Task design fostering construction of limit confirming examples as means of argumentation

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Tasks that require students to construct examples that meet certain constraints are known to be used in mathematics education. It is also well established that while examples are not proofs (for general statements), they have a supporting role in the preliminary stages of making sense of a certain mathematical phenomenon. In this study we examine a task design in which students are required to submit a supporting example to their explanation negating an existential statement. We introduce the idea of limit confirming examples and present their use in an analytical geometry task in a 10th grade class in Italy, along with the explanations they support in negating an existential statement. The results show that the design was effective in fostering the construction of limit confirming examples that could be considered as means of argumentation in the initial parts of proof construction.

Keywords: Reasoning with examples, limit confirming examples, automatic online assessment,

Theoretical Background

Alongside with computerized environments in mathematics education recent developments of online assessment platforms enable students to submit open ended tasks that are automatically assessed (Olsher, Yerushalmy, & Chazan, 2016). One challenge in designing tasks for these environments is to design means of mathematical argumentation that could be automatically analyzed (Yerushalmy, Nagari-Haddif, & Olsher, 2017).

One form of open ended that elicits different characteristics in student answers, and serves means of argumentation about them is by providing examples (Buchbinder & Zaslavsky, 2009). Examples could serve as inductive of general example-based arguments (Dreyfus, Nardi, & Leikin, 2012), thus providing an initial step in the proving process. Although there are limitations to the use of empirical examples as proof (Zaslavsky, 2018), research recognizes merit in arguing why characteristics of a certain example would work for any other one as well (ibid).

Students use at times systematic exploration of examples as a proving strategy, referring to their examples as cases, and then *prove by cases* (Buchbinder, 2018). When constructing cases, the literature recognizes several strategies that could lead to constructing refutations. One of these strategies, generating *limit cases*, might be constructed by creating an auxiliary problem in which the condition of the initial problem is transitioned to the limit (Balk, 1971), and is also referred to as *extreme cases* (Clement, 1991) or *boundary cases* (Ellis, Lockwood, Williams, Dogan, & Knuth, 2013).

In this paper, we present a task design aimed at fostering students' construction of a specific type of examples - *limit confirming examples*. We introduce this term as a theoretical result of our a-priori analysis of the task (for this reason, we will define and exemplify this concept after presenting the task). Our hypothesis is that limit confirming examples represent possible effective means of argumentation (Stylianides, Beida, & Morselli, 2016) towards persuasion that certain characteristics could not co-exist in a specific mathematical context. This study is part of a wider collaboration for studying the design and use of online formative assessment activities using the STEP platform[1] (Olsher, et al., 2016).

Methodology

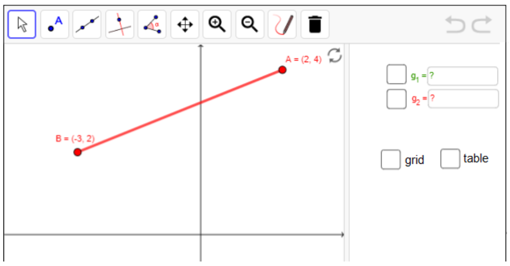
We adopted a design-based research approach (Cobb, et al., 2003), characterized by cycles of design, enactment, analysis and redesign. The pilot study on which this paper is focused has been developed within the first cycle of design. In particular, here we present aspects of both the design phase and the analysis phase.

The participants were 25 secondary Italian students, from a 10th grade class (students aged 15-16) of a scientific lyceum, in Italy. We focus on the students' resolution of a task that is part of an online activity within the context of analytical geometry, specifically lines intersecting a segment. The online activity was proposed to the group of students at the end of the school year, when they had already studied some basis of analytic geometry (in particular, coordinates of point, equations of lines, conditions of perpendicularity and parallelism).

The activity comprised three tasks designed as interactive diagrams describing a geometrical context on a Cartesian axis, using the STEP platform. The interactive diagrams were constructed using GeoGebra, and enabled the participants to construct or drag a set of elements in the diagram, according to the predefined characteristics determined by the designers of the task. The context was described in the task. The participants needed to submit examples satisfying different conditions. Students had to complete the whole activity within one hour.

The task

The task requires the students to consider the segment connecting the points A(2,4) and B(-3,2) and the family of lines y=mx (see figure 1), and addresses the following existential statement: "*There are two lines of the family perpendicular to each other and both intersecting the segment AB*". The students are asked to state if the statement is true or false, then, in case they think it is true, to submit the equations of two lines that satisfy it, or, in case they think it is false, to explain why and submit a screenshot that supports their choice.



[1] Seeing the Entire Picture - STEP – is a formative assessment platform developed at the University of Haifa's Center for Mathematics Education Research and Innovation (MERI). For more detail about this platform, see www.visustep.com.

Figure 1: Applet accompanying task requiring an explanation and a supporting example

The existential statement could be formulated as follows: "*There are two lines that satisfy all of the following three properties: (A) the two lines belong to the family y=mx; (B) the two lines are perpendicular to each other; (C) the two lines intersect the segment AB*". Therefore, it could be thought as $A \wedge B \wedge C$, that is an intersection of three conditions. Proving that this statement is false requires to prove that the universal statement $\neg(A \wedge B \wedge C)$ is true, which is logically equivalent to each of the following statements: $(A \wedge B) \rightarrow \neg C$; $(B \wedge C) \rightarrow \neg A$; $(A \wedge C) \rightarrow \neg B$

So, in order to prove that the existential statement "*There are two lines of the family perpendicular to each other and both intersecting the segment AB*" is false, it is enough to prove one of the following universal statements:

1) "*If two lines belong to the family y=mx and are perpendicular to each other, then they do not both intersect the segment AB*" ($(A \wedge B) \rightarrow \neg C$)

2) "*If two lines belong to the family y=mx and both intersect the segment AB, then they are not perpendicular to each other*" ($(A \wedge C) \rightarrow \neg B$)

3) "*If two lines both intersect the segment AB and are perpendicular to each other, then they do not both belong to the family y=mx*" ($(B \wedge C) \rightarrow \neg A$)

Students should be aware of the fact that identifying a *confirming example* for each of the universal statements 1, 2, and 3 is not enough to prove them. Yet, students' choice of the examples that should support their claim that an existential statement is false could represent an important sign to highlight their awareness about what kind of examples could be the starting point for the construction of argumentations.

Limit confirming examples

We define *limit confirming examples* (for a universal statement) as specific limit, or boundary examples that incorporate within them all other possible confirming examples. The characteristic of incorporating all possible confirming examples makes limit confirming examples effective supports for construction of complete argumentation about the truthfulness of a universal statement because they foster the activation of a "domino effect", enabling students to highlight why all the other possible examples that can be constructed will confirm the statement (in the following, we refer to the confirming examples that are not limit confirming examples as *not-limit confirming examples*).

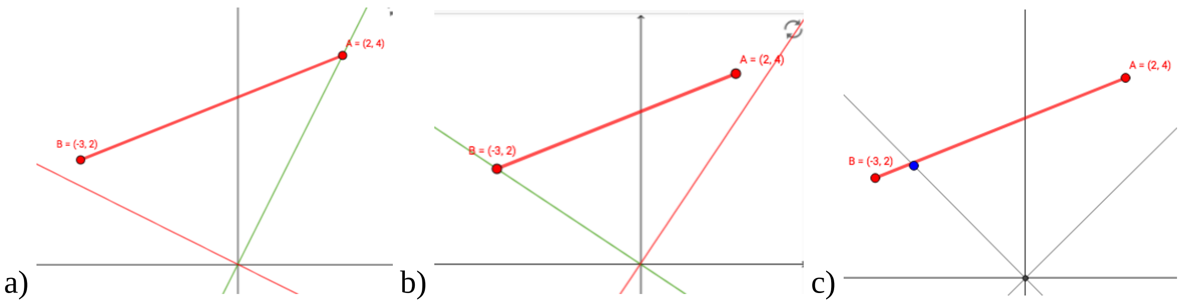


Figure 2: Limit confirming examples (a, b) and a not-limit confirming example (c) for statement 1

We refer to the task introduced in the previous paragraph to exemplify this idea. There are two limit confirming examples (Figure 2, a and b) for statement 1 ("*If two lines belong to the family y=mx and are perpendicular to each other, then they do not both intersect the segment AB*").

These two examples are characterized by the fact that one of the two lines intersects the segment at one of its edges. They are *limit confirming examples* because, if we consider other examples constructed in the same way (two lines passing through the origin and perpendicular to each other), if one line intersects the segment in a point that is not an extreme for AB, the other line certainly does not intersect AB, as highlighted in Figure 2 (c), a screenshot submitted by a student.

As regards statement 2 ("*If two lines belong to the family y=mx and both intersect the segment AB, then they are not perpendicular to each other*"), there is one *limit confirming example* (Figure 3a) characterized by the fact that the two lines considered are those that intersect the segment in its extreme points. We can consider it a *limit confirming example* because all the other couples of lines intersecting the segment and belonging to the family (Figure 3b) form an angle that is smaller than the one highlighted in figure 3a, meaning that the two lines are not perpendicular.

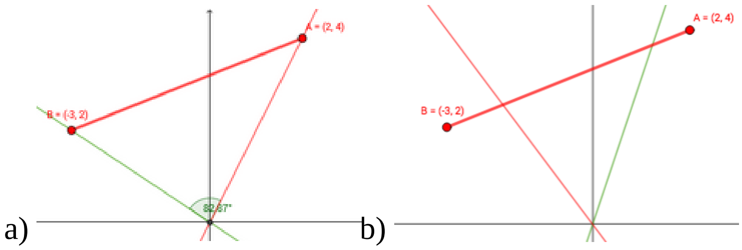


Figure 3: Limit and non-limit confirming example for statement 2

The limit confirming example (Figure 4a) for statement 3 ("*If two lines both intersect the segment AB and are perpendicular to each other, then they do not both belong to the family y=mx*") is characterized by the fact that the two lines considered are perpendicular to each other, intersect the segment AB in its edges and intersect each other in the point that is at a minimal distance from the origin (that is the point of intersection between the circumference whose diameter is AB and the line passing through the origin and the center of this circumference). This is a *limit confirming example* because all the other possible couples of lines perpendicular to each other and intersecting the segment AB intersect each other in a point of the circumference that is at a greater distance from the origin (Figure 4b) or in a point inside the circumference whose diameter is AB (Figure 4c), so they do not belong to the family y=mx.

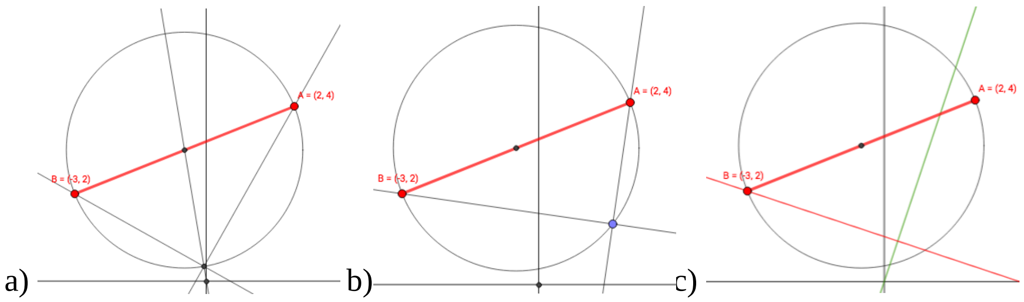


Figure 4: The limit confirming example for statement 3 (a) and two not-limit confirming example incorporated within it (b, c)

Methodology of analysis

Some important elements of the context must be added before presenting the methodology of analysis and the results. First of all, we did not share with students the logical analysis of the statement, that is the identification of A, B , C and the reflection on the logical equivalence of $\neg(\boldsymbol{A} \wedge \boldsymbol{B} \wedge \boldsymbol{C})$ and each of the statements $(\boldsymbol{A} \wedge \boldsymbol{B}) \rightarrow \neg\boldsymbol{C}$; $(\boldsymbol{B} \wedge \boldsymbol{C}) \rightarrow \neg\boldsymbol{A}$; $(\boldsymbol{A} \wedge \boldsymbol{C}) \rightarrow \neg\boldsymbol{B}$. Moreover, we did not explain what limit-confirming examples are, neither we asked them to find out examples with specific characteristics.

The aim of our analysis was, on one side, to highlight if the design of the task was effective in fostering students' construction of limit-confirming examples and, on the other side, to detect, in students' answers, elements that could contribute to the construction of complete argumentations. Our units of analysis were the answers given by students to the task, including three main aspects: (1) students' claims about the truthfulness of the statement (level 1); (2) students' choices of the examples to be sent to support their claims (level 2); (3) students' verbal arguments to justify their claims (level 3)**.**.

As regards the second level of analysis, we: (a) initially distinguished between the answers characterized by the submission of a limit confirming example and the answers characterized by the submission of a not-limit confirming example; and (b) subsequently distinguished, among the limit confirming examples sent by students, between the categories of limit confirming examples presented in the previous paragraph (that is those referred to statement 1, or 2, or 3).

The analysis of the verbal arguments sent by students to support their claims (level 3) was developed at a qualitative level, focusing on: (a) the reference to the chosen example and the coherence/incoherence between the choice of the example and the content of the verbal argument; (b) hints of students' awareness about the role that the examples they chose could play to support the construction of a complete argumentation to justify their claim.

Results

As regards level 1 of analysis, all students correctly stated that the statement is false.

Focusing on the examples they sent (level 2 of analysis), most of them (22 out of 25) submitted a limit confirming example. Among the 3 remaining students, two submitted a not-limit confirming example, while the third one did not submit any example. Further analysis of students' choice of limit confirming examples shows that most of them (17) submitted the limit confirming example related to statement 2 (Figure 3a); 3 students submitted a limit confirming example related to statement 1 (Figure 2); one student sent the limit confirming example referred to statement 3 (Figure 4a); one student sent a limit confirming example related to statement 1, containing also the sketch of the circumference whose diameter is AB.

In the following, we present some results of the analysis of students' verbal arguments (level 3).

Among the 17 students submitting a limit confirming example related to statement 2 (Figure 3a), 7 show some form of consideration to the fact that their example incorporates all other possible examples. Some of the students explicitly refer to the other possible examples incorporated into the limit confirming example, as shown in Figure 5.

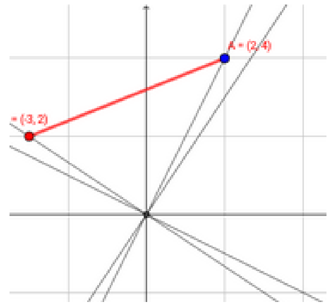
	*It is impossible that two lines of equation y=mx, perpendicular to each other, intersect the segment AB because the value of m for the line that intersects the segment in B is -3/2, while the value of m for the line that intersects the segment in A is ½. If the two lines were perpendicular, the product between the two values of m should be -1, but it is -1/3. And no other lines that satisfy these conditions exist, because the other lines that intersect the segment AB form an angle that is even smaller than the one formed by the lines that intersect AB in the extreme points.*

Figure 5: The screenshot submitted by S8 and the corresponding argumentation

Other answers show an implicit reference to the set of examples that can be generated starting from the submitted limit confirming example. For example, the argument proposed by S9 (Figure 6) refers to the idea of "widest angle", making an implicit reference to the set of possible couples of lines passing through the origin and intersecting AB.

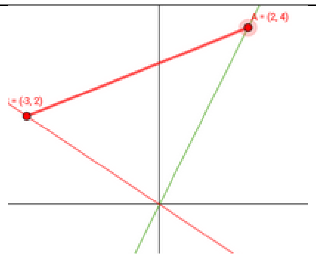
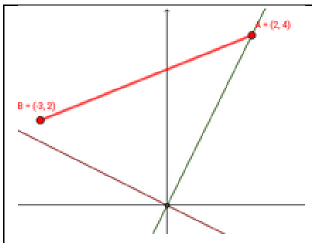
	*Since the lines should pass through the origin, and since the lines that pass through the extreme points of the segment (AB) form the widest angle, and since this angle, referring to the formula of perpendicularity, is not right, the claim is false.*

Figure 6: The screenshot submitted by S9 and the corresponding explanation

The other 10 students that submitted a limit confirming example related to statement 2 either did not submit an explanation (1 student), or submitted explanations with no evidence about of the role of a limit confirming example. These incomplete argumentations included either a repeated statement about the claim not being true, or an additional verbal description of the submitted example, or referred to aspects not directly linked to the provided example.

The choice of the limit confirming example to be submitted seems to influence the construction of the related argumentation. In fact, none of the three students who sent a limit confirming example referred to statement 1 (figure 2) show some form of consideration to the fact that their example incorporates all other possible examples: One of them do not submit an explanation. The second student reformulated the claim about the initial statement not being true, and the third student (S16, Figure 7) referred to the relation between the slopes of two perpendicular lines, without focusing on the possibility of creating other examples starting from the limit confirming one.

	*In order that a line is perpendicular, the values of m should be anti-reciprocal. That is: it is impossible, according to the data. The line passing through the origin and A has equation y=2x. So its perpendicular must have equation y=-0.5x (passing through the origin) and it has to pass through B. It is easy to verify that it is not*

	true.

Figure 7: The screenshot submitted by S16 and the corresponding explanation

The single explanation submitted along with a limit confirming example referred to statement 3 (S18, Figure 8) does not highlight a consideration to the fact that their example incorporates all other possible examples.

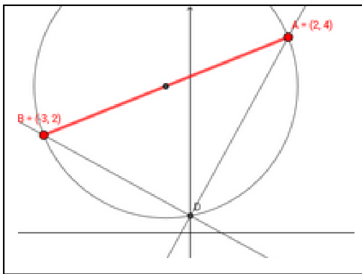
	*The two lines of the family that intersect the segment AB, are perpendicular to each other but they do not pass through the origin. Considering the circumference whose diameter is AB, the angle D is right, for a known theorem.*

Figure 8: The screenshot submitted by S18 and the corresponding explanation

Conclusions

The most unambiguous outcome of this limited study is that 22 out of 25 students (88 percent) submitted limit confirming examples to support an explanation that disproves an existential statement in the form $A \wedge B \wedge C$. Disproving this statement is equivalent to proving one of the universal statements $(A \wedge B) \rightarrow \neg C; (B \wedge C) \rightarrow \neg A; (A \wedge C) \rightarrow \neg B$. The students provided the limit confirming examples without being previously introduced to these ideas, mostly since students should be aware that a confirming example is not enough for a proof (Buchbinder & Zaslavsky, 2009; Zaslavsky, 2018), in the same way in which a non-confirming example for the existential statement is not enough to disprove it. Yet, identifying limit confirming examples for one of the statements $(A \wedge B) \rightarrow \neg C; (B \wedge C) \rightarrow \neg A; (A \wedge C) \rightarrow \neg B$ was something that most students could come up with on their own as a supporting example. This result supports the notion that these examples might be an easier starting point in the argumentation process, opening the way to the construction of a complete argumentation somewhat aligned with example-based arguments (Dreyfus, Nardi, & Leikin, 2012) as means of argumentation (Stylianides, Beida, & Morselli, 2016) in an initial stage of the proving process.

Since many students that submitted a limit confirming example were not able to construct rich argumentations, our study also shows that, although choosing to submit a limit confirming example could be promising in fostering students' construction of complete argumentations, it is not sufficient. While the topic of the tasks in this study is analytical geometry, and used 3 properties, we believe that additional mathematical strands could be explored, while not necessarily limiting the number of properties to 3, but our final conclusions in terms of task design are driven straight from the finding of this study and the tasks studied: We suggest the following criteria for the creation of effective tasks aimed at developing learners' abilities in creating and using limit confirming examples: (a) analysis of complex false existential statements involving at least 3 properties to be satisfied by the elements of a precise domain; (b) possible deconstruction of the statement to corresponding universal statements to be proven; (c) requirement of argumentation and a supporting example about the truthfulness of one of these universal statements.

Limitations and future steps of the research

This research included a single task and a small sample of students. In order to further investigate and reaffirm the findings more tasks should be developed using this design, and tested in different settings. Furthermore, the use of the STEP environment could support the teacher in automatically having the student submissions categorized, thus providing grounds for research about teacher use.

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