^\infty(\Omega), \qquad (5.13)$$
$$|\nabla\zeta_n|^{p-2}\nabla\zeta_n \to |\nabla\zeta|^{p-2}\nabla\zeta \text{ weakly in } (L^{p'}(\Omega))^N,$$

and ζ is the unique weak solution of (5.7). Now we want to prove that ζ_n converges to ζ strongly in $W_0^{1,p}(\Omega)$. In order to obtain this, by Lemma 5 in [**16**], it is sufficient to prove the following

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla\zeta_n|^{p-2}\nabla\zeta_n - |\nabla\zeta|^{p-2}\nabla\zeta\right)\cdot\nabla(\zeta_n - \zeta) = 0. \qquad (5.14)$$

We have that

$$\int_\Omega \left(|\nabla\zeta_n|^{p-2}\nabla\zeta_n - |\nabla\zeta|^{p-2}\nabla\zeta\right)\cdot\nabla(\zeta_n-\zeta) = \int_\Omega |\nabla\zeta_n|^p - \int_\Omega |\nabla\zeta|^{p-2}\nabla\zeta\cdot\nabla\zeta_n - \int_\Omega |\nabla\zeta_n|^{p-2}\nabla\zeta_n\cdot\nabla\zeta + \|\zeta\|^p_{W_0^{1,p}(\Omega)}. \tag{5.15}$$

The second and the third term on the right hand side of (5.15) converge, by (5.13), to $\|\zeta\|^p_{W_0^{1,p}(\Omega)}$. Then it is sufficient to prove that

$$\lim_{n\to\infty} \|\zeta_n\|^p_{W_0^{1,p}(\Omega)} = \|\zeta\|^p_{W_0^{1,p}(\Omega)}. \tag{5.16}$$

Since ζ_n is equal to $T(v_n) \geq 0$, we have that

$$\int_\Omega |\nabla\zeta_n|^p = \int_\Omega |v_n|^r\zeta_n^{\theta+1}.$$

By (5.12) and Vitali's theorem, we deduce that

$$\lim_{n\to\infty}\int_\Omega |v_n|^r\zeta_n^{\theta+1} = \int_\Omega |v|^r\zeta^{\theta+1} = \|\zeta\|^p_{W_0^{1,p}(\Omega)},$$

so that (5.16) is true and (5.14) is proved. Hence we have proved that if ψ_n converges to ψ weakly in $W_0^{1,p}(\Omega)$ then $\zeta_n = T(S(\psi_n))$ converges to $\zeta = T(S(\psi))$ strongly in $W_0^{1,p}(\Omega)$. As a consequence we have that $T\circ S$ is a continuous operator and that $T(S(\overline{B_R(0)}))\subset W_0^{1,p}(\Omega)$ is a compact subset. Then there exists, by Schauder's fixed point theorem, a function φ in $W_0^{1,p}(\Omega)$ such that $\varphi = T(S(\varphi))$ and, since $T(v)\geq 0$ for every v in $W_0^{1,p}(\Omega)$, φ is nonnegative . Moreover let $u = S(\varphi)$, we have that u is a minimum for I_1 and φ is a maximum for I_2. Hence (u,φ) is a saddle point of J defined by (5.3) and a weak solution of (5.1). □

5.3. Existence and regularizing effect in the case $\theta = 0$

In this section we assume $\theta = 0$ and we study the regularizing effect on the existence of finite energy solutions of both equations even if the data do not belong to the dual space. We recall that the assumption on θ implies that we deal with the system (5.2). We consider the datum f in $L^{(r+1)'}(\Omega)$ and a sequence $\{f_n\}$ such that

$$f_n \in L^\infty(\Omega),\ |f_n| \leq |f|\ \forall n\in\mathbb{N} \text{ and } f_n \to f \text{ strongly in } L^{(r+1)'}(\Omega).$$

By Proposition 5.1, there exists (u_n,φ_n) in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ that satisfies

$$\begin{cases} -\text{div}(|\nabla u_n|^{p-2}\nabla u_n) + A\varphi_n|u_n|^{r-2}u_n = f_n, & (i),\\ -\text{div}(|\nabla\varphi_n|^{p-2}\nabla\varphi_n) = |u_n|^r, & (ii), \end{cases} \tag{5.17}$$

with $\varphi_n \geq 0$, u_n and φ_n in $L^\infty(\Omega)$. Choosing u_n as test function in (i) and φ_n in (ii) of (5.17) we have

$$\int_\Omega |\nabla u_n|^p + A\int_\Omega \varphi_n|u_n|^r = \int_\Omega f_n u_n, \qquad \int_\Omega |\nabla\varphi_n|^p = \int_\Omega |u_n|^r\varphi_n.$$

Then

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla \varphi_n|^p \le C \int_\Omega f_n u_n. \tag{5.18}$$

Choosing $u_n^+ = u_n \chi_{\{u_n \ge 0\}}$ as test function in (ii) we obtain

$$\int_\Omega |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla u_n^+ = \int_\Omega |u_n|^r u_n^+ = \int_\Omega |u_n^+|^{r+1}. \tag{5.19}$$

For the term on the left hand side of (5.19) we have, by Young's inequality and (5.18), that

$$\begin{aligned} \int_\Omega |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla u_n^+ &\le \frac{1}{p'} \int_\Omega |\nabla \varphi_n|^p + \frac{1}{p} \int_\Omega |\nabla u_n^+|^p \\ &\le \frac{1}{p'} \int_\Omega |\nabla \varphi_n|^p + \frac{1}{p} \int_\Omega |\nabla u_n|^p \le C \int_\Omega f_n u_n. \end{aligned} \tag{5.20}$$

Putting together (5.19) and (5.20), we obtain

$$\int_\Omega |u_n^+|^{r+1} \le C \int_\Omega f_n u_n.$$

In the same way, using $u_n^- = -u_n \chi_{\{u_n < 0\}}$ as test function in (ii), we have

$$\int_\Omega |u_n^-|^{r+1} \le C \int_\Omega f_n u_n,$$

so that

$$\int_\Omega |u_n|^{r+1} = \int_\Omega |u_n^+|^{r+1} + \int_\Omega |u_n^-|^{r+1} \le C \int_\Omega f_n u_n \le C \int_\Omega |f||u_n|. \tag{5.21}$$

Then, applying Hölder inequality to the right hand side of (5.21) with exponents $(r+1)'$ and $r+1$, we deduce

$$\|u_n\|_{L^{r+1}(\Omega)} \le C \|f\|_{L^{(r+1)'}(\Omega)}^{\frac{1}{r}}. \tag{5.22}$$

This implies, by (5.18) and Hölder's inequality, that

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla \varphi_n|^p \le C \|f\|_{L^{(r+1)'}(\Omega)} \|u_n\|_{L^{r+1}(\Omega)} \le C \|f\|_{L^{(r+1)'}(\Omega)}^{\frac{r+1}{r}}, \tag{5.23}$$

and

$$\int_\Omega \varphi_n |u_n|^r \le C \|f\|_{L^{(r+1)'}(\Omega)}^{\frac{r+1}{r}}. \tag{5.24}$$

As a consequence of (5.22), (5.23) and (5.24), we have the following lemma.

LEMMA 5.2. *Let f in $L^{(r+1)'}(\Omega)$, and let $A > 0$ and $r > 1$. Then the weak solution (u_n, φ_n) of (5.17) is such that*

$$\|u_n\|_{L^{r+1}(\Omega)} + \|u_n\|_{W_0^{1,p}(\Omega)} + \|\varphi_n\|_{W_0^{1,p}(\Omega)} + \int_\Omega \varphi_n |u_n|^r \le C(f),$$

where $C(f)$ is a positive constant depending only on $\|f\|_{L^{(r+1)'}(\Omega)}$.

The above lemma implies that there exist subsequences still indexed by u_n and φ_n and functions u and φ belonging to $W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & u_n \to u \text{ weakly in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega, \\ & u_n \to u \text{ weakly in } L^{r+1}(\Omega), \text{ and strongly in } L^q(\Omega)\ \forall q < \max\{r+1, p^*\}, \\ & \varphi_n \to \varphi \text{ weakly in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega. \end{aligned} \tag{5.25}$$

By applying these convergence results, we can prove the following existence theorem.

THEOREM 5.3. *Let $A > 0$, and let $r > 1$ and f in $L^m(\Omega)$, with $m \geq (r+1)'$. Then there exists a weak solution (u, φ) of system* (5.2)*, with u and φ in $W_0^{1,p}(\Omega)$.*

The proof is a consequence of the proof of Theorem 5.8 in the case $\theta = 0$. We deduce, by Theorem 5.3, the regularizing effect for the solutions of (5.2). We assume

$$(r+1)' < (p^*)' \Leftrightarrow r > \frac{N(p-1)+p}{N-p} \quad \text{and} \quad f \in L^m(\Omega), \text{ with } m \geq (r+1)'. \tag{5.26}$$

REMARK 5.4. *Under these assumptions we note that, if $m \geq (p^*)'$, thanks to the results in* [**14**]*, we have that u belongs to $W_0^{1,p}(\Omega) \cap L^t(\Omega)$, with $t := \frac{Nm(p-1)}{N-pm}$. Then, if $\frac{t}{r} < (p^*)'$, that is $m < m_1 := \frac{Npr}{N(p-1)^2 + p(p-1) + p^2 r}$, φ belongs to $W_0^{1,p}(\Omega)$ even if the datum of the second equation of* (5.2) *does not belongs to the dual space. We verify that $m_1 > (p^*)'$. Since*

$$m_1 = \frac{pNr}{N(p-1)^2 + p(p-1) + p^2 r} > (p^*)' = \frac{Np}{N(p-1)+p} \Leftrightarrow r > p^* - 1,$$

it follows thanks to (5.26)*. Moreover we have that, if $m < (p^*)'$ (i.e. the datum f does not belong to $W^{-1,p'}(\Omega)$), then u belongs to $W_0^{1,p}(\Omega)$. Hence we have a regularizing effect due to the system: the functions u and φ belong to $W_0^{1,p}(\Omega)$ because of the coupling between the equations. This fact does not follow on being solutions of the single equations.*

We now prove summability results for u.

PROPOSITION 5.5. *Under the assumptions* (5.26)*, the weak solution u of* (5.2)*, given by Theorem 5.3, belongs to $L^s(\Omega)$, with $s = \frac{m(pr+p-1)}{m(p-1)+1}$.*

PROOF. We recall that u is obtained from (5.25) and that (u_n, φ_n) is a weak solution of the system (5.17). Choosing $(u_n^+)^\gamma$ as test function in (ii) of (5.17), with $\gamma \geq 1$, we have

$$\gamma \int_\Omega |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla u_n^+ (u_n^+)^{\gamma-1} = \int_\Omega (u_n^+)^{r+\gamma}. \tag{5.27}$$

Applying Young's inequality to the left hand side of (5.27) we obtain, by Lemma 5.2, that

$$\gamma \int_\Omega |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla u_n^+ (u_n^+)^{\gamma-1} \le C \int_\Omega |\nabla \varphi_n|^p + C \int_\Omega |\nabla u_n|^p (u_n^+)^{p(\gamma-1)} = C(f) + C \int_\Omega |\nabla u_n|^p (u_n^+)^{p\gamma - p}. \tag{5.28}$$

Now using $(u_n^+)^{p\gamma-p+1}$ as test function in (i) of (5.17) we have, by Hölder's inequality, that

$$\int_\Omega |\nabla u_n^+|^p (u_n^+)^{p\gamma-p} \le C \int_\Omega |\nabla u_n^+|^p (u_n^+)^{p\gamma-p} + C \int_\Omega \varphi_n (u_n^+)^{r+p\gamma-p} \le C \int_\Omega f_n (u_n^+)^{p\gamma-p+1} \le C\|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n^+)^{m'(p\gamma-p+1)} \right)^{\frac{1}{m'}}. \tag{5.29}$$

As a consequence of (5.27), (5.28) and (5.29) we obtain

$$\int_\Omega (u_n^+)^{r+\gamma} \le C(f) + C\|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n^+)^{m'(p\gamma-p+1)} \right)^{\frac{1}{m'}}. \tag{5.30}$$

Imposing $r + \gamma = m'(p\gamma - p + 1)$ we have

$$\gamma = \frac{r(m-1) + m(p-1)}{m(p-1)+1} \quad \text{and} \quad s := r + \gamma = \frac{m(pr+p-1)}{m(p-1)+1}.$$

We verify that $\gamma \ge 1$:

$$\gamma = \frac{r(m-1) + m(p-1)}{m(p-1)+1} \ge 1 \Leftrightarrow m \ge \frac{r+1}{r} = (r+1)',$$

which it is true by (5.26). Then, by (5.30), we deduce

$$\|u_n^+\|_{L^s(\Omega)} \le C(f),$$

where $C(f)$ is a positive constant depending only on $\|f\|_{L^m(\Omega)}$. In the same way we obtain, using u_n^- as test function, that

$$\|u_n^-\|_{L^s(\Omega)} \le C(f).$$

Then we have

$$\|u_n\|_{L^s(\Omega)} = \|u_n^+\|_{L^s(\Omega)} + \|u_n^-\|_{L^s(\Omega)} \le C(f),$$

and u_n converges to u weakly in $L^s(\Omega)$, so that $u \in L^s(\Omega)$. □

REMARK 5.6. *Comparing this summability result on u with the result contained in* (5.25) *we observe that*

$$s = \frac{m(pr+p-1)}{m(p-1)+1} \ge r+1 \Leftrightarrow m \ge \frac{r+1}{r} = (r+1)',$$

then, if (5.26) *holds, $L^s(\Omega) \subset L^{r+1}(\Omega)$. Moreover, if $m \ge (p^*)'$, it follows from* **[14]** *that u belongs to $L^t(\Omega)$, with $t = \dfrac{Nm(p-1)}{N-pm}$. We have that*

$$s \ge t \Leftrightarrow m \le m_1.$$

Summarizing we obtain that the best summability results for u are

$$u \in L^s(\Omega), \quad \textit{if } (r+1)' \le m < m_1, \tag{5.31}$$

and

$$u \in L^t(\Omega), \quad \textit{if } m \ge m_1.$$

Then we note, by (5.31)*, that we have also a regularizing effect for the summability of the solution u.*

5.4. Existence and regularizing effect in the dual case

We prove now the existence theorem for a weak solution of (5.1) for $\theta \ge 0$ and f belonging to $L^{(p^*)'}(\Omega)$. Let $\{f_n\}$ be a sequence that satisfies

$$f_n \in L^\infty(\Omega),\ |f_n| \le |f|\ \forall n \in \mathbb{N} \text{ and } f_n \to f \text{ strongly in } L^{(p^*)'}(\Omega).$$

Then, by Proposition 5.1, there exists a solution (u_n, φ_n) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ of the system

$$\begin{cases} -\mathrm{div}(|\nabla u_n|^{p-2}\nabla u_n) + A\varphi_n^{\theta+1}|u_n|^{r-2}u_n = f_n, & (I), \\ -\mathrm{div}(|\nabla \varphi_n|^{p-2}\nabla \varphi_n) = |u_n|^r \varphi_n^\theta, & (II), \end{cases} \tag{5.32}$$

with $\varphi_n \ge 0$, u_n and φ_n in $L^\infty(\Omega)$. Choosing u_n as test function in (I) and φ_n in (II) we have

$$\int_\Omega |\nabla u_n|^p + A\int_\Omega \varphi_n^{\theta+1}|u_n|^r = \int_\Omega f_n u_n, \qquad \int_\Omega |\nabla \varphi_n|^p = \int_\Omega |u_n|^r \varphi_n^{\theta+1}. \tag{5.33}$$

Then

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla \varphi_n|^p \le C\int_\Omega f_n u_n. \tag{5.34}$$

We obtain, by (5.34) and applying Hölder's inequality and the Sobolev embedding theorem, that

$$\begin{aligned} \int_\Omega |\nabla u_n|^p &\le \int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla \varphi_n|^p \le C\int_\Omega f_n u_n \\ &\le C\|f\|_{L^{(p^*)'}(\Omega)}\|u_n\|_{L^{p^*}(\Omega)} \le C\|f\|_{L^{(p^*)'}(\Omega)}\|u_n\|_{W_0^{1,p}(\Omega)}, \end{aligned}$$

so that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{1}{p-1}} \quad \text{and} \quad \|\varphi_n\|_{W_0^{1,p}(\Omega)} \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{1}{p-1}}. \tag{5.35}$$

Moreover, by (5.33), we deduce

$$\int_\Omega \varphi_n^{\theta+1}|u_n|^r \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{p}{p-1}}. \tag{5.36}$$

Choosing u_n^+ as test function in (II) we obtain

$$\int_\Omega |\nabla \varphi_n|^{p-2}\nabla\varphi_n \cdot \nabla u_n^+ = \int_\Omega |u_n|^r u_n^+ \varphi_n^\theta = \int_\Omega |u_n^+|^{r+1}\varphi_n^\theta.$$

Using Young's inequality and (5.35), we find

$$\begin{aligned}\int_\Omega |u_n^+|^{r+1}\varphi_n^\theta &= \int_\Omega |\nabla\varphi_n|^{p-2}\nabla\varphi_n\cdot\nabla u_n^+ \le \frac{1}{p'}\int_\Omega|\nabla\varphi_n|^p + \frac{1}{p}\int_\Omega|\nabla u_n^+|^p\\ &\le \frac{1}{p'}\int_\Omega|\nabla\varphi_n|^p + \frac{1}{p}\int_\Omega|\nabla u_n|^p \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{p}{p-1}}.\end{aligned}$$

In the same way, choosing u_n^- as test function in (II), we deduce

$$\int_\Omega |u_n^-|^{r+1}\varphi_n^\theta \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{p}{p-1}},$$

so that

$$\int_\Omega |u_n|^{r+1}\varphi_n^\theta = \int_\Omega |u_n^+|^{r+1}\varphi_n^\theta + \int_\Omega |u_n^-|^{r+1}\varphi_n^\theta \le C\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{p}{p-1}}. \tag{5.37}$$

As a consequence of (5.35), (5.36) and (5.37), we have the following lemma.

LEMMA 5.7. *Let f in $L^{(p^*)'}(\Omega)$, and let $A>0$, $r>1$ and $0\le\theta<p-1$. Then the weak solution (u_n,φ_n) of (5.32), given by Proposition 5.1, is such that*

$$\|u_n\|_{W_0^{1,p}(\Omega)} + \|\varphi_n\|_{W_0^{1,p}(\Omega)} + \int_\Omega \varphi_n^{\theta+1}|u_n|^r + \int_\Omega |u_n|^{r+1}\varphi_n^\theta \le C(f)\,,$$

where $C(f)$ is a positive constant depending only on $\|f\|_{L^{(p^*)'}(\Omega)}$.

Once again, by Lemma 5.7, there exist subsequences still indexed by u_n and φ_n and functions u and φ in $W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&u_n \to u \text{ weakly in } W_0^{1,p}(\Omega), \text{ strongly in } L^q(\Omega), \text{ with } q<p^*, \text{ and a.e. in } \Omega,\\ &\varphi_n \to \varphi \text{ weakly in } W_0^{1,p}(\Omega), \text{ strongly in } L^q(\Omega), \text{ with } q<p^*, \text{ and a.e. in } \Omega.\end{aligned} \tag{5.38}$$

THEOREM 5.8. *Let $A>0$, and let $r>1$, $0\le\theta<p-1$ and f in $L^m(\Omega)$, with $m\ge(p^*)'$. Then there exists a weak solution (u,φ) in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ of system (5.1).*

PROOF. Let u and φ be the functions defined in (5.38). We want to pass to the limit in (II) of (5.32). We recall that φ_n satisfies

$$\int_\Omega |\nabla\varphi_n|^{p-2}\nabla\varphi_n\cdot\nabla\psi = \int_\Omega |u_n|^r\varphi_n^\theta\psi, \quad \forall\psi\in W_0^{1,p}(\Omega). \tag{5.39}$$

We want to prove that $|u_n|^r\varphi_n^\theta$ strongly converges to $|u|^r\varphi^\theta$ in $L^1(\Omega)$. Fix $\sigma>0$ and let $E\subset\Omega$. By Lemma 5.7 there exists $\overline{k}\in\mathbb{N}$ such that

$$\begin{aligned}\int_E |u_n|^r\varphi_n^\theta &= \int_{E\cap\{|u_n|\le\overline{k}\}} |u_n|^r\varphi_n^\theta + \int_{E\cap\{|u_n|>\overline{k}\}} |u_n|^r\varphi_n^\theta \le \overline{k}^r\int_E\varphi_n^\theta + \frac{1}{\overline{k}}\int_{\{|u_n|>\overline{k}\}}|u_n|^{r+1}\varphi_n^\theta\\ &\le \overline{k}^r\int_E\varphi_n^\theta + \frac{C(f)}{\overline{k}} \le \overline{k}^r\int_E\varphi_n^\theta + \frac{\sigma}{2}.\end{aligned}$$

Since, by (5.38), φ_n^θ strongly converges to φ^θ in $L^1(\Omega)$, applying Vitali's theorem, there exists $\delta > 0$ such that $|E| < \delta$ and

$$\int_E |u_n|^r \varphi_n^\theta \leq \overline{k}^r \int_E \varphi_n^\theta + \frac{\sigma}{2} \leq \sigma.$$

Then, once again using Vitali's theorem, we have

$$|u_n|^r \varphi_n^\theta \to |u|^r \varphi^\theta \text{ strongly in } L^1(\Omega). \tag{5.40}$$

Hence, by Theorem 2.1 in **[15]**, we obtain that $\nabla \varphi_n$ converges $\nabla \varphi$ almost everywhere in Ω. Moreover

$$\| |\nabla \varphi_n|^{p-2} \nabla \varphi_n \|_{(L^{p'}(\Omega))^N} \leq \|\varphi_n\|^{p-1}_{W_0^{1,p}(\Omega)} \leq C(f),$$

so that

$$|\nabla \varphi_n|^{p-2} \nabla \varphi_n \to |\nabla \varphi|^{p-2} \nabla \varphi \text{ weakly in } (L^{p'}(\Omega))^N. \tag{5.41}$$

Fix ψ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have, by (5.41), that

$$\lim_{n\to\infty} \int_\Omega |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla \psi = \int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \psi.$$

On the other hand, by (5.40) and Vitali's theorem, we find

$$\lim_{n\to\infty} \int_\Omega |u_n|^r \varphi_n^\theta \psi = \int_\Omega |u|^r \varphi^\theta \psi.$$

By passing to the limit in (5.39), we obtain that

$$\int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \psi = \int_\Omega |u|^r \varphi^\theta \psi, \quad \forall \psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{5.42}$$

Let η belong to $W_0^{1,p}(\Omega)$. Choosing $\psi = T_k(\eta)$ as test function in (5.42), we obtain

$$\int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla T_k(\eta) = \int_\Omega |u|^r \varphi^\theta T_k(\eta). \tag{5.43}$$

We have that $|\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla T_k(\eta)$ converges to $|\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \eta$ almost everywhere in Ω and that

$$\left| |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla T_k(\eta) \right| \leq \left| |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \eta \right|,$$

with $\left| |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \eta \right|$ in $L^1(\Omega)$. Then, by Lebesgue's theorem, we deduce

$$\lim_{k\to\infty} \int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla T_k(\eta) = \int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \eta. \tag{5.44}$$

Now we want to let k to infinity on the right hand side of (5.43). We recall that

$$|u|^r \varphi^\theta T_k(\eta) = |u|^r \varphi^\theta T_k(\eta^+) - |u|^r \varphi^\theta T_k(\eta^-),$$

where $|u|^r\varphi^\theta T_k(\eta^+)$ and $|u|^r\varphi^\theta T_k(\eta^-)$ are nonnegative functions increasing in k. We have that $|u|^r\varphi^\theta T_k(\eta^+)$ converges to $|u|^r\varphi^\theta\eta^+$ and $|u|^r\varphi^\theta T_k(\eta^-)$ converges to $|u|^r\varphi^\theta\eta^-$ almost everywhere in Ω. It follows from Beppo Levi's theorem that

$$\lim_{k\to\infty}\int_\Omega |u|^r\varphi^\theta T_k(\eta^+) = \int_\Omega |u|^r\varphi^\theta\eta^+ \text{ and } \lim_{k\to\infty}\int_\Omega |u|^r\varphi^\theta T_k(\eta^-) = \int_\Omega |u|^r\varphi^\theta\eta^-,$$

so that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |u|^r\varphi^\theta T_k(\eta) &= \lim_{k\to\infty}\int_\Omega |u|^r\varphi^\theta T_k(\eta^+) - \lim_{k\to\infty}\int_\Omega |u|^r\varphi^\theta T_k(\eta^-)\\ &= \int_\Omega |u|^r\varphi^\theta\eta^+ - \int_\Omega |u|^r\varphi^\theta\eta^- = \int_\Omega |u|^r\varphi^\theta\eta.\end{aligned}\tag{5.45}$$

Letting k to infinity in (5.43), by (5.44) and (5.45), we obtain

$$\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\cdot\nabla\eta = \int_\Omega |u|^r\varphi^\theta\eta, \quad \forall\eta\in W_0^{1,p}(\Omega).$$

Then φ in $W_0^{1,p}(\Omega)$ is a weak solution of the second equation of (5.1). Now we want to pass to the limit in (I) of (5.32). We have that u_n satisfies

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi + A\int_\Omega \varphi_n^{\theta+1}|u_n|^{r-2}u_n\psi = \int_\Omega f_n\psi, \quad \forall\psi\in W_0^{1,p}(\Omega).\tag{5.46}$$

Fix $\varepsilon>0$. Choosing $\psi = \dfrac{T_\varepsilon(G_k(u_n))}{\varepsilon}$ in (5.46), we obtain

$$\frac{1}{\varepsilon}\int_{\{k\le|u_n|\le k+\varepsilon\}}|\nabla u_n|^p + A\int_{\{|u_n|\ge k\}}\varphi_n^{\theta+1}|u_n|^{r-2}u_n\frac{T_\varepsilon(G_k(u_n))}{\varepsilon} = \int_{\{|u_n|\ge k\}} f_n\frac{T_\varepsilon(G_k(u_n))}{\varepsilon}.$$

Dropping the first nonnegative term, we have

$$\begin{aligned}A\int_{\{|u_n|\ge k+\varepsilon\}}\varphi_n^{\theta+1}|u_n|^{r-1} &\le A\int_{\{|u_n|\ge k\}}\varphi_n^{\theta+1}|u_n|^{r-2}u_n\frac{T_\varepsilon(G_k(u_n))}{\varepsilon}\\ &\le \int_{\{|u_n|\ge k\}}|f_n|\left|\frac{T_\varepsilon(G_k(u_n))}{\varepsilon}\right| \le \int_{\{|u_n|\ge k\}}|f_n|,\end{aligned}$$

so that

$$A\int_{\{|u_n|\ge k+\varepsilon\}}\varphi_n^{\theta+1}|u_n|^{r-1} \le \int_{\{|u_n|\ge k\}}|f|.$$

Letting ε tend to zero, by Beppo Levi's theorem, we obtain

$$\int_{\{|u_n|\ge k\}}\varphi_n^{\theta+1}|u_n|^{r-1} \le \frac{1}{A}\int_{\{|u_n|\ge k\}}|f|.\tag{5.47}$$

Once again fix $\sigma > 0$ and let $E \subset \Omega$. By (5.47), we have

$$\int_E \varphi_n^{\theta+1}|u_n|^{r-1} = \int_{E\cap\{|u_n|\le k\}} \varphi_n^{\theta+1}|u_n|^{r-1} + \int_{E\cap\{|u_n|>k\}} \varphi_n^{\theta+1}|u_n|^{r-1}$$
$$\le k^{r-1}\int_E \varphi_n^{\theta+1} + \frac{1}{A}\int_{\{|u_n|\ge k\}} |f|.$$

As a consequence of (5.38) and applying Vitali's theorem, there exist $\tilde{k}$ and $\delta > 0$, with $|E| < \delta$, such that

$$\frac{1}{A}\int_{\{|u_n|\ge \tilde{k}\}} |f| \le \frac{\sigma}{2} \quad \text{and} \quad \tilde{k}^{r-1}\int_E \varphi_n^{\theta+1} \le \frac{\sigma}{2},$$

uniformly in n. Then we deduce

$$\int_E \varphi_n^{\theta+1}|u_n|^{r-1} \le \sigma, \tag{5.48}$$

uniformly in n. We recall that, by (5.38), $\varphi_n^{\theta+1}|u_n|^{r-1}$ converges to $\varphi^{\theta+1}|u|^{r-1}$ almost everywhere in Ω. Thanks to (5.48), applying Vitali's theorem, we obtain that

$$\varphi_n^{\theta+1}|u_n|^{r-1} \to \varphi^{\theta+1}|u|^{r-1} \text{ strongly in } L^1(\Omega). \tag{5.49}$$

We have that

$$-\text{div}(|\nabla u_n|^{p-2}\nabla u_n) = -A\varphi_n^{\theta+1}|u_n|^{r-2}u_n + f_n =: g_n,$$

and, by the assumptions on f and (5.49), that $\|g_n\|_{L^1(\Omega)} \le C$. Applying Theorem 2.1 in **[15]**, we obtain that ∇u_n converges to ∇u almost everywhere in Ω. Moreover

$$\||\nabla u_n|^{p-2}\nabla u_n\|_{(L^{p'}(\Omega))^N} \le \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \le C(f),$$

then

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \text{ weakly in } (L^{p'}(\Omega))^N. \tag{5.50}$$

By passing to the limit as n tends to infinity in (5.46), by (5.49) and (5.50), and applying Lebesgue's theorem, we deduce that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi + A\int_\Omega \varphi^{\theta+1}|u|^{r-2}u\psi = \int_\Omega f\psi, \quad \forall\psi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Proceeding as when we passed to the limit in (II), we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v + A\int_\Omega \varphi^{\theta+1}|u|^{r-2}uv = \int_\Omega fv, \quad \forall v \in W_0^{1,p}(\Omega).$$

Then u in $W_0^{1,p}(\Omega)$ is a weak solution of the first equation of (5.1) and (u,φ) is a weak solution of (5.1). ☐

REMARK 5.9. *We want to stress the fact that, in order to prove this theorem, we only used the results* (5.38) *obtained as consequence of the estimates in Lemma 5.7. Since the results* (5.25) *are analogous, proceeding in the same way we can prove, as said before, Theorem 5.3.*

Remark 5.10. *We observe that, thanks to the results in* **[17]**, *the second equation of* (5.1) *admits a weak solution in* $W_0^{1,p}(\Omega)$ *if* $|u|^r \in L^s(\Omega)$, *with* $s \geq \left(\frac{p^*}{\theta+1}\right)'$. *We recall that* u *belongs to* $L^t(\Omega)$, *with* $t = \frac{Nm(p-1)}{N-pm}$. *Then, if* $\frac{t}{r} < \left(\frac{p^*}{\theta+1}\right)'$, *we deduce once again a regularizing effect on* φ *due to the coupling in the system. We have that*

$$\frac{t}{r} < \left(\frac{p^*}{\theta+1}\right)' \Leftrightarrow m < m_2 := \frac{Npr}{N(p-1)^2 + p(p-1) + p^2 r - \theta(p-1)(N-p)}.$$

For this to be possible we must have that $r > p^* - 1 - \theta$. *We stress the fact that for* $\theta = 0$ *we recover the regularizing effect on* φ *observed in Remark 5.4.*

In this case ($\theta > 0$) we are not able to prove a regularizing effect on the existence of a finite energy solution for the first equation of (5.1). We feel that this is an obstacle only due to the method used, and that the following conjecture should be true.

Conjecture 5.11. *Let* $A > 0$, *and let* $r > 1$ *and* $0 \leq \theta < p-1$. *Then there exists* $1 < \underline{m} < (p^*)'$ *such that if* f *belongs to* $L^m(\Omega)$, *with* $m \geq \underline{m}$, *then there exists a weak solution* (u, φ) *in* $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of system* (5.1).

For instance if we assume that $|u_n| \leq c\,\varphi_n$ in Ω, for some $c > 0$, we are able to prove that this conjecture is true with $\underline{m} = (r+1+\theta)'$ and $r > p^* - 1 - \theta$. Indeed, if we consider the approximate problem (5.32), choosing u_n^+ as test function in (II), we obtain

$$\text{(5.51)} \qquad \int_\Omega |\nabla\varphi_n|^{p-2}\nabla\varphi_n \cdot \nabla u_n^+ = \int_\Omega |u_n|^r \varphi_n^\theta u_n^+ = \int_\Omega |u_n^+|^{r+1}\varphi_n^\theta \geq \frac{1}{c^\theta}\int_\Omega |u_n^+|^{r+1+\theta}.$$

So, by Young's inequality, using (5.34) and applying Hölder's inequality, we deduce from (5.51) that

$$\begin{aligned} \frac{1}{c^\theta}\int_\Omega |u_n^+|^{r+1+\theta} &\leq \int_\Omega |\nabla\varphi_n|^{p-2}\nabla\varphi_n \cdot \nabla u_n^+ \leq \frac{1}{p'}\int_\Omega |\nabla\varphi_n|^p + \frac{1}{p}\int_\Omega |\nabla u_n^+|^p \\ &\leq \frac{1}{p'}\int_\Omega |\nabla\varphi_n|^p + \frac{1}{p}\int_\Omega |\nabla u_n|^p \leq C\int_\Omega |f||u_n| \leq C\|f\|_{L^{(r+1+\theta)'}(\Omega)}\|u_n\|_{L^{r+1+\theta}(\Omega)}. \end{aligned}$$

Thus we have, once again, that

$$\|u_n\|_{W_0^{1,p}(\Omega)} + \|\varphi_n\|_{W_0^{1,p}(\Omega)} + \int_\Omega \varphi_n^{\theta+1}|u_n|^r + \int_\Omega |u_n|^{r+1}\varphi_n^\theta \leq C(f),$$

where $C(f)$ is a positive constant depending only on $\|f\|_{L^{(r+1+\theta)'}(\Omega)}$.
Thanks to these estimates it follows from Remark 5.9 that we can pass to the limit in (5.32). Hence we have proved our conjecture with $\underline{m} = (r+1+\theta)'$.
We note that for $\theta = 0$ we obtain $\underline{m} = (r+1)'$, that is, exactly, the result stated in Theorem 5.3.

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