SIGN-CHANGING RADIAL SOLUTIONS FOR THE SCHRÖDINGER-POISSON-SLATER PROBLEM.

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ABSTRACT. We consider the Schrödinger-Poisson-Slater (SPS) system in $\mathbb{R}^3$ and a nonlocal SPS type equation in balls of $\mathbb{R}^3$ with Dirichlet boundary conditions. We show that for every $k \in \mathbb{N}$ each problem considered admits a nodal radially symmetric solution which changes sign exactly k times in the radial variable.

Moreover when the domain is the ball of $\mathbb{R}^3$ we obtain the existence of radial global solutions for the associated nonlocal parabolic problem having $k+1$ nodal regions at every time.

1. INTRODUCTION

We consider the Schrödinger-Poisson-Slater (SPS) problem in $\mathbb{R}^3$

$$(1) \qquad \begin{cases} -\Delta u + u + \phi u - |u|^{q-1}u = 0 & \text{in } \mathbb{R}^3 \\ -\Delta \phi = 4\pi u^2 & \text{in } \mathbb{R}^3 \\ \lim_{|x|\to+\infty} \phi(x) = 0 \end{cases}$$

From a physical point of view systems like (1) appear is semiconductor theory to model the evolution of an electron ensemble in a semiconductor crystal (see [20, 21, 6, 26]). In this context the Poisson potential ϕ comes from the repulsive interactions among electrons while the nonlinear term $|u|^{p-1}u$ is introduced as a correction to the repulsive Poisson potential to explain different phenomena observed from experimentations (for instance in simulations with superlattices structures). In particular in the case $p = \frac{5}{3}$ this term is usually known as *Slater correction*, since it comes from a term that was first introduced by Slater (1951) as a local approximation for the exchange term in the Hartree-Fock equations (see [27], [14]). For this reasons we refer to system (1) as Schrödinger-Poisson-Slater system (SPS).

A different justification of system (1) can be found also in [4] where it is proposed as a model formally describing the interaction of a charged particle with its own electrostatic field.

From a mathematical point of view system (1) shows several difficulties, the nonlinear nature of it being due both to the power-type nonlinearity in the first equation and to the coupling, and has been object of many investigation in the last years (we recall among others the papers [1, 2, 4, 9, 11, 12, 13, 19, 24], see also [18]).

As shown by recent results the structure of the solution set of (1) depends strongly on the value of q of the power-type nonlinearity.

For $q \leq 2$ and $q \geq 5$ system (1) doesn't admit any nontrivial solution (see [12, 19, 24]), while when $q \in (2, 5)$ existence and multiplicity results have been proved using variational techniques.

Key words and phrases. Schrödinger-Poisson-Slater system, nodal solutions, parabolic problem, dynamical approach.

Precisely in [11, 19, 24] the existence of at least one nontrivial radial solution is proved while in [1] they show the existence of infinitely many radial solutions.

For completeness we recall that mostly these existence results are obtained through min-max procedures, and one needs to restrict the energy functional to the natural constrained of the radial functions to overcome the problem of the lack of compactness of the Sobolev embeddings in the unbounded domain $\mathbb{R}^3$. We also point out that in [1] no information about the sign of the solutions is given. On the other hand in [2] the existence of a positive ground state solution has been proved but it is still an open problem whether it is radial or not.

In the present paper we analyze more deeply the structure of the radial bound states set for problem (1).

We show the existence of infinitely many radially symmetric sign-changing solutions which are distinguished by the number of nodal regions, more precisely we prove the existence of radial solutions which have a prescribed number of nodal domains. As far as we know this is the first paper where existence of sign-changing solution is proved for problem (1).

Our proof combines a dynamical and topological approach together with a limit procedure and it is mainly inspired by [30]. Up to our knowledge this is the first time that such a different approach (not variational) is used in the context of the Schrödinger-Poisson-Slater problems.

We point out that one cannot simply apply the well known Nehari's method [22] of piecing together positive and negative solutions on alternating annuli (see also [3]); indeed it is known that for $q = 3$ the Nehari manifold's arguments for problem (1) fail, moreover, even for different values of the exponent q, the procedure would be quite involved since the solution of the Poisson equation on each annulus would need global informations on u.

Our main result is the following

Theorem 1.1. *Let $q \in [3, 5)$. For every integer $k \geq 2$, (1) admits a couple of radial solutions $(\pm u, \phi)$ such that $\pm u$ changes sign precisely $k-1$ times in the radial variable.*

In order to obtain this result we will study first the existence of sign changing radial solutions for the following semilinear elliptic equation with Dirichlet boundary condition

$$\begin{cases} -\Delta u + u + u \int_{\mathbb{B}_R} \frac{u^2(y)}{|x-y|} dy - |u|^{q-1}u = 0 & \text{in } \mathbb{B}_R \\ u = 0 & \text{on } \partial\mathbb{B}_R \end{cases} \tag{2}$$

where $\mathbb{B}_R$ is the ball of radius R in $\mathbb{R}^3$. This problem, with both local and nonlocal nonlinearities, has been also investigated in [25] when $q \in (1, 2)$.

We recall that solutions of (2) are critical points of the energy functional $E : H_0^1(\mathbb{B}_R) \to \mathbb{R}$ given by

$$E(u) = \frac{1}{2}\int |\nabla u(x)|^2 dx + \frac{1}{2}\int u(x)^2 dx + \frac{1}{4}\int\int \frac{u^2(x)u^2(y)}{|x-y|} dxdy - \frac{1}{q+1}\int |u|^{q+1}(x) dx$$

Our second main result is then the analogous of Theorem 1.1 for problem (2):

Theorem 1.2. *Let $q \in [3,5)$. For every $R \geq 1$ and every integer $k \geq 2$, (2) admits a couple of radial solutions $\pm u$ changing sign precisely $k-1$ times in the radial variable. Moreover there exists a constant $C_k > 0$, independent of R, such that $E(\pm u) \leq C_k$.*

The proof of theorem 1.2 relies on a dynamical method. We find solution of the elliptic problem (2) looking for equilibria in the ω-limit set of trajectories of the associated parabolic problem, namely of

$$(3) \qquad \begin{cases} u_t - \Delta u + u + u \int_{\mathbb{B}_R} \frac{u^2(y)}{|x-y|} dy - |u|^{q-1}u = 0 & \text{in } \mathbb{B}_R \times [0,\infty) \\ u = 0 \quad \text{on } \partial\mathbb{B}_R \times [0,\infty) \\ u(\cdot,0) = u_0 \text{ on } \mathbb{B}_R \end{cases} .$$

This nonlocal initial boundary value parabolic problem has been studied in [17] by the author himself.

Solutions of elliptic equations via the corresponding parabolic flow can be found in the literature (we recall among others [7, 23, 10]), in particular here we follow an approach introduced in [30] in the contest of symmetric systems of two coupled Schrödinger equations but which can be adapted, with proper modifications, also to scalar equations with odd nonlinearities, like our problem (2).

This method consists in selecting special initial data on the boundary of the domain of attraction of an asymptotically stable equilibrium in order to obtain equilibria with a fixed number of changes of sign.

It relies on a crucial monotonicity property of the semilinear parabolic problem (3) stating that the number of zeros is not increasing along the flow and combines the study of the parabolic flow with a topological argument based on the use of the Krasnoselskii genus.

We underline that, for fixed k, the solutions found in Theorem 1.2 satisfy an energy bound independent on the radius R of the domain. This is the starting point to prove Theorem 1.1, through a limit procedure on the radius of the domain.

Moreover, as a byproduct in the proof of Theorem 1.2 we obtain also the following result related to the existence of sign-changing global solutions for the parabolic problem (3) that we believe to be of independent interest

Theorem 1.3. *Let $q \in [3,5)$. For any integer $k \geq 2$ there exists a couple $\pm u_k : \mathbb{B}_R \times [0,\infty) \to \mathbb{R}$ of global radial solutions of (3) such that, for all $t \geq 0$ $\pm u_k(\cdot,t)$ has exactly $k-1$ changes of sign in the radial variable.*

We point out that all the results in the present paper are obtained for values $q \in [3,5)$, the case $q \in (2,3)$ being still open.

This is due to the fact that first the existence of solutions for the parabolic problem (3) when $q \in (2,3)$ is still open (see [17]), second the geometric properties of the energy functional depend strongly on the value of q (see the proof of Proposition 6.1 where we need to restrict to the case $q \geq 3$). We recall also that it is still an open problem whether the (PS) property holds or not in the gap $q \in (2,3)$, precisely it has not yet been proved the existence of a bounded Palais-Smale sequence when q belongs to this gap.

Our difficulties seem to be strictly related to the ones one finds when dealing with this open problem.

We briefly describe the paper's organization.

In Section 2 we collect some notations and preliminaries. In Section 3 we recall the properties of the nonlocal parabolic problem (3) (local and global existence results, regularity, compactness properties) which have been studied in [17] by the author himself. Moreover we prove the monotonicity of the number of zeros along the parabolic flow.

In Section 4 we define a family of finite dimensional spaces $(W_k)_{k\in\mathbb{N}}$ such that the restriction of the energy functional on it is unbounded from below and bounded from above uniformly on the radius R of the domain. The use of W_k will be crucial for the proof of Theorem 1.2, in particular to obtain uniform estimates of the energy.

Section 5 is therefore devoted to the proofs of Theorems 1.2 and 1.3.

Section 6 contains the proof of Theorem 1.1 which is obtained through a limit procedure on the radius of the balls $\mathbb{B}_R$. In particular we controll the number of zeros while passing to the limit using ODE techniques, Strauss Lemma and maximum principles.

2. Notations and preliminaries

Let us fix some notations.

$\mathbb{B}_R := \left\{x \in \mathbb{R}^3 : |x| < R\right\}$ is the ball of $\mathbb{R}^3$ of radius R.

For an open $\Omega \subseteq \mathbb{R}^3$, $(L^r(\Omega), \|\cdot\|_{L^3(\Omega)})$ is the usual Lebesgue space, we may also write the norm simply as $\|\cdot\|_r$ when there is no misunderstanding about the integration set.

$(W^{s,r}(\Omega), \|\cdot\|_{W^{s,r}(\Omega)})$ and $W^{s,r}_0(\Omega)$ are the usual Sobolev spaces we may also write the norm simply as $\|\cdot\|_{s,r}$ when there is no misunderstanding about the integration set.

In particular we write $H^1(\Omega)$ resp. $H^1_0(\Omega)$ instead of $W^{1,2}(\Omega)$ resp. $W^{1,2}_0(\Omega)$ and in this case we may denote the norm simply with $\|u\| := \|u\|_{H^1_0(\Omega)} = \int_\Omega(|\nabla u|^2 + |u|^2)dx$.

$C^{m,\alpha}(\Omega)$ is the subspaces of $C^m(\Omega)$ consisting of functions whose m-th order partial derivatives are *locally Hölder continuous with exponent α in Ω.*

If Ω is bounded then we denote by $(C^{m,\alpha}(\bar{\Omega}), \|u\|_{C^{m,\alpha}(\bar{\Omega})})$ the Banach space of all the functions belonging to $C^m(\bar{\Omega})$ whose m-th order partial derivatives are *uniformly Hölder continuous with exponent α in $\bar{\Omega}$.* endowed with the usual norm.

$D^{1,2}(\mathbb{R}^3)$ is the closure of $C^\infty_0(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}} = \|\nabla u\|_{L^2(\mathbb{R}^3)}$.

The following facts are known (see for instance [16, Lemma 0.3.1]) and [24, Lemma 2.1]):

Lemma 2.1. *For any $v \in H^1(\mathbb{R}^3)$ let*

$$\phi_v(x) := \int_{\mathbb{R}^3} \frac{v^2(y)}{|x-y|} dy.$$

Then

i) $\phi_v \in D^{1,2}(\mathbb{R}^3)$ *and there exists* $C > 0$ *(independent of* v*) such that*

$$\|\phi_v\|_{D^{1,2}} \leq C\|v\|^2.$$

Hence in particular there exists $C > 0$ *(independent of* v*) such that*

$$\iint \frac{w^2(x)v^2(y)}{|x-y|} dxdx \leq C\|w\|^2\|v\|^2 \qquad \forall w \in H^1(\mathbb{R}^3). \tag{4}$$

ii) ϕ_v *is the unique weak solution in* $D^{1,2}(\mathbb{R}^3)$ *of the equation* $-\Delta\phi_v = 4\pi v^2$ *in* $\mathbb{R}^3$.

iii) *If* v *is radial, then* ϕ_v *is radial and has the following expression*

$$\phi_v(r) = \frac{1}{r}\int_0^{+\infty} v^2(s)s\min\{r,s\}ds.$$

iv) *Let* $v_n, v \in H^1(\mathbb{R}^3)$*, radial and satisfying* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^3)$*, then* $\phi_{v_n} \to \phi_v$ *in* $D^{1,2}(\mathbb{R}^3)$.

Here and in the following for $v \in H_0^1(\mathbb{B}_R)$, we write again v also for the trivial extension of v in the whole $\mathbb{R}^3$, which belongs to $H^1(\mathbb{R}^3)$, one has that the nonlocal term which appears in the equation (2) coincides with the restriction of ϕ_v to $\mathbb{B}_R$.

Hence if $u \in H_0^1(\mathbb{B}_R)$ is a solution of (2) then the couple $(u, \phi_u|_{\mathbb{B}_R})$ is a solution of the SPS system in the ball

$$\begin{cases} -\Delta u + u + \phi u - |u|^{q-1}u = 0 & \text{in } \mathbb{B}_R \\ -\Delta\phi = 4\pi u^2 & \text{in } \mathbb{B}_R \\ u = 0 & \text{on } \partial\mathbb{B}_R \end{cases}$$

We underline that ϕ_u is not 0 on $\partial\mathbb{B}_R$.

Remark 2.2. *If* $v \in W_0^{1,p}(\mathbb{B}_R)$ *,* $p > 3$ *then* $\phi_v \in C^{2,\alpha}(\bar{\mathbb{B}}_R)$ *and satisfies* $-\Delta\phi_v = 4\pi v^2$ *in* $\mathbb{B}_R$.

Indeed by Sobolev embedding ($p > 3$*)* $v \in W_0^{1,p}(\mathbb{B}_R) \hookrightarrow C^{0,\alpha}(\bar{\mathbb{B}}_R)$ *and* $v = 0$ *on* $\partial\mathbb{B}_R$*. Let* $\tilde{v}$ *be the trivial extension of* v *in* $\mathbb{B}_{2R}$ *then* $\tilde{v}^2 \in C^{0,\alpha}(\bar{\mathbb{B}}_{2R})$ *and so (see* [15, Lemma 4.2 and 4.4]*)* $\phi_v \in C^{2,\alpha}(\bar{\mathbb{B}}_R)$ *and satisfies* $-\Delta\phi_v = 4\pi v^2$ *in* $\mathbb{B}_R$.

3. The associated parabolic problem

We consider the semilinear nonlocal parabolic initial-boundary-value problem (IBVP) associated with (2) which has been studied in [17]:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u = F(u) & \text{in } \mathbb{B}_R \times (0,+\infty), \\ u = 0 & \text{in } \partial\mathbb{B}_R \times (0,+\infty), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{B}_R, \end{cases} \tag{5}$$

where the nonlinearity is

$$F(u) := |u|^{q-1}u - u\int_{\mathbb{B}_R} \frac{1}{|x-y|}u^2(y)dy, \qquad q \in [3,5).$$

Next we fix $3 < p < \infty$ and we consider the function spaces

$$X = \left\{u \in W_0^{1,p}(\mathbb{B}_R) : u \text{ radially symmetric}\right\},$$
$$Y = \{u \in C^1(\bar{\mathbb{B}}_R) : u \text{ is radial and } u = 0 \text{ on } \partial\mathbb{B}_R\}.$$

We have the embedding $Y \hookrightarrow X \hookrightarrow C(\bar{\mathbb{B}}_R)$.

We recall the following result related to local existence and regularity which can be found in [17, Theorem 1.1].

Theorem 3.1. *For every $u_0 \in X$ the IBVP (5) has a unique (mild) solution $u(t) = \varphi(t, u_0) \in C([0, T), X)$ with maximal existence time $T := T(u_0) > 0$ which is a classical solution for $t \in (0, T)$.*

The set $\mathcal{G} := \{(t, u_0) : t \in [0, T(u_0))\}$ is open in $[0, \infty) \times X$, and $\varphi : \mathcal{G} \to X$ is a semiflow on X.

Moreover the following continuity property with respect to the initial datum in stronger norm holds (see [17, Theorem 1.1-iv)])

Proposition 3.2. *For every $u_0 \in X$ and every $t \in (0, T(u_0))$ there is a neighborhood $U \subset X$ of u_0 in X such that $T(u) > t$ for $u \in U$, and $\varphi(t, \cdot) : (U, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a continuous map.*

In the following we will often write $\varphi^t(u)$ instead of $\varphi(t, u)$.

The energy E is strictly decreasing along nonconstant trajectories $t \mapsto \varphi^t(u_0)$ in X. In fact for a classical solution of (5) we have

$$
\begin{aligned}
\dot{E} &= \frac{d}{dt}E(u) = \frac{d}{dt}\int \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{q+1}|u|^{q+1} + \frac{1}{4}\phi_u u^2\right) dx \\
&= \int \left(\nabla u \nabla u_t + u u_t - |u|^{q-1} u u_t + \phi_u u u_t\right) dx \\
&= \int \left(-\Delta u + u - |u|^{q-1} u + \phi_u u\right) u_t dx \\
&= -\int u_t^2 dx = -\|u_t\|_2^2, \qquad (6)
\end{aligned}
$$

this property is crucial in order to prove the following global existence result and compactness property (see [17, Theorem 1.4] for the proof)

Theorem 3.3. *Let $u_0 \in X$ and $T = T(u_0)$ be such that the function $t \mapsto E(\varphi^t(u_0))$ is bounded from below in $(0, T.)$ Then $T = \infty$ and for every $\delta > 0$ the set $\{\varphi^t(u_0) : t \geq \delta\}$ is bounded in $W^{s_1,p}(\mathbb{B}_R)$ for every $s_1 \in [1, 2)$ and hence relatively compact in $C^1(\bar{\mathbb{B}}_R)$.*

Corollary 3.4. *Let $u_0 \in X$ and $T = T(u_0)$ be such that the function $t \mapsto E(\varphi^t(u_0))$ is bounded from below in $(0, T.)$ Then $T = \infty$ and the ω-limit set*

$$\omega(u_0) = \cap_{t>0} Clos_Y\left(\{\varphi^s(u_0) \ : \ s \geq t\}\right)$$

is a nonempty compact subset of Y consisting of radial solutions of (2).

Next we show that the number of nodal regions of a solution of (5) is non increasing along the flow.
Given $u \in X$ we define the number of sign changes in the radial variable $i(u)$ of u as the maximal $k \in \mathbb{N} \cup \{0, \infty\}$ such that there exist points $x_1, .., x_{k+1} \in \mathbb{B}_R$ with $0 \leq |x_1| < .. < |x_{k+1}| < R$ and $u(x_i)u(x_{i+1}) < 0$ for $i = 1, .., k$.

Lemma 3.5. *Let $u_0 \in X$ and $T = T(u_0)$. Then $t \mapsto i(\varphi^t(u_0))$ is non increasing in $t \in [0, T)$.*

Proof. In view of the semiflow properties, it suffices to show the inequality $i(\varphi^\tau(u_0)) \leq i(u_0)$ for a fixed $\tau \in (0,T)$. Since $u(t) := \varphi^t(u_0)$ satisfies the equation

$$u_t - \Delta u + f(x,t)u = 0 \text{ in } \mathbb{B}_R \times (0,\tau]$$

where $f(x,t) := 1 + \phi_u(x,t) - |u(x,t)|^{q-1}$, to prove the result we can follow the arguments in [8, Theorem 2.1] (see also [30, Lemma 2.5]). The only thing which remains to be proved is therefore that f is bounded in $\mathbb{B}_R \times [0,\tau]$.

On this scope we observe that $|u|^{q-1}$ is continuous and hence bounded in $\bar{\mathbb{B}}_R \times [0,\tau]$ (indeed $X \hookrightarrow C^{0,\alpha}_r$, $u(\cdot,t) \in X$ for every t and $t \mapsto u(t,x) \in C([0,T),X)$).
We show now that $\phi_u(x,t) = \int_{\mathbb{B}_R} \frac{u^2(y,t)}{|x-y|} dy$ is continuous and hence bounded in $\bar{\mathbb{B}}_R \times [0,\tau]$. The continuity of the function $x \mapsto \phi_u(x,t)$ in $\bar{\mathbb{B}}_R$ for fixed $t \in [0,\tau]$ is trivial (see Remark 2.2). We fix now $x \in \bar{\mathbb{B}}_R$ and show that the function $t \mapsto \phi_u(x,t)$ is continuous in $[0,\tau]$.

Let $(t_n)_n \subset [0,\tau]$ $t_n \to_n t_0$, we want to prove that $\phi_u(x,t_n) \to_n \phi_u(x,t_0)$. This follows from the following:

$$\begin{aligned}\lim_n |\phi_u(x,t_n) - \phi_u(x,t_0)| &= \lim_n \left| \int_{\mathbb{B}_R} \left(\frac{u^2(y,t_n)}{|x-y|} - \frac{u^2(y,t_0)}{|x-y|} \right) dy \right| \\ &\leq \lim_n \max_{y \in \bar{\mathbb{B}}_R} \left| u^2(y,t_n) - u^2(y,t_0) \right| \int_{\mathbb{B}_R} \frac{1}{|x-y|} dy = 0\end{aligned}$$

□

4. A finite dimensional subspace of X

In this section we define, for any fixed integer $k \geq 2$ a convenient k-dimensional subspace $W_k \subset X$ such that the restriction to it of the energy functional $E : H^1_0(\mathbb{B}_R) \to \mathbb{R}$

$$E(u) = \frac{1}{2}\int |\nabla u(x)|^2 dx + \frac{1}{2}\int u(x)^2 dx + \frac{1}{4}\int\int \frac{u^2(x)u^2(y)}{|x-y|} dxdy - \frac{1}{q+1}\int |u|^{q+1}(x) dx$$

has some "good properties" (it is unbounded below "at infinity" and bounded from above uniformly in the radius R of the domain).

As we will see, we need to pay particular attention to the case $q = 3$, since the geometric properties of the energy E are different (see condition (8) below).

The spaces W_k will be useful in next section to prove our results.

First for $0 < a < b$ we define the annulus $\mathbb{A}_{a,b} := \{x \in \mathbb{R}^3 : a < |x| < b\}$.

Then we fix k radial functions $w_i \in C^2(\mathbb{R}^3)$ $i = 1,..,k$ with disjoint supports which satisfy the following properties:

$$\begin{cases} w_1 > 0 & \text{in } \mathbb{B}_{\frac{1}{k}} \\ w_1 = 0 & \text{in } \mathbb{R}^3 \setminus \mathbb{B}_{\frac{1}{k}} \end{cases}$$

$$\begin{cases} w_i > 0 & \text{in } \mathbb{A}_{\frac{i-1}{k},\frac{i}{k}} & i = 2,..,k \\ w_i = 0 & \text{in } \mathbb{R}^3 \setminus \mathbb{A}_{\frac{i-1}{k},\frac{i}{k}} & i = 2,..,k \end{cases}$$

We can always assume that

$$\|w_i\|^2 = \|w_i\|^{q+1}_{q+1} \qquad i = 1,..,k \tag{7}$$

(if not we just rescale w_i).

We also define

$$M_k := \max_{i=1,..,k} \|w_i\|^2 > 0.$$

Last for $q = 3$ we require also that

$$M_k < \frac{1}{k} \tag{8}$$

Lemma 4.1. *Let*

$$W_k := \left\{ w = w_{(t_1,..,t_k)} := \sum_{j=1}^{k} t_j w_j \quad : \ (t_1,..,t_k) \in \mathbb{R}^k l \right\} \ \subset \ X.$$

Then there exists $C_k > 0$ such that

$$E(w) \le C_k \quad \textit{for all } w \in W_k. \tag{9}$$

Moreover

$$\lim_{\substack{\|w\| \to \infty \\ w \in W_k}} E(w) = -\infty \tag{10}$$

Proof. Let $w \in W_k$, then $w = \sum_{j=1}^{k} t_j w_j$ where $t_j \in \mathbb{R}$, $j = 1,..,k$, hence using the inequality (4)

$$\begin{aligned}
E(w) \ &= \ \sum_{j=1}^{k} \left[\frac{t_j^2}{2} \|w_j\|^2 - \frac{|t_j|^{q+1}}{q+1} \|w_j\|_{q+1}^{q+1} \right] + \frac{1}{4} \sum_{i,j=1}^{k} t_j^2 t_i^2 \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{w_j^2(x) w_i^2(y)}{|x-y|} dx dy \\
&\le \sum_{j=1}^{k} \left[\frac{t_j^2}{2} \|w_j\|^2 - \frac{|t_j|^{q+1}}{q+1} \|w_j\|^2 \right] + \frac{1}{4} \sum_{i,j=1}^{k} t_j^2 t_i^2 \|w_j\|^2 \|w_i\|^2 \qquad (\|w_j\|_{q+1}^{q+1} = \|w_j\|^2) \\
&\le \sum_{j=1}^{k} \left[\frac{t_j^2}{2} \|w_j\|^2 - \frac{|t_j|^{q+1}}{q+1} \|w_j\|^2 \right] + \frac{k}{4} \sum_{j=1}^{k} t_j^4 \|w_j\|^4 \\
&= \sum_{j=1}^{k} \|w_j\|^2 \left[\frac{t_j^2}{2} + \frac{k}{4} t_j^4 \|w_j\|^2 - \frac{|t_j|^{q+1}}{q+1} \right] \\
&\le \sum_{j=1}^{k} \|w_j\|^2 \left[\frac{t_j^2}{2} + \frac{k M_k}{4} t_j^4 - \frac{|t_j|^{q+1}}{q+1} \right] \qquad (11)
\end{aligned}$$

Let now be $q > 3$, then putting $0 < G_k := \max_{s \in [0,+\infty)} g_k(s)$, where $g_k(s) := \left[\frac{s^2}{2} + \frac{k M_k}{4} s^4 - \frac{|s|^{q+1}}{q+1} \right]$, from (11) it follows that

$$E(w) \le \sum_{j=1}^{k} \|w_j\|^2 G_k \le k M_k G_k.$$

Moreover, again from (11) we have that

$$\lim_{\|w\| \to +\infty} E(w) = -\infty,$$

since $+\infty \leftarrow \|w\| = \sum_{j=1}^{k} |t_j| \|w_j\|$ iff there exists (at least one) $J \in \{1,..,k\}$ s.t. $|t_J| \to +\infty$.

If $q = 3$ then $g_k(s) = \frac{s^2}{2} - s^4 \frac{1 - k M_k}{4}$ where $1 - k M_k > 0$ by our choice in (8) and we proceed in a similar way. $\square$

Remark 4.2. *Each function w_j $j = 1, .., k$ has support contained in $\mathbb{B}_1$, hence in $\mathbb{B}_R$ for any $R \geq 1$. Therefore the choice of the space W_k as well as the results in the lemma above are independent on the radius R of the domain of problem* (2).

5. Proof of Theorem 1.2 and Theorem 1.3

The proof relies on a dynamical method: we find solutions of the elliptic problem (2) looking for equilibria in the ω-limit sets of trajectories of the autonomous parabolic problem (3). In order to obtain equilibria with a fixed number of changes of sign we need to select in a proper way the initial condition. This is done following an approach first introduced in [30] in the contest of symmetric systems of two coupled Schrödinger equations.

This method consists in selecting special initial data on the boundary of the domain of attraction of an asymptotically stable equilibrium.

It relies on the crucial monotonicity property for the number of zeros along the flow (Lemma 3.5) and combines the study of the parabolic flow with a topological argument based on the use of the Krasnonelskii genus. For completeness we will repeat here the main arguments of [30] adapted to our scalar case.

Since the energy functional E is strictly decreasing along nonconstant trajectories (see (6)) and 0 is a strict local minimum for it, it follows that the constant solution $u \equiv 0$ is asymptotically stable in X.

Let $\mathcal{A}_*$ be its domain of attraction

$$\mathcal{A}_* := \{u \in X : T(u) = +\infty \text{ and } \varphi^t \to 0 \text{ in } X \text{ as } t \to +\infty\}.$$

The asymptotic stability of 0, the semiflow properties of solutions of (5) and the continuous dependence of solutions on initial data (Corollary 3.2) imply that the set $\mathcal{A}_*$ is a relatively open neighborhood of 0 in X.

As in [30] we denote with $\partial\mathcal{A}_*$ the relative boundary of the set $\mathcal{A}^*$ in X.

Since $\mathcal{A}_*$ is open and 0 is asymptotically stable, the continuous dependence of the semiflow φ on the initial values implies that $\partial\mathcal{A}_*$ is positively invariant under φ.
Moreover $E(u) \geq 0$ for every $u \in \mathcal{A}_*$ since E is decreasing along trajectories, and hence, by continuity, this is true also for every $u \in \partial\mathcal{A}_*$. As a consequence by Corollary 3.4 one has that the solution is global for every initial value $u \in \partial\mathcal{A}_*$, the ω-limit set is nonempty and $\omega(u) \subset \partial\mathcal{A}_*$.

Since the ω-limit consists of radial solutions of the elliptic problem (2),our aim is to select suitable initial conditions u on $\partial\mathcal{A}_*$ in a way that any element in $\omega(u)$ has $k-1$ changes of sign. Following [30] we define therefore the closed subset of X

$$\mathcal{A}_k := \{u \in \partial\mathcal{A}_* : i(u) \leq k-1\}.$$

Lemma 3.5 and the positive invariance of $\partial\mathcal{A}_*$ for the flow φ imply that $\mathcal{A}_k$ is a positively invariant set for the flow φ.
Using a topological argument similar to the one in [30], we prove hence the existence of a certain $\bar{u} \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ such that $\omega(\bar{u}) \subset \mathcal{A}_k \setminus \mathcal{A}_{k-1}$, for every $k \geq 2$.

On this scope let's observe that the parabolic problem (5) has an odd nonlinearity hence the semiflow φ^t is odd and the sets $\partial\mathcal{A}_*$ and $\mathcal{A}_k$, $k \geq 1$ are symmetric with

respect to the origin.
For a closed symmetric subset $B \subset \partial\mathcal{A}_*$ we denote by $\gamma(B)$ the usual Krasnoselsii genus and we recall some of the properties we will need:

Lemma 5.1. *Let $A, B \subset \partial\mathcal{A}_*$ be closed and symmetric.*
(i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(ii) *If $h : A \to \partial\mathcal{A}_*$ is continuous and odd, then $\gamma(A) \leq \gamma(\overline{h(A)})$.*
(iii) *If S is a bounded symmetric neighborhood of the origin in a k-dimensional normed vector space and $v : \partial S \to \partial\mathcal{A}_*$ is continuous and odd, then $\gamma(v(\partial S)) \geq k$.*

Let
$$\mathcal{O} := \mathcal{A}_* \cap W_k,$$
where W_k is the k-dimensional subspace of X defined at Section 4 (Lemma 4.1). $\mathcal{O}$ is a symmetric, bounded (from (10)) open neighborhood of 0 on W_k.

Lemma 5.2. *$\partial\mathcal{O} \subset \mathcal{A}_k$ and $\gamma(\partial\mathcal{O}) = \gamma(\mathcal{A}_k) = k$*

Proof. The inclusion $\partial\mathcal{O} \subset \mathcal{A}_k$ is a consequence of the definition of W_k. Let us compute the genus. From the property (iii) of the genus it follows immediately that $\gamma(\partial\mathcal{O}) \geq k$. Moreover, adapting the arguments in the proof of [30, Lemma 3.3], one can prove that $\gamma(\mathcal{A}_k) \leq k$. The conclusion comes from the monotonicity property (i) of the genus. □

We define also the closed subsets of $\partial\mathcal{A}_*$
$$\mathcal{C}^t_{k-1} := \{u \in \partial\mathcal{A}_* : \varphi^t(u) \in \mathcal{A}_{k-1}\} \quad \text{for } t > 0.$$

Lemma 5.3. *$\mathcal{A}_{k-1} \subset \mathcal{C}^t_{k-1}$ and $\gamma(\mathcal{C}^t_{k-1}) = \gamma(\mathcal{A}_{k-1}) = k-1$ for every $t > 0$.*

Proof. The proof is trivial once we know that $\gamma(\mathcal{C}^t_{k-1}) \leq k-1$.

This is a consequence of the property (ii) of the genus, indeed $\gamma(\mathcal{C}^t_{k-1}) \leq \gamma(\overline{\varphi^t(\mathcal{C}^t_{k-1})})$ since the map $\varphi^t : \mathcal{C}^t_{k-1} \to \partial\mathcal{A}_*$ is continuous and odd, and $\gamma(\overline{\varphi^t(\mathcal{C}^t_{k-1})}) \leq k-1$ because of the inclusion $\overline{\varphi^t(\mathcal{C}^t_{k-1})} \subset \mathcal{A}_{k-1}$. □

Proposition 5.4. *There exists $\bar{u} \in \partial\mathcal{O} \setminus \mathcal{A}_{k-1}$ such that $\emptyset \neq \omega(\bar{u}) \subset \mathcal{A}_k \setminus \mathcal{A}_{k-1}$.*

Proof. From Lemma 5.2 and Lemma 5.3 $\emptyset \neq \partial\mathcal{O} \setminus \mathcal{C}^t_{k-1} \subset \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ for every $t > 0$.
In particular for any positive integer n there exists $u_n \in \partial\mathcal{O} \setminus \mathcal{C}^n_{k-1}$ and, since $\partial\mathcal{O}$ is compact, we may pass to a subsequence such that $u_n \to \bar{u} \in \partial\mathcal{O}$ as $n \to \infty$.
Obviously $\omega(\bar{u}) \subset \mathcal{A}_k$, as in [30] we now prove that $\omega(\bar{u}) \subset \mathcal{A}_k \setminus \mathcal{A}_{k-1}$.

On this scope we define the sets
$$Y_k := \{u \in Y : i(u) \leq k-1\}$$
(see Section 3 for the definition of the space Y). By construction (using the continuity property in Proposition 3.2) one has that $\varphi^t(\bar{u}) \notin Int_Y(Y_{k-1})$ for every $t > 0$, which implies that $\omega(\bar{u}) \cap Int_Y(Y_{k-1}) = \emptyset$.

On the other hand if we assume by contradiction that $\omega(\bar{u}) \cap \mathcal{A}_{k-1} \neq \emptyset$ than, since $\omega(\bar{u})$ consists of radial solutions of (2), one can easily show (with arguments similar to the ones in [30, Lemma 3.1]) that $\omega(\bar{u}) \cap Int_Y(Y_{k-1}) \neq \emptyset$, reaching a contradiction. □

The proof of Theorem 1.2 follows from Proposition 5.4 taking any $u \in w(\bar{u})$. u

is a radial solution for the elliptic problem (2) with exactly $k-1$ changes of sign. Moreover, since the energy is non-increasing along trajectories and using the energy estimate in W_k, (see (9)) it satisfies

$$E(u) \leq E(\bar{u}) \leq C_k.$$

As a byproduct we obtain also the proof of Theorem 1.3 related to the existence of global solutions of the parabolic problem (5) with the same fixed number of nodal regions along the flow.

Indeed $u_k(t) := \varphi^t(\bar{u})$ is a global solution of the parabolic problem and, from the positive invariance of the sets $\mathcal{A}_k$, it follows that $u_k(t) \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ for every $t \geq 0$.

6. Proof of Theorem 1.1

For fixed $k \geq 2$, let $R_n \geq 1, n \in \mathbb{N}$ such that $R_n \to +\infty$ as $n \to +\infty$, let $u_n \in H_0^1(\mathbb{B}_n)$ be a radial weak solution of the Dirichlet problem in the ball $\mathbb{B}_n = \mathbb{B}_{R_n}$

$$\begin{cases} -\Delta u_n + u_n + u_n \int \frac{u_n^2(y)}{|x-y|} dy - |u_n|^{q-1} u_n = 0 & \text{in } \mathbb{B}_n \\ u_n = 0 \quad \text{on } \partial\mathbb{B}_n \end{cases} \tag{12}$$

with precisely $(k-1)$ changes of sign in the radial variable and which satisfies the energy uniform bound $E(u_n) \leq M_k$ (from Theorem 1.2).

Proposition 6.1 (Uniform H^1-bound)**.** *There exists $D_k > 0$ (independent of n) such that*

$$\|u_n\| \leq D_k \quad \forall n \in \mathbb{N}.$$

Proof. Since u_n is a solution it satisfies

$$E'(u_n)(u_n) = 0,$$

hence, since $q \geq 3$

$$M_k \geq E(u_n) = E(u_n) - \frac{1}{4} E'(u_n)(u_n) = \frac{1}{4}\|u_n\|^2 + \frac{q-3}{4(q+1)}\|u_n\|_{q+1}^{q+1} \geq \frac{1}{4}\|u_n\|^2,$$

namely

$$\|u_n\| \leq 2\sqrt{M_k}.$$

□

Lemma 6.2 (Regularity of u_n and uniform $C^{2,\alpha}_{loc}$-bound)**.** $u_n \in C^{2,\alpha}(\bar{\mathbb{B}}_n)$ *and it is a classical solution of the Dirichlet problem* (12)*. In particular for any $R > 0$ there exist $n_R \in \mathbb{N}$ and $C_R > 0$ such that*

$$u_n \in C^{2,\alpha}(\bar{\mathbb{B}}_R) \quad \text{and} \quad \|u_n\|_{C^{2,\alpha}(\bar{\mathbb{B}}_R)} \leq C_R \quad \text{for all } n \geq n_R.$$

(The constant C_R is independent of n but depends on R, q, α, D_k).

Proof. Let

$$\phi_n(x) := \phi_{u_n}(x) = \int \frac{u_n^2(y)}{|x-y|} dy.$$

STEP 1 $\phi_n \in C^{0,\alpha}(\mathbb{R}^3)$. *Moreover for any $R > 0$ there exists $C = C(\alpha, R, D_k) > 0$ such that*

$$\|\phi_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_R)} \leq C \ \forall n \in \mathbb{N}.$$

Since $u_n \in H^1(\mathbb{R}^3)$, it can be proved that $\phi_n \in D^{1,2}(\mathbb{R}^3)$ and that the following bound holds

$$\|\phi_n\|_{D^{1,2}} \leq C\|u_n\|^2 \tag{13}$$

where $C > 0$ is independent of $n \in \mathbb{N}$ (see Lemma (2.1)).

Moreover, by the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we deduce that $u_n^2 \in L^3(\mathbb{R}^3)$, hence (using for instance [15, Theorem 9.9] in any domain Ω which contains $\bar{\mathbb{B}}_n$) $\phi_n \in W^{2,3}_{loc}(\mathbb{R}^3)$ and $-\Delta\phi_n = u_n^2$ a.e. in $\mathbb{R}^3$.

Therefore from [15, Theorem 9.11], Sobolev embeddings and (13) we obtain

$$\begin{aligned}\|\phi_n\|_{W^{2,3}(\mathbb{B}_R)} &\leq C(R)\left(\|\phi_n\|_{L^3(\mathbb{B}_{2R})} + \|u_n^2\|_{L^3(\mathbb{B}_{2R})}\right)\\ &\leq C(R)\left(\|\phi_n\|_{D^{1,2}} + \|u_n\|^2\right)\\ &\leq C(R)(1+C)\|u_n\|^2.\end{aligned}$$

The conclusion follows from the Sobolev embedding $W^{2,3}(\mathbb{B}_R) \hookrightarrow C^{0,\alpha}(\bar{\mathbb{B}}_R)$ and Proposition 6.1.

STEP 2 $u_n \in C^{0,\alpha}(\mathbb{R}^3)$. *In particular for any $R > 0$ there exists $C = C(q, \alpha, R, D_k) > 0$ such that*

$$u_n \in C^{0,\alpha}(\bar{\mathbb{B}}_R) \quad \text{and} \quad \|u_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_R)} \leq C \quad \forall n$$

(here u_n stays for its trivial extension).

u_n is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u_n = f_n & \text{in } \mathbb{B}_n \\ u_n = 0 & \text{on } \partial\mathbb{B}_n \end{cases} \tag{14}$$

where $f_n := |u_n|^{q-1}u_n - (1+\phi_n)u_n \in L^{2^*/q}(\mathbb{B}_n)$ ($\phi_n \in C^0(\bar{\mathbb{B}}_n)$ by Step 1). By L^p-regularity it follows that $u_n \in W^{2,2^*/q}(\mathbb{B}_n)$ and so by a classical bootstrap argument $u_n \in W^{2,p}(\mathbb{B}_n)$ for a certain $p > \frac{N}{2}$; by Sobolev embeddings we conclude that $u_n \in C^{0,\alpha}(\bar{\mathbb{B}}_n)$. Substituting u_n by its trivial extension (remember that $u_n = 0$ on $\partial\mathbb{B}_n$) one obtains that $u_n \in C^{0,\alpha}(\mathbb{R}^3)$.

We prove now the uniform $C^{0,\alpha}_{loc}$-estimate. We denote by C any constant which doesn't depend on $n \in \mathbb{N}$ but which may eventually depend on q, α, R, D_k and which may vary from line to line.
Let us observe that for each fixed $R > 0$, there exists $n_R \in \mathbb{N}$ such that $\mathbb{B}_{2R} \subseteq \mathbb{B}_n$ $\forall n \geq n_R$. Therefore from Sobolev embeddings ($p > \frac{N}{2}$) we have for $n \geq n_R$

$$\|u_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_R)} \leq C\|u_n\|_{W^{2,p}(\mathbb{B}_R)}; \tag{15}$$

moreover from L^p-estimates (cfr. [15, Theorem 9.11]) and Step 1

$$\begin{aligned}\|u_n\|_{W^{2,p}(\mathbb{B}_R)} &\leq C\left(\|u_n\|_{L^p(\mathbb{B}_{2R})} + \|f_n\|_{L^p(\mathbb{B}_{2R})}\right)\\ &\leq C\left(\|u_n\|_{L^p(\mathbb{B}_{2R})} + \|u_n\|^q_{L^{qp}(\mathbb{B}_{2R})} + (1 + \|\phi_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_{2R})})\|u_n\|_{L^p(\mathbb{B}_{2R})}\right)\\ &\leq C\left(\|u_n\|^q_{L^{qp}(\mathbb{B}_{2R})} + \|u_n\|_{L^{pq}(\mathbb{B}_{2R})}\right) \quad (pq > p) \qquad (16)\end{aligned}$$

Remember now that $p \geq \frac{2^*}{q}$ is obtained after a finite number $m \geq 0$ of iterations (bootstrap procedure) starting from $\frac{2^*}{q}$. If $p = \frac{2^*}{q}$ we can conclude directly from (15) and (16), using Sobolev embeddings and Proposition 6.1, indeed

$$\|u_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_R)} \leq C\left(\|u_n\|^q + \|u_n\|\right) \leq C\left(D_k^q + D_k\right) = C.$$

Otherwise, if $p > \frac{2^*}{q}$ then it is easy to verify that we reduce to the previous case iterating m times the estimate (16) together with the Sobolev embeddings $W^{2,\frac{Ns}{N+2s}} \hookrightarrow L^s$, for opportune $s > 0$. We remark that at each step the constant involved is independent of n.

The previous argument gives the local estimate only definitely for $n \geq n_R$; To extend the result also to the first $n_R - 1$ elements it is enough to substitute u_n with its trivial extension for each $n \leq n_R - 1$ (in this case $\mathbb{B}_n \subset \mathbb{B}_{2R}$).

STEP 3 *Conclusion.*

u_n is a weak solution of the Dirichlet problem (14), where $f_n \in C^{0,\alpha}(\bar{\mathbb{B}}_n)$ (from Step 1 and Step 2). By elliptic regularity we conclude that $u_n \in C^{2,\alpha}(\bar{\mathbb{B}}_n)$ and it is a classical solution of the Dirichlet problem.

To prove the uniform $C^{2,\alpha}_{loc}$-estimate we fix any $R > 0$ and let $n_R \in \mathbb{N}$ such that $\mathbb{B}_{2R} \subseteq \mathbb{B}_n$ $\forall n \geq n_R$. Hence for $n \geq n_R$

$$\|u_n\|_{C^{2,\alpha}(\bar{\mathbb{B}}_R)} \leq C \left(\|u_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_{2R})} + \|f_n\|_{C^{0,\alpha}(\bar{\mathbb{B}}_{2R})} \right) \leq C,$$

where we have used the Schauder estimates (see for instance [15, Theorem 4.6]) and the uniform local bounds from Step 1 and Step 2. □

Proposition 6.3. *There exists $u \in C^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, radial such that*

$$u_{n_k} \to u \text{ in } C^2_{loc}(\mathbb{R}^3).$$

Moreover u is a solution of

$$-\Delta u + u + u \int \frac{u^2(y)}{|x-y|} dy - |u|^{q-1} u = 0 \quad \text{in } \mathbb{R}^3 \tag{17}$$

Proof. Since u_n is bounded in $H^1(\mathbb{R}^3)$ (Proposition 6.1), one can extract a subsequence of u_n, again denoted by u_n, such that u_n converges weakly in $H^1(\mathbb{R}^3)$ and almost everywhere in $\mathbb{R}^3$ to a function u. Observe that $u \in H^1(\mathbb{R}^3)$ is spherically symmetric. Moreover, since for any $R > 0$ the sequence u_n is definitely bounded in $C^{2,\alpha}(\mathbb{B}_R)$ (Lemma 6.2), by Arzela's theorem and a standard diagonal process one can also prove that $u \in C^2(\mathbb{R}^3)$ and that u_n converges to u in $C^2_{loc}(\mathbb{R}^3)$.

In order to prove that u satisfies equation (17) it is enough to pass to the limit for a.e. $x \in \mathbb{R}^3$ into the equation pointwise satisfied by u_n definitely (by Lemma 6.2)

$$-\Delta u_n(x) + u_n(x) + u_n(x)\phi_n(x) - |u_n(x)|^{q-1} u_n(x) = 0$$

To this scope let's observe that from the compactness result iv) in Lemma 2.1 one can extract a subsequence of ϕ_n, again denoted by ϕ_n, such that ϕ_n converges (strongly in $D^{1,2}(\mathbb{R}^3)$ and hence, by Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$) almost everywhere in $\mathbb{R}^3$ to the function $\phi_u = \int \frac{u^2(y)}{|x-y|} dy$. □

Lemma 6.4. *$\phi_n \in C^{2,\alpha}(\mathbb{R}^3)$ and it is a classical solution of the equation $-\Delta \phi_n = u_n^2$ in $\mathbb{R}^3$. Moreover for any $R > 0$ there exists $D_R > 0$ such that*

$$\phi_n \in C^{2,\alpha}(\bar{\mathbb{B}}_R) \quad \text{and} \quad \|\phi_n\|_{C^{2,\alpha}(\bar{\mathbb{B}}_R)} \leq D_R \quad \text{for all } n \in \mathbb{N}.$$

(The constant D_R is independent of n but depends on R, q, α, D_k).

Proof. One can easily prove that $u_n^2 \in C^{0,\alpha}(\mathbb{R}^3)$ and it is bounded (see Lemma 6.2-Step 2). Therefore it follows that $\phi_n \in C^2(\mathbb{R}^3)$ and satisfies the equation $-\Delta \phi_n = u_n^2$ in $\mathbb{R}^3$ (using for instance the known regularity result about Newtonian potentials in [15, Lemma 4.2] in any domain Ω which contains $\bar{\mathbb{B}}_n$).

The conclusion follows from interior Hölder estimates for solutions of the Poisson's equation (for instance [15, Theorem 4.6]), namely $\phi_n \in C^{2,\alpha}(\mathbb{R}^3)$ and, fixed any $R > 0$ one has

$$\|\phi_n\|_{C^{2,\alpha}(\bar{\mathbb{B}}_R)} \leq C\left(\|\phi_n\|_{C^0(\bar{\mathbb{B}}_{2R})} + \|u_n^2\|_{C^{0,\alpha}(\bar{\mathbb{B}}_{2R})}\right) \leq C \quad \forall n$$

where the last inequality follows from Step 1 and Step 2 in Lemma 6.2. □

Proposition 6.5. $\phi_u := \int \frac{u^2(y)}{|x-y|}dy \in C^2(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3)$ *and it is a classical radial solution of the equation* $-\Delta\phi_u = 4\pi u^2$ *in* $\mathbb{R}^3$.

Proof. From Lemma 6.4, using Arzela's theorem and a standard diagonal process, one can extract a subsequence of ϕ_n, again denoted by ϕ_n, which converges in $C^2_{loc}(\mathbb{R}^3)$ to a function $w \in C^2(\mathbb{R}^3)$. Moreover (Proposition 6.3) one can extract a subsequence of u_n, again denoted by u_n, which converges pointwise in $\mathbb{R}^3$ to the function u. Therefore passing to the limit for a.e. x in $\mathbb{R}^3$ into the equation $-\Delta\phi_n(x) = 4\pi u_n^2(x)$, one can prove that w is a classical solution of the equation $-\Delta w = 4\pi u^2$ in $\mathbb{R}^3$.

Last it is clear that w coincides with ϕ_u since we already know that $\phi_n \to \phi_u$ a.e. in $\mathbb{R}^3$ (see the proof of Proposition 6.3).

To conclude we observe that $\phi_u \in D^{1,2}(\mathbb{R}^3)$ and it's radial because $u \in H^1(\mathbb{R}^3)$ and it's radial (see Lemma 2.1). □

Propositions 6.3 and 6.5 yield the existence of a radial solution (u, ϕ_u) for system (1).

Next we prove that u is nontrivial and has exactly $(k-1)$ changes of sign in the radial variable.

Lemma 6.6. *Let* $\bar{r} > 0$ *be a positive local maximum or a negative local minimum point for* $r \mapsto u(r)$ *(resp.* $r \mapsto u_n(r)$*). Then*

$$|u(\bar{r})| \ \textit{(resp.} \ \ |u_n(\bar{r})|\textit{)} \quad \geq 1$$

Proof. Let $w = u$ (resp. u_n), so writing the equation (12) (resp. (17)) in polar coordinates:

$$-w''(\bar{r}) - \frac{2}{\bar{r}}w'(\bar{r}) = w(\bar{r})\left[|w(\bar{r})|^{q-1} - (1 + \phi_w(\bar{r}))\right].$$

If $\bar{r} > 0$ is a local maximum (resp. a local minimum) point for w then

$$w(\bar{r})\left[|w(\bar{r})|^{q-1} - (1 + \phi_w(\bar{r}))\right] \geq 0 \quad \text{(resp. } \leq 0\text{)}$$

hence, since $w(\bar{r}) > 0$ (resp. $w(\bar{r}) < 0$)

$$|w(\bar{r})|^{q-1} \geq (1 + \phi_w(\bar{r})) \geq 1$$

namely the thesis. □

Lemma 6.7. *There exists* $R > 0$ *such that any positive local maximum or negative local minimum point* $r \geq 0$ *for the function* $r \mapsto u(r)$ *or* $r \mapsto u_n(r)$*, satisfies*

$$r \leq R$$

(R *doesn't depend on* u *nor* n *but may depend on* k*).*

Proof. By Strauss Lemma[1] (see [5]) and the uniform bound on the H^1 norms there exists $R > 0$ such that

$$|u(x)|, |u_n(x)| < \frac{1}{2}, \quad \text{for any } \ x \in \mathbb{R}^3 \setminus \mathbb{B}_R, \quad \text{for any } n \tag{18}$$

We prove the result for u_n. The proof for u can be done in a similar way.

For small n such that $R_n \leq R$ the result is trivial (since $u_n \equiv 0$ in $\mathbb{R}^3 \setminus \mathbb{B}_R$). Hence let's consider n such that $R_n > R$ and by contradiction let $r_n > R$ be a maximum (resp. a minimum) for u_n with $u_n(r_n) > 0$ (resp. $u_n(r_n) < 0$). Hence by Lemma 6.6

$$|u_n(r_n)| \geq 1$$

which contradicts (18). □

Lemma 6.8 (Properties of u)**.** *Let $R > 0$ be as in Lemma 6.7.*

i) *$u \neq 0$;*
ii) *u changes sign;*
iii) *let $\bar{r}$ such that $u(\bar{r}) = 0$, then $\frac{\partial u}{\partial r}(\bar{r}) \neq 0$;*
iv) *let $\bar{r}$ such that $u(\bar{r}) = 0$, then $\bar{r} \in (0, R)$ and it is isolated.*
v) *in every subinterval where $r \mapsto u(r)$ changes sign precisely once, $r \mapsto u_n(r)$ also changes sign precisely once for large n.*

Proof. To prove i) we assume by contradiction that $u \equiv 0$. Then, since u_n converges to u in $C^2_{loc}(\mathbb{R}^3)$ (Proposition 6.3), in particular it follows that

$$\max_{x \in \bar{\mathbb{B}}_R} |u_n(x) - u(x)| = \max_{x \in \bar{\mathbb{B}}_R} |u_n(x)| \to_n 0. \tag{19}$$

But we know that for each $n \in \mathbb{N}$ the function $r \mapsto u_n(r)$ changes sign exactly $(k-1)$ times and it is regular, hence it has at least $(k-1)$ positive maximum/negative minimum points and moreover (Lemma 6.7) these points are all inside $\mathbb{B}_R$. Therefore, from Lemma 6.6 it follows that

$$\max_{x \in \bar{\mathbb{B}}_R} |u_n(x)| \geq 1 \quad \forall n \in \mathbb{N},$$

which contradicts (19).

To prove ii) we assume by contradiction that $u \geq 0$. Then $u > 0$ by the strong maximum principle and hence there exists $C > 0$ such that $u \geq C$ in $\bar{\mathbb{B}}_R$. Since u_n converges to u in $C^2_{loc}(\mathbb{R}^3)$ (Proposition 6.3), one has that $u_n \geq \frac{C}{2} > 0$ in $\bar{\mathbb{B}}_R$ definitely. Which is absurd since u_n changes sign is $\mathbb{B}_R$.

iii) is a direct consequence of the Hopf's boundary Lemma while iv) follows immediately from iii) and Lemma 6.7.

Last we prove v). Let (a, b) be an interval where $r \mapsto u(r)$ changes sign precisely once and let $\bar{r} \in (a, b)$ be the unique point in (a, b) such that $u(\bar{r}) = 0$.

We want to prove that u_n changes sign precisely once in (a, b) for large n.

By Lemma 6.7 we may restrict w.l.o.g to the case $(a, b) \subseteq (0, R)$, in particular (a, b) is a bounded interval.

Since u_n converges uniformly to u on compact intervals (Proposition 6.3), one easily deduces that for n large u_n changes sign at least once in (a, b).

---

[1]Strauss Lemma: let $N \geq 2$; every radial function $u \in H^1(R^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$ and such that

$$|U(x)| \leq C_N |x|^{(1-N)/2} \|u\|_{H^1(\mathbb{R}^N)} \ \text{ for } \ |x| \geq R_N$$

where C_N and R_N depend only on the dimension N

On the other hand from iii) we know that $\frac{\partial u}{\partial r}(\bar{r}) \neq 0$, w.l.o.g. we may assume for instance that $\frac{\partial u}{\partial r}(\bar{r}) > 0$. By continuity $\frac{\partial u}{\partial r}(r) \geq \alpha > 0$ in a neighborhood $O_{\bar{r}} \subseteq (a,b)$ of $\bar{r}$, and from the C^2_{loc}-convergence, it follows that for large n

$$\frac{\partial u_n}{\partial r}(r) \geq \frac{\alpha}{2} > 0 \quad \forall r \in O_{\bar{r}},$$

namely for large n the function $r \mapsto u_n(r)$ is strictly monotone in $O_{\bar{r}}$.

Moreover by assumption, $|u(r)| \geq \beta > 0$ for $r \in (a,b) \setminus O_{\bar{r}}$, therefore from the C^2_{loc}-convergence also $|u_n(r)| \geq \frac{\beta}{2} > 0$ for $r \in (a,b) \setminus O_{\bar{r}}$, for large n.

Hence we can conclude that for large n the function $r \mapsto u_n(r)$ changes sign inside $O_{\bar{r}}$ and exactly once because of the strict monotonicity. □

Proposition 6.9. *u changes sign precisely $(k-1)$ times in the radial variable.*

Proof. From Lemma 6.8 we know that there exists an integer $m \geq 1$ such that the function $r \mapsto u(r)$ changes sign m times. In particular there exist m isolated points $0 < r_1 < r_2 < .. < r_m < R$ such that $u(r_i) = 0$ for any $i = 1,..,m$ and $u(r) \neq 0$ for $r \neq r_i$ $i = 1,..,m$.

Let us define the partition $0 = x_0 < x_1 < ... < x_{m-1} < x_m = R$ of $(0,R)$ where $x_i = (r_i + r_{i+1})/2$, $i = 1,..,m-1$; and let us consider the subintervals $I_k := (x_k, x_{k+1})$ for any $k = 0,..,m-1$.

By construction the function $r \mapsto u(r)$ changes sign exactly once in each I_k and so, by point v) in Lemma 6.8, it follows that for n large the function $r \mapsto u_n(r)$ changes sign exactly once in each I_k.

Therefore in the interval $(0,R)$ the function $r \mapsto u_n(r)$ changes sign exactly m times for n large and by Lemma 6.7 it changes sign exactly m times at all. As a consequence $m = k-1$. □

Theorem 1.1 is a direct consequence of Propositions 6.3, 6.5 and 6.9.

Acknowledgments

The author would like to express her sincere gratitude to professor Tobias Weth for bringing to her attention paper [30] as well as for his many helpful advices and fruitful discussions during her stay in the Goethe Universität of Frankfurt-am-Main.

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