

THE ACTION OF A COXETER ELEMENT ON AN AFFINE ROOT SYSTEM

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ABSTRACT. The characterization of orbits of roots under the action of a Coxeter element is a fundamental tool in the study of finite root systems and their reflection groups. This paper develops the analogous tool in the affine setting, adding detail and uniformity to a result of Dlab and Ringel.

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1. INTRODUCTION

Among the most important early results in the study of finite reflection groups is the description of orbits of roots (or reflecting hyperplanes) under the action of a Coxeter element—the product of a permutation of the simple reflections. This description was proved uniformly by Steinberg [30], who analyzed the action of Coxeter elements on the Coxeter plane—a certain plane first considered by Coxeter in [11]. (See also [25].) Steinberg’s uniform construction of the Coxeter plane was based on a careful analysis of eigenvalues and eigenvectors which led to (and was motivated by) a uniform proof of the formula nh for the number of roots, where n is the rank of the root system and h is its *Coxeter number* (the order of a Coxeter element).

We quote a version of this result that gives a *transversal* of the orbits (a collection consisting of exactly one element from each of the orbits). Let Φ be a root system of rank n , and let c be any Coxeter element in the associated Weyl group. We label the simple roots so that $c = s_1 \cdots s_n$ where s_i is the reflection with respect to the simple root α_i of Φ . Define

$$(1.1) \quad \vec{\Psi}^c := \{\alpha_1, s_1\alpha_2, \dots, s_1 \cdots s_{n-1}\alpha_n\},$$

$$(1.2) \quad \overleftarrow{\Psi}^c := \{\alpha_n, s_n\alpha_{n-1}, \dots, s_n \cdots s_2\alpha_1\},$$

and write Ψ^c for the union of the two sets. The following is [5, Proposition VI.1.33].

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Theorem 1.1. *Suppose Φ is an irreducible finite root system and c is a Coxeter element in the associated Weyl group W . There are exactly n c -orbits in Φ . The set $\vec{\Psi}^c$ is a transversal of these orbits. The set $\overleftarrow{\Psi}^c$ is also a transversal of these orbits. Each orbit has cardinality equal to the Coxeter number h .*

Dlab and Ringel [12] proved an analogous result for the action of a Coxeter element c on an affine root system Φ . Here, we improve on [12] by arguing uniformly (rather than type-by-type in the classification of affine root systems) and by clarifying some details for finite orbits. Our immediate motivation is to support an almost-positive roots model [28] for cluster algebras of affine type. (The finite-type model [9, 13, 14, 22, 32] uses Theorem 1.1 by way of [5, Exercise V§6.2].)

The arguments in [12] for infinite orbits are easy and uniform, and ours are the same. The difficulty lies in the treatment of finite orbits. Just as Steinberg's uniform proof of Theorem 1.1 rests on an analysis of eigenvalues and eigenvectors, our analysis of finite orbits relies on a characterization, in Proposition 3.1, of the eigenvalues and eigenvectors of a Coxeter element in a Weyl group of affine type. (Previous work on the affine eigenvalue problem, with different emphases, includes [1, 3, 10, 16, 31].) We will see that c has a linearly independent set of $(n-1)$ eigenvectors, spanning a hyperplane that we call U^c . We define Υ^c to be $\Phi \cap U^c$. Writing Φ_{fin} for the finite root system associated to Φ , we write Υ_{fin}^c for $\Phi_{\text{fin}} \cap U^c$. We will see in Proposition 4.4 that Υ_{fin}^c is a root system of rank $n-2$. There is canonical system Ξ_{fin}^c of simple roots for Υ_{fin}^c (the unique simple system for Υ_{fin}^c that is a subset of Φ^+).

Let δ be the positive imaginary root closest to the origin. Let $\text{aff} \in \{1, \dots, n\}$ be the index such that α_{aff} is the unique simple root of Φ that is not in Φ_{fin} . To keep track of the location of the letter s_{aff} in the expression $s_1 \cdots s_n$ for c , we let $c_{\triangleleft} = s_1 \cdots s_{\text{aff}-1}$ and let $c_{\triangleright} = s_{\text{aff}+1} \cdots s_n$, so that $c = c_{\triangleleft} s_{\text{aff}} c_{\triangleright}$. We will see in Proposition 4.4 that the roots in Ξ_{fin}^c can be ordered $\beta_1, \dots, \beta_{n-2}$ so that $t_{\beta_1} \cdots t_{\beta_{n-2}} = c_{\triangleleft} t_{\theta} c_{\triangleright}$, where θ is the highest root or highest short root in Φ_{fin}^+ as explained in Section 2 and t_{α} stands for the reflection orthogonal to a root α . For such an ordering of Ξ_{fin}^c , define $\Omega^c := \{\beta_1, t_{\beta_1} \beta_2, \dots, t_{\beta_1} \cdots t_{\beta_{n-3}} \beta_{n-2}\}$. For any $\beta \in \Omega^c$, let $\kappa(\beta)$ be the smallest positive integer such that $\kappa(\beta)\delta - \beta$ is a root. (The existence of $\kappa(\beta)$ will be established as part of the proof of the main theorem, where we will also see that $\kappa(\beta)$ is constant on components of Υ_{fin}^c .)

We now state the result on c -orbits in affine type.

Theorem 1.2 (Cf. Dlab-Ringel [12, Chapter 1]). *Suppose Φ is an affine root system and c is a Coxeter element in the associated Weyl group W .*

- (1) *There are exactly $2n$ infinite c -orbits in Φ . The set Ψ^c is a transversal of these orbits.*
- (2) *The c -orbit of a root $\beta \in \Phi$ is finite if and only if $\beta \in U^c$.*
- (3) *Every imaginary root is fixed by c .*
- (4) *For Φ of rank 2, there are no finite c -orbits of real roots. For larger rank, there are infinitely many finite c -orbits of real roots and the set $\{\beta + m \cdot \kappa(\beta)\delta : \beta \in \Omega^c, m \in \mathbb{Z}\}$ is a transversal of them.*
- (5) *Each finite c -orbit contains either only positive roots or only negative roots. In particular, the c -orbit of a real root $\beta + m \cdot \kappa(\beta)\delta$ for $\beta \in \Omega^c$ consists of positive roots if and only if $m \geq 0$.*
- (6) *A finite c -orbit intersects Φ_{fin}^+ if and only if it intersects Ω^c .*

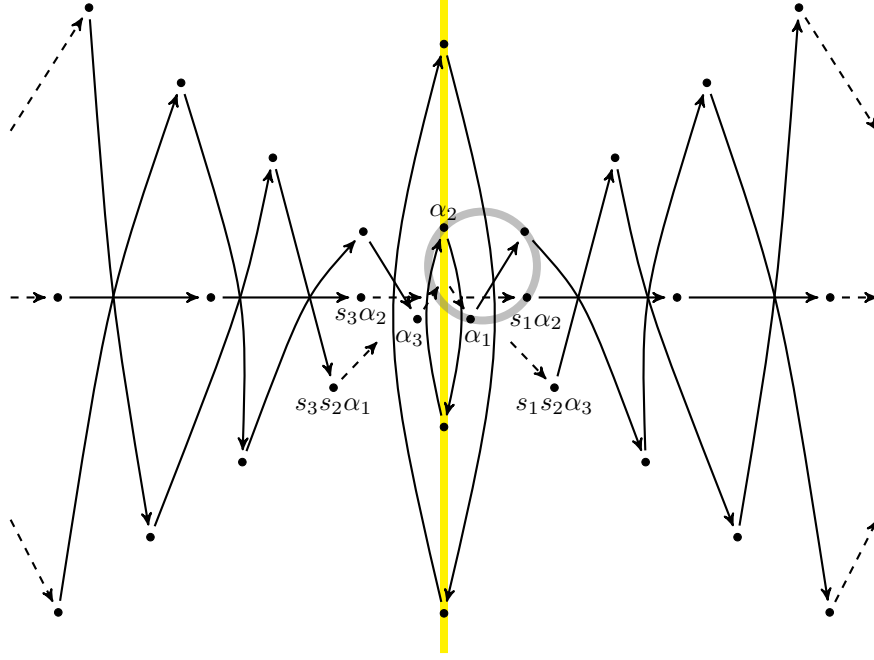


FIGURE 1. The action of c on positive roots, for Φ and c as in Example 1.3

Example 1.3. Consider the Cartan matrix $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$ corresponding to the Dynkin diagram $\begin{array}{c} \bullet \\ \leftarrow \bullet \rightleftarrows \bullet \rightarrow \\ \bullet \end{array}$ of type $D_3^{(2)}$. (The associated root system is not a standard affine root system in the sense of Example 1.4.) Let $c = s_1 s_2 s_3$. In the basis of simple roots $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, c is given by the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}$. Its eigenvectors are $\alpha_1 + \alpha_3$ with eigenvalue -1 , and $\delta = \alpha_1 + \alpha_2 + \alpha_3$ with eigenvalue 1 . (Looking forward to Proposition 3.1, the generalized 1-eigenvector γ_c is $\alpha_1 + \frac{1}{2}\alpha_2$.)

There are 6 infinite c -orbits and a transversal of them is $\Psi^c = \overleftarrow{\Psi}^c \cup \overrightarrow{\Psi}^c$ with

$$\overleftarrow{\Psi}^c = \{s_3 s_2 \alpha_1 = \alpha_1 + \alpha_2 + 2\alpha_3, \quad s_3 \alpha_2 = \alpha_2 + 2\alpha_3, \quad \alpha_3\}$$

$$\overrightarrow{\Psi}^c = \{\alpha_1, \quad s_1 \alpha_2 = 2\alpha_1 + \alpha_2, \quad s_1 s_2 \alpha_3 = 2\alpha_1 + \alpha_2 + \alpha_3\}.$$

We have $\Omega^c = \{\alpha_2\}$, and the c -orbit of α_2 is $\{\alpha_2, 2\alpha_1 + \alpha_2 + 2\alpha_3\}$. Since $\delta - \alpha_2 = \alpha_1 + \alpha_3$ is not a root but $2\delta - \alpha_2 = 2\alpha_1 + \alpha_2 + 2\alpha_3$ is a root, we have $\kappa(\alpha_2) = 2$. The finite orbits are the 2δ -translates of this orbit.

We depict parts of some orbits in Fig. 1. The figure is obtained by drawing a ray from the origin through each root, taking the intersection with a unit sphere centered at 0 and stereographically projecting the result from the direction δ . We plot only positive roots of small height. Negative roots would be concentrated in the center of the picture, while positive roots of greater height would be further out towards the edge of the figure. The imaginary root δ is the point at infinity. The yellow line indicates $U^c = \text{Span}\{\alpha_1 + \alpha_3, \delta\}$, and the gray circle indicates $\text{Span}\{\Phi_{\text{fin}}\}$. Arrows give the action of c on roots. Arrows to or from roots that do not appear in the figure are shown dotted.

Example 1.4. Table 1 gives the data necessary to describe finite orbits for the standard affine root systems and a particular choice of Coxeter element. (These are the root systems that are obtained from finite root systems by the usual construction. See Section 2.) We name the types as in [18, §4.8]. The choice of Coxeter element $c = s_1 \cdots s_n$ is given by the labeling of nodes in the second column. In every case α_{aff} is α_n so that $c = c_{\triangleleft} s_{\text{aff}}$, and we draw the affine node in a different color. We also label the nodes in the diagram of Υ_{fin}^c according to Proposition 4.4, i.e. so that $c_{\triangleleft} t_{\theta} = s_1 \cdots s_{n-1} t_{\theta} = t_{\beta_1} \cdots t_{\beta_{n-2}}$. In each case, the integer $\kappa(\beta)$ is 1 for all $\beta \in \Omega^c$.

Type	Diagram of Φ	Diagram of Υ_{fin}^c	Simple roots of Υ_{fin}^c
$A_1^{(1)}$			
$A_{n-1}^{(1)}$ ($n \geq 3$) $k \neq n$			$\beta_j = \alpha_{j+1}$
$B_{n-1}^{(1)}$ ($n \geq 4$)			$\beta_{n-2} = \sum_{i=1}^{n-1} \alpha_i$ $\beta_j = \alpha_{j+1}$
$C_{n-1}^{(1)}$ ($n \geq 3$)			$\beta_j = \alpha_{j+1}$
$D_{n-1}^{(1)}$ ($n \geq 5$)			$\beta_{n-3} = \alpha_1 + \sum_{i=3}^{n-1} \alpha_i$ $\beta_{n-2} = \alpha_2 + \sum_{i=3}^{n-1} \alpha_i$ $\beta_j = \alpha_{j+2}$
$E_6^{(1)}$			$\beta_1 = \alpha_4 + \alpha_5$ $\beta_2 = \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6$ $\beta_3 = \alpha_2 + \alpha_5$ $\beta_4 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ $\beta_5 = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$
$E_7^{(1)}$			$\beta_1 = \alpha_4 + \alpha_5$ $\beta_2 = \alpha_1 + \alpha_5 + \alpha_6$ $\beta_3 = \sum_{i=2}^7 \alpha_i$ $\beta_4 = \sum_{i=3}^6 \alpha_i$ $\beta_5 = \alpha_1 + \sum_{i=4}^7 \alpha_i$ $\beta_6 = \alpha_1 + \alpha_5 + \sum_{i=3}^7 \alpha_i$
$E_8^{(1)}$			$\beta_1 = \alpha_3 + \alpha_4 + \alpha_5$ $\beta_2 = \alpha_1 + \sum_{i=4}^6 \alpha_i$ $\beta_3 = \sum_{i=2}^7 \alpha_i$ $\beta_4 = \alpha_1 + \sum_{i=3}^8 \alpha_i$ $\beta_5 = \alpha_1 + \alpha_4 + \sum_{i=3}^7 \alpha_i$ $\beta_6 = \alpha_4 + \alpha_5 + \sum_{i=1}^8 \alpha_i$ $\beta_7 = \alpha_4 + \sum_{i=3}^6 \alpha_i + \sum_{i=1}^8 \alpha_i$
$F_4^{(1)}$			$\beta_1 = \alpha_2 + \alpha_3$ $\beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ $\beta_3 = 2\alpha_2 + \alpha_3 + \alpha_4$
$G_2^{(1)}$			$\beta_1 = \alpha_1 + \alpha_2$

TABLE 1. Standard affine root systems and their finite orbits

Remark 1.5. Details about c -orbits and Υ_{fin}^c in nonstandard affine root systems can be obtained from the information in Table 1 by rescaling. See Section 2. Furthermore, every Coxeter element in an affine Weyl group can be obtained from a Coxeter element described in Table 1 via source-sink moves (Proposition 2.6). Each source-sink move conjugates the Coxeter element by a simple reflection. Thus, complete details on orbits of any affine root system under the action of any Coxeter element can be recovered from the details in Table 1 by conjugation and rescaling.

Remark 1.6. The sets $\overrightarrow{\Psi}^c$ and $\overleftarrow{\Psi}^c$ depend only on c , not on a choice of reduced word for c . Each contains n distinct positive roots. Theorem 1.2, Ψ^c implies that $\overrightarrow{\Psi}^c$ and $\overleftarrow{\Psi}^c$ are disjoint when Φ is affine.

Remark 1.7. The set Υ^c is, in a certain sense, a “root subsystem” of Φ , but illustrates the strange rank behavior that can occur when one considers the intersections of infinite root systems with subspaces. (See [26, Remark 2.13].) The set Υ^c can be obtained from Υ_{fin}^c by following the usual affinization procedure that adjoins δ . However, this procedure, when applied to reducible finite root systems, gives the same result as extending each irreducible component using a new vector δ_i and then identifying all of the δ_i with δ . The difference in rank between the original root system and the extended root system is (in effect) the number of irreducible components, even though only one extra dimension has been added. Looking backward to Table 1, we see that the rank of Υ^c can vary from $n-2$ to $n+1$ because Υ_{fin}^c has between 0 and 3 connected components.

Remark 1.8. We will see in Proposition 4.6 that the sizes of finite orbits of real roots are the ranks of irreducible components of Υ^c , which are one greater than the ranks of components of Υ_{fin}^c . Table 1 reveals that Υ_{fin}^c has at most 3 irreducible components, so there are at most three different sizes of finite orbits of real roots.

Remark 1.9. The difference between Theorem 1.2 and the Dlab-Ringel result (as it can be pieced together from [12, Proposition 1.9] and the preceding and following remarks) is in Theorem 1.2(4). In place of $\kappa(\beta)$, Dlab-Ringel uses the *tier number* r (the parenthesized superscript in the notation of [18, Chapter 4]). In every case, $\kappa(\beta) \leq r$, but this inequality may be strict. When $\kappa(\beta) < r$, the Dlab-Ringel formulation must replace Ω^c by a larger set. When $r = 1$ (the standard types discussed in Example 1.4), we must have $\kappa(\beta) = r = 1$. One can check that $\kappa(\beta) = r$ in types $D_n^{(2)}$ and $D_4^{(3)}$ and that $\kappa(\beta) = 1 < r$ in type $A_{2n-2}^{(2)}$. The remaining types are more complicated: In type $A_{2n-3}^{(2)}$, we have $\kappa(\beta) = 2$ for β in the rank-1 component of Υ_{fin}^c and $\kappa(\beta) = 1$ in the other component. In type $E_6^{(2)}$, $\beta(\kappa) = 1$ in the rank-1 component and $\beta(\kappa) = 2$ in the rank-2 component.

Remark 1.10. Several of the ideas of this paper can also be found in [23]. Some translation is necessary, since [23] works with affine Coxeter groups as groups of Euclidean motions rather than as Weyl groups of affine Kac-Moody root systems. For example, Υ_{fin}^c appears in [23] as the “horizontal root system.” See [23, Definition 6.1] and compare the third column of Table 1 with [23, Table 1].

2. ROOT SYSTEMS AND COXETER GROUPS

A *symmetrizable (generalized) Cartan matrix* is a square integer matrix $A = [a_{ij}]_{1 \leq i, j \leq n}$ with diagonal entries 2 and nonpositive off-diagonal entries, such that there exist positive real numbers d_i with $d_i a_{ij} = d_j a_{ji}$ for all i, j .

Let V be a complex vector space with basis $\Pi = \{\alpha_1, \dots, \alpha_n\}$. We only care about the real span of Π , except when we consider eigenvectors, so it is safe to think of V as a real vector space, passing to the complexification when necessary. The elements of Π are the **simple roots**. Define the **simple co-roots** to be $\alpha_i^\vee = d_i^{-1}\alpha_i$.

Let K be the symmetric bilinear form on V defined by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. For each $i = 1, \dots, n$, the **simple reflection** s_i is the linear map given on the basis of simple roots by $s_i(\alpha_j) = \alpha_j - K(\alpha_i^\vee, \alpha_j)\alpha_i$. On the basis of simple co-roots s_i acts as $s_i(\alpha_j^\vee) = \alpha_j^\vee - K(\alpha_j^\vee, \alpha_i)\alpha_i^\vee$. The group W generated by $S = \{s_i : i = 1, \dots, n\}$ is called the **Weyl group**. Each element of W is a symmetry of K .

The **real root system** Φ^{re} is the set of vectors $w\alpha_i$ for $w \in W$ and $i = 1, \dots, n$ (called **real roots**). There is a larger set $\Phi \supseteq \Phi^{\text{re}}$, called the **root system**, which is strictly larger than Φ^{re} if and only if Φ^{re} is infinite. We describe the **imaginary roots** (the elements of $\Phi \setminus \Phi^{\text{re}}$) below in the case where A is of affine type.

Each root in Φ is either **positive** (in the nonnegative linear span of Π) or **negative** (in the nonpositive linear span of Π). Each real root β has an associated co-root $\beta^\vee = \frac{2}{K(\beta, \beta)}\beta$ and defines a **reflection** t_β on V given by $t_\beta x = x - K(\beta^\vee, x)\beta$ for every $x \in V$. Every reflection in W is t_β for a unique positive real root β . The notation $[\beta : \alpha_i]$ means the α_i -coefficient of β in the basis of simple roots.

A Cartan matrix A is called **reducible** if it can be reindexed to have a nontrivial block-diagonal decomposition, in which case each diagonal block is a Cartan matrix (a component of A). Otherwise, it is **irreducible**. The root system Φ and Weyl group W associated to A are accordingly reducible or irreducible.

A **Coxeter element** c is the product of any permutation of S . It is possible for different permutations of S to have the same product c . An element $s \in S$ is **initial in** c if there exists a permutation of S whose product is c and whose first entry is s . Similarly, s is **final in** c if c is the product of a permutation ending in s . When s is initial or final in c , the element scs is also a Coxeter element for W . The operation of passing from c to scs is called a **source-sink move**. As in the introduction, we assume that A has been indexed so that $c = s_1 \dots s_n$.

We use c to define a skew-symmetric bilinear form on V by

$$(2.1) \quad \omega_c(\alpha_i^\vee, \alpha_j) = \begin{cases} a_{ij} & \text{if } i > j, \\ 0 & \text{if } i = j, \text{ or} \\ -a_{ij} & \text{if } i < j. \end{cases}$$

The following lemma is [26, Lemma 3.8].

Lemma 2.1. *If s is initial or final in c , then $\omega_c(\alpha, \beta) = \omega_{scs}(s\alpha, s\beta)$ for all α and β in V .*

If $w \in W$ is the product $r_1 \dots r_k$ with each $r_i \in S$, then the expression $r_1 \dots r_k$ is called **reduced** if every other expression $w = p_1 \dots p_\ell$ with each $p_i \in S$ has $\ell \geq k$. An expression $r_1 \dots r_k$ is reduced if and only if its **left reflections** $t_i = r_1 \dots r_i \dots r_1$ are all distinct. When $r_1 \dots r_k$ is reduced, the set of left reflections depends only on w and is called the set of **inversions** of w . Furthermore, writing $\gamma_i \in \Pi$ for the simple root associated to r_i , the expression $r_1 \dots r_k$ is reduced if and only if the roots $r_1 \dots r_{i-1}\gamma_i$ are distinct and are all positive. When $r_1 \dots r_k$ is reduced, these are exactly the positive roots associated to the inversions of w .

The following result is due to [19] in broad generality and to [29] in full generality. (See also [15, 20].) Write $(s_1 \dots s_n)^k$ for the k -fold concatenation of $s_1 \dots s_n$ with itself.

Theorem 2.2. *If W is infinite and irreducible, then the word $(s_1 \cdots s_n)^k$ is reduced for all $k \geq 1$.*

The Weyl group W is finite if and only if K is positive definite. We gather some results about the finite case. The next proposition follows from [17, Lemma 3.16].

Proposition 2.3. *If W is finite and c is a Coxeter element of W , then the fixed space of the action of c on V is $\{0\}$.*

The following is a result of Brady and Watt [6, 7].

Proposition 2.4. *Suppose W is a finite Coxeter group, w is an element of W and $t_1 \cdots t_k$ is an expression for w as a product of reflections, minimizing k among all such expressions. Then $\{\gamma_1, \dots, \gamma_k\}$ is a basis for the orthogonal complement of the fixed space of the action of w on V , where γ_i is the positive root such that $t_i = t_{\gamma_i}$.*

Proposition 2.4 can be obtained by combining [24, Theorem 2.2(i)] (another restatement of results of [6, 7]), which says that the fixed space of w is $\bigcap_{i=1}^k \gamma_i^\perp$, with Carter's Lemma, which states that $\{\gamma_1, \dots, \gamma_k\}$ is linearly independent. (See [8] and also [2, Lemma 2.4.5].) The orthogonal complement the proposition refers to is with respect to the Euclidean form preserved by W .

A (**standard**) **parabolic subgroup** of a Coxeter group W is a subgroup generated by some subset of S . The following fact is one direction of [4, Lemma 1.4.3].

Lemma 2.5. *Suppose c is a Coxeter element in a finite Coxeter group W of rank n . If $t_1 \cdots t_n$ is an expression for c as a product of reflections in W , then for each i , there exists a standard parabolic subgroup W' of W of rank i such that $t_1 \cdots t_i$ is conjugate in W to a Coxeter element of W' .*

We say that A , Φ and W are of **affine type** if K is positive semidefinite and not positive definite and if the restriction of K to $\text{Span}\{\alpha_i : i \in J\}$ is positive definite for all $J \not\subseteq [1, n]$. In this case, A , W and Φ are in particular irreducible. Background on affine root systems can be found in [18, 21]. We continue to let n be the rank of Φ , even when Φ is affine (despite a common convention where affine root systems have rank $n + 1$).

If A is of affine type, then there exists $\text{aff} \in \{1, \dots, n\}$ such that, writing W_{fin} for the subgroup of W generated by $S \setminus \{s_{\text{aff}}\}$, the group W is isomorphic to a semidirect product of W_{fin} with the lattice generated by $\{\alpha_i^\vee : i \neq \text{aff}\}$. The choice of aff may not be unique, but we fix a choice.

We write V_{fin} for the subspace of V spanned by $\Pi \setminus \{\alpha_{\text{aff}}\}$. We write Φ_{fin} for $\Phi \cap V_{\text{fin}}$. This is an indecomposable finite root system.

Some affine root systems arise from indecomposable finite root systems through a standard construction (e.g. [21, Proposition 2.1]). These are the **standard** affine root system, shown in Table Aff 1 of [18, Chapter 4] and in Table 1 of the present paper. Every affine root system Φ is a **rescaling** of a unique standard affine root system Φ' . This means that every root of Φ is a positive scaling of a root in Φ' and that the bilinear form K associated to Φ and Φ' coincide. In particular, both root systems define the same Weyl group, and the scaling factors relating roots are constant on W -orbits of roots.

When Φ is of affine type, the kernel of K is one-dimensional. The intersection of Φ with this kernel is the set of imaginary roots and is of the form $\{x\delta : x \in \mathbb{Z} \setminus \{0\}\}$, where δ is a positive imaginary root. Because δ is in the kernel of K , it is fixed by every element of W .

Every real root in Φ is a positive scaling of $\beta + k\delta$ for some $\beta \in \Phi_{\text{fin}}$ and $k \in \mathbb{Z}$. In some (nonstandard) affine root systems and for some roots $\beta \in \Phi_{\text{fin}}$ and integers k , $\beta + k\delta$ is not a positive scaling of a root in Φ . In the standard affine root systems, every $\beta + k\delta$ is a root. A root that is a positive rescaling of $\beta + k\delta$ is a positive root if and only if either k is positive or $k = 0$ and β is positive. See [18, Proposition 6.3].

The expansion of the imaginary root δ in the basis of simple roots has strictly positive coordinates. In particular, $[\delta : \alpha_{\text{aff}}]$ is positive. (To be precise $[\delta : \alpha_{\text{aff}}] = 1$ in all affine types except for $A_{2k}^{(2)}$ where it is 2.) The vector $\theta = \delta - [\delta : \alpha_{\text{aff}}]\alpha_{\text{aff}}$ is a positive root in Φ_{fin} . (Usually, including in the standard affine root systems, θ is the highest root of Φ_{fin} , but in some affine root systems, it is the highest short root [18, Proposition 6.4].) In any case, α_{aff} is a positive scaling of $\delta - \theta$. Since $K(\delta, x) = 0$ for any $x \in V$, the action of s_{aff} on x is:

$$(2.2) \quad s_{\text{aff}}x = t_{\theta}x + K(\theta^{\vee}, x)\delta.$$

To conclude the section, we quote a result on conjugacy classes of affine Coxeter elements. In [30, 4.1], it is shown that all Coxeter elements in a finite Coxeter group are conjugate, via source-sink moves. The argument given there applies to all affine types except for $A_{n-1}^{(1)}$, where there are multiple conjugacy classes. The classification of conjugacy classes in type $A_{n-1}^{(1)}$ is well known, and we quote it as part of the following proposition.

Proposition 2.6. *If W is an affine Weyl group not of type $A_{n-1}^{(1)}$, then any two Coxeter elements of W are conjugate in W . If W is of type $A_{n-1}^{(1)}$, then there is one conjugacy class for each $k \in [1, n-1]$, represented by the Coxeter element $s_1 \cdots s_n$ with $(s_1 s_{k+1})^3 = (s_k s_n)^3 = 1$ and $(s_i s_{i+1})^3 = 1$ for $i \neq k, n$. The conjugations can be carried out by a sequence of source-sink moves.*

3. EIGENVALUES AND EIGENVECTORS OF AFFINE COXETER ELEMENTS

Let $c = s_1 \cdots s_n$ be a Coxeter element. As in the introduction, let $c_{\triangleleft} = s_1 \cdots s_{\text{aff}-1}$ and let $c_{\triangleright} = s_{\text{aff}+1} \cdots s_n$. The elements c_{\triangleleft} and c_{\triangleright} depend on how c is expressed as the product of a permutation of S (for example if $s_{\text{aff}-1}$ commutes with s_{aff}) but the statements and arguments are correct regardless of the choice. We write $c = c_{\triangleleft} s_{\text{aff}} c_{\triangleright}$ to refer to this definition of c_{\triangleleft} and c_{\triangleright} .

Proposition 3.1. *Let $c = c_{\triangleleft} s_{\text{aff}} c_{\triangleright}$ be a Coxeter element in an affine Weyl group.*

- (1) *c has eigenvalue 1 with algebraic multiplicity 2 and geometric multiplicity 1. The imaginary root δ is a 1-eigenvector.*
- (2) *There exists a unique generalized 1-eigenvector γ_c contained in the subspace V_{fin} of V . (This means that $(c-1)\gamma_c = \delta$.) The vector γ_c is the unique vector in V_{fin} fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ and having $K(\theta^{\vee}, c_{\triangleright} \gamma_c) = 1$.*
- (3) *There is a linearly independent set of $n-2$ eigenvectors of c with eigenvalues λ roots of unity but $\lambda \neq 1$.*
- (4) *The direct sum U^c of all eigenspaces of c is $\{v \in V : K(\gamma_c, v) = 0\}$.*
- (5) *For $\lambda \neq 1$, a vector $v \in V_{\text{fin}}$ is a λ -eigenvector of $c_{\triangleleft} t_{\theta} c_{\triangleright}$ if and only if $v + \frac{K(\theta^{\vee}, c_{\triangleright} v)}{(\lambda-1)} \delta$ is a λ -eigenvector of c .*

Proof. Applying c_{\triangleleft} to the left of both sides of (2.2), keeping in mind that δ is fixed by the action of W , and setting $x = c_{\triangleright} v$ for $v \in V_{\text{fin}}$, we obtain

$$(3.1) \quad cv = c_{\triangleleft} t_{\theta} c_{\triangleright} v + K(\theta^{\vee}, c_{\triangleright} v)\delta$$

Thus $c_{\triangleleft} t_{\theta} c_{\triangleright} v = \lambda v$ if and only if $cv - K(\theta^{\vee}, c_{\triangleright} v)\delta = \lambda v$. Adding the vector $\frac{\lambda}{\lambda-1}K(\theta^{\vee}, c_{\triangleright} v)\delta$ to both sides of the latter and again keeping in mind that δ is fixed by the action of W , we obtain $c(v + \frac{K(\theta^{\vee}, c_{\triangleright} v)}{(\lambda-1)}\delta) = \lambda(v + \frac{K(\theta^{\vee}, c_{\triangleright} v)}{(\lambda-1)}\delta)$. We have established (5).

We next check that the fixed space of $c_{\triangleleft} t_{\theta} c_{\triangleright}$ in V_{fin} is at least one-dimensional. Since $c_{\triangleleft} t_{\theta} c_{\triangleright}$ is in the finite Coxeter group W_{fin} , and since K restricts to a Euclidean form on V_{fin} we can use Proposition 2.4. Therefore to establish the claim (since the rank of W_{fin} is $n-1$) it suffices to check that a shortest expression for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ as a product of reflections in W_{fin} has length at most $n-2$. The same proposition implies that a shortest expression for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ has length at most $n-1$. But since the determinant of a reflection is -1 , every expression for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ as product of reflections has the same length modulo 2. The expression $c_{\triangleleft} t_{\theta} c_{\triangleright}$ is a product of n reflections in W_{fin} ($n-1$ simple reflections and t_{θ}) and we conclude that a shortest expression for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ has length at most $n-2$ as desired.

Given a vector $v \in V_{\text{fin}}$, we compute $c_{\triangleleft} t_{\theta} c_{\triangleright} v = c_{\triangleleft}(c_{\triangleright} v - K(\theta^{\vee}, c_{\triangleright} v)\theta)$, so v is fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ if and only if $c_{\triangleleft}(c_{\triangleright} v - K(\theta^{\vee}, c_{\triangleright} v)\theta) = v$ if and only if $(c_{\triangleleft} c_{\triangleright} - 1)v = K(\theta^{\vee}, c_{\triangleright} v)c_{\triangleleft}\theta$. But $c_{\triangleleft} c_{\triangleright}$ is a Coxeter element of W_{fin} and thus Proposition 2.3 says that the fixed space of $c_{\triangleleft} c_{\triangleright}$ on V_{fin} is $\{0\}$. Thus $(c_{\triangleleft} c_{\triangleright} - 1)$ is invertible on V_{fin} , so v is fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ if and only if $v = K(\theta^{\vee}, c_{\triangleright} v)(c_{\triangleleft} c_{\triangleright} - 1)^{-1}c_{\triangleleft}\theta$. In particular, the fixed space of $c_{\triangleleft} t_{\theta} c_{\triangleright}$ is contained in $\mathbb{R}(c_{\triangleleft} c_{\triangleright} - 1)^{-1}c_{\triangleleft}\theta$, but since the fixed space is at least one dimensional, it is exactly the line $\mathbb{R}(c_{\triangleleft} c_{\triangleright} - 1)^{-1}c_{\triangleleft}\theta$. Furthermore, each nonzero vector v fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ has $K(\theta^{\vee}, c_{\triangleright} v) \neq 0$.

The element $c_{\triangleleft} t_{\theta} c_{\triangleright}$ is an orthogonal transformation of the Euclidean vector space V_{fin} , so it has eigenvalues which are roots of unity and also has a basis of eigenvectors. We have seen that only one of these eigenvectors has eigenvalue 1. Since δ is not in V_{fin} and in light of (5), the map $v \mapsto v + \frac{K(\theta^{\vee}, c_{\triangleright} v)}{(\lambda-1)}\delta$ takes the $n-2$ non-fixed eigenvectors in a basis of eigenvectors of $c_{\triangleleft} t_{\theta} c_{\triangleright}$ to a linearly independent set of eigenvectors of c . We have proved (3).

Furthermore, we see that c has the eigenvalue 1 with algebraic multiplicity 2. The vector δ is a 1-eigenvector, because it is fixed by the action of c . A nonzero vector $v \in V_{\text{fin}}$ fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ has $K(\theta^{\vee}, c_{\triangleright} v) \neq 0$, so in particular, there indeed exists a unique vector $\gamma_c \in V_{\text{fin}}$ fixed by $c_{\triangleleft} t_{\theta} c_{\triangleright}$ and having $K(\theta^{\vee}, c_{\triangleright} \gamma_c) = 1$. By (2.2) and because δ is fixed by W , we compute $(c-1)\gamma_c = c_{\triangleleft} t_{\theta} c_{\triangleright} \gamma_c + K(\theta^{\vee}, c_{\triangleright} \gamma_c)\delta - \gamma_c = \delta$. We conclude that the eigenvalue 1 has geometric multiplicity 1 and that γ_c is a generalized 1-eigenvector. We have proved (1). A generalized 1-eigenvector is unique up to adding a multiple of the 1-eigenvector δ , and (2) follows.

By (1) and (3), c has a linearly independent set of $n-1$ eigenvectors. Thus to prove (4), it is enough to show that if v is an eigenvector of c , then $K(\gamma_c, v) = 0$. If v has eigenvalue λ , then $K(\gamma_c, v) = K(c\gamma_c, cv) = K(\gamma_c + \delta, \lambda v) = \lambda K(\gamma_c, v)$. If $\lambda \neq 1$, then we conclude that $K(\gamma_c, v) = 0$. If $\lambda = 1$, then v is a multiple of δ , so again $K(\gamma_c, v) = 0$. \square

We now describe how γ_c transforms under source-sink moves.

Proposition 3.2. *Let Φ be an affine root system and let c be a Coxeter element. If s is initial or final in c , then*

$$\gamma_{scs} = \begin{cases} s\gamma_c & \text{if } s \neq s_{\text{aff}} \\ t_{\theta}\gamma_c & \text{if } s = s_{\text{aff}} \end{cases}$$

Proof. By Proposition 3.1(2), γ_{scs} is the unique generalized 1-eigenvector of scs contained in V_{fin} . Since γ_c is a generalized 1-eigenvector of c and δ is fixed by s , we have $(scs - 1)(s\gamma_c) = s(c - 1)\gamma_c = s\delta = \delta$, so $s\gamma_c$ is a generalized 1-eigenvector for scs . Now γ_c is in V_{fin} , so $s\gamma_c$ is in V_{fin} if $s \neq s_{\text{aff}}$. If $s = s_{\text{aff}}$, then (2.2) says that $t_\theta\gamma_c$ differs from $s\gamma_c$ by a multiple of δ , so $t_\theta\gamma_c$ is also a generalized 1-eigenvector for scs . Also, $t_\theta\gamma_c$ is in V_{fin} . \square

Combining this result with (2.2) and Proposition 3.1(4) we get the following.

Corollary 3.3. *If s is initial or final in c , then $U^{scs} = sU^c$.*

Continuing on the topic of eigenvectors of c , we identify and characterize the 1-eigenspace of the action of c on the space dual to V . We write V^* for the dual space to V and $\langle \cdot, \cdot \rangle$ for the canonical pairing between V^* and V . As usual, the action of W on V^* , dual to its action on V , is given by $\langle w\phi, v \rangle := \langle \phi, w^{-1}v \rangle$. Denote by ϕ_c the element of V^* defined by $\langle \phi_c, v \rangle = K(\gamma_c, v)$ for all $v \in V$.

Lemma 3.4. *The fixed space of the action of c on V^* is $\mathbb{R}\phi_c$.*

Proof. We make a straightforward computation, using the definition of ϕ_c , the fact that K is invariant under the action of W , Proposition 3.1(2), and the fact that $K(\delta, v) = 0$ for all $v \in V$. Specifically, for v in V , we calculate

$$\langle c\phi_c, v \rangle = \langle \phi_c, c^{-1}v \rangle = K(\gamma_c, c^{-1}v) = K(c\gamma_c, v) = K(\delta + \gamma_c, v) = K(\gamma_c, v) = \langle \phi_c, v \rangle.$$

The fixed space of c acting on V^* is a line because the fixed space of c on V is a line. Proposition 3.1(4) implies that $\phi_c \neq 0$, so it spans the fixed space. \square

We now relate ϕ_c to a vector in V^* that plays a key role in [27]. Following [27, Section 4.4], we write x_c for the linear functional $\omega_c(\delta, \cdot)$ in V^* .

Lemma 3.5. *The vector x_c is fixed by c and is a negative scaling of ϕ_c .*

Proof. Using first the definition of x_c , then Lemma 2.1 (n times), then the fact that δ is fixed by the action of W , for any vector v in V , we calculate

$$\langle cx_c, v \rangle = \langle x_c, c^{-1}v \rangle = \omega_c(\delta, c^{-1}v) = \omega_c(c\delta, v) = \omega_c(\delta, v) = \langle x_c, v \rangle.$$

Thus x_c is fixed by the action of c on V^* . By Lemma 3.4, x_c and ϕ_c agree up to scaling. To determine the scale factor, we again write $c = c_{\triangleleft} s_{\text{aff}} c_{\triangleright}$ and recall from the definition of γ_c that $K(\theta^\vee, c_{\triangleright}\gamma_c) = 1$. Using the invariance of K , we have $K(c_{\triangleright}^{-1}\theta^\vee, \gamma_c) = 1$ or in other words $\langle \phi_c, c_{\triangleright}^{-1}\theta^\vee \rangle = 1$. Again using Lemma 2.1 several times, then the W -invariance of δ , then the fact that ω_c is skew-symmetric, we calculate $\langle x_c, c_{\triangleright}^{-1}\theta^\vee \rangle$ to be

$$\omega_c(\delta, c_{\triangleright}^{-1}\theta^\vee) = \omega_{c_{\triangleright}c_{\triangleleft}s_{\text{aff}}}(\delta, \theta^\vee) = \omega_{c_{\triangleright}c_{\triangleleft}s_{\text{aff}}}(f\alpha_{\text{aff}} + \theta, \theta^\vee) = f\omega_{c_{\triangleright}c_{\triangleleft}s_{\text{aff}}}(\alpha_{\text{aff}}, \theta^\vee),$$

with $f = 1$ except in type $A_{2k}^{(2)}$, where $f = 2$. Now $\omega_{c_{\triangleright}c_{\triangleleft}s_{\text{aff}}}(\alpha_{\text{aff}}, \theta^\vee)$ equals $-\omega_{c_{\triangleright}c_{\triangleleft}s_{\text{aff}}}(\theta^\vee, \alpha_{\text{aff}})$, which by (2.1) is negative. \square

We make note of a formula for x_c that is obtained by using (2.1) to evaluate x_c on each simple co-root. We write $\{\rho_i : i = 1, \dots, n\}$ for the **fundamental weights**, the basis of V^* that is dual to the basis Π^\vee of co-roots.

Lemma 3.6. $x_c = \sum_{1 \leq i < j \leq n} ([\delta : \alpha_j] a_{ij} \rho_i - [\delta : \alpha_i] a_{ji} \rho_j).$

4. c -ORBITS OF ROOTS

In this section, we prove Theorem 1.2. We begin with a technical lemma.

Lemma 4.1 (Cf. [12, Lemma 1.6]). *Suppose β is a positive root in Φ . Then $c\beta$ is negative if and only if $\beta \in \overleftarrow{\Psi}^c$, in which case $-c\beta \in \overrightarrow{\Psi}^c$. Also $c^{-1}\beta$ is negative if and only if $\beta \in \overrightarrow{\Psi}^c$, in which case $-c^{-1}\beta \in \overleftarrow{\Psi}^c$.*

Proof. A root β changes sign under the action of a simple reflection s_i if and only if $\beta \in \{\pm\alpha_i\}$. Thus $c\beta$ is negative if and only if there exists an index $i \in \{1, \dots, n\}$ such that $c = c_1 s_i c_2$ and $c_2\beta = \alpha_i$, so that $\beta = c_2^{-1}\alpha_i \in \overleftarrow{\Psi}^c$. In this case, $-c\beta = -c_1(-\alpha_i) \in \overrightarrow{\Psi}^c$. Similarly, $c^{-1}\beta$ is negative if and only if there exists $i \in \{1, \dots, n\}$ with $c = c_1 s_i c_2$ and $c_1^{-1}\beta = \alpha_i$, so that $\beta = c_1\alpha_i \in \overrightarrow{\Psi}^c$. In this case, $-c^{-1}\beta = -c_2^{-1}(-\alpha_i) \in \overleftarrow{\Psi}^c$. \square

Lemma 4.1 leads to the following proposition.

Proposition 4.2 (Cf. [12, Lemma 1.7]). *If $\beta \in \Phi \setminus U^c$ then there exists $k \in \mathbb{Z}$ such that $c^k\beta \in \Psi^c$.*

Proof. We will show that $\{c^k\beta : k \in \mathbb{Z}\}$ contains both positive and negative roots. Then Lemma 4.1 completes the proof.

We write $\beta = a\gamma_c + b\delta + v$ with $v \in U^c \cap V_{\text{fin}}$. Since $\beta \in \Phi \setminus U^c$, we have $a \neq 0$. By Proposition 3.1(5), c acts on U^c with the same eigenvalues as $c_{\triangleleft} t_{\theta} c_{\triangleright}$. The latter is of finite order, so there is a positive integer ℓ such that $c^\ell v = v$. Using Proposition 3.1(2) to compute $c\gamma_c$, we see that $c^{m\ell}\beta$ is $a\gamma_c + (b + mla)\delta + v$ for any $m \in \mathbb{Z}$.

Since $a\gamma_c + v \in V_{\text{fin}}$, we have $[a\gamma_c + v : \alpha_{\text{aff}}] = 0$. Thus $[c^{m\ell}\beta : \alpha_{\text{aff}}]$ equals $(mla + b)[\delta : \alpha_{\text{aff}}]$. Since $la \neq 0$, by varying m , the expression $(mla + b)[\delta : \alpha_{\text{aff}}]$ can be made positive or negative. \square

Proof of Theorem 1.2(2). The proof of Proposition 4.2 also establishes that for $\beta \in \Phi \setminus U^c$, the set $\{[c^m\beta : \alpha_{\text{aff}}] : m \in \mathbb{Z}\}$ is infinite and therefore the orbit of β is infinite. On the other hand, Proposition 3.1(5) implies that c is diagonalizable on U^c with eigenvectors that are roots of unity. Thus c has finite order on U^c . \square

Proposition 4.2 is also a step in the direction of proving Theorem 1.2(1). To complete the proof, we need to show that the elements of Ψ^c are in distinct infinite orbits. We will see that this assertion is essentially equivalent to Theorem 2.2.

Proof of Theorem 1.2(1). For any k , index the simple roots modulo n to write the word $(s_1 \cdots s_n)^k$ from Theorem 2.2 as $s_1 s_2 \cdots s_{kn}$. Consider the kn roots $\alpha_1, s_1\alpha_2, s_1 s_2 \alpha_3$, etc. The conclusion of Theorem 2.2 is that these roots are all distinct and positive. But these roots are exactly the roots $\overrightarrow{\Psi}^c \cup c\overrightarrow{\Psi}^c \cup \dots \cup c^{k-1}\overrightarrow{\Psi}^c$, which is therefore a disjoint union of k sets of size n . This is true for any $k \geq 0$, and we conclude: first, that the c -orbit of any element of $\overrightarrow{\Psi}^c$ is infinite; second, that $c^m\alpha$ is a positive root for any $\alpha \in \overrightarrow{\Psi}^c$ and $m \geq 0$; and third, that there do not exist $\alpha, \beta \in \overrightarrow{\Psi}^c$ and $m \geq 0$ such that $\beta = c^m\alpha$, unless $\alpha = \beta$ and $m = 0$. Since $\overleftarrow{\Psi}^c = \overrightarrow{\Psi}^{c^{-1}}$, we can also conclude that the c -orbit of any element of $\overleftarrow{\Psi}^c$ is infinite, that $c^m\alpha$ is positive for $\alpha \in \overleftarrow{\Psi}^c$ and $m \leq 0$, and that there do not exist $\alpha, \beta \in \overleftarrow{\Psi}^c$ and $m \leq 0$ such that $\beta = c^m\alpha$, unless $\alpha = \beta$ and $m = 0$. It remains only to rule out the possibility that there exist $\alpha \in \overrightarrow{\Psi}^c$ and $\beta \in \overleftarrow{\Psi}^c$ such that $\beta = c^m\alpha$ for some integer m . But

in this case, if $m \geq 0$, then $c^{m+1}\alpha$ is a negative root by Lemma 4.1, contradicting the fact established above that $c^{m+1}\alpha$ is positive. If $m < 0$, then we reach a similar contradiction because $\alpha \in \overleftarrow{\Psi}^{c^{-1}}$, $\beta \in \overrightarrow{\Psi}^{c^{-1}}$ and $\alpha = c^{-m}\beta$. \square

We pause in the proof of Theorem 1.2 to make a useful observation that follows from the proof of Theorem 1.2(1).

Proposition 4.3 (Cf. [12, Proposition 1.9]). *The orbits of roots in $\overrightarrow{\Psi}^c$ are separated from the orbits of roots in $\overleftarrow{\Psi}^c$ by the hyperplane U^c . Specifically, $K(\gamma_c, \beta) > 0$ for $\beta \in c^m \overrightarrow{\Psi}^c$ and $m \in \mathbb{Z}$, while $K(\gamma_c, \beta) < 0$ for $\beta \in c^m \overleftarrow{\Psi}^c$ and $m \in \mathbb{Z}$.*

Proof. Lemma 4.1 implies that a root β is in the c -orbit of an element of $\overleftarrow{\Psi}^c$ if and only if $-\beta$ is in the c -orbit of an element of $\overrightarrow{\Psi}^c$. In particular it suffices to establish the claim for positive roots. The set $\bigcup_{m \geq 0} c^m \overrightarrow{\Psi}^c$ is $\{s_1 \cdots s_{k-1} \alpha_k : k \geq 1\}$ with indices interpreted modulo n . Half of the proof is to show that $K(\gamma_c, s_1 \cdots s_{k-1} \alpha_k) > 0$ for all k . We argue by induction, with c allowed to vary. If $k = 0$, then $K(\gamma_c, \alpha_1)$ is a negative multiple of $\omega_c(\delta, \alpha_1)$ by Lemma 3.5. Since W is irreducible (because it is affine), we see from (2.1) that $\omega_c(\delta, \alpha_1) < 0$. If $k > 0$, then by induction $K(\gamma_{s_1 c s_1}, s_2 \cdots s_{k-1} \alpha_k) > 0$. By Proposition 3.2 and (2.2), γ_c differs from $s_1 \gamma_{s_1 c s_1}$ by a multiple of δ . Since K is W -invariant and since $K(\delta, \cdot) = 0$, we see that $K(\gamma_c, s_1 \cdots s_{k-1} \alpha_k) > 0$. The other half of the proof is symmetric, switching c with c^{-1} . \square

The remaining parts of Theorem 1.2 concern finite orbits. One of them, Theorem 1.2(3), is trivial because the action of W fixes all imaginary roots. Another of them is now easily proved: Theorem 1.2(1) combines with Lemma 4.1 to imply that roots only change sign within infinite orbits, and Theorem 1.2(5) follows.

It remains to prove Theorem 1.2(4,6). The easiest assertion left to prove is that there are infinitely many finite c -orbits of real roots, or equivalently (in light of Theorem 1.2(2)) that there are infinitely many real roots in U^c . We begin by showing that there exist real roots in U^c , specifically, roots in $\Phi_{\text{fin}} \cap U^c$. Recall the notation Υ_{fin}^c for the intersection $\Phi_{\text{fin}} \cap U^c$. Since U^c is a subspace, Υ_{fin}^c is a (necessarily finite) root subsystem of Φ_{fin} , but *a priori* it may be empty. To the contrary, we will see that it is as large as it could be given the dimension of $U^c \cap V_{\text{fin}}$. Understanding the structure of Υ_{fin}^c will turn out to be critically important.

Proposition 4.4. *Let Υ_{fin}^c be the finite root system $\Phi_{\text{fin}} \cap U^c$ and write $c = c_{\triangleleft} s_{\text{aff}} c_{\triangleright}$.*

- (1) Υ_{fin}^c has rank $n - 2$.
- (2) The irreducible components of Υ_{fin}^c are all of finite type A .
- (3) The roots in the symple system Ξ_{fin}^c of Υ_{fin}^c can be ordered $\beta_1, \dots, \beta_{n-2}$ so that $t_{\beta_1} \cdots t_{\beta_{n-2}} = c_{\triangleleft} t_{\theta} c_{\triangleright}$.
- (4) The order $\beta_1, \dots, \beta_{n-2}$ of (3) may not be unique, but as long as $t_{\beta_1} \cdots t_{\beta_{n-2}}$ is $c_{\triangleleft} t_{\theta} c_{\triangleright}$, the set $\Omega^c = \{\beta_1, t_{\beta_1} \beta_2, \dots, t_{\beta_1} \cdots t_{\beta_{n-2}} \beta_{n-2}\}$ does not depend on the order.

Proof. We already showed (in the proof of Proposition 3.1) that the fixed space of $c_{\triangleleft} t_{\theta} c_{\triangleright}$ (in V_{fin}) is spanned by γ_c . Since there exists some minimal expression for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ as a product of reflections in W_{fin} , Proposition 2.4 implies that Φ_{fin} contains a basis for $\{v \in V_{\text{fin}} : K(\gamma_c, v) = 0\} = U^c \cap V_{\text{fin}}$. This space is $(n - 2)$ -dimensional, so the basis has $n - 2$ vectors, and therefore Υ_{fin}^c has rank $n - 2$, and we have proven (1).

Furthermore, when we write the expression $t_1 \cdots t_{n-2}$ for $c_{\triangleleft} t_{\theta} c_{\triangleright}$, we can define t_{n-1} to be the reflection $c_{\triangleright}^{-1} t_{\theta} c_{\triangleright}$, so that $t_1 \cdots t_{n-1} = c_{\triangleleft} c_{\triangleright}$. Since $c_{\triangleleft} c_{\triangleright}$ is a Coxeter element for W_{fin} , Lemma 2.5 says that $t_1 \cdots t_{n-2}$ is conjugate to a Coxeter element of some parabolic subgroup of W_{fin} .

Define $\text{Proj}_{\text{fin}} : V \rightarrow V_{\text{fin}}$ to be the linear map that fixes V_{fin} pointwise and sends δ to zero. By (3.1), we conclude that $c_{\triangleleft} t_{\theta} c_{\triangleright}$ and $\text{Proj}_{\text{fin}} \circ c$ coincide, as maps on V_{fin} .

Now consider any irreducible component Θ of Υ_{fin}^c and any expression $t_1 \cdots t_{n-2}$ for $c_{\triangleleft} t_{\theta} c_{\triangleright}$ as a product of $n-2$ reflections in W_{fin} . Since the reflections t_i all correspond to roots in Υ_{fin}^c and since Θ is an irreducible component, we can, by transposing commuting reflections if necessary, assume that the reflections corresponding to roots in Θ are t_1, \dots, t_k for some k . Since $t_1 \cdots t_{n-2}$ is conjugate to a Coxeter element of some (possibly reducible) parabolic subgroup of W_{fin} the same conjugation takes $t_1 \cdots t_k$ to a Coxeter element of a (typically smaller and always irreducible) parabolic subgroup. The elements $t_1 \cdots t_k$, $c_{\triangleleft} t_{\theta} c_{\triangleright}$, and $\text{Proj}_{\text{fin}} \circ c$ all have the same action on Θ .

Let $\tilde{\Theta}$ be the root subsystem $\Phi \cap \text{Span}(\Theta \cup \{\delta\})$ of Φ . Then $\tilde{\Theta}$ is an affine root system whose associated finite root system is Θ . The action of c fixes $\tilde{\Theta}$ as a set and takes positive roots to positive roots, so the canonical simple system for $\tilde{\Theta}$ is a union of c -orbits. (If some power of c maps a root in the canonical simple system to a root not in the canonical simple system, then that power of c maps the canonical simple system to a different simple system defining the same positive roots, and that is a contradiction.) These orbits are the same size as $t_1 \cdots t_k$ -orbits of roots in Θ , and since $t_1 \cdots t_k$ is conjugate to a Coxeter element, Theorem 1.1 says that each orbit has size h , the Coxeter number of Θ . The Coxeter number is always strictly greater than the rank and the canonical simple system of $\tilde{\Theta}$ consists of $k+1$ roots. We conclude that the canonical simple system is a single c -orbit.

We now consider what the type of $\tilde{\Theta}$ might be. The element c restricts to an automorphism of its Dynkin diagram, and this automorphism has a single orbit. Since each affine Dynkin diagram is either a tree or a cycle (the latter with only single edges), we conclude that $\tilde{\Theta}$ is of type $A_k^{(1)}$. Thus Θ is of type A_k . (Alternately, we reason based on the fact that the Coxeter number of Θ is one more than its rank.) This completes the proof of (2).

Numbering the simple roots of Θ as β_1 through β_k linearly along the Dynkin diagram, and numbering the additional simple root of $\tilde{\Theta}$ as β_0 , the action of c on $\tilde{\Theta}$ is by the diagram automorphism $i \mapsto i+1 \pmod{k+1}$. Thus $t_1 \cdots t_k$ acts by $\beta_i \mapsto \beta_{i+1}$ for $i = 1, \dots, k-1$ and by $\beta_k \mapsto -(\beta_1 + \cdots + \beta_k) \mapsto \beta_1$. The same action is accomplished by the Coxeter element $t_{\beta_1} \cdots t_{\beta_k}$ of Θ , and we conclude that $t_1 \cdots t_k = t_{\beta_1} \cdots t_{\beta_k}$. Since Θ is an irreducible component of Υ_{fin}^c (and since reflections in different irreducible components commute), by induction of the number of irreducible components, we see that an appropriate numbering of the simple roots of Υ_{fin}^c yields $t_1 \cdots t_{n-2} = t_{\beta_1} \cdots t_{\beta_{n-2}}$. This is (3).

Finally, we prove (4). Since $\{\beta_i : i = 1, \dots, n-2\}$ is a simple system for Υ_{fin}^c , the element $c_{\triangleleft} t_{\theta} c_{\triangleright}$ is a Coxeter element for the associated Weyl group. Each orderings on the simple system such that $t_{\beta_1} \cdots t_{\beta_{n-2}} = \text{Proj}_{\text{fin}} \circ c$ is a reduced word for $\text{Proj}_{\text{fin}} \circ c$. The set Ω^c is the set of positive roots associated to the inversions of $\text{Proj}_{\text{fin}} \circ c$ and is thus independent of the choice of reduced word. \square

Remark 4.5. Since the sub-root system Υ_{fin}^c has only type- A components, there is a straightforward characterization of the roots $\{\beta_j\}_{j \in [1, n-2]}$ for any choice of

Coxeter element c . These are the roots in $\{\alpha \in \Phi_{\text{fin}}^+ : K(\gamma_c, \alpha) = 0\}$ that are not sums of other roots in the same set.

The proof of Proposition 4.4 also establishes the following fact:

Proposition 4.6. *Each component of Υ^c is of affine type $A^{(1)}$. The action of c on each component is to rotate the Dynkin diagram of the component, taking each node to an adjacent node (or when the component has rank two, to transpose the two nodes of the Dynkin diagram).*

Proof of Theorem 1.2(4,6). If Φ is of rank 2, then $U^c = \mathbb{R}\delta$, so Theorem 1.2(2) implies that there are no finite c -orbits of real roots. If Φ is of rank at least 3, then Υ_{fin}^c has rank at least 1, and thus has at least one irreducible component. In particular, Υ^c contains at least one infinite root subsystem (called $\tilde{\Theta}$ in the proof of Proposition 4.4(2)) so there are infinitely many finite c -orbits.

Let Θ be an irreducible component of Υ_{fin}^c and write $\tilde{\Theta} = \Phi \cap \text{Span}(\Theta \cup \{\delta\})$. Ordering the canonical simple system of Θ as β_1, \dots, β_k as in the proof of Proposition 4.4, the intersection $\Omega^c \cap \Theta$ is

$$\{\beta_1, t_{\beta_1}\beta_2, \dots, t_{\beta_1} \cdots t_{\beta_{k-1}}\beta_k\} = \{\beta_1, \beta_1 + \beta_2, \dots, \beta_1 + \cdots + \beta_k\}.$$

Write β_0 for the root $c\beta_k$. In light of Proposition 4.4(2) and Proposition 4.6, we see that $\beta_0 + \beta_1 + \cdots + \beta_k$ is in the line spanned by δ , and thus equals $\kappa\delta$ for some positive integer κ . Furthermore, we see that $\kappa(\beta) = \kappa$ for every $\beta \in \Omega^c \cap \Theta$.

The Coxeter element c acts by $\beta_i \mapsto \beta_{i+1} \pmod{k+1}$. Since the positive roots of Θ are of the form $\beta_i + \cdots + \beta_j$ for $1 \leq i \leq j \leq k$, we see that every positive root of Θ is in the c -orbit of exactly one root in $\Omega^c \cap \Theta$. The positive roots of $\tilde{\Theta}$ are positive roots of Θ plus nonnegative multiples of $\kappa\delta$ or negative roots of Θ plus positive multiples of $\kappa\delta$. Thus every positive root of $\tilde{\Theta}$ is contained in the c -orbit of exactly one root of the form $\beta + m \cdot \kappa\delta$ with $\beta \in \Omega^c \cap \Theta$ and $m \geq 0$. Applying the antipodal map, we see that every negative root of $\tilde{\Theta}$ is in the c -orbit of exactly one root of the form $-\beta + m \cdot \kappa\delta$ with $\beta \in \Omega^c \cap \Theta$ and $m \leq 0$. Since $-\beta + \kappa\delta$ is in the c -orbit of some element of $\Omega^c \cap \Theta$, we see that every negative root of $\tilde{\Theta}$ is in the c -orbit of exactly one root of the form $\beta + m \cdot \kappa(\beta)\delta$ with $\beta \in \Omega^c \cap \Theta$ and $m < 0$.

We have proved Theorem 1.2(4,6) in the case where Υ_{fin}^c has exactly one irreducible component. Since roots in different irreducible components are orthogonal, the general case of (4,6) follows easily. \square

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