LOCAL AND GLOBAL SOLUTIONS FOR SOME PARABOLIC NONLOCAL PROBLEMS

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ABSTRACT. We study local and global existence of solutions for some semilinear parabolic initial boundary value problems with autonomous nonlinearities having a "Newtonian" nonlocal term.

1. INTRODUCTION

We consider the following semilinear parabolic initial-boundary-value problems (IBVP)

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u = F_i(u) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{in } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{on } \Omega, \end{cases} \tag{1}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^3$ and F_i, $i = 1, 2, 3$ is one of the following autonomous nonlinearities

$$F_1(u)(x) := \phi_u(x)u(x) \qquad x \in \Omega, \tag{2}$$

$$F_2(u)(x) := |u(x)|^{q-1}u(x) + \phi_u(x)u(x) \qquad x \in \Omega, \tag{3}$$

$$F_3(u)(x) := |u(x)|^{q-1}u(x) - \phi_u(x)u(x) \qquad x \in \Omega, \tag{4}$$

$q \in (1, 5)$ and ϕ_u is the "Newtonian" nonlocal term:

$$\phi_u(x) := \int_\Omega \frac{1}{|x-y|} u^2(y) dy \qquad x \in \mathbb{R}^3. \tag{5}$$

Elliptic problems with the nonlocal term $\phi_u u$ have been object of many investigations in the last years (Schrödinger-Newton problem, Schrödinger-Poisson-Slater problem). In particular in [5] the author itself studies nodal solutions for the nonlocal elliptic problem corresponding to (1) with nonlinearity (4) via a dynamical approach. From this work the interest in the parabolic problems (1) arises naturally.

Up to our knowledge, this is the first time that parabolic problems with nonlinearities involving the nonlocal term (5) are studied. Different kinds of superlinear nonlocal nonlinearities have been exploited for instance in [8], moreover the semilinear parabolic problem with the power-type nonlinearity has been extensively

Key words and phrases. Nonlocal parabolic problem, Newtonian nonlinearity, local existence, global existence, a priori estimates.

studied (see for instance [1, 2, 4, 7]).

As we will see a key ingredient to handle this nonlocal term is the Hardy-Littlewood-Sobolev inequality (see Lemma 2.1).

This inequality is crucial to prove the local existence of solutions (see the proof of Lemma 3.1). Moreover thanks to the same inequality we are also able to obtain a "polynomial bound" for the nonlinearities which, together with an opportune *a priori* bound for the solutions, eventually leads us to obtain global existence and compactness results.
We anticipate that, concerning the a priori bound for the solutions, the cases of "combined" nonlinearity, namely nonlinearity (3) or (4), are the most delicate to be studied, since the a priori bound we need depends also on the value of q.

Indeed, as we will see, in order to cover different values of q, we need to combine the techniques we previously used for the "pure Newtonian" case (namely nonlinearity (2)) with some more refined arguments, which among other things involve once more the Hardy-Littlewood-Sobolev inequality. Moreover the case of nonlinearity (3) is particularly critical and we obtain a priori bounds only restricting the value of the exponent q to the range $[3, 5)$.

Our first main result is the following local existence and regularity theorem

Theorem 1.1. *Let $p > 3$. For every $u_0 \in W_0^{1,p}(\Omega)$ the IBVP (1) has a unique L^p-solution $u(t) = \varphi(t, u_0)$ with maximal existence time $T := T(u_0) > 0$.*
Moreover

i) *$u \in C^1((0,T), L^p(\Omega)) \cap C((0,T), W^{2,p}(\Omega)) \cap C^{\frac{1-\lambda}{2}}([0,T), W^{\lambda,p}(\Omega))$ for every $\lambda \in [0,1]$;*

ii) *for each $t_1 \in (0, T(u_0))$ the solution u satisfies the integral equation*
$$u(t) = e^{-(t-t_1)A_p}u(t_1) + \int_{t_1}^{t} e^{-(t-s)A_p}F(u(s))ds \qquad t \in [t_1, T(u_0)) \tag{6}$$
where $A_p := -\Delta + Id : W_0^{2,p}(\Omega) \subset L^p(\Omega) \to L^p(\Omega)$;

iii) *the set $\mathcal{G} := \{(t, u_0) \in [0,\infty) \times W_0^{1,p}(\Omega) : t \in [0, T(u_0))\}$ is open in $[0,\infty) \times W_0^{1,p}(\Omega)$, $\varphi : \mathcal{G} \to W_0^{1,p}(\Omega)$ is a semiflow on $W_0^{1,p}(\Omega)$;*

iv) *$u \in C^{\frac{2-\lambda}{2}}((0,T), W^{\lambda,p}(\Omega))$ for any $\lambda \in [1,2)$. Moreover for every $u_0 \in W_0^{1,p}(\Omega)$ and every $t \in (0, T(u_0))$ there is a neighborhood $U \subset W_0^{1,p}(\Omega)$ of u_0 in $W_0^{1,p}(\Omega)$ and a positive constant C such that $T(\tilde{u}_0) > t$ for $\tilde{u}_0 \in U$, and*
$$\|\varphi(t, \tilde{u}_0) - \varphi(t, u_0)\|_{\lambda,p} \leq C\|\tilde{u}_0 - u_0\|_{1,p}$$

v) *u is a classical solution for $t \in (0,T)$.*

We denote by $E_i : W_0^{1,2}(\Omega) \to \mathbb{R}$, $i = 1, 2, 3$ the energy functionals in case of nonlinearity (2), (3) and (4) respectively
$$E_1(u) := \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + u(x)^2)dx - \frac{1}{4}\int_\Omega \phi_u(x)u^2(x)dx$$

$$E_2(u) := \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + u(x)^2)dx - \frac{1}{4}\int_\Omega \phi_u(x)u^2(x)dx - \frac{1}{q+1}\int_\Omega |u(x)|^{q+1}dx$$

$$E_3(u) := \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + u(x)^2)dx + \frac{1}{4}\int_\Omega \phi_u(x)u^2(x)dx - \frac{1}{q+1}\int_\Omega |u(x)|^{q+1}dx.$$

Next results are about global existence and compactness in case of nonlinearity (2), (3) and (4) respectively:

Theorem 1.2. *Let $p > 3$.*
Let $u_0 \in W_0^{1,p}(\Omega)$, $u(t) = \varphi(t, u_0)$ be the solution of (1) and (2) and $T = T(u_0)$. If

$$t \mapsto E_1(u(t)) \quad \textit{is bounded from below on } (0, T)$$

then

- $T = +\infty$
- *for every $\delta > 0$, $\sup_{t\geq\delta} \|u(t)\|_{s,p} < \infty$ for every $s \in [1, 2)$ and the set $\{u(t) : t \geq \delta\}$ is relatively compact in $C^1(\bar{\Omega})$.*

Theorem 1.3. *Let $p > 3$.*
Let $u_0 \in W_0^{1,p}(\Omega)$, $u(t) = \varphi(t, u_0)$ be the solution of (1) and (3) and $T = T(u_0)$. Let $q \in (1, 2^* - 1)$. If

$$t \mapsto E_2(u(t)) \quad \textit{is bounded from below on } (0, T)$$

then the conclusions of Theorem 1.2 are true.

Theorem 1.4. *Let $p > 3$.*
Let $u_0 \in W_0^{1,p}(\Omega)$, $u(t) = \varphi(t, u_0)$ be the solution of (1) and (4) and $T = T(u_0)$. Let $q \in [3, 2^* - 1)$. If

$$t \mapsto E_3(u(t)) \quad \textit{is bounded from below on } (0, T)$$

then the conclusions of Theorem 1.2 are true.

The paper is organized as follows.
Section 2 is for notations and preliminaries, in particular we recall the Hardy-Littlewood-Sobolev inequality involving the nonlocal term (see Lemma 2.1).
Section 3 is related to the proof of the local existence result.
The main section of this paper is Section 4. Here we prove polynomial bounds for the nonlinearity F_i (Lemma 4.1) and a priori bounds for the solutions (Proposition 4.2).
Last Section 5 contains the proof of Theorem 1.2, Theorem 1.3 and Theorem 1.4.

2. Notations and preliminaries

Let us fix some notations.
$L^r(\Omega)$, $W^{s,r}(\Omega)$, $W_0^{s,r}(\Omega)$ are the usual Lebesgue and the Sobolev or Sobolev-Slobodeckii spaces.
$C^{k+\alpha}(\bar{\Omega})$, with $k \in \mathbb{N}$ and $\alpha \in (0,1)$ is the Banach space of all the functions belonging to $C^k(\bar{\Omega})$ whose k-th order partial derivatives are uniformly α-Hölder continuous in $\bar{\Omega}$.
We denote by $\|\cdot\|_r, \|\cdot\|_{s,r}, \|\cdot\|_{C^{k+\alpha}}$ the usual norms in $L^r(\Omega)$, $W^{s,r}(\Omega), C^{k+\alpha}(\bar{\Omega})$ respectively.
Let p' denote the dual exponent to $p \in (1,\infty)$, namely $\frac{1}{p}+\frac{1}{p'}=1$.
By C we denote various positive constants which may vary from step to step.

We shall use the following interpolation embeddings

$$W^{1,2}(J,L^2(\Omega)) \cap L^{\beta(q+1)}(J,L^{q+1}(\Omega)) \hookrightarrow L^\infty(J,L^a(\Omega)), \tag{7}$$

$$W^{1,2}(J,L^2(\Omega)) \cap L^4(J,W^{1,2}(\Omega)) \hookrightarrow L^\infty(J,L^\rho(\Omega)), \tag{8}$$

where J is a compact interval, $a \in (1, q+1-\frac{q-1}{\beta+1})$ and $\rho \in (1,\frac{18}{5})$ (for the proof see [4, Appendix] or [8]).

We also recall that the maximal Sobolev regularity property holds for problem (1) (see [3, p. 188] or [9, Appendix E, p. 470]). Hence, given a compact interval J and $F_i \in L^a(J,L^b(\Omega))$, $1<a,b<\infty$, the solution u of (1) satisfies

$$\|u\|_{W^{1,a}(J,L^b(\Omega))} + \|u\|_{L^a(J,W^{2,b}(\Omega))} \le C\left(\|u_0\|_{W^{s,b}(\Omega)} + \|F_i\|_{L^a(J,L^b(\Omega))}\right), \tag{9}$$

provided $s > 2(1-1/a)$.

The following lemma will play a crucial role in the proof of the a priori estimates in Section 4

Lemma 2.1. *If $u \in L^{2m}(\Omega)$ then ϕ_u is well-defined and belongs to $L^r(\mathbb{R}^3)$ with*

$$\|\phi_u\|_{L^r(\mathbb{R}^3)} \le C\|u\|_{2m}^2, \quad \frac{1}{m}+\frac{1}{3}=1+\frac{1}{r}, \quad 1<m,r<\infty. \tag{10}$$

Proof. Let $\tilde{u}$ the trivial extension (to 0) of u in $\mathbb{R}^3$, then $\tilde{u} \in L^{2m}(\mathbb{R}^3)$ and by Hardy-Littlewood-Sobolev inequality (see [6, Theorem 4.3]) one has the following

$$\left\|\left(\frac{1}{|x|} * \tilde{u}^2\right)\right\|_{L^r(\mathbb{R}^3)} \le C\|\tilde{u}\|_{L^{2m}(\mathbb{R}^3)}^2, \quad \frac{1}{m}+\frac{1}{3}=1+\frac{1}{r}, \quad 1<m,r<\infty.$$

The conclusion comes observing that $\|\tilde{u}\|_{L^{2m}(\mathbb{R}^3)} = \|u\|_{2m}$ and that

$$\left(\frac{1}{|x|} * \tilde{u}^2\right)(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|}\tilde{u}^2(y)dy = \int_\Omega \frac{1}{|x-y|}u^2(y)dy = \phi_u(x).$$

□

3. Proof of Theorem 1.1

Lemma 3.1. *Let $p > 3$ and $\tau > \frac{3}{p}$ then $F_i : W^{\tau,p}(\Omega) \to L^p(\Omega)$, $i = 1, 2, 3$ is locally Lipschitz continuous.*

Proof. That $W^{\tau,p}(\Omega) \ni u \mapsto |u|^{q-1}u \in L^p(\Omega)$ is well defined and locally Lipschitz can be found in [1, Proposition 15.4].
Here we show that $W^{\tau,p}(\Omega) \ni u \mapsto \phi_u u$ is well defined and locally Lipschitz continuous.

Let $u \in W^{\tau,p}(\Omega)$, first we prove that $\phi_u u \in L^p(\Omega)$. Indeed from the continuous embedding $W^{\tau,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ (since $\tau > \frac{3}{p}$) and using (10) with $m := \frac{3p}{2p+3}$ (indeed $m > 1$ since $p > 3$) we get

$$\begin{aligned}
\|\phi_u u\|_p &\leq \|u\|_{C^0}\|\phi_u\|_p \\
&\leq C\|u\|_{\tau,p}\|\phi_u\|_p \\
&\leq C\|u\|_{\tau,p}\|u\|_{2m}^2 \\
&\leq C\|u\|_{\tau,p}\|u\|_{C^0}^2 \\
&\leq C\|u\|_{\tau,p}^3 < \infty.
\end{aligned}$$

Next, using again inequality (10) with $m := \frac{3p}{2p+3}$ and the continuous embedding $W^{\tau,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we prove the locally Lipschitz continuity. Indeed letting $u_i \in W^{\tau,p}(\Omega)$, $i = 1, 2$ $\|u_i\|_{\tau,p} \leq R$, then

$$\begin{aligned}
\|u_1\phi_{u_1} - u_2\phi_{u_2}\|_p &\leq \|\phi_{u_1}(u_1 - u_2)\|_p + \|u_2(\phi_{u_1} - \phi_{u_2})\|_p \\
&\leq \|u_1 - u_2\|_{C^0}\|\phi_{u_1}\|_p + \|u_2\|_{C^0}\|\phi_{u_1} - \phi_{u_2}\|_p \\
&\leq \|u_1 - u_2\|_{\tau,p}\|\phi_{u_1}\|_p + \|u_2\|_{\tau,p}\|\phi_w\|_p \qquad (\text{where } w^2 := |u_1^2 - u_2^2|) \\
&\leq C\|u_1 - u_2\|_{\tau,p}\|u_1\|_{2m}^2 + C\|u_2\|_{\tau,p}\|u_1^2 - u_2^2\|_m \qquad (\text{by } (10)) \\
&\leq C\|u_1 - u_2\|_{\tau,p}\|u_1\|_{C^0}^2 + C\|u_2\|_{\tau,p}\|(u_1 - u_2)(u_1 + u_2)\|_m \\
&\leq C\|u_1 - u_2\|_{\tau,p}\|u_1\|_{\tau,p}^2 + C\|u_2\|_{\tau,p}(\|u_1\|_{C^0} + \|u_2\|_{C^0})\|u_1 - u_2\|_{C^0} \\
&\leq C\|u_1 - u_2\|_{\tau,p}.
\end{aligned}$$

□

Next we prove Theorem 1.1.

Proof of Theorem 1.1. The first part of the proof can be derived from abstract results of Amann concerning local existence and regularity for semilinear parabolic IBVP (see [1]).
Following Amann's notation we set $\mathscr{A}(t) \equiv \mathscr{A} = -\Delta + Id$ and $\mathscr{B}(t) \equiv \mathscr{B} = Id$.
Since $p > 3$, we can choose $\tau \in (\frac{3}{p}, 1)$ in Lemma 3.1. As a consequence the hypothesis "$\mathscr{H}(p, s, \tau, \sigma, l)$" in [1] with values $s = 0$, $\sigma = 1$ and $l \geq 2$ is satisfied. Hence the local existence, the integral equation (6) and the semiflow properties of φ follow directly applying for each fixed $T > 0$ Theorem 15.1 and Corollary 15.2 in [1] (we point out that $\mathscr{A}$, $\mathscr{B}$ and F_i are not depending on t).
The further regularity as well as the continuous dependence property in stronger norm in point iv) can be derived from Theorem 51.7-Example 51.4 in the Appendix E of the book [9] (see in particular Remark 51.8 (iii)).
Last to prove that the solution is classical one can adapt the arguments in the Example 51.9 of [9], we give here only a brief sketch of it and we refer to [9] for further details. Basically one considers the parabolic problem as a linear problem

with Hölder continuous right-hand side and applies parabolic Schauder estimates. It's not difficult to show the Hölder continuity of the RHS once one knows that $u \in C^{\rho}((0,T), C^{1+\rho}(\bar{\Omega}))$ for a certain $\rho \in (0,1)$. And this last result follows from point iv) choosing $\lambda := 2 - 2\rho$, for $\rho \in (0,1)$ and sufficiently small in order to have the embedding $W^{\lambda,p}(\Omega) \hookrightarrow C^{1+\rho}(\bar{\Omega})$. □

Remark 3.2. *We point out that the local existence results in Theorem 1.1 hold actually for every $q \geq 1$ and not only for $q \in (1,5)$.*

4. A priori estimates for solutions

Throughout this section $p > 3$. Moreover we consider a fixed $u_0 \in W_0^{1,p}(\Omega)$ and, for $i = 1,2,3$ we let $u_i(t) = \varphi_i(t, u_0)$ be the solution of (1) with the nonlinearity F_i defined respectively in (2), (3) and (4) and $T = T(u_0)$ be its maximal existence time. If no confusion seems likely, then we may shortly write u instead of u_i and also F, E instead of F_i, E_i.

Next Lemma gives a polynomial bound for the nonlinearity F_i and it is proved using Lemma 2.1.

Lemma 4.1. *Let $1 < r < \infty$. Then*

$$\|F_1(v)\|_r \leq C \||v|^3\|_r \quad \forall\ v \in L^{3r}(\Omega); \tag{11}$$

moreover for $i = 2,3$

$$\|F_i(v)\|_r \leq C\,(1 + \||v|^{\kappa}\|_r) \quad \forall v \in L^{\kappa r}(\Omega) \tag{12}$$

where $\kappa := \max\{q, 3\}$.

Proof. First we prove (11). We fix $\alpha > \max\{\frac{3}{2}, \frac{3}{r}\}$ and we define $m = \frac{3r\alpha}{2r\alpha+3}$ ($m > 1$ since $\alpha > \frac{3}{r}$). From Hölder inequality, Lemma 2.1 and Sobolev embeddings we obtain

$$\begin{aligned}
\|F_1(v)\|_r &= \|v(|x|^{-1} * v^2)\|_r \\
&\overset{\text{Hölder}}{\leq} C\|v\|_{r\alpha'}\ \|(|x|^{-1} * v^2)\|_{r\alpha} \\
&\overset{\text{Lemma 2.1}}{\leq} C\|v\|_{r\alpha'}\ \|v\|_{2m}^2 \\
&\overset{\text{Sobolev emb.}}{\leq} C\|v\|_{3r}\ \|v\|_3^2 \\
&\overset{\text{Sobolev emb.}}{\leq} C\|v\|_{3r}^3.
\end{aligned}$$

where we have used the fact that $2m < 3$ and $\alpha' < 3$ (since by definition $\alpha > \frac{3}{2}$). Inequality (12) follows in a similar way, indeed for $i = 2, 3$ one has

$$
\begin{aligned}
\|F_i(v)\|_r &\leq \||v|^q\|_r + \|v(|x|^{-1} * v^2)\|_r \\
&\overset{\text{Hölder}}{\leq} \||v|^q\|_r + C\|v\|_{r\alpha'} \; \|(|x|^{-1} * v^2)\|_{r\alpha} \\
&\overset{\text{Lemma 2.1}}{\leq} \|v\|_{qr}^q + C\|v\|_{r\alpha'} \; \|v\|_{2m}^2 \\
&\overset{\text{Sobolev emb.}}{\leq} \|v\|_{\kappa r}^q + C\|v\|_{\kappa r} \; \|v\|_{\kappa}^2 \\
&\overset{\text{Sobolev emb.}}{\leq} \|v\|_{\kappa r}^q + C\|v\|_{\kappa r}^3 \\
&\leq \tilde{C}(1 + \|v\|_{\kappa r}^{\kappa})
\end{aligned} \tag{13}
$$

where we have used the fact that $\alpha' < \kappa$ (indeed $\alpha' < 3$) and also that $2m < \kappa$ (since $2m < 3$). □

Next result is an a priori bound for u_i, $i = 1, 2, 3$ in a proper L^a-norm under the additional condition that the energy functional E_i stays bounded from below along the trajectory.

Proposition 4.2. *Let $i = 1, 2$ or 3 and assume that $t \mapsto E_i(u_i(t))$ is bounded from below on $(0, T)$. Moreover let $q \in (1, 2^* - 1)$ when $i = 2$ and $q \in [3, 2^* - 1)$ when $i = 3$. Then*

$$\sup_{(0,T)} \|u_i(t)\|_a < \infty, \qquad \textit{for all} \;\; a < \frac{18}{5}. \tag{14}$$

Moreover for $i = 2, 3$ and $q \geq 3$ if $i = 2$, $q > 3$ if $i = 3$, one has for every $\delta > 0$

$$\sup_{[\delta,T)} \|u_i(t)\|_a < \infty, \qquad \textit{for all} \;\; a < q + 1. \tag{15}$$

Remark 4.3. (i) *A priori estimates for solutions of parabolic equations with a power-type nonlinearity have been proved among others by* [4, 7]*. Moreover we refer to* [8] *for a priori bounds relating to solutions of more general superlinear parabolic problems, subcritical where also some nonlocal problem has been studied. Anyway as far as we know Proposition 4.2 is the first result in this direction for nonlinearities that involve a Newtonian nonlocal term* (5)*.*

We point out that differently with [4, 7] *here we are not assuming that the solution is a priori global (indeed in next section we will need estimates* (14) *and* (15) *exactly to prove that the solutions are global).*

As a consequence we need here to impose the additional condition on the lower bound of the energy along the trajectory (a condition that is instead obtained for free in [4, 7] *by a blow-up argument for global solutions).*

For this reason the bound for $\|u_i(t)\|_a$ we obtain in Proposition 4.2 can not depend simply on the norm of the initial condition (and on δ of course) like in [4] *or* [7] *, but its dependence on the initial condition u_0 must be more complicated. Hence in our proof we fix u_0 and we do not investigate the way the bound depends on its norm.*

We underline that in [8] *no condition on the lower bound of the energy is imposed and the solutions are not assumed to be a priori global. Anyway a blow-up argument gives in this case a lower bound on the energy only for* $t < T - \lambda$ *where* $\lambda > 0$ *and so the a priori bounds obtained in* [8] *are of the form* $\|u(t)\|_a < C_\lambda$ *for* $t < T - \lambda$.

Last we remark that the assumption we add on the lower bound of the energy is totally reasonable, see for instance [5] *where the trajectories under consideration lie in sets on which the energy stays strictly positive.*

(ii) *Observe that when* $i = 3$ *no a priori bound is obtained for* $q < 3$, *for this reason in Theorem 1.4 we will restrict to the case* $q \geq 3$.

Moreover for both the "combined" nonlinearity, namely when $i = 2, 3$, *the bound* (15) *holds only when* q *is big enough* ($q \geq 3$ *and* $q > 3$ *respectively), while for the other values of* q *we only have the weaker bound* (14) *(same bound as in the "pure Newtonian" case). Anyway, as we will see in Section 5, in order to obtain global existence the bound* (14) *is sufficient when considering small* q *(precisely* $q < 17/5$*), while only for bigger values of* q *we need a stronger bound like* (15). *Hence, combining both the results we are eventually able to cover all the different values of* q *considered in Theorem 1.3 and 1.4 respectively.*

(iii) *The proof is divided into two main parts. The first part, concerning the case of a priori global solutions* ($T = \infty$) *follows quite immediately from a monotonicity property of the energy functional along the flow. The second part, concerning the case* $T < \infty$, *needs a more careful analysis.*

Precisely the proof of (14) *is mainly inspired to arguments in* [4, 7].

To prove the stronger bound (15) *we need a bootstrap argument involving the maximal Sobolev regularity property in the spirit of* [7, 8]. *Besides due to the presence of the nonlocal Newtonian term we will need once more the Hardy-Littlewood-Sobolev inequality (see the proof of Lemma 4.6).*

In order to prove Proposition 4.2 we need the following lemma

Lemma 4.4. *Let* $i = 1, 2, 3$. *The function* $t \mapsto E_i(u_i(t))$ *is continuous in* $[0, T)$ *and* C^1 *in* $(0, T)$, *moreover*

$$\frac{d}{dt} E_i(u_i(t)) = -\|u_{it}(t)\|_2^2 \quad \text{for } t \in (0, T), \tag{16}$$

$$E_i(u_i(t)) \leq E_i(u_0) \quad \text{for } t \in [0, T). \tag{17}$$

Furthermore

$$\|u_i(t)\|_{1,2}^2 \leq C(1 + \|u_i(t) u_{it}(t)\|_1) \quad \text{for } t \in (0, T), \tag{18}$$

where when $i = 3$ *we are also assuming that* $q \geq 3$.
Moreover, for $i = 2, 3$

$$\|u_i(t)\|_{q+1}^{q+1} \leq C(1 + \|u_i(t) u_{it}(t)\|_1) \quad \text{for } t \in (0, T), \tag{19}$$

where when $i = 3$ *we are also assuming that* $q > 3$.

Proof. The continuity and the differentiability of $t \mapsto E_i(u_i(t))$ follow from the embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ and the fact $u_i \in C^0([0, T), W_0^{1,p}(\Omega))$ and also that

u_i is a classical solution on $(0,T)$. The (16) follows differentiating and integrating by parts, for instance in case $i=3$ we get

$$\begin{aligned}
\dot{E}_3 &= \frac{d}{dt}E_3(u_3) = \frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\nabla u_3|^2 + \frac{1}{2}|u_3|^2 - \frac{1}{q+1}|u_3|^{q+1} + \frac{1}{4}\phi_{u_3}u_3^2\right) dx \\
&= \int_\Omega \left(\nabla u_3 \nabla u_{3t} + u u_t - |u_3|^{q-1}u_3 u_{3t} + \phi_{u_3}u_3 u_{3t}\right) dx \\
&= \int_\Omega \left(-\Delta u_3 + u_3 - |u_3|^{q-1}u_3 + \phi_{u_3}u_3\right) u_{3t} dx \\
&= -\int_\Omega u_{3t}^2 dx = -\|u_{3t}\|_2^2.
\end{aligned}$$

When $i=1,2$ the proof is similar.

To prove (17) just observe that $t \mapsto E_i(u_i(t))$ is decreasing for $t>0$ (by (16)) and continuous in $t=0$.

Next we prove (18) in the three cases $i=1,2,3$.

Multiplying the equation $u_{1t} = \Delta u_1 - u_1 + \phi_{u_1}u_1$ by u_1 and integrating by parts we obtain

$$\|u_1(t)\|_{1,2}^2 \leq \|u_1(t)u_{1t}(t)\|_1 + 4E(u_1(t)),$$

hence the conclusion in case $i=1$ follows from (17).

Similarly, multiplying the equation $u_{2t} = \Delta u_2 - u_2 + |u_2|^{q-1}u_2 + \phi_{u_2}u_2$ by u_2 and integrating by parts we obtain for any constant $K>2$

$$\begin{aligned}
\|u_2(t)\|_{1,2}^2 \leq\ & \frac{2}{K-2}\|u_2(t)u_{2t}(t)\|_1 + \frac{2K}{K-2}E(u_2(t)) - \frac{2(q+1-K)}{(K-2)(q+1)}\|u_2(t)\|_{q+1}^{q+1} \\
& -\frac{4-K}{2(K-2)}\int_\Omega \phi_{u_2}(t)u_2^2(t),
\end{aligned}$$

hence taking $K := \min\{4, q+1\}$, since $\phi_{u_2} \geq 0$ by definition, we have

$$\|u_2(t)\|_{1,2}^2 \leq \frac{2}{K-2}\|u_2(t)u_{2t}(t)\|_1 + \frac{2K}{K-2}E(u_2(t))$$

and the conclusion in case $i=2$ follows again from (17).

Last we consider the case $i=3$. In this case, multiplying the equation $u_{3t} = \Delta u_3 - u_3 + |u_3|^{q-1}u_3 - \phi_{u_3}u_3$ by u_3 and integrating by parts we obtain for any constant $K>2$

$$\begin{aligned}
\|u_3(t)\|_{1,2}^2 \leq\ & \frac{2}{K-2}\|u_3(t)u_{3t}(t)\|_1 + \frac{2K}{K-2}E(u_3(t)) - \frac{2(q+1-K)}{(K-2)(q+1)}\|u_3(t)\|_{q+1}^{q+1} \\
& +\frac{4-K}{2(K-2)}\int_\Omega \phi_{u_3}(t)u_3^2(t), \qquad (20)
\end{aligned}$$

hence, taking any $K \in [4, q+1]$ ($q \geq 3$ by assumption) we get

$$\|u_3(t)\|_{1,2}^2 \leq \frac{2}{K-2}\|u_3(t)u_{3t}(t)\|_1 + \frac{2K}{K-2}E(u_3(t)),$$

and so the conclusion comes from (17).

The proof of (19) follows in a similar way.

For $i = 2$ we multiply the equation $u_{2t} = \Delta u_2 - u_2 + |u_2|^{q-1}u_2 + \phi_{u_2}u_2$ by u_2, integrate by parts and use (18) obtaining directly

$$\begin{aligned} \|u_2(t)\|_{q+1}^{q+1} &\leq \|u_2(t)u_{2t}(t)\|_1 + \|u_2(t)\|_{1,2}^2 - \int_\Omega \phi_{u_2}(t)u_2^2(t) \\ &\leq \|u_2(t)u_{2t}(t)\|_1 + \|u_2(t)\|_{1,2}^2 \\ &\leq C(1 + \|u_2(t)u_{2t}(t)\|_1). \end{aligned}$$

The case $i = 3$ follows from (20) taking any $K \in [4, q+1)$ (indeed $q > 3$ by assumption) and using once more (17). □

Proof of Proposition 4.2. Case $T = \infty$.

The C^1 function $(0, +\infty) \ni t \mapsto E_i(u_i(t))$ is decreasing because of (16) and bounded from below by assumption, hence $E_i(u_i(t)) \searrow c \in \mathbb{R}$ and $\frac{d}{dt}E_i(u_i(t)) \to 0$ as $t \to +\infty$.
Combining this with (16) and (18), we get from Hölder inequality

$$\|u_i(t)\|_{1,2}^2 \leq C(1 + \|u_i(t)\|_2\|u_{it}(t)\|_2) \leq C(1 + \|u_i(t)\|_{1,2}), \text{ for } t \in (0, +\infty)$$

namely

$$\|u_i(t)\|_{1,2} \leq C \quad \forall t \in [0, +\infty),$$

(we can consider $t = 0$ because $u_i \in C^0([0, T), W_0^{1,2}(\Omega)))$ which implies, by Sobolev embedding

$$\|u_i(t)\|_{2^*} \leq C \quad \forall t \in [0, +\infty).$$

and hence (14) and (15) for $i = 1, 2, 3$ respectively. □

From now on we consider $T < \infty$.

Lemma 4.5. *Assume that $t \mapsto E_i(u_i(t))$ is bounded from below on $(0, T)$. Let also $T < \infty$. Then*

$$\sup_{t \geq 0} \|u_i(t)\|_2 < \infty \tag{21}$$

$$\int_0^T \|u_{it}(t)\|_2^2 dt \leq C \tag{22}$$

Proof. We put $E_{inf} := \inf_{t \geq 0} E_i(u_i(t)) > -\infty$. The following argument is similar to the one used in [10, p. 89]. Let $h(t) := \|u_i(t)\|_2^2$. By Hölder inequality and (16) we have for $t \in (0, T)$

$$\frac{d}{dt}h = 2\int_\Omega u_i u_{it} dx \leq 2\|u_i\|_2\|u_{it}\|_2 \leq \|u_i\|_2^2 + \|u_{it}\|_2^2 = h - \dot{E}_i$$

and so

$$\frac{d}{dt}(e^{-t}h(t)) = e^{-t}[\dot{h} - h] \leq -e^{-t}\dot{E}_i \leq -\dot{E}_i.$$

Integrating in $0 < \delta < t$ we have

$$\begin{aligned} e^{-t}h(t) &= e^{-\delta}h(\delta) + \int_\delta^t \frac{d}{ds}(e^{-s}h(s))ds \le e^{-\delta}h(\delta) + \int_\delta^t -\dot{E}_i(u_i(s))ds \\ &= e^{-\delta}h(\delta) - E_i(u_i(t)) + E_i(u_i(\delta)) \le e^{-\delta}h(\delta) - E_{inf} + E_i(u_i(\delta)) \end{aligned}$$

Passing to the limit for $\delta \to 0^+$ we obtain

$$e^{-t}h(t) \le h(0) - E_{inf} + E_i(u_0) := C,$$

namely

$$h(t) \le e^t C \le e^T C := C_T \text{ for } t \in [0, T),$$

which proves (21). The bound in (22) follows immediately from (16):

$$\int_0^T \|u_{it}(t)\|_2^2 dt = \int_0^T (-\dot{E}_i(t))dt = E_i(u_0) - \lim_{t \to T} E_i(t) \le E_i(u_0) - E_{inf} = C(> 0).$$

□

Lemma 4.6. *Let $i = 2$ or 3 and $q \ge 3$ if $i = 2$, $q > 3$ if $i = 3$. Assume also that $t \mapsto E_i(u_i(t))$ is bounded from below on $(0, T)$ and that $T < \infty$. Let $\beta \ge 2$ and $\delta > 0$. Then*

$$\int_\delta^T \|u_i(t)\|_{q+1}^{\beta(q+1)} dt < \infty \tag{23}$$

and

$$\sup_{[\delta, T)} \|u_i(t)\|_a < \infty, \qquad \text{for all } a < a(\beta), \tag{24}$$

where

$$a(\beta) = q + 1 - \frac{q-1}{\beta+1}.$$

Proof. We divide the proof in the following steps:

STEP 1 We show that (23) implies (24).

STEP 2 We prove that (23) is true for $\beta = 2$ and $\delta = 0$.

STEP 3 We prove the validity of (23) for any $\beta > 2$ (and any $\delta > 0$) through a bootstrap argument.

STEP 1. Putting together (22) and (23) we have

$$\int_\delta^T \left(\|u_{it}(t)\|_2^2 + \|u_i(t)\|_{q+1}^{\beta(q+1)} \right) dt < \infty, \tag{25}$$

and so (24) is a consequence of the interpolation embedding (7).

STEP 2. We prove that (23) holds with $\beta = 2$ and $\delta = 0$. From (19) (indeed we are assuming $q > 3$ when $i = 3$) and using Hölder inequality we have

$$\|u_i(t)\|_{q+1}^{q+1} \le C(1 + \|u_i(t)u_{it}(t)\|_1) \le C(1 + \|u_i(t)\|_2 \|u_{it}\|_2) \quad \forall t > 0,$$

so, raising it to the power two and using (21) it follows

$$\|u_i(t)\|_{q+1}^{2(q+1)} \le C(1 + \|u_i(t)\|_2^2 \|u_{it}\|_2^2) \le C(1 + \|u_{it}(t)\|_2^2) \quad \forall t > 0.$$

Integrating, using (22) and the assumption $T<\infty$ we get the conclusion

$$\int_0^T \|u_i(t)\|_{q+1}^{2(q+1)}dt \leq C\int_0^T (1+\|u_{it}(t)\|_2^2)dt<\infty.$$

STEP 3. We want to prove that the validity of (23) for some $\beta\geq 2$ implies (23) for some $\tilde{\beta}>\beta$. Moreover we want the difference $\tilde{\beta}-\beta$ to be bounded below by a positive uniform constant for all β so that, performing a bootstrap procedure, after finitely many steps, we end up with some $\tilde{\beta}$ big enough.

First observe that from inequality (12) taking $r:=\frac{q+1}{q}$ it follows

$$\|F_i(u_i(t))\|_{\frac{q+1}{q}} \leq C\left(1+\|\,|u_i(t)|^{\kappa}\,\|_{\frac{q+1}{q}}\right) = C\left(1+\|u_i(t)\|_{q+1}^{\frac{q}{q+1}(q+1)}\right), \tag{26}$$

where $\kappa:=\max\{q,3\}=q$, since we assumed $q\geq 3$.

Hence the proof follows applying to our case the abstract bootstrap Lemma 2.2-(i) in [8]. Indeed, our problem possesses the maximal Sobolev regularity property (see (9)) and so, taking $\mathcal{G}(u):=C(1+\|u\|_{q+1}^{q+1})$ for a suitable positive constant C, one can show that the inequality (26) above, together with (19), (23) and (22), implies that $\mathcal{G}(u_i(t))$ satisfies all the assumptions required in [8].
Anyway for completeness we briefly repeat here the main points of the proof adapting the abstract setting of [8] to our case.

Hence, fix any $\delta>0$ and let (23) be true for some $\beta\geq 2$. Then by STEP 1 we know that (24) is true for all $a<a(\beta)$ (notice that $2<a(\beta)<q+1$). Choose $a>2$ and denote

$$\theta=\frac{q+1}{q-1}\frac{a-2}{a},$$

then $\theta\in(0,1)$ and using (19), Hölder inequality, (24) and the interpolation inequality , we obtain

$$\begin{aligned}\|u_i(t)\|_{q+1}^{q+1} &\leq C(1+\|u_i(t)u_{it}(t)\|_1)\leq C(1+\|u_i(t)\|_a\|u_{it}(t)\|_{a'})\\ &\leq C(1+\|u_{it}(t)\|_{a'})\leq C(1+\|u_{it}(t)\|_{\frac{q+1}{q}}^{\theta}\|u_{it}(t)\|_2^{1-\theta}) \quad \text{for all } t>0.\end{aligned} \tag{27}$$

We now fix any $h\in(0,2)$ and take $\tilde{\beta}=\beta+h$. Than it is easy to show that for a is sufficiently close to $a(\beta)$ one has

$$s:=\frac{2}{(1-\theta)\tilde{\beta}}>1. \tag{28}$$

Notice also that $\theta\tilde{\beta}s'>1$. We raise (27) to the power $\tilde{\beta}$ and integrate it from δ to T, use Hölder inequality with exponents s and s', (22), the maximal Sobolev regularity (9) to get

$$\begin{aligned}
\int_\delta^T \|u_i(t)\|_{q+1}^{\tilde\beta(q+1)}dt &\le C\left(1+\int_\delta^T \|u_{it}(t)\|_{\frac{q+1}{q}}^{\theta\tilde\beta}\|u_{it}(t)\|_2^{(1-\theta)\tilde\beta}dt\right)\\
&\overset{\text{(Hölder ineq.)}}{\le} C\left[1+\left(\int_\delta^T \|u_{it}(t)\|_{\frac{q+1}{q}}^{\theta\tilde\beta s'}dt\right)^{\frac{1}{s'}}\left(\int_\delta^T \|u_{it}(t)\|_2^{(1-\theta)\tilde\beta s}dt\right)^{\frac{1}{s}}\right]\\
&= C\left[1+\left(\int_\delta^T \|u_{it}(t)\|_{\frac{q+1}{q}}^{\theta\tilde\beta s'}dt\right)^{\frac{1}{s'}}\left(\int_\delta^T \|u_{it}(t)\|_2^{2}dt\right)^{\frac{1}{s}}\right]\\
&\overset{(22)}{\le} C\left[1+\left(\int_\delta^T \|u_{it}(t)\|_{\frac{q+1}{q}}^{\theta\tilde\beta s'}dt\right)^{\frac{1}{s'}}\right]\\
&\overset{(9)}{\le} C\left[1+\left(\int_\delta^T \|F_i(u_i(t))\|_{\frac{q+1}{q}}^{\theta\tilde\beta s'}dt+\|u_i(\delta)\|_{2,\frac{q+1}{q}}^{\theta\tilde\beta s'}\right)^{\frac{1}{s'}}\right]\\
&\le C\left[1+\left(\int_\delta^T \|F_i(u_i(t))\|_{\frac{q+1}{q}}^{\theta\tilde\beta s'}dt\right)^{\frac{1}{s'}}\right]. \qquad (29)
\end{aligned}$$

Last substituting (26) into (29) we get

$$\int_\delta^T \|u_i(t)\|_{q+1}^{\tilde\beta(q+1)}dt \le C\left[1+\left(\int_\delta^T \|u_i(t)\|_{q+1}^{\frac{q\theta\tilde\beta s'}{q+1}(q+1)}dt\right)^{\frac{1}{s'}}\right],$$

which implies the bound (23) with β replaced by $\tilde\beta$, since by our choice of h and for a sufficiently close to $a(\beta)$ one can show that

$$\theta s' \le \frac{q+1}{q}. \qquad (30)$$

□

We are now in the position to conclude the proof of Proposition 4.2.

Proof of Proposition 4.2. Case $T<\infty$.

First we prove the bound (14). From (18) (indeed when $i=3$ we are assuming $q\ge 3$) and using Hölder inequality we have

$$\|u_i(t)\|_{1,2}^2 \le C(1+\|u_i(t)u_{it}(t)\|_1) \le C(1+\|u_i(t)\|_2\|u_{it}\|_2) \quad \forall t>0,$$

so, raising it to the power two and using (21) it follows

$$\|u_i(t)\|_{1,2}^4 \le C(1+\|u_i(t)\|_2^2\|u_{it}\|_2^2) \le C(1+\|u_{it}(t)\|_2^2) \quad \forall t>0.$$

Integrating, using (22) and the assumption $T<\infty$ we get

$$\int_0^T \|u_i(t)\|_{1,2}^4 dt \le C\int_0^T (1+\|u_{it}(t)\|_2^2)dt < \infty. \qquad (31)$$

Putting together (22) and (31) we have

$$\int_0^T \left(\|u_{it}(t)\|_2^2 + \|u_i(t)\|_{1,2}^4\right) dt < \infty, \tag{32}$$

and so from the interpolation embedding (8) we obtain the estimate (14):

$$\sup_{(0,T)} \|u_i(t)\|_a < \infty \qquad \text{for all } a < \frac{18}{5}.$$

We now prove (15). Hence we restrict to the cases $i = 2, 3$ and assume $q \geq 3$ when $i = 2$ and $q > 3$ when $i = 3$. The estimate (15) follows then from Lemma 4.6, taking β big enough in (24), since $a(\beta) \overset{\beta \to +\infty}{\longrightarrow} q + 1$. $\square$

5. Proof of theorems 1.2, 1.3 and 1.4

The proof uses Amann's abstract results for global existence and relative compactness (see [2]). Precisely we can apply to our case [2, Theorem 5.1] combined together with [2, Proposition 3.4].

In view of these general results, we have global solutions and boundedness in $W^{s,p}$-norm for every $s \in [1, 2)$, provided that for every $\delta > 0$ the following "polynomial bound" for $F_i(u_i(t))$ and a priori bound for $\|u_i(t)\|_{p_0}$ hold for certain $1 \leq \gamma_0 < 1 + \frac{2}{3}p_0$:

$$\|F_i(u_i(t))\|_p \leq C\,(1 + \||u_i(t)|^{\gamma_0}\|_p) \qquad t \in [\delta, T), \tag{33}$$

$$\sup_{[\delta,T)} \|u_i(t)\|_{p_0} < \infty. \tag{34}$$

When $i = 1$ from Lemma 4.1-(11) and Proposition 4.2-(14) we know that (33) and (34) are satisfied taking $\gamma_0 = 3$, and any $p_0 \in (3, \frac{18}{5})$.

Next we consider $i = 2, 3$. In this case Lemma 4.1-(12) gives (33) with $\gamma_0 = \max\{3, q\}$, hence it's enough to prove (34) for any $p_0 > \max\{3, \frac{3}{2}(q-1)\}$.

If $q < \frac{17}{5}$, namely if $\max\{3, \frac{3}{2}(q-1)\} < \frac{18}{5}$, then the conclusion follows directly from Proposition 4.2-(14).

Precisely the estimate (14), for any fixed $q \in (1, \frac{17}{5})$ in case $i = 2$ or $q \in [3, \frac{17}{5})$ in case $i = 3$, implies (34) for any $p_0 \in (\max\{3, \frac{3}{2}(q-1)\}, \frac{18}{5})$.

In order to consider bigger values of q we need to use Proposition 4.2- (15).

Indeed for any fixed $q \in [\frac{17}{5}, 2^* - 1)$ estimate (15) gives (34) for any $p_0 \in (\frac{3}{2}(q-1), q+1)$, where $\frac{3}{2}(q-1) < q + 1$ since $q < 2^* - 1$ by assumption.

It remains to prove the relative compactness property. From the boundedness of the $W^{s,p}$-norm and the compactness of the embedding $W^{x,p}(\Omega) \hookrightarrow W^{y,p}(\Omega)$ for $y < x$ it follows that the set $\{u_i(t) : t \geq \delta\}$ is relatively compact in $W^{s,p}(\Omega)$ for every

$s \in [1, 2)$. Hence the conclusion follows from the embedding $W^{s,p}(\Omega) \hookrightarrow C^1(\bar{\Omega})$ choosing s sufficiently close to 2.

Acknowledgments

The author would like to thank professor Tobias Weth for many enriching discussions and useful suggestions.

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